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A family of finite p-groups satisfying Carlson's depth conjecture

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Abstract

Let $p > 3$ be a prime number and let r be an integer with $1 < r < p-1$. For each r, let moreover G_r denote the unique quotient of the maximal class pro-p group of size p^{r+1}. We show that the mod-p cohomology ring of G_r has depth one and that, in turn, it satisfies the equalities in Carlson's depth conjecture [2]. This is the first family of finite p-groups for which Carlson's depth conjecture has been verified besides p-groups of abelian type mod-p cohomology or extraspecial p-groups. Moreover, this computation is possible without first describing the structure of the cohomology ring.

KEYWORDS
depth, finite p-groups, mod-p cohomology ring

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1 | INTRODUCTION

Let p be a prime number, let G be a finite p-group and let $\mathbb{F}_p$ denote the finite field of p elements with trivial G-action. Then, the mod-p cohomology ring $\mathrm{H}^*(G;\mathbb{F}_p)$ is a finitely generated, graded-commutative $\mathbb{F}_p$-algebra (see [6, Corollary 7.4.6]), and so many ring-theoretic notions can be defined; Krull dimension, associated primes and depth, among others. Some of the aforementioned concepts have a group-theoretic interpretation; for instance, the Krull dimension $\dim \mathrm{H}^*(G;\mathbb{F}_p)$ of $\mathrm{H}^*(G;\mathbb{F}_p)$ equals the p-rank $\mathrm{rk}_p\, G$ of G, i.e., the largest integer $s \geq 1$ such that G contains an elementary abelian subgroup of rank s. However, the depth of $\mathrm{H}^*(G;\mathbb{F}_p)$, written as depth $\mathrm{H}^*(G;\mathbb{F}_p)$, is the length of the longest regular sequence in $\mathrm{H}^*(G;\mathbb{F}_p)$, and it seems to be far more difficult to compute. There are, nevertheless, lower and upper bounds for this number. For instance, Duflot [5] proved that the depth of $\mathrm{H}^*(G;\mathbb{F}_p)$ is at least as big as the p-rank of the centre $Z(G)$ of G, i.e., $\operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p) \geq \mathrm{rk}_p\, Z(G)$, and Notbohm [20] proved that for every elementary abelian subgroup E of G with centralizer $C_G(E)$ in G, the inequality $\operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p) \leq \operatorname{depth} \mathrm{H}^*(C_G(E);\mathbb{F}_p)$ holds. In [2], Carlson provided further upper bounds for the depth (see Theorem 2.3) and stated a conjecture that still remains open (see Conjecture 2.4).

The aim of the present work is to compute the depth of the mod-p cohomology rings of certain quotients of the maximal class pro-p group that moreover satisfy the equalities in the aforementioned conjecture. Let p be an odd prime number, let $\mathbb{Z}_p$ denote the ring of p-adic integers and let ζ be a primitive p-th root of unity. Consider the cyclotomic extension $\mathbb{Z}_p[\zeta]$ of degree $p-1$ and note that its additive group is isomorphic to $\mathbb{Z}_p^{p-1}$. The cyclic group $C_p = \langle\sigma\rangle$ acts on $\mathbb{Z}_p[\zeta]$ via multiplication by ζ, i.e., for any $x \in \mathbb{Z}_p[\zeta]$, the action is given as $x^\sigma = \zeta x$. Using the ordered basis $1, \zeta, \dots, \zeta^{p-2}$ in $\mathbb{Z}_p[\zeta] \cong \mathbb{Z}_p^{p-1}$, this action is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -1 & -1 & -1 & \dots & -1 \end{pmatrix}.$$

We form the semidirect product $S = C_p \ltimes \mathbb{Z}_p^{p-1}$, which is the unique pro-p group of maximal nilpotency class. Note that this is the analogue of the infinite dihedral pro-2 group for the p odd case. Moreover, S is a uniserial p-adic space group with cyclic point group C_p (compare [16, Section 7.4]). We write $[x,_k \sigma] = [x, \sigma, \overset{k}{\dots}, \sigma]$ for the iterated group commutator. Set $T_0 = \mathbb{Z}_p[\zeta]$ and define, for each integer $i \geq 1$,

$$T_i = (\zeta - 1)^i \mathbb{Z}_p[\zeta] = [T_0,_i \sigma] = \gamma_{i+1}(S).$$

These subgroups are all the C_p-invariant subgroups of T_0, and the successive quotients satisfy

$$T_i/T_{i+1} \cong \mathbb{Z}_p[\zeta]/(\zeta - 1)\mathbb{Z}_p[\zeta] \cong C_p.$$

Hence, $|T_0 : T_i| = p^i$ for every $i \geq 0$. For each integer $r > 0$, consider the quotient $T_0/T_r = \mathbb{Z}_p[\zeta]/(\zeta-1)^r\mathbb{Z}_p[\zeta]$. Since the subgroups T_r are C_p-invariant, we can form the semidirect product

$$G_r = C_p \ltimes T_0/T_r. \tag{1.1}$$

The finite p-groups G_r have size p^{r+1}.

For each integer r with $1 < r < p-1$, we can choose a minimal generating set for T_0/T_r as follows,

$$a_1 = 1 + T_r, \quad a_2 = (\zeta - 1) + T_r, \quad \dots, \quad a_r = (\zeta - 1)^{r-1} + T_r.$$

Using the multiplicative notation, we obtain

$$T_0/T_r = \langle a_1, \dots, a_r \rangle \cong C_p \times \overset{r}{\cdots} \times C_p,$$

and thus,

$$G_r = C_p \ltimes T_0/T_r \cong C_p \ltimes (C_p \times \overset{r}{\cdots} \times C_p).$$

The finite p-groups G_r have size p^{r+1} and exponent p. Note that in particular, G_2 is the extraspecial group of size p^3 and exponent p. We state the main result.

Theorem 1.1 (Main Theorem). *Let $p > 3$ be a prime number, let r be an integer with $1 < r < p-1$ and let G_r be given as in (1.1). Then,* $\operatorname{depth} \mathrm{H}^*(G_r; \mathbb{F}_p) = \omega_d(G_r) = 1$.

For each prime p, if $r = p-1$, then G_r has size p^{r+1}, has exponent p and is of maximal nilpotency class; while if $r > p-1$, then G_r has size p^{r+1} and exponent bigger than p. By Proposition 4.1, we in particular obtain that, for $p \geq 3$ and $r \geq p-1$, the inequality $\operatorname{depth} \mathrm{H}^*(G_r; \mathbb{F}_p) \leq 2$ holds. We observed that if we mimic the construction of the mod-p cohomology class θ_r in Section 5.1 for such p-groups, it is no longer true that its restriction to the mod-p cohomology of the centralizer of all elementary abelian subgroups of G_r of rank 2 vanishes. Moreover, for the $p = 3$ and $r = 2$ case, G_2 is the extraspecial 3-group of order 27 and exponent 3, and it is known that the depth of its mod-3 cohomology ring is 2 (compare [15] and [18]). We believe that this phenomena will occur with more generality and we propose the following conjecture.

Conjecture 1.2. *Let p be an odd prime, let $r \geq p-1$ be an integer, and let*

$$G_r = C_p \ltimes T_0/T_r$$

be as in (1.1). Then $\mathrm{H}^*(G_r;\mathbb{F}_p)$ has depth 2.

The above conjecture is known to be true for the particular cases where $p = 3$ and $r = 2$ or $r = 3$. In these two cases the mod-p cohomology rings have been calculated using computational tools (see [12]). Another argument supporting the conjecture is that for a fixed prime p and $r \geq p-1$, the groups G_r have isomorphic mod-p cohomology groups; not as rings, but as $\mathbb{F}_p$-modules (see [7]). This last isomorphism comes from a universal object described in the category of cochain complexes together with a quasi-isomorphism that induces an isomorphism at the level of spectral sequences.

Notation. Throughout, let p be an odd prime number and let G denote a finite p-group. A G-module A will be a right $\mathbb{F}_pG$-module. For such G-modules, we shall use additive notation in Sections 2 and 3, and multiplicative notation in Section 5, for our convenience. Moreover, if $a \in A$ and $g \in G$, we write a^g to denote the action of g on a.

Let A be a G-module and let $P_* \longrightarrow \mathbb{F}_p$ be a projective resolution of the trivial G-module $\mathbb{F}_p$, then for every $n \geq 0$, the n-th cohomology group $\mathrm{H}^n(G;A)$ is defined as $\mathrm{Ext}^n(\mathbb{F}_p, A) = \mathrm{H}^n(\mathrm{Hom}_G(P_*, A))$. Let $K \leq G$ be a subgroup of G and let $\iota : K \longrightarrow G$ denote an inclusion map. This map induces the restriction map $\mathrm{res}^G_K : \mathrm{H}^*(G;A) \longrightarrow \mathrm{H}^*(K;A)$ in cohomology.

Group commutators are given as $[g,h] = g^{-1}h^{-1}gh = g^{-1}g^h$ and for every $k \geq 1$, iterated commutators are written as $[x, y, \overset{k}{\ldots}, y] = [x,_k y]$, where we use left normed group commutators, i.e., $[x,y,z] = [[x,y],z]$. Also, the k-th term of the lower central series of G is denoted by $\gamma_k(G) = [G, \overset{k}{\ldots}, G]$.

2 | PRELIMINARIES

2.1 | Depth

In this section we give background on the depth of mod-p cohomology rings of finite p-groups and we also state one of the key results for the proof of Theorem 1.1.

Let $n \geq 1$ be an integer and let $x_1, \ldots, x_n \in \mathrm{H}^*(G;\mathbb{F}_p)$. We say that the sequence $x_1, \ldots, x_n$ is *regular* if x_1 is not a zero divisor in $\mathrm{H}^*(G;\mathbb{F}_p)$ and, for every $i = 2, \ldots, n$, the element x_i is not a zero divisor in the quotient $\mathrm{H}^*(G;\mathbb{F}_p)/(x_1, \ldots, x_{i-1})$, where $(x_1, \ldots, x_{i-1})$ denotes the ideal generated by the elements $x_1, \ldots, x_{i-1}$ in $\mathrm{H}^*(G;\mathbb{F}_p)$.

Definition 2.1. The *depth* of $\mathrm{H}^*(G;\mathbb{F}_p)$, denoted by $\operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p)$, is the maximal length of a regular sequence in $\mathrm{H}^*(G;\mathbb{F}_p)$.

Recall that a prime ideal $\mathfrak{p} \subseteq \mathrm{H}^*(G;\mathbb{F}_p)$ is an *associated prime* of $\mathrm{H}^*(G;\mathbb{F}_p)$ if, for some $\varphi \in \mathrm{H}^*(G;\mathbb{F}_p)$, it is of the form

$$\mathfrak{p} = \{\psi \in \mathrm{H}^*(G;\mathbb{F}_p) \mid \varphi \cup \psi = 0\}.$$

The set of all associated primes of $\mathrm{H}^*(G;\mathbb{F}_p)$ is denoted by $\operatorname{Ass} \mathrm{H}^*(G;\mathbb{F}_p)$. It is known that for every $\mathfrak{p} \in \operatorname{Ass} \mathrm{H}^*(G;\mathbb{F}_p)$, the following inequality, $\operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p) \leq \dim \mathrm{H}^*(G;\mathbb{F}_p)/\mathfrak{p}$ holds ([3, Proposition 12.2.5]). In particular, we have

$$\operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p) \leq \dim \mathrm{H}^*(G;\mathbb{F}_p). \tag{2.1}$$

When the two values coincide, the mod-p cohomology ring is said to be *Cohen–Macaulay*. We recall the lower bound for the depth of $\mathrm{H}^*(G;\mathbb{F}_p)$ by Duflot [5],

$$1 \leq \mathrm{rk}_p Z(G) \leq \operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p). \tag{2.2}$$

Before stating the crucial result for our construction, we introduce the concept of detection in cohomology.

Definition 2.2. Let G be a finite p-group and let $\mathcal{H}$ be a collection of subgroups of G. We say that $\mathrm{H}^*(G;\mathbb{F}_p)$ is *detected* by $\mathcal{H}$ if

$$\bigcap_{H\in\mathcal{H}} \operatorname{Ker} \operatorname{res}^G_H = 0.$$

Given a finite p-group G and a subgroup $E \leq G$, let $C_G(E)$ denote the centralizer of E in G. For $s \geq 1$, define:

$$\mathcal{H}_s(G) = \{C_G(E) \mid E \text{ is an elementary abelian subgroup of } G,\ \operatorname{rk}_p E = s\},$$

$$\omega_a(G) = \min\{\dim \mathrm{H}^*(G;\mathbb{F}_p)/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} \mathrm{H}^*(G;\mathbb{F}_p)\},$$

$$\omega_d(G) = \max\{s \geq 1 \mid \mathrm{H}^*(G;\mathbb{F}_p) \text{ is detected by } \mathcal{H}_s(G)\}.$$

Theorem 2.3 ([2]). *Let G be a finite p-group. Then, the following inequalities hold:*

$$\operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p) \leq \omega_a(G) \leq \omega_d(G).$$

In fact, in the same article, Carlson conjectured that the previous inequalities are actual equalities.

Conjecture 2.4 (Carlson). *Let G be a finite p-group. Then,*

$$\operatorname{depth} \mathrm{H}^*(G;\mathbb{F}_p) = \omega_a(G) = \omega_d(G).$$

A particular case of the above conjecture was proven by Green in [8, Theorem 0.1] and Theorem 2.3 was generalized in the context of compact Lie groups (see [14, Theorem 2.30] and [13, Theorem 2.13]) and saturated fusion systems (see [9, Theorem 4.16]).

2.2 | Yoneda and crossed extensions

Let G be a finite p-group. We describe the mod-p cohomology ring $\mathrm{H}^*(G;\mathbb{F}_p)$ first in terms of Yoneda extensions, and then in terms of crossed extensions. For a more detailed account of these topics, we refer to [17, Chapter III] and [19]; and [10], [11] and [19], respectively.

Definition 2.5. Let A and B be G-modules. For every integer $n \geq 1$, a *Yoneda n-fold extension φ of B by A* is an exact sequence of G-modules of the form

$$\varphi : 0 \longrightarrow A \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow B \longrightarrow 0.$$

We can define an equivalence relation on the set of all n-fold Yoneda extensions of B by A, and denote by $\mathrm{YExt}^n(B,A)$ the set of all such extensions up to equivalence. Then, $\mathrm{YExt}^n(B,A)$ with the Baer sum is an abelian group.

Given $\varphi \in \mathrm{YExt}^n(B,A)$, we denote by $\alpha_*\varphi \in \mathrm{YExt}^n(B,A')$ the pushout of φ via a G-module homomorphism $\alpha : A \to A'$, and by $\beta^*\varphi \in \mathrm{YExt}^n(B',A)$ the pullback via $\beta : B' \to B$.

We will now move on to crossed extensions.

Definition 2.6. Let M_1 and M_2 be groups with M_1 acting on M_2. A *crossed module* is a group homomorphism $\rho : M_2 \to M_1$ satisfying the following properties:

(i) $y_2^{\rho(y_2')} = y_2^{y_2'}$ for all $y_2, y_2' \in M_2$, and
(ii) $\rho(y_2^{y_1}) = \rho(y_2)^{y_1}$ for all $y_1 \in M_1$ and $y_2 \in M_2$.

Definition 2.7. Let $n \geq 1$ be an integer and let A be a G-module. A *crossed* n-fold extension ψ of G by A is an exact sequence of groups of the form

$$\psi : 0 \longrightarrow A \xrightarrow{\rho_n} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{\rho_1} M_1 \longrightarrow G \longrightarrow 1,$$

satisfying the following conditions:

(i) $\rho_1 : M_2 \longrightarrow M_1$ is a crossed module,
(ii) M_i is a G-module for every $i = 3, \dots, n$, and
(iii) ρ_i is a G-module homomorphism for every $i = 2, \dots, n$.

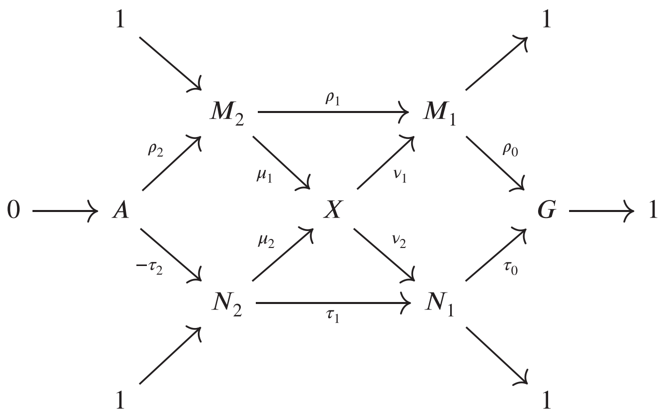
We can define an equivalence relation on crossed n-fold extensions of G by A as for Yoneda extensions. We will denote by $\mathrm{XExt}^n(G, A)$ the set of all crossed n-fold extensions of G by A up to equivalence, which is an abelian group endowed with the Baer sum of crossed extensions.

For the $n = 2$ case, we can use the following characterization of equivalent crossed extensions.

Proposition 2.8 ([10, Lemma 2.5]). *Let G be a finite p-group and let A be a G-module. Then, two crossed 2-fold extensions of G by A*

$$\psi : \ 0 \longrightarrow A \xrightarrow{\rho_2} M_2 \xrightarrow{\rho_1} M_1 \xrightarrow{\rho_0} G \longrightarrow 1 \quad \text{and} \quad \psi' : \ 0 \longrightarrow A \xrightarrow{\tau_2} N_2 \xrightarrow{\tau_1} N_1 \xrightarrow{\tau_0} G \longrightarrow 1$$

are equivalent if and only if there exist a group X and a commutative diagram



(2.3)

satisfying the following properties:

(a) $-\tau_2 : A \longrightarrow N_2$ is given by $(-\tau_2)(a) = \tau_2(-a)$ for $a \in A$,
(b) the diagonals are short exact sequences,
(c) $\mu_1 \circ \rho_2(A) = \mu_1(M_2) \cap \mu_2(N_2)$, and
(d) conjugation in X coincides with the actions of both M_1 on M_2 and N_1 on N_2.

Analogous to Yoneda extensions, for an integer $n \geq 1$, given an n-crossed extension $\varphi \in \mathrm{XExt}^n(G, A)$ and a G-module homomorphism $\alpha : A \longrightarrow A'$, we denote by $\alpha_* \varphi \in \mathrm{XExt}^n(G, A')$ the pushout of φ via α, and given a group homomorphism $\beta : G' \longrightarrow G$ we denote by $\beta^* \varphi \in \mathrm{XExt}^n(G', A)$ the pullback of φ via β (see [10, Proposition 4.1]).

Theorem 2.9 ([17, Theorem 6.4], [10, Theorem 4.5]). *Let G be a finite p-group. For every G-module A and every integer $n \geq 1$, there are group isomorphisms*

$$\mathrm{H}^{n+1}(G; A) \cong \mathrm{YExt}^{n+1}(\mathbb{F}_p, A) \cong \mathrm{XExt}^n(G, A)$$

that are natural in both G and A.

3 | PRODUCT BETWEEN EXTENSIONS

3.1 | Product of Yoneda extensions and crossed extensions

It is well known that, given two Yoneda extensions $\varphi \in \mathrm{YExt}^n(B, A)$ and $\varphi' \in \mathrm{YExt}^m(C, B)$, we can define their Yoneda product $\varphi \cup \varphi' \in \mathrm{YExt}^{n+m}(C, A)$ by splicing them together. In $\mathrm{H}^*(G; \mathbb{F}_p)$, this product coincides with the usual cup product of cohomology classes.

We proceed now to define the analogous Yoneda product of a Yoneda extension and a crossed extension.

Definition 3.1. Let G be a finite p-group, let A and B be G-modules and let $n, m \geq 1$ be integers. Given a Yoneda n-fold extension class $\varphi \in \mathrm{YExt}^n(A, B)$ represented by

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow A \longrightarrow 0,$$

and a crossed m-fold extension class $\psi \in \mathrm{XExt}^m(G, A)$ represented by

$$0 \longrightarrow A \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1,$$

we define their *Yoneda product* $\varphi \cup \psi$ as the extension

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1.$$

Remark 3.2. It can be readily checked that

$$\mathrm{YExt}^n(A, B) \otimes \mathrm{XExt}^m(G, A) \longrightarrow \mathrm{XExt}^{n+m}(G, B)$$

given by $(\varphi, \psi) \mapsto \varphi \cup \psi$ is a well defined bilinear pairing by following the analogous proofs for the Yoneda product of two Yoneda extensions, see [17, Section III.5].

3.2 | Yoneda and cup products coincide

In order to show that the Yoneda product of Yoneda extensions with crossed extensions coincides with the usual cup product, we will follow a construction by Conrad [4], giving an explicit correspondence between crossed extensions and Yoneda extensions.

Let G be a finite p-group and let A be a G-module. Let $\psi \in \mathrm{XExt}^n(G, A)$ be a class represented by a crossed n-fold extension

$$0 \longrightarrow A \xrightarrow{\rho_n} M_n \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{\rho_1} M_1 \xrightarrow{\rho_0} G \longrightarrow 1,$$

with M_2 abelian (see [10, Proposition 2.7] for the existence of such a representative). Consider the G-module $\mathrm{Im}\, \rho_1 \leq M_1$. Then, we have an extension $\psi_0 \in \mathrm{XExt}^1(G, \mathrm{Im}\, \rho_1)$ of the form

$$\psi_0 : \quad 0 \longrightarrow \mathrm{Im}\, \rho_1 \longrightarrow M_1 \longrightarrow G \longrightarrow 1.$$

Now, we can embed $\mathrm{Im}\, \rho_1$ into an injective G-module I. As I is injective, we have that $\mathrm{XExt}^1(G, I) \cong \mathrm{H}^2(G, I) = 0$, and so the pushout of ψ_0 via the embedding of $\mathrm{Im}\, \rho_1$ into I splits, i.e., there is a group homomorphism $\Phi : M_1 \longrightarrow G \ltimes I$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\, \rho_1 & \longrightarrow & M_1 & \xrightarrow{\rho_0} & G & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow \Phi & & \big\| & & \\ 0 & \longrightarrow & I & \longrightarrow & G \ltimes I & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

We can find a group homomorphism $\nu : M_1 \longrightarrow G$ and a map $\chi : M_1 \longrightarrow I$ that for every $x, y \in M_1$ satisfies

$$\chi(xy) = \chi(x)^{\nu(y)}\chi(y), \tag{3.1}$$

such that, for every $x \in M_1$, we can write

$$\Phi(x) = \big(\nu(x), \chi(x)\big).$$

Moreover, if we denote by $\pi : I \longrightarrow I/\operatorname{Im}\rho_1$ the canonical projection, there is a unique map $\tau : G \longrightarrow I/\operatorname{Im}\rho_1$ such that $\tau\circ\nu = \pi\circ\chi$. Furthermore, because χ satisfies (3.1) and $\nu = \rho_0$ is surjective, we have that for every $g, h \in G$,

$$\tau(gh) = \tau(g)^h + \tau(h),$$

and so τ is a 1-cocycle. Hence, τ can be represented as a cohomology class in $\mathrm{H}^1\big(G, I/\operatorname{Im}\rho_1\big) \cong \mathrm{YExt}^1\big(\mathbb{F}_p, I/\operatorname{Im}\rho_1\big)$ by a Yoneda extension of the form

$$0 \longrightarrow I/\operatorname{Im}\rho_1 \longrightarrow E_\tau \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

Remark 3.3. The choices of the G-module I and the cocycle τ, and consequently E_τ, only depend on $\operatorname{Im}\rho_1 \leq M_1$.

Finally, we can construct the element $\mathrm{Y}(\psi) \in \mathrm{YExt}^{n+1}\big(\mathbb{F}_p, A\big)$ given by the Yoneda extension

$$0 \longrightarrow A \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_2 \longrightarrow I \longrightarrow E_\tau \longrightarrow \mathbb{F}_p \longrightarrow 0. \tag{3.2}$$

This construction gives rise to a group isomorphism

$$\mathrm{Y} : \mathrm{XExt}^n(G, A) \longrightarrow \mathrm{YExt}^{n+1}\big(\mathbb{F}_p, A\big).$$

Proposition 3.4. *Let G be a finite p-group and let $n, m \geq 1$ be integers. Then, the Yoneda product*

$$\mathrm{YExt}^n(A, B) \otimes \mathrm{XExt}^m(G, A) \longrightarrow \mathrm{XExt}^{n+m}(G, B)$$

coincides with the Yoneda product

$$\mathrm{YExt}^n(A, B) \otimes \mathrm{YExt}^{m+1}\big(\mathbb{F}_p, A\big) \longrightarrow \mathrm{YExt}^{n+m+1}\big(\mathbb{F}_p, B\big).$$

In particular, if $A = B = \mathbb{F}_p$, the above product coincides with the cup product

$$\cup : \ \mathrm{H}^n\big(G; \mathbb{F}_p\big) \otimes \mathrm{H}^{m+1}\big(G; \mathbb{F}_p\big) \longrightarrow \mathrm{H}^{n+m+1}\big(G; \mathbb{F}_p\big).$$

Proof. Let $\varphi \in \mathrm{YExt}^n(A, B)$ be a class represented by an extension

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{\mu_0} A \longrightarrow 0,$$

and let $\psi \in \mathrm{XExt}^m(G, A)$ be a class represented by an extension

$$0 \longrightarrow A \xrightarrow{\rho_m} M_m \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{\rho_1} M_1 \longrightarrow G \longrightarrow 1,$$

with M_2 abelian. We need to prove that $\mathrm{Y}(\varphi \cup \psi) = \varphi \cup \mathrm{Y}(\psi)$.

For $m = 1$, we have that $\psi \in \mathrm{XExt}^1(G, A)$ is represented by a crossed 1-fold extension of the form

$$0 \longrightarrow A \xrightarrow{\rho_1} M_1 \longrightarrow G \longrightarrow 1.$$

Then, $\varphi \cup \psi$ is given by the crossed $(n+1)$-fold extension

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{\gamma_1} M_1 \longrightarrow G \longrightarrow 1,$$

where $\gamma_1 = \rho_1 \circ \mu_0$. Now, we have that $\operatorname{Im}\gamma_1 = \operatorname{Im}\rho_1$, and so we can once again use the same I and τ in the construction of both $\mathrm{Y}(\psi)$ and $\mathrm{Y}(\varphi \cup \psi)$. Therefore, both $\varphi \cup \mathrm{Y}(\psi)$ and $\mathrm{Y}(\varphi \cup \psi)$ are given by the same extension

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow I \longrightarrow E_\tau \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

For $m > 1$, by (3.2), the extension $\mathrm{Y}(\psi) \in \mathrm{YExt}^{m+1}(\mathbb{F}_p, A)$ is of the form

$$0 \longrightarrow A \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_2 \longrightarrow I \longrightarrow E_\tau \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

and $\varphi \cup \psi \in \mathrm{XExt}^{n+m}(G, A)$ is represented by the crossed $(n+m)$-fold extension

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow G \longrightarrow 1.$$

By Remark 3.3, we can use the same I and τ in the construction of $\mathrm{Y}(\psi)$. Therefore, $\mathrm{Y}(\varphi \cup \psi) \in \mathrm{YExt}^{n+m+1}(G, A)$ is represented by

$$0 \longrightarrow B \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_2 \longrightarrow I \longrightarrow E_\tau \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

which coincides with $\varphi \cup \mathrm{Y}(\psi)$.

Finally, if $A = B = \mathbb{F}_p$ then the Yoneda product of Yoneda extensions coincides with the cup product of cohomology classes. □

4 | FINITE p-GROUPS OF MOD-p COHOMOLOGY DEPTH AT MOST 2

Until the end of Section 4, we assume that p is an odd prime number and that, for each integer $r > 1$, G_r denotes the finite p-group described in (1.1). If we write $r = (p-1)\cdot n + m$ with $n, m \geq 0$ integers such that $m < p-1$, then G_r can be described as a semidirect product

$$G_r = C_p \ltimes \left(C_{p^{n+1}} \times \overset{m}{\cdots} \times C_{p^{n+1}} \times C_{p^n} \times \overset{p-m-1}{\cdots} \times C_{p^n}\right) = C_p \ltimes T_0/T_r,$$

where T_0/T_r is the maximal abelian p-subgroup of G_r. In particular, for $r < p-1$, the group G_r can be described as the semidirect product (1.1),

$$G_r = C_p \ltimes \left(C_p \times \overset{r}{\cdots} \times C_p\right) = C_p \ltimes T_0/T_r,$$

where T_0/T_r is the maximal elementary abelian p-subgroup of G_r.

Proposition 4.1. *For every integer $r > 1$, the following inequalities hold:*

$$1 \leq \operatorname{depth} \mathrm{H}^*\left(G_r; \mathbb{F}_p\right) \leq 2.$$

Proof. The inequality $1 \leq \operatorname{depth} \mathrm{H}^*\left(G_r; \mathbb{F}_p\right)$ holds by (2.2). Suppose that $p = 3$. Then, for every $r > 1$, we have that

$$\mathrm{rk}_p(G_r) = 2 = \dim \mathrm{H}^*\left(G_r; \mathbb{F}_p\right),$$

and by (2.1), we conclude that $\operatorname{depth} \mathrm{H}^*\left(G_r; \mathbb{F}_p\right) \leq 2$.

Now, suppose that $p \geq 5$. It can be readily checked that, for any $r > 1$, every elementary abelian p-subgroup E of G_r with $\mathrm{rk}_p(E) = 3$ satisfies that $E \leq T_0/T_r$, and the centralizer is $C_{G_r}(E) = T_0/T_r$. Therefore, for every E as above, its centralizer in G_r is contained in the proper subgroup T_0/T_r of G_r. Hence, by [2, Corollary 2.4], we conclude that depth $\mathrm{H}^*(G_r; \mathbb{F}_p) < 3$. □

5 | FINITE p-GROUPS OF DEPTH ONE MOD-p COHOMOLOGY

Until the end of Section 5, we assume that $p > 3$ is a prime number, that r is an arbitrary but fixed integer satisfying $1 < r < p - 1$ and that G_r denotes the finite p-group described in (1.1). This group is generated by the elements $\sigma, a_1, \dots, a_r$ satisfying the following relations:

- $\sigma^p = a_i^p = [a_i, a_j] = [a_r, \sigma] = 1$, for $i = 1, \dots, r$ and $j = 1, \dots, r-1$,
- $[a_j, \sigma] = a_{j+1}$ for $j = 1, \dots, r-1$.

The aim of this section is to prove Theorem 1.1. To show the result, we construct a non-trivial mod-p cohomology class in $\mathrm{H}^*(G_r; \mathbb{F}_p)$ that restricts trivially to the mod-p cohomologies of the centralizers of all rank 2 elementary abelian subgroups of G_r. Then, $\omega_d(G_r) = 1$ and Theorem 2.3 yields that depth $\mathrm{H}^*(G_r; \mathbb{F}_p) = 1$.

5.1 | Construction

We follow the assumptions in the Notation. In this section, we construct a cohomology class $\theta_r \in \mathrm{H}^3(G_r; \mathbb{F}_p)$ that is a cup product of a Yoneda 1-fold extension and a crossed 1-fold extension.

We start by defining a cohomology class $\sigma^* \in \mathrm{H}^1(G_r; \mathbb{F}_p) = \mathrm{Hom}(G_r, \mathbb{F}_p)$. To that aim, consider the homomorphism $\sigma^* : G_r \longrightarrow \mathbb{F}_p$ satisfying

$$\sigma^*(\sigma) = 1, \quad \sigma^*(a_1) = \cdots = \sigma^*(a_r) = 0. \tag{5.1}$$

The class σ^* can be represented by the Yoneda extension

$$1 \longrightarrow C_p = \langle a_{r+2} \rangle \longrightarrow C_p \times C_p \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow 1,$$

where the action of G_r on $C_p \times C_p = \langle a_{r+1}, a_{r+2} \rangle$ is described by

$$\text{for } \ g \in G_r, \ \text{ set } \ a_{r+1}^g = a_{r+1} a_{r+2}^{\sigma^*(g)}, \quad a_{r+2}^g = a_{r+2}.$$

We continue by defining a crossed 1-fold extension $\eta_r \in \mathrm{H}^2(G_r; \mathbb{F}_p)$ as follows. Let

$$\lambda_r : T_0/T_{r+1} \times T_0/T_{r+1} \longrightarrow T_0/T_{r+1}$$

be the alternating bilinear map satisfying

$$\lambda_r(a_{r-1}, a_r) = a_{r+1} \ \text{ and } \ \lambda_r(a_i, a_j) = 0, \ \text{for all } i < j \text{ with } (i, j) \neq (r-1, r).$$

Now, define $(T_0/T_{r+1})_{\lambda_r}$ to be the group with underlying set T_0/T_{r+1} and with group operation given by

$$\text{for } \ x, y \in T_0/T_{r+1} \ \text{ we define } \ x \cdot_{\lambda_r} y = xy\lambda_r(x, y)^{1/2}.$$

Finally, define the p-group $\widehat{G}_r = C_p \ltimes (T_0/T_{r+1})_{\lambda_r}$ of size $|\widehat{G}_r| = p^{r+2}$ and exponent p.

Let $\eta_r \in \mathrm{H}^2(G_r, \mathbb{F}_p)$ be the cohomology class represented by the crossed 1-fold extension

$$1 \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow \widehat{G}_r \longrightarrow G_r \longrightarrow 1. \tag{5.2}$$

Then, we define the cohomology class $\theta_r = \sigma^* \cup \eta_r \in \mathrm{H}^3(G_r; \mathbb{F}_p)$, which is represented by the crossed 2-fold extension

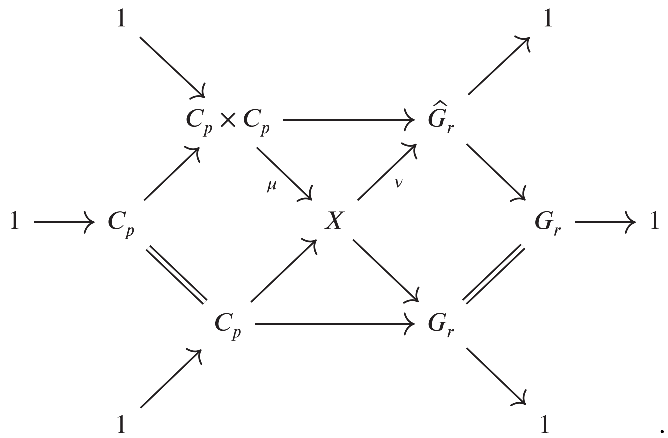
$$1 \longrightarrow C_p \longrightarrow C_p \times C_p \longrightarrow \widehat{G}_r \longrightarrow G_r \longrightarrow 1. \tag{5.3}$$

5.2 | Non-triviality

In the present section we prove the following result.

Proposition 5.1. *The cohomology class θ_r constructed in (5.3) is non-trivial.*

Proof. Assume by contradiction that $\theta_r = 0$. Then, by Proposition 2.8 there exists a group X such that the following diagram commutes:



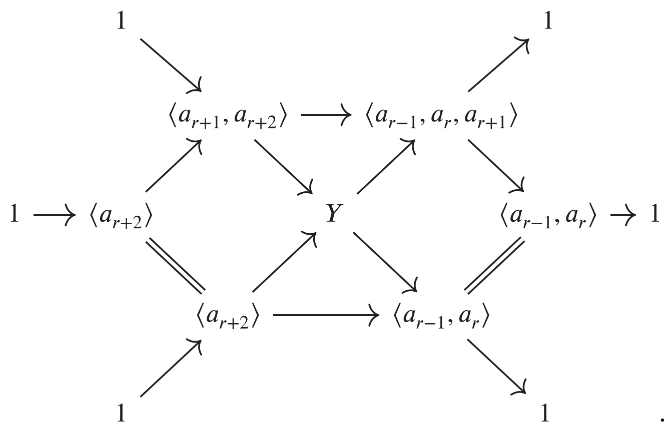
We have that $X = \langle \bar{\sigma}, \bar{a}_1, \dots, \bar{a}_{r+2} \rangle$ with elements $\bar{\sigma}, \bar{a}_1, \dots, \bar{a}_{r+1}, \bar{a}_{r+2} \in X$ that satisfy

$$\bar{a}_{r+2} = \mu(a_{r+2}), \quad \nu(\bar{\sigma}) = \sigma \quad \text{and} \quad \nu(\bar{a}_i) = a_i \text{ for all } i = 1, \dots, r+1,$$

and we have $Z(X) = \langle \bar{a}_{r+2} \rangle$ and $\gamma_r(X) = \langle \bar{a}_r, \bar{a}_{r+1}, \bar{a}_{r+2} \rangle$. Consider the normal subgroup

$$Y = \langle \bar{a}_{r-1}, \bar{a}_r, \bar{a}_{r+1}, \bar{a}_{r+2} \rangle \trianglelefteq X,$$

which fits into the following commutative diagram:



Then, we have that $Z(Y) = \langle \bar{a}_{r+1}, \bar{a}_{r+2} \rangle$, and moreover,

$$\left[\bar{\sigma}, Y, \gamma_r(X)\right] = \left[\gamma_r(X), \gamma_r(X)\right] = 1 \text{ and } \left[\gamma_r(X), \bar{\sigma}, Y\right] = \left[Z(Y), Y\right] = 1.$$

Therefore, the three subgroup lemma (see [21, 5.1.10]) leads us to the conclusion that $\left[Y, \gamma_r(X), \bar{\sigma}\right] = 1$. Nevertheless, a direct computation shows that

$$\left[Y, \gamma_r(X), \bar{\sigma}\right] = \left[Z(Y), \bar{\sigma}\right] = Z(X) \neq 1,$$

which gives a contradiction. Hence, $\theta_r \neq 0$. □

5.3 | Trivial restriction

In this section we show that for every elementary abelian subgroup E of G_r of p-rank $\mathrm{rk}_p E = 2$, the image of θ_r via the restriction map,

$$\mathrm{res}^{G_r}_{C_{G_r}(E)} : \mathrm{H}^3\left(G_r; \mathbb{F}_p\right) \longrightarrow \mathrm{H}^3\left(C_{G_r}(E); \mathbb{F}_p\right),$$

is trivial, i.e., $\mathrm{res}^{G_r}_{C_{G_r}(E)} \theta_r = 0$. This will imply that the cohomology class θ_r is not detected by $\mathcal{H}_2\left(G_r\right)$.

Proposition 5.2. *Let $E \leq G_r$ be an elementary abelian subgroup with $\mathrm{rk}_p E = 2$. Then, $\mathrm{res}^{G_r}_{C_G(E)} \theta_r = 0$. Consequently, $\omega_d(G) = 1$.*

Proof. There are two types of elementary abelian subgroups $E \leq G_r$, either $E \leq \langle a_1, \dots, a_r \rangle$ or $E \nleq \langle a_1, \dots, a_r \rangle$. Assume first that $E \leq \langle a_1, \dots, a_r \rangle$. Then, $C_{G_r}(E) = \langle a_1, \dots, a_r \rangle$ and we have that $\mathrm{res}^{G_r}_{C_{G_r}(E)} \sigma^* = 0$. Therefore,

$$\mathrm{res}^{G_r}_{C_{G_r}(E)} \theta_r = \left(\mathrm{res}^{G_r}_{C_r(E)} \sigma^*\right) \cup \left(\mathrm{res}^{G_r}_{C_{G_r}(E)} \eta_r\right) = 0.$$

Assume now that $E \nleq \langle a_1, \dots, a_r \rangle$. Then, $E = \langle b, a_r \rangle$ with $b = \sigma x$ for some $x \in \langle a_1, \dots, a_{r-1} \rangle$, and $C_{G_r}(E) = E$. Moreover, $\mathrm{res}^{G_r}_{C_G(E)} \eta_r$ is represented by the extension that is obtained by taking the pullback of η_r via the inclusion $E \longrightarrow G_r$, as illustrated in the following diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle a_{r+1} \rangle & \longrightarrow & \hat{E} = \langle b, a_r, a_{r+1} \rangle & \longrightarrow & E & \longrightarrow & 1 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \langle a_{r+1} \rangle & \longrightarrow & \hat{G}_r & \longrightarrow & G_r & \longrightarrow & 1
\end{array}$$

Observe that $\hat{E} \cong C_p \ltimes \left(C_p \times C_p\right)$ is the extraspecial group of order p^3 and exponent p. Hence, $\mathrm{res}^{G_r}_{C_{G_r}(E)} \eta_r$ is represented by the extension

$$1 \longrightarrow C_p = \langle a_{r+1} \rangle \longrightarrow \hat{E} = C_p \ltimes \left(C_p \times C_p\right) \longrightarrow C_p \times C_p = \langle b, a_r \rangle \longrightarrow 1. \tag{5.4}$$

Define, similar to (5.1), $a_r^*, b^* \in \mathrm{Hom}\left(G_r, \mathbb{F}_p\right)$. It can be readily checked (following the construction in [1, Section IV.3]) that the extension class of (5.4) coincides with the cup-product $b^* \cup a_r^*$, and so $\mathrm{res}^{G_r}_{C_{G_r}(E)} \eta_r = b^* \cup a_r^*$. Consequently,

$$\mathrm{res}^{G_r}_{C_{G_r}(E)} \theta_r = \left(\mathrm{res}^{G_r}_{C_{G_r}(E)} \sigma^*\right) \cup b^* \cup a_r^* = 0,$$

as the product of any three elements of degree one is trivial in $\mathrm{H}^3(E;\mathbb{F}_p)$. In particular, this means that $\mathrm{H}^*(G_r;\mathbb{F}_p)$ is not detected by $\mathcal{H}_2(G_r)$ and $\omega_d(G_r) = 1$. □

Proof of Theorem 1.1. By (2.2), we know that $1 \leq \operatorname{depth} \mathrm{H}^*(G_r;\mathbb{F}_p)$, and Proposition 5.2 yields that $\omega_d(G) = 1$. Then, by Theorem 2.3, we conclude that $\operatorname{depth} \mathrm{H}^*(G_r;\mathbb{F}_p) = \omega_d(G_r) = 1$. □

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