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Physical Principles in Option Pricing

Analysis and valuation based on the Black-Scholes model

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1 Introduction

Nowadays the world of finance is increasingly based on mathematical and physical models. That is why many financial companies hire mathematicians and physicists to build and analyse those models. The branch of finance that deals with these tasks is known as quantitative finance.

It is said that quantitative finance was born in 1900 when Louis de Bachelier published his Ph.D. thesis *The theory of speculation.* He introduced the concept of Brownian motion to approximate asset prices random path. The asset could be, for instance, a stock. From that moment, many other theories have been developed in an attempt to predict the behaviour of financial markets and instruments. In 1973 Fischer Black and Myron Scholes introduced the Black-Scholes model to price derivatives within their publication *The pricing of options and corporate liabilities.* They were awarded in 1997 with a Nobel Prize for this work. Although it had certain limitations, The Black-Scholes model became a reference for many other subsequent models. The constant changes in markets require the evolution of models and the appearance of new updated ones. The main focus field has always been to manage the volatility and most of the built models are based on stochastic processes [1].

This project explores an application of physics to study financial systems. In particular, we will focus on the physical principles in option pricing. Options are financial instruments whose price depends on the price of an underlying asset. This means that we will have to model the behaviour of the underlying asset to be able to model the price of the option. Some concepts like Brownian motion and diffusion process can be extrapolated to those tasks as we will see. There is more than one model to price options, but we will only analyse the Black-Scholes model as it can be related to physical concepts. This model makes certain assumptions that are not very realistic in real markets, but it gives a good qualitative overview of how options are priced. In fact, many subsequent more sophisticated models are based on it.

First of all, we will introduce in Section 3 some basic financial notions which are necessary to understand the rest of the project. It is very important to assimilate what is an option and how does it work.

In Section 4 we will explain the physical concepts. This includes the definition of Brownian motion, its relation with diffusion processes, finding a solution to the diffusion equation and making a mathematical formulation of the Brownian motion (this is known as a Wiener process). These aspects will be extrapolated later to finance throughout the project.

Once the financial and physical concepts are clear, we will try to model the behaviour of the underlying asset price in Section 5. We will assume a random walk model for it. This model is based on the definition of Wiener process made in Section 4.3. We will find out that the asset price follows a Geometric Brownian motion and that the probability density function of its logarithm suffers a diffusion process. However, we will previously need to introduce a mathematical concept called Itô's Lemma.

Knowing the behaviour of the underlying asset price, we will be able to build the Black-Scholes model that leads to the partial differential equation used to price options. This is done in Section 6.

Our next step will be to find a solution to the Black-Scholes equation to price European options in Section 7. This is known as the Black-Scholes Formula. We will transform the Black-Scholes equation into a diffusion equation and use the solution found in Section

4.2. The problem of valuing the option will be treated as a diffusion process. We will also analyse a numerical example for the derived solution.

In Section 8 we will study the case of pricing American options. We will see that we cannot explicitly solve the Black-Scholes equation and that we have to deal with a free boundary problem. We will transform the free boundary problem into a linear complementarity problem to eliminate the dependence on the free boundary. After that, we will write the problem using the finite-difference formulation and the Crank-Nicolson scheme. An algorithm which includes the LU method will be built to solve the problem. Finally, we will translate this algorithm into our own Python code and run some simulations.

To finish the project, we will make some conclusions based on the analysed topics.

2 Objectives

The main objective of the project is to establish a connection between physics and pinance, analysing the physical principles in Option Pricing. Moreover, the projects also has the following particular objectives:

1. Learn that the asset price behaviour can be modelled with a Brownian motion and visualize the diffusion process suffered by the probability density function of its logarithm.
2. Derive the Black-Scholes equation used to price options and solve it as a diffusion process to price European options.
3. Understand the free boundary problem for American options and solve it using the Crank-Nicolson formulation and an algorithm based on the LU method.
4. Write our own Python code with the mentioned algorithm and run a simulation with a numerical example to analyse results.
5. Deepen in mathematical resolutions, understanding the mathematical developments beyond the level of detail given by the bibliography.

3 Basic financial concepts

Before we start developing our project, we need to introduce some basic financial concepts that will help us to understand the rest of the topics.

3.1 What is an option?

An **option** is a contract between two parties, the holder and the writer, on an **underlying asset**. The holder pays a compensation (the **premium**) to the writer to have the right, but not the obligation, to buy or sell the underlying asset at an agreed price (the **strike** or **exercise price**) by a specific date (**expiration date** or **maturity**) [2]. In case the holder wanted to exercise his right to buy or sell the underlying asset, the writer would have the obligation to do the opposite movement, that is, sell or buy it. At the same time, the holder of the option can sell his right of execution to a third party in exchange of a premium before the expiration date. The underlying asset is usually a stock or a bond, but it could also be an index, an interest rate or even commodities. To clarify concepts, we are going to explain them one by one:

- The **option** is the contract that depends on the **underlying** asset, which can be a stock, a bond,...
- The **holder** of the option is the one who has the right to execute the option to buy or sell the underlying asset. The holder can also sell that right to another person.
- The **writer** of the option is the one who receives a premium for writing the option but later depends on what the holder decides.
- The **premium** is the value that has to be paid to acquire the position of holder, that is, it is the price of the option. Pricing an option means finding the premium.
- The **strike** or **exercise price** is the agreed amount of money that the holder pays/receives when executing the option to buy/sell the underlying asset.
- The **expiration date** or **maturity** is the date by which the holder can exercise the option.

There are various types of options attending to different parameters. The first main differentiation is made between call and put options. Call options give the holder the right to buy the underlying asset, whereas put options give the chance to sell it. The simplest ones are the European call and put options, where the holder can exercise his right just at expiration date. There are also American call and put options. In this case, the right to buy or sell the underlying asset can be executed at any time before expiration date. European and American options are typically called Vanilla options if they have no other special condition to exercise the option. Apart from Vanilla options, there are other types of options like Exotic options, where the conditions of execution are different [3]. However, we are not going to work with them in this project.

3.2 Vanilla options

Vanilla options are the simplest type of options and are usually traded on an exchange (marketplace where financial instruments are traded). They are divided into call and put options.

3.2.1 Call options

As it has been explained before, **call options** give the holder the right to buy the underlying asset. Depending on whether it is American or European, the purchase can be done before or just at expiration date. Call options are executed when the price of the underlying asset is higher than the strike price. This means we are buying it cheaper than the actual price.

We are now going to see an example to better understand how do call options work. Let us consider a holder A and a writer B. They agree on an European call option over a stock whose current price is 100 \$. The expiration date is 6 months and the strike price is also 100 \$. The holder pays 5 \$ to the writer as the premium for the option. As it is an European option, there are two possible scenarios at expiration date. If the stock price goes up to, for instance, 110 \$, the holder would execute his right to buy it for 100 \$. He would get a payoff of 10 \$ and a profit of 5 \$ taking into consideration the 5 \$ he had paid as the premium. If, on the contrary, the stock value goes down to 90 \$, the holder would not buy the stock for 100 \$. In that case, he would lose the 5 \$ initially paid to the writer. If the option were American instead of European, it could be exercised at any time before expiration date, but we will see later that it should only be done at maturity for call options.

We can deduce and plot in Figure 1.a a general expression for the payoff and profit of an European call option at maturity from the point of view of the holder [3].

$$\text{Payoff}: \ \max\{S(T)-K,0\}, \qquad \text{Profit}: \ \max\{S(T)-K,0\}-C(t_i,T,K) \tag{1}$$

where the 0 represents the case of no execution and

- T is the expiration date or maturity
- K is the exercise or strike price
- S(T) is the price of the underlying asset at maturity
- C(t_i,T,K) is the price of the call option (the premium) when it is bought at initial time t_i

Regarding Figure 1.a, we can visualize that the option has to be exercised in some cases despite the profit is negative. In those situations, exercising the option means reducing the loss of money.

In our example, we had that $K = 100$ \$, t_i=0, T= 0.5 year, $C(t_i,T,K) = 5$ \$. If we consider the first case in which the stock price raised to 110 \$, then $S(T) = 110$ \$ and so the payoff and profit are

$$\text{Payoff}: \ \max\{110-100,0\} = 10 \ \$, \qquad \text{Profit}: \ \max\{110-100,0\}-5 = 5 \ \$ \tag{2}$$

and, in the other case, where the stock price fell to 90 \$,

$$\text{Payoff}: \ \max\{90-100,0\} = 0 \ \$, \qquad \text{Profit}: \ \max\{90-100,0\}-5 = -5 \ \$ \tag{3}$$

We can also define the payoff function in general at a time t for both European and American call options from the point of view of the holder:

$$\text{Payoff: } \max\{S(t)-K,0\} \tag{4}$$

The shape of this payoff function will be the same as the shape of the payoff function in Figure 1.a, but for another $S(t)$.

3.2.2 Put options

Put options grant the holder the right to sell the underlying asset. Once again, depending on if it is American or European, the sale can be done before or at maturity. In this case, the holder would only sell the underlying asset if its price went below the strike price.

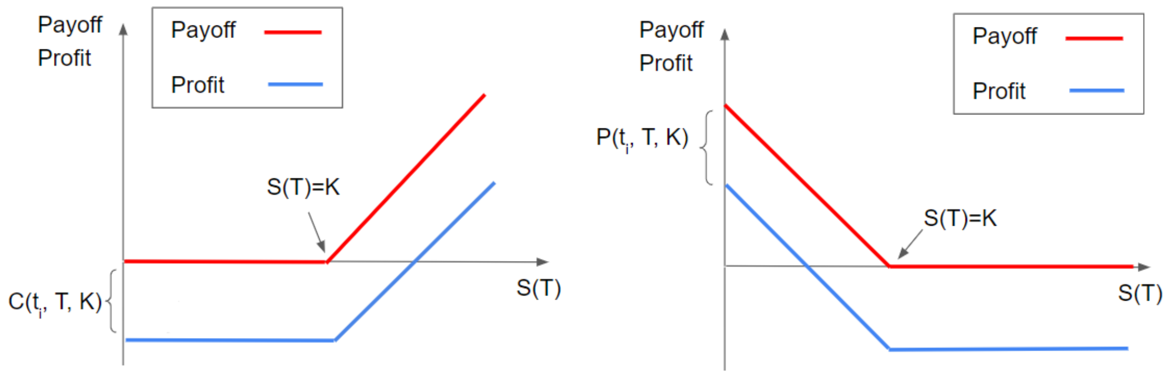
We can also gather an expression for the payoff and profit of an European put option at expiration date from the point of view of the holder and plot them in in Figure 1.b [3].

$$\text{Payoff}: \ \max\{K - S(T), 0\}, \qquad \text{Profit}: \ \max\{K - S(T), 0\} - P(t_i, T, K) \tag{5}$$

The variables K, T, t_i, $S(T)$ are the same as in the case of the call option and $P(t_i, T, K)$ is the price of the put option (the premium) when it is bought at initial time t_i. For European and American put options, the general payoff function at a time t from the point of view of the holder would be:

$$\text{Payoff: } \max\{K - S(t), 0\} \tag{6}$$

Once more, the shape of this payoff function will be the same as the shape of the payoff function in Figure 1.b, but for another $S(t)$.



(a) Call option

(b) Put option

Figure 1: Payoff function and profit function for European option at time T from the point of view of the holder.

3.3 Interest rate

When borrowing money or depositing it in a bank, there is a charge or benefit for the operation at the end. The parameter that measures this change of money is the **interest rate**. For instance, if an individual introduces an amount of money $X(t_0)$ in the bank at time t_0, the money at time t considering a continuous and constant interest rate r will be [4]:

$$X(t) = X(t_0)e^{r(t-t_0)} \tag{7}$$

This relation can also be expressed in differential form:

$$\frac{dX}{X} = rdt \tag{8}$$

The opposite calculation is known as the present value:

$$X(t_0) = X(t)e^{-r(t-t_0)} \tag{9}$$

When an investment has zero risk, the interest rate is called risk-free rate.

3.4 Arbitrage

The concept of **arbitrage** refers to the possibility of making instantaneous risk-less profit with an investment. This is obviously not a desirable situation in financial markets. In fact, most of the financial theories are developed assuming the absence of arbitrage.

It is sometimes possible to make risk-less profit in an investment, for instance, depositing money in the bank at a risk-free rate (8). However, this is not a situation of arbitrage since the profit is not instantaneous [4].

3.5 Dividends

Dividends are earnings that someone receives for owning an asset. The typical case is the company that distributes some of its earnings as dividends between its shareholders. Dividends are received at a specific time or periodically. In our topic of study, dividends will be given by the underlying asset.

3.6 Portfolio

A **portfolio** is a set of financial instruments like options, stocks, bonds or commodities, which aims to provide benefits. Portfolios typically tend to diversify investments in order to reduce the risk of loss [5].

3.7 Long position, short position

When talking about options, a **long position** refers to the situation of being the holder of an option. By contrast, a **short position** means that the investor sells the right of execution of the option.

In the case of stocks, the long position is to buy the stock, but the short position, which is usually known as **short selling**, consists on selling a stock that the investor does not own. The objective of short selling is to benefit from a fall in the price of the stock. The investor borrows a stock and immediately sells it to another person and, after some time, pays the lender the price of the stock at that time. If the price of the stock has fallen, the investor obtains profit [6].

4 Brownian motion

Once we have the main financial concepts, we are going to explain now several physical notions that will be used throughout the project. First of all, we need to known what is a Brownian motion and how does it govern a diffusion process. We will also find a solution to the diffusion equation. Moreover, we are going to introduce the mathematical formulation of the Brownian motion, known as the Wiener process.

4.1 Definition

The **Brownian motion** is the movement of particles in a fluid, which can be a gas or a liquid. Collisions with the molecules of the fluid make this movement random and unpredictable.

It was the botanist Robert Brown the first to observe the movement in 1827 when looking at pollen grains suspended in water. That is why it is called Brownian motion. He discovered minuscule particles randomly moving in a water drop. From that moment, many qualitative hypothesis were proposed by scientists, but it was not until 1905 when Albert Einstein developed a quantitative model. He based his theory in three main principles [7]:

- The existence of the particles.
- The movement of the particles in a fluid is due to the enormous number of collisions with the fluid molecules.
- The movement of the molecules is so complex that it can just be probabilistically described as a result of many independent hits.

As a consequence of the huge number of collisions per unit time that suffers a particle, Einstein studied the problem as a whole rather than individually for each particle. He worked with a density function of the particles $u(x, \tau)$ and discovered that it satisfied the diffusion equation [8]:

$$\frac{\partial u}{\partial \tau} = D\frac{\partial^2 u}{\partial x^2} \tag{10}$$

where x is the position variable that follows a Brownian motion and τ is the time. This equation suggests us that diffusion processes are governed by the Brownian motion. The density function of the particles suffers a diffusion as a result of the Brownian motion followed by the particles. The diffusion process can be seen as the macroscopic manifestation of the microscopic Brownian motion of particles in a fluid [9]. We are now going to analyse the diffusion equation in detail because it will be essential throughout the project.

4.2 Diffusion equation

The **diffusion equation** is a partial differential equation with several applications in physics, engineering and even finance as we will see in this work. It is widely used in physics to model the flow of heat in a continuous medium [4]. That is why it is sometimes known as the heat equation. The diffusion equation gives the time evolution (τ) of the probability density function u of a variable x that follows a Brownian motion, as we have recently seen. We are going to derive a solution for the equation.

The diffusion equation (10) is a linear, second order and parabolic equation. We are considering the forward equation in which the system evolves from initial time to the future. There are several solutions and among them one interesting solution can be obtained using the Fourier transform. The solution $u(x,\tau)$ can be written in terms of the inverse Fourier transform [10]:

$$u(x,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,\tau)e^{-ikx}dk \tag{11}$$

If we substitute this expression in the diffusion equation (10), we get an ordinary differential equation in terms of $\hat{u}(k,\tau)$:

$$\frac{\partial \hat{u}(k,\tau)}{\partial \tau} = -k^2 D\hat{u}(k,\tau) \tag{12}$$

Integrating equation (12)

$$\hat{u}(k,\tau) = \hat{u}_0 e^{-k^2 D\tau} \tag{13}$$

we reach to an expression for $\hat{u}(k,\tau)$ where $\hat{u}_0$ is an integration constant. Introducing this result in the inverse Fourier transform (11), we get a solution $u(x,\tau)$:

$$\begin{aligned} u(x,\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_0 e^{-k^2 D\tau}e^{-ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_0 e^{-D\tau\left[\left(k+\frac{ix}{2D\tau}\right)^2+\frac{x^2}{4D^2\tau^2}\right]}dk = \\ &= \frac{\hat{u}_0}{2\pi}e^{-\frac{x^2}{4D\tau}}\int_{-\infty}^{\infty} e^{-D\tau\left(k+\frac{ix}{2D\tau}\right)^2}dk = \frac{\hat{u}_0}{2\pi}e^{-\frac{x^2}{4D\tau}}\sqrt{\frac{\pi}{D\tau}} = \frac{\hat{u}_0}{2\sqrt{\pi D\tau}}e^{-\frac{x^2}{4D\tau}} \end{aligned} \tag{14}$$

To solve the integral we have completed the square in the exponent and later used $\int_{-\infty}^{\infty} e^{-\alpha y^2}dy = \sqrt{\pi/\alpha}$. This solution of the diffusion equation is known as the **Green's function** and is of particular relevance. We are going to denote it $G(x,\tau)$ ($\hat{G}_0$ is the same as $\hat{u}_0$) [10].

$$G(x,\tau) = \frac{\hat{G}_0}{2\sqrt{\pi D\tau}}e^{-\frac{x^2}{4D\tau}} \tag{15}$$

Green's function is a Gaussian function which spreads out with time since its variance depends on τ. The variance is $2D\tau$, so it also depends on D. This means that the greater D is, the more the function spreads out. In the limit where $\tau \longrightarrow 0$, the Gaussian becomes a Dirac's delta:

$$\lim_{\tau\longrightarrow 0}\frac{\hat{G}_0}{2\sqrt{\pi D\tau}}e^{-\frac{x^2}{4D\tau}} = \delta(x) \tag{16}$$

These two situations suggest that Green's function can reflect, for instance, the behaviour of heat injected at a single point. When we inject heat at time $\tau = 0$ at a point x_1, the heat is concentrated at that point and the function $G(x - x_1, \tau = 0)$ is a Dirac's delta $\delta(x - x_1)$. As time goes by, the heat spreads out from x_1 according to $G(x - x_1, \tau)$ as part of the diffusion process.

If we injected the same amount of heat at two points x_1 and x_2, the solution to the diffusion process would be proportional to $G(x - x_1, \tau) + G(x - x_2, \tau)$. Another possible situation could be injecting heat according to a smooth distribution $\rho(x')$. In this case, the solution to the diffusion process, $H(x,\tau)$, would be the sum over all points x' of every Green's function, weighted by $\rho(x')$ [10]:

$$H(x,\tau) = \int_{-\infty}^{\infty} G(x-x',\tau)\rho(x')dx' \tag{17}$$

4.3 Wiener process

Nowadays the Brownian motion is widely used to study stochastic processes. In 1918 Norbert Wiener made a mathematical formulation of the Brownian motion known as the **Wiener process** W. He stated that a Wiener process is a stochastic process continuous in time and that is characterized by the following aspects [11]:

- At initial time $W(0) = 0$
- For $0 \leq s \leq t$, $W(t) - W(s)$ is normally distributed with mean value 0 and variance $t - s$: $W(t) - W(s) \sim N(0, t-s)$
- The increments of W are independent: for any times $0 \leq t_1 < t_2 < ... < t_n$, $W(t_2) - W(t_1)$, $W(t_3) - W(t_2)$, ..., $W(t_n) - W(t_{n-1})$ are independent
- The sample paths $W(t)$ are continuous functions of time t

5 Asset pricing

The first step in our way to price options is to analyse the behaviour of the underlying asset and the changes in its price. Trying to price an asset has always been one of the most challenging tasks in the world of finance. In fact, there is no perfect method to do it. In this context, we have two opposed types of theories that attempt to predict the price of and asset: deterministic and non-deterministic. Deterministic theories believe that asset price does not completely follow a random walk and that thanks to past information of prices, together with some technical analysis, it can be predicted to some extent, for instance using Machine Learning. Conversely, non-deterministic models maintain that asset prices evolve according to random walks and that they cannot be predicted [12]. This project develops a random walk model that is widely used for option pricing. It is based on the definition of Wiener process we have given.

5.1 Random walk model

The **random walk model** aims to predict the changes in the price of an asset along time. It is influenced by Bachelier's Ph.D. thesis *The theory of speculation*, which, as we have mentioned before, uses the Brownian motion to approximate asset prices. The random walk model assumes that the efficient market hypothesis (EMH) is verified. This means that past data is completely displayed in the price of an asset and that any new information about the asset is immediately reflected in its price.

It is more interesting to analyze the return of the asset price rather than the absolute change. The return refers to the relative change in the price, this is, if S is the price of the asset, dS is the absolute change and dS/S the return.

Let us suppose now that the price of an asset is $S(t)$ at time t. After a small time interval dt the asset price would become $S + dS$. We want an expression for the return

dS/S. This expression can be divided into two parts: one deterministic and the other non-deterministic [4].

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{18}$$

The deterministic part, μdt, reflects the return on money invested in a risk-free bank. μ is the drift rate or average rate of growth of S. Depending on the case, it is constant or it can be a function of S and t. This term of the return corresponds to the expression (8), but this time the rate of growth is an average.

The non-deterministic part of the return, σdW, refers to the external factors that cannot be predicted. W is the Wiener process or Brownian Motion previously explained. σ is a parameter to scale the level of randomness generated from the Brownian Motion, called the **volatility**.

Expression (18) is a stochastic differential equation (SDE). The first term shows the general trend of the asset price and the second term reflects the random variations around that tendency. One interesting way of writing the Brownian motion is:

$$dW = z\sqrt{dt} \tag{19}$$

where z is a standard normal distribution with mean value 0 and variance 1: $z \sim N(0,1)$. We knew that dW had mean value 0 as it is a Wiener process, so this condition is kept with the change. The term $\sqrt{dt}$ ensures that the variance of dW is dt (another condition from the definition of W). Introducing expression (19) into (18), we get that the return of the asset price is:

$$\frac{dS}{S} = \mu dt + \sigma z\sqrt{dt} \tag{20}$$

Before we continue with our study of the asset price, we are going to introduce a mathematical concept that we need to find a solution for the SDE (20). We refer to Itô's Lemma.

5.2 Itô's Lemma

Itô's Lemma is a rule used to find the differential of a function that depends on time and a stochastic process. We are going to derive it.

Let us consider a function $f(t, X(t))$ that depends on time t and a stochastic process $X(t)$. This process $X(t)$ verifies a general stochastic differential equation

$$dX = \lambda dt + \beta dW \tag{21}$$

where λ is the drift, β the volatility and W a Brownian motion.

We can write the differential of $f(t, X(t))$ based on Taylor's expansion:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2 + \frac{\partial^2 f}{\partial t \partial X}dtdX + \dots \tag{22}$$

If we suppose a continuous time model, we can consider that $dt \longrightarrow 0$. In this context, we can discard some terms in (22). The term of $(dt)^2$ is really small compared to the others so we can get rid of it. We can manipulate the term of $dtdX$ to analyse its order:

$$dtdX = dt(\lambda dt + \beta dW) = \lambda(dt)^2 + \beta dtdW \tag{23}$$

Once again we can cancel the term of $(dt)^2$ and, remembering that $dW \sim \sqrt{dt}$, we have that $dtdW \sim dt\sqrt{dt}$, so we can also take out the term of $dtdW$ as it is smaller than the others. After these approximations, the differential of the function is:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2 \tag{24}$$

If we now calculate $(dX)^2$:

$$(dX)^2 = (\lambda dt + \beta dW)^2 = \lambda^2(dt)^2 + 2\lambda\beta dtdW + \beta^2(dW)^2 \tag{25}$$

The terms of $(dt)^2$ and $dtdW$ are negligible one more time. For the remaining term, we switch dW by the expression (19):

$$(dX)^2 = \beta^2(dW)^2 = \beta^2(z\sqrt{dt})^2 \tag{26}$$

We now apply the following argument: the variance of dW has to be dt according to the second feature explained in Section 4.3 for the Wiener process definition. The variance of dW is also obtained through the formula:

$$\mathrm{Var}[dW] = E[(dW)^2] - E[(dW)]^2 \tag{27}$$

The expected value of dW, $E[(dW)]$, is 0 because the mean value of z is also 0. Therefore, the variance is:

$$\mathrm{Var}[dW] = E[(dW)^2] = dt \tag{28}$$

Using this result for relation (25)

$$(dX)^2 = \beta^2 dt \tag{29}$$

and substituting $(dX)^2$ in df (24), we get the following expression:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\beta^2\frac{\partial^2 f}{\partial X^2}dt \tag{30}$$

This relation is known as Itô's Lemma and is essential for our next steps.

5.3 Geometric Brownian motion

We are now going to use Itô's Lemma to find a solution for the SDE (18). Let us consider a function $f(S) = \ln(S)$ that only depends on the asset price $S(t)$. By analogy, $S(t)$ corresponds to $X(t)$ in the previous derivation. Comparing expressions (18) and (21), we know that, in this case, $\lambda = \mu S$ and $\beta = \sigma S$.

If we apply Itô's Lemma to the function $f(S)$, we have that [4]:

$$df = 0 + \frac{\partial f}{\partial S}dS + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}dt = \frac{1}{S}dS - \frac{1}{2}\sigma^2 S^2\frac{1}{S^2}dt \tag{31}$$

We can now introduce relation (18) for dS into (31) and get that the logarithm of the asset price also has Brownian motion:

$$df = d(\ln(S)) = \frac{1}{S}(\mu S dt + \sigma S dW) - \frac{1}{2}\sigma^2 dt = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW \tag{32}$$

If we integrate both sides of this equation for a given time interval t, we reach to:

$$\ln S - \ln S_0 = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) dt + \int_0^t \sigma dW \tag{33}$$

where $S \equiv S_t$. In case μ and σ are constants, we can obtain an explicit result for the right-hand side of the expression:

$$\ln S - \ln S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma(W(t) - W(0)) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}z(t) \tag{34}$$

Here we have used relation (19) for $W(t) - W(0)$. As we can see, $f - f_0 = \ln S - \ln S_0$ has a normal distribution. The mean value and the variance are the following:

$$E\left[\ln \frac{S}{S_0}\right] = E\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}z(t)\right] = \left(\mu - \frac{1}{2}\sigma^2\right) t \tag{35}$$

$$\mathrm{Var}\left[\ln \frac{S}{S_0}\right] = \mathrm{Var}\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}z(t)\right] = \mathrm{Var}\left[\sigma\sqrt{t}z(t)\right] = \sigma^2 t \tag{36}$$

With this results, we can build the probability density function of $f - f_0$:

$$\phi(f - f_0, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\left[f - f_0 - \left(\mu - \frac{1}{2}\sigma^2\right)t\right]^2 / 2\sigma^2 t} \tag{37}$$

Comparing this expression with the solution we found for the diffusion equation, that is, the Green's function (15), we can see they have the same form. In fact, (37) is a Green's function. This suggests us that the evolution of $\phi(f - f_0, t)$ as the time interval t grows is like a diffusion process and makes sense because it depends on a variable $f = \ln S$ which, as we have seen in (32), follows a Brownian motion. The variance (36) depends on time interval t and on the volatility σ. Consequently, the Gaussian probability density function (37) spreads out as time interval increases and as the volatility increases. On the other hand, the mean value (35) depends on the average rate of growth μ, on time interval t and on the volatility σ. In this case, the mean value moves to the right as t increases, but the volatility contributes oppositely to the movement.

In Figure 2 we can visualize the probability density function (37) for different values of time interval t and volatility σ. The time is measured in years (0.25 year = 3 months) and the volatility in $1/\sqrt{\text{year}}$. The average rate of growth is $\mu = 0.05$ (5%) for all cases. We can see that the greater the time interval and the volatility are, the more the function "diffuses" and so it is more difficult to predict the return of the asset price. This makes sense because it is harder to make predictions in a distant future and with a greater volatility (level of randomness). For the lateral movement of the function we cannot deduce a general tendency since time and volatility row in the opposite direction. However, we can deduce that if the mean value is negative, then the value of the asset price is expected to decrease after time interval t, and if it is positive, the asset price is expected to increase.

If the logarithm of the asset price has a normal distribution, then the asset price S has a lognormal distribution. Equation (34) can also be written in the exponential form:

$$S = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}z(t)} \tag{38}$$

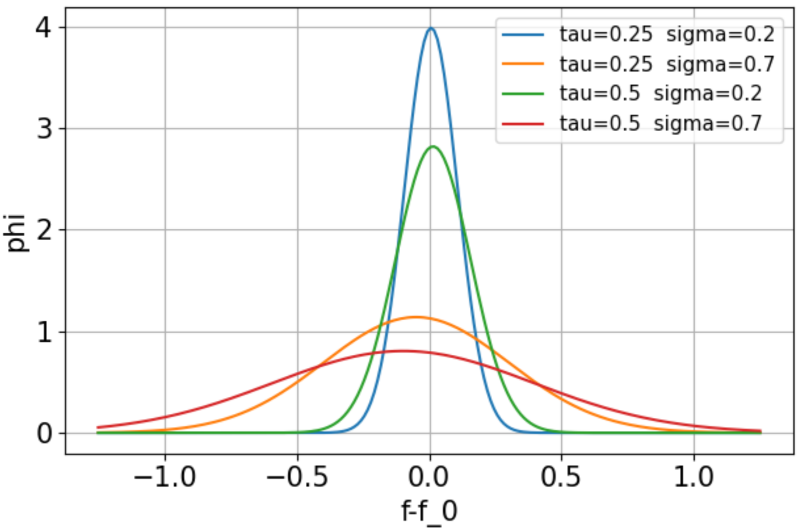


Figure 2: Probability density function of $f - f_0$ for different values of t and σ and for $\mu = 0.1$ in all cases.

This expression shows that the asset price follows a **Geometric Brownian motion**. The probability density function of S can easily be obtained manipulating the probability density function of $f - f_0$ (37):

$$\begin{aligned} \phi(f - f_0, t) d(f - f_0) &= \phi\left(\ln \frac{S}{S_0}, t\right) d\left(\ln \frac{S}{S_0}\right) = \\ &= \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\left[\ln \frac{S}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right]^2 / 2\sigma^2 t} \frac{dS}{S} = \Psi(S, t) dS \end{aligned} \tag{39}$$

$$\Psi(S, t) = \frac{1}{S\sqrt{2\pi\sigma^2 t}} e^{-\left[\ln \frac{S}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right]^2 / 2\sigma^2 t} \tag{40}$$

We are going to try to find the meaning of μ and σ. Let us calculate the expected value of the asset price S:

$$E[S] = \int_{-\infty}^{\infty} S\Psi(S, t) dS = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} e^{-\left[\ln \frac{S}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right]^2 / 2\sigma^2 t} dS \tag{41}$$

If we make the transformation $y = \ln S/S_0$, then $y \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$ and we get:

$$E[S] = S_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\left[y - \left(\mu - \frac{1}{2}\sigma^2\right)t\right]^2 / 2\sigma^2 t} e^y dy = S_0 E[e^y] = S_0 M_y(1) \tag{42}$$

$M_y(1)$ is the moment generating function of y, $M_y(q)$, with $q = 1$. Taking into consideration that given a normal distribution X with mean a and variance b^2, the moment generating function is $M_X(q) = E[e^{qX}] = e^{aq} e^{b^2 q^2/2}$, by analogy we reach to:

$$E[S] = S_0 M_y(1) = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t} e^{\frac{1}{2}\sigma^2 t} = S_0 e^{\mu t} \tag{43}$$

From expression (43) we can state that, as we had seen building the random walk model, μ is the average rate of growth of the asset price S (remember (7)). On the other hand, we had obtained before that the variance of the normal distribution of $\ln S/S_0$ was $\sigma^2 t$. This result shows that the square of the volatility, σ^2, is the variance per unit time of the distribution [3].

6 The Black-Scholes model

Once we have studied the behaviour of the asset price, we are going to apply the Black-Scholes analysis that leads to the partial differential equation used to price options. The **Black-Scholes model** was developed in 1973 by Fischer Black and Myron Scholes and became a reference for many other subsequent models. They received a Nobel Prize in 1997 for their work. Nevertheless, as we have said before, the model makes certain assumptions that are not very realistic in real markets, but qualitatively is a great first approximation. Before introducing it, we are going to explain the assumptions [4]:

- The underlying asset price follows the Geometric Brownian motion. This does not mean that the Black-Scholes analysis cannot be applied with any other model that is not the one of the random walk previously explained.
- The risk-free interest rate r and the volatility σ are known during the life of the option.
- The underlying asset pays no dividends.
- It is possible to buy or sell a fractional number of the underlying asset and short selling is allowed.
- There is no possibility of arbitrage.
- There are no transaction costs for buying or selling an option or underlying asset.

Let us suppose that the price of an option $V(S,t)$ (does not matter if it is a call or put option) depends only on the underlying asset price S and time t. Applying Itô's Lemma (30), taking into consideration that $X \equiv S$, $\lambda \equiv \mu S$ and $\beta \equiv \sigma S$, we can write [4]:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \tag{44}$$

If we introduce relation (18) for dS into (44), we reach to the expression that gives the random walk followed by the option price V:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW \tag{45}$$

Now we are going to build a portfolio made up of one option (long position) and the short sell of Δ shares of the underlying asset. The value of the portfolio, Π, in this case is:

$$\Pi = V - \Delta S \tag{46}$$

and the differential of this value will be:

$$d\Pi = dV - \Delta dS \tag{47}$$

Substituting expressions (18) and (45) into (47), we get that the value of the portfolio Π also follows a random walk:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mu\Delta S\right)dt + \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dW \tag{48}$$

Regarding this expression, we can choose Δ so that we eliminate the random component containing dW:

$$\Delta = \frac{\partial V}{\partial S} \tag{49}$$

As we can see, in this case Δ is the rate of change of the option price V with respect to the underlying asset price S. The result is that the change in the value of the portfolio is deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{50}$$

We are now going to consider some arguments related to arbitrage. We suppose that there are no transaction costs for buying or selling an option or underlying asset, as said before. According to relation (8), when investing an amount Π at a risk-free rate r, the return of the investment after a time t will be:

$$\frac{d\Pi}{\Pi} = r dt \tag{51}$$

As a result, the change in Π is:

$$d\Pi = \Pi r dt \tag{52}$$

The right-hand side of (50) has to be equal to the right-hand side of (52) so that there is no arbitrage. If it was greater, anyone could borrow an amount Π at a risk-free rate r and invest it in the portfolio. The result would be a riskless profit because the increment in the value of the portfolio (right-hand side of (50)) is greater than the amount that has to be returned for borrowing (right-hand side of (52)). This means that there is arbitrage. If, in contrast, the right-hand side of (50) was smaller than the right-hand side of (52), it would be possible to short sell the portfolio (sell the option V and buy Δ shares of S) and invest that Π amount in a bank at a risk-free rate r. Once again, there is riskless profit and, as a result, arbitrage.

As we have stated that there is no place for arbitrage, we must impose that:

$$\Pi r dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{53}$$

Dividing by dt at both sides and introducing expressions (46) and (49) into (53):

$$\left(V - \frac{\partial V}{\partial S} S \right) r = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \tag{54}$$

we reach to:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{55}$$

This is known as the **Black-Scholes partial differential equation (PDE)**. It is a linear, parabolic and backward in time equation. Any option whose price only depends on time t and the underlying asset price S, must verify this equation, as long as the assumptions made so far are verified. Consequently, solving the Black-Scholes equation gives way to pricing options.

It is interesting to remark that μ, the average rate of growth of the underlying asset price S, does not appear in the equation. The only parameter from the stochastic differential equation (18) that affects the price of the option is the volatility σ. This means that, although there might be a discrepancy between people in the estimation of μ, the price of the option would be the same anyway [4].

7 The Black-Scholes formula for European options

7.1 Derivation

We have already found a PDE to price options, so now it is time to look for solutions that satisfy it. The **Black-Scholes formula** is a well-known expression that gives price to European options. It can be obtained solving the Black-Scholes equation (55) with boundary and final conditions.

We are going to derive the formula for an European put option $P(S,t)$. Remember that an European put option granted its holder the right to sell the underlying asset only at expiration date T. We start from the Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \tag{56}$$

The equation is backward in time due to the opposite sign of term $\partial P/\partial t$ with respect to the other partial derivatives $\partial^2 P/\partial S^2$ and $\partial P/\partial S$. For that reason, we need a final condition to solve the equation, this is, a condition at expiration date T. Regarding expression (5) for the payoff of an European put option, we determine that the option price at T has to be equal to the payoff to avoid arbitrage [4]:

$$P(S,T) = \max\{K - S, 0\} \tag{57}$$

We also require boundary conditions. According to the final condition (57), when the asset price is 0 at expiration date T, the price of the option is K: $P(0,T) = K$. If we want to determine $P(0,t)$, we just have to calculate the present value (9) of $P(0,T)$. Assuming a constant interest rate r, we get that the first boundary condition is:

$$P(0,t) = Ke^{-r(T-t)} \tag{58}$$

On the other hand, when the asset price tends to ∞, the put option is very unlikely to be exercised, so it loses its value and the second boundary condition is:

$$P(S,t) \longrightarrow 0, \quad \text{as } S \longrightarrow \infty \tag{59}$$

Once we have the final and boundary conditions, we are going to solve the equation. We are going to make some transformations to try to convert the Black-Scholes equation for the put option (56) into a diffusion equation(10). The solutions for the forward diffusion equation are already known for us. However, we have seen that the Black-Scholes is backward. Since we want to convert it into a forward diffusion equation, we introduce a new time variable $\tau = T - t$. Let us now assume the following transformations [10]:

$$x = \ln\frac{S}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau \tag{60}$$

$$P(S,t) = p(x,\tau) \tag{61}$$

$$p(x,\tau) = e^{-r\tau} g(x,\tau) \tag{62}$$

With this modifications, we have that

$$\begin{aligned} \frac{\partial P}{\partial t} = -\left(\frac{\partial p}{\partial \tau} + \frac{\partial p}{\partial x}\frac{\partial x}{\partial \tau}\right) &= -\frac{\partial\left(e^{-r\tau}g\right)}{\partial \tau} - \frac{\partial\left(e^{-r\tau}g\right)}{\partial x}\frac{\partial x}{\partial \tau} \\ &= -e^{-r\tau}\left[-rg + \frac{\partial g}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial g}{\partial x}\right] \end{aligned} \tag{63}$$

$$\frac{\partial P}{\partial S} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial S} = e^{-r\tau}\frac{\partial g}{\partial x}\frac{1}{S} \tag{64}$$

$$\begin{aligned} \frac{\partial^2 P}{\partial S^2} = \frac{\partial}{\partial S}\left(e^{-r\tau}\frac{\partial g}{\partial x}\frac{1}{S}\right) &= \frac{\partial}{\partial S}\left(e^{-r\tau}\frac{\partial g}{\partial x}\right)\frac{1}{S} + e^{-r\tau}\frac{\partial g}{\partial x}\left(-\frac{1}{S^2}\right) = \\ = \frac{\partial}{\partial x}\left(e^{-r\tau}\frac{\partial g}{\partial x}\right)\frac{\partial x}{\partial S}\frac{1}{S} - e^{-r\tau}\frac{\partial g}{\partial x}\frac{1}{S^2} &= \frac{e^{-r\tau}}{S^2}\left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial x}\right) \end{aligned} \tag{65}$$

and so equation (56) becomes

$$e^{-r\tau}\left[-rg + \frac{\partial g}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial g}{\partial x}\right] = \frac{1}{2}\sigma^2 S^2 \frac{e^{-r\tau}}{S^2}\left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial x}\right) + rS\frac{e^{-r\tau}}{S}\frac{\partial g}{\partial x} - re^{-r\tau}g \tag{66}$$

Simplifying we get:

$$\frac{\partial g}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2} \tag{67}$$

We have reached a diffusion equation as we wanted. We had previously found that Green's function (15) is a fundamental solution to this equation, so, in this case, we can write (we take $\hat{G}_0 = 1$):

$$g(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{x^2}{2\sigma^2\tau}} \tag{68}$$

Adding the factor $e^{-r\tau}$ given by (62) to this result, we get Green's function for the Black-Scholes equation [10]:

$$G(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{x^2}{2\sigma^2\tau} - r\tau} \tag{69}$$

If we reverse the transformations made, it can be verified that (69) satisfies equation (56). However, it does not satisfy all the conditions. For instance, the final condition (57), which turns into an initial condition with the change $\tau = T - t$, is not verified as $G(x,0) \longrightarrow \delta(x)$.

In Section 4.2 we saw that thanks to the linearity of the diffusion equation, there was another way to obtain a solution using an initial distribution and Green's function.

We refer to expression (17). This means that we can turn our problem of valuing a put option into a diffusion problem where the final condition (57) (initial condition in time framework τ) is the initial distribution. Green's function acts as a propagator, backwards in time t and forward in time τ, of the initial distribution. This initial distribution has the following expression:

$$e^{r\tau}\max\{K-S,0\}|_{\tau=0}=\max\{K-S,0\} \tag{70}$$

The first factor $e^{r\tau}$ comes from transformation (62) and the second one, which is $\max\{K-S,0\}$, from the final condition. If we introduce a variable $x'=\ln(S/K)$, which corresponds to x when $\tau=0$, the initial distribution can be written as

$$\max\{K-S,0\}=K\max\{1-\frac{S}{K},0\}=K\max\{1-e^{x'},0\} \quad \text{at } \tau=0 \tag{71}$$

Now that we have the initial distribution, we can calculate its time τ evolution integrating the contribution of all points x' with their Green's function as in (17). Each point x' represents a possible value of S at $\tau=0$ (at $t=T$). Green's function $G(x-x',\tau)$ acts as the propagator from x' to x [10].

$$p(x,\tau)=\int_{-\infty}^{\infty}K\max\{1-e^{x'},0\}G(x-x',\tau)dx' \tag{72}$$

Regarding the initial distribution (71), we can see that it can be divided into two parts: $K(1-e^{x'})$ when $x'<0$ and 0 when $x'>0$. This means that we can just integrate (72) from $-\infty$ to 0.

$$\begin{aligned}
p(x,\tau)&=\int_{-\infty}^{0}K\left(1-e^{x'}\right)G(x-x',\tau)dx'=\int_{-\infty}^{0}K\left(1-e^{x'}\right)\frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-\frac{(x-x')^2}{2\sigma^2\tau}-r\tau}dx'\\
&=\frac{Ke^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{0}e^{-\frac{(x'-x)^2}{2\sigma^2\tau}}dx'-\int_{-\infty}^{0}e^{-\frac{(x'-x)^2}{2\sigma^2\tau}+x'}dx'\right)\\
&=\frac{Ke^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{0}e^{-\frac{(x'-x)^2}{2\sigma^2\tau}}dx'-\int_{-\infty}^{0}e^{-\frac{\left[(x'-(x+\sigma^2\tau)\right]^2+x^2-(x+\sigma^2\tau)^2}{2\sigma^2\tau}}dx'\right)\\
&=\frac{Ke^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\left(\int_{-\infty}^{0}e^{-\frac{(x'-x)^2}{2\sigma^2\tau}}dx'-e^{x+\frac{\sigma^2\tau}{2}}\int_{-\infty}^{0}e^{-\frac{(x'-x-\sigma^2\tau)^2}{2\sigma^2\tau}}dx'\right)=I_1+I_2
\end{aligned} \tag{73}$$

In the second term we have completed the square in the exponent. Before we continue integrating, we need to introduce the standard normal cumulative distribution function:

$$N(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-\frac{1}{2}z^2}dz \tag{74}$$

This function represents the probability of a random variable Y, which has a normal distribution, to be less or equal to y. Once we have this expression, we are going to use it in (73).

$$I_1=\frac{Ke^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{0}e^{-\frac{(x'-x)^2}{2\sigma^2\tau}}dx'=\frac{Ke^{-r\tau}}{\sqrt{2\pi}}\int_{-\infty}^{-x/\sqrt{\sigma^2\tau}}e^{-\frac{z^2}{2}}dz=Ke^{-r\tau}N\left(-\frac{x}{\sqrt{\sigma^2\tau}}\right) \tag{75}$$

To solve this term, we have made a change of variable $z = \frac{x'-x}{\sqrt{\sigma^2\tau}}$. For the second term, the transformation is $z = \frac{x'-x-\sigma^2\tau}{\sqrt{\sigma^2\tau}}$.

$$\begin{aligned} I_2 &= -\frac{Ke^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} e^{x+\frac{\sigma^2\tau}{2}} \int_{-\infty}^{0} e^{-\frac{(x'-x-\sigma^2\tau)^2}{2\sigma^2\tau}} dx' = -Ke^{-r\tau} e^{x+\frac{\sigma^2\tau}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(-x-\sigma^2\tau)/\sqrt{\sigma^2\tau}} e^{-\frac{z^2}{2}} dz \\ &= -Ke^{-r\tau} e^{x+\frac{\sigma^2\tau}{2}} N\left(-\frac{x+\sigma^2\tau}{\sqrt{\sigma^2\tau}}\right) \end{aligned} \tag{76}$$

Joining both terms, we reach:

$$p(x,\tau) = I_1 + I_2 = Ke^{-r\tau} N\left(-\frac{x}{\sqrt{\sigma^2\tau}}\right) - Ke^{-r\tau} e^{x+\frac{\sigma^2\tau}{2}} N\left(-\frac{x+\sigma^2\tau}{\sqrt{\sigma^2\tau}}\right) \tag{77}$$

If we now replace x and τ by their original expressions, we get that [10]:

$$P(S,t) = Ke^{-r(T-t)} N\left(-\frac{\ln\frac{S}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}\right) - SN\left(-\frac{\ln\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}\right) \tag{78}$$

This is the well-known Black-Scholes formula for a put option. It gives price to European put options as a function of the underlying asset price and time. Remember that the formula was derived considering constant interest rate r and volatility σ during the life of the option. For American put options this formula is not valid because they can be exercised at any time before maturity T and so, as we will see later, the boundary conditions are not the same.

The Black-Scholes formula for a call option can be obtained following the same process, but with different boundary and final conditions. The formula in that case is [10]:

$$C(S,t) = SN\left(\frac{\ln\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}\right) - Ke^{-r(T-t)} N\left(\frac{\ln\frac{S}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}}\right) \tag{79}$$

and the final and boundary conditions are [4]

$$C(S,T) = \max\{S-K, 0\} \tag{80}$$

$$C(0,t) = 0 \qquad C(S,t) \sim S \quad \text{as } S \longrightarrow \infty \tag{81}$$

7.2 Numerical example

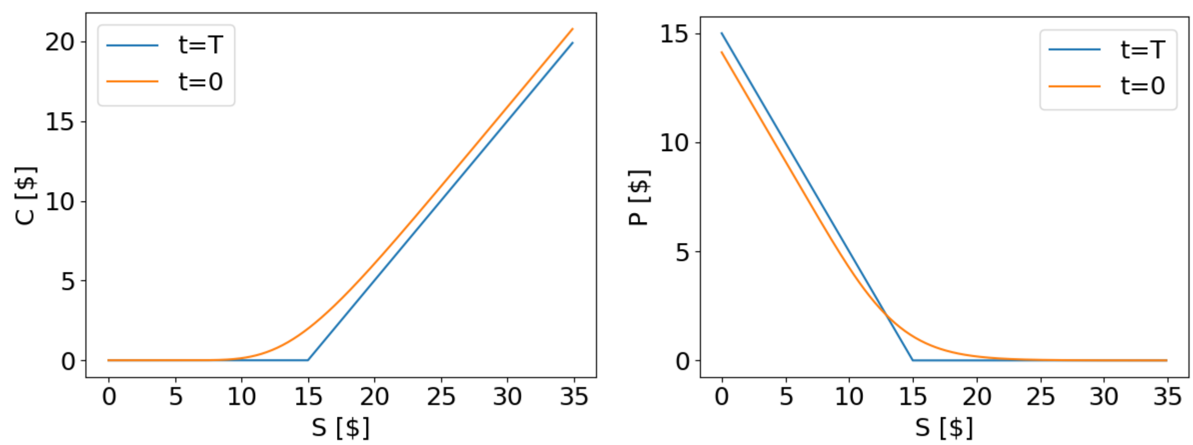
We are now going to use both Black-Scholes formulas (78)(79) for a numerical example. We consider the following parameters:

- Risk-free interest rate r=0.08 (8%)
- Volatility σ=0.3 $\sqrt{year}^{-1}$ (30%)
- Expiration date $T = 0.75$ year (9 months)
- Strike price $K = 15\$$

We first plot in Figure 3.a the value of the European call option C as a function of the price of the underlying asset S for a given time $t = 0$. The plot also contains the value of the option at expiration $t = T$, which according to (80), is equal to the payoff function. Regarding the curve for $t = 0$, we can see that when S tends to 0, the option price C also approaches 0, and as S increases considerably, the value of the option linearly grows with S with unitary slope. This means that the boundary conditions (81) are verified.

On the other hand, we represent in Figure 3.b the value of the European put option P as a function of the price of the underlying asset S for a given time $t = 0$. Once again, the plot also contains the value of the option at expiration $t = T$, which is equal to the payoff function as the final condition (57) enforces. In the case of $t = 0$, when the underlying asset price S is 0, the value of the option is slightly smaller that the strike price $K = 15\$$, particularly $Ke^{-r(T-t)}$. This value is the same as the boundary condition (58). When S increases significantly, the option price P tends to 0, the same way as in the other boundary condition (59).

As we have seen, in both cases the final and boundary conditions are satisfied, so the behaviour of the options value was the expected.



(a) $C(S)$ for $t = 0$ and $t = T$

(b) $P(S)$ for $t = 0$ and $t = T$.

Figure 3: $C(S)$ and $P(S)$ for $t = 0$ and $t = T$.

8 American options

We have solved the Black-Scholes equation for European call and put options, but we still do not have the way to price American options. We have seen that we can not explicitly solve the equation for American options due to the possibility of early exercise. It is time now to analyse the problem of pricing American options and find a solution for it.

8.1 General concepts

American options, unlike European options, can be exercised at any time before expiration date. We are soon going to see that the possibility of early exercise changes the boundary conditions, so the Black-Scholes formula does not apply to those cases. In fact, the possibility of early exercise gives the holder more flexibility, so we could expect the value of an American option to be higher than the value of an European option. This can be shown using arbitrage arguments.

We are going to analyse the case of a put option. There are values of the underlying asset price S for which the value of an European put option is less than the payoff function $\max\{K-S,0\}$. As a result, if we consider that the price of an American put option is the same as the European put option, we can buy an American option for P and immediately exercise it for K, obtaining a profit of $K-S-P$ without risk. There is arbitrage. As an example, we consider the case of $S=0$. According to the payoff function $\max\{K-S,0\}$, the payoff is K. If we remember that $S=0$ was a boundary condition (58) with option value $Ke^{-r(T-t)}$, we can see that the payoff function is greater than the value of the option, $K > Ke^{-r(T-t)}$, so we are in the situation described before. This can be visualised in Figure 3.b. To avoid arbitrage, we must impose the following condition for American put options [4]:

$$P(S,t) \geq \max\{K-S,0\} \tag{82}$$

Similarly, for American call options the condition would be:

$$C(S,t) \geq \max\{S-K,0\} \tag{83}$$

At expiration date, the price of American options has to be equal to the payoff function, that is, the price of the American options is equal to the price of the European options given by the Black-Scholes formula.

$$P(S,T) = \max\{K-S,0\} \qquad C(S,T) = \max\{S-K,0\} \tag{84}$$

American call options are special since they should never be exercised before maturity unless they pay dividends. For now, we have always considered options with no dividends and we are going to continue doing the same. This is just a remark to clarify that American call options are just exercised at expiration date. The reason for this is that early exercise requires the immediate payment of the strike price K and it is more profitable to keep that money, for instance in the bank, with its risk-free interest rate until maturity. This can shown building two portfolios as it is explained in Ref[13]:

- Portfolio A: American call option C + money in the bank $Ke^{-r(T-t)}$
- Portfolio B: one share S

If we exercise the call option in portfolio A at early time $t_{ex} < T$ because $S > K$, the payoff will be $S - K + Ke^{-r(T-t_{ex})}$, which is smaller than S in portfolio B. If otherwise we exercise the option at T, the payoff of A will be $\max(S-K,0)+K = \max(S,K)$, which is always greater than or equal to S in B. Consequently, non-dividend paying American call options should only be exercised at maturity and its price is the same as European call options.

We are now going to focus on the study of American put options since for them early exercise is optimal in some cases.

8.2 American put options

As early exercise is possible for American put options, there must also be some values of S for which the exercise of the option is optimal before maturity. At each time t, there are two regions for S: one with the values of S for which the option should be exercised and the other with the values of S for which the option should be hold. The point that marks the boundary between those two regions at time t is referred as the **optimal exercise price** and it is denoted by $S_f(t)$. This optimal exercise price is unknown to us. That is why the problem of pricing an American put option is called a **free boundary problem**. We have to solve a problem divided in two regions where we do not know where is the boundary. We will have have to find out this boundary as part of the solution.

Free boundary problems are really common in physics. For instance, the Stefan problem that describes the joint evolution of a liquid and a solid phase is a free boundary problem [14]. Another typical example is the obstacle problem, which consists in finding the equilibrium configuration of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle [15].

We are now going to formulate the free boundary problem for American put options. Let us consider a put option with value $P(S,t)$. As we have seen before, the value of the option verifies that [4]

$$P(S,t) \geq \max\{K-S,0\} \tag{85}$$

and the final condition is

$$P(S,T) = \max\{K-S,0\} \tag{86}$$

An American put option should be early exercised at a time $t < T$ when the value of S is lower than or equal to the optimal exercise price $S_f(t)$. In those cases, the price of the option must be $P(S,t) = \max\{K-S,0\}$. Conversely, if $S > S_f$, the option should be hold since its price is $P(S,t) > \max\{K-S,0\}$ and it is more profitable to sell the option than to execute it. When $S > S_f$, the option price follows the Black-Scholes equation. The combination of this two facts makes the Black-Scholes equation (56) become an inequality.

We can now analyse how the Black-Scholes equation becomes an inequality. We build the same portfolio as in (46), with the same value of delta (49). For the put option, it would be:

$$\Pi = P - \frac{\partial P}{\partial S} S \tag{87}$$

As early exercise is possible with American options, the arbitrage argument used for European options, where the return of the portfolio had to be equal to the return of money deposited in the bank, is not valid. This means that expression (53) is not an equality anymore. In this case, all we can say is that the return of the portfolio cannot be greater than the money invested in the bank:

$$\Pi r dt \geq \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} \right) dt \tag{88}$$

If we introduce the value of the portfolio (87) into (88), we reach to:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0 \tag{89}$$

We can see that the Black-Scholes equation has become an inequality for American put options. The expression is an equality when the option should be hold and an inequality when the exercise is optimal, so we can write that for the region $0 \leq S < S_f(t)$:

$$P(S,t) = \max\{K - S, 0\} = K - S \tag{90}$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0 \tag{91}$$

and for the other region, $S_f(t) < S < \infty$:

$$P(S,t) > K - S \tag{92}$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \tag{93}$$

We now need to impose two conditions at the free boundary $S_f(t)$. We suppose that $S_f(t)$ is smaller than the strike price K so as to have a positive payoff. The first condition comes from the continuity of $P(S,t)$ at $S = S_f(t)$:

$$P(S_f(t), t) = \max\{K - S_f(t), 0\} = K - S_f(t) \tag{94}$$

The second one, from the continuity of the rate of change or delta (49) of $P(S,t)$ at $S = S_f(t)$:

$$\left. \frac{\partial P}{\partial S} \right|_{S=S_f} = -1 \tag{95}$$

To understand the reason for this second condition, we are going to consider the two other possible scenarios: $\partial P/\partial S < -1$ and $\partial P/\partial S > -1$. In the first case, an increase in S from $S_f(t)$ implies a decrease of $P(S,t)$ below $\max\{K - S, 0\}$. According to (85), $P(S,t)$ cannot be smaller than $\max\{K - S, 0\}$, so this situation is not possible. In the second case, a decrease in S from $S_f(t)$ induces a misvalued increment of the option value, giving rise to possibilities of arbitrage. This is not a desired situation. As a result, the only possible scenario is that $\partial P/\partial S = -1$ [4].

We have already formulated the free boundary problem to price American put options. We have the problem divided into two regions and we have two conditions at the boundary of the regions. We also have a final condition.

8.3 Solution to the free boundary problem

Now is time to find a way to solve the free boundary problem. For that, we need to use numerical analysis. However, first of all we are going to reduce the problem to what is known as a lineal complementarity problem. It basically consists on rewriting the problem in such a way that the explicit dependence on the free boundary is eliminated. We solve the problem without the influence of the free boundary condition and later recover it from the solution.

It is easier to work with a diffusion equation rather than the Black-Scholes equation since it has less partial derivative terms. That is why we are going to make again a transformation to the Black-Scholes equation, but this time is going to be different to the one used to derive the Black-Scholes formula. We start from equation (56) and we apply the following modifications [4]:

$$t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \tag{96}$$

$$x = \ln \frac{S}{K} \tag{97}$$

$$P(S,t) = Kp(x,\tau) \tag{98}$$

With these modifications we have that

$$\frac{\partial P}{\partial t} = K\frac{\partial p}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{K}{2}\sigma^2\frac{\partial p}{\partial \tau} \tag{99}$$

$$\frac{\partial P}{\partial S} = K\frac{\partial p}{\partial x}\frac{\partial x}{\partial S} = e^{-x}\frac{\partial p}{\partial x} \tag{100}$$

$$\frac{\partial^2 P}{\partial S^2} = \frac{\partial}{\partial S}\left(e^{-x}\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^{-x}\frac{\partial p}{\partial x}\right)\frac{\partial x}{\partial S} = \frac{e^{-2x}}{K}\left(\frac{\partial^2 p}{\partial x^2} - \frac{\partial p}{\partial x}\right) \tag{101}$$

and the Black-Scholes equation becomes

$$\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial x^2} + (q-1)\frac{\partial p}{\partial x} - qp \tag{102}$$

where $q = 2r/\sigma^2$. We still need to apply one more transformation before we reach the diffusion equation:

$$p(x,\tau) = e^{ax+b\tau}u(x,\tau) \tag{103}$$

$$\frac{\partial p}{\partial \tau} = e^{ax+b\tau}\left(bu + \frac{\partial u}{\partial \tau}\right) \tag{104}$$

$$\frac{\partial p}{\partial x} = e^{ax+b\tau}\left(au + \frac{\partial u}{\partial x}\right) \tag{105}$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial x}\left[e^{ax+b\tau}\left(au + \frac{\partial u}{\partial x}\right)\right] = e^{ax+b\tau}\left(2a\frac{\partial u}{\partial x} + a^2u + \frac{\partial^2 u}{\partial x^2}\right) \tag{106}$$

Introducing these terms into (102) we get:

$$bu + \frac{\partial u}{\partial \tau} = a^2 u + 2a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (q-1)\left(au + \frac{\partial u}{\partial x}\right) - qu \tag{107}$$

We can now choose a and b in such a way the terms of u and $\partial u/\partial x$ are canceled:

$$\begin{cases} b = a^2 + (q-1)a - q \\ 0 = 2a + \quad (q-1) \end{cases} \tag{108}$$

Solving the system of equations we find the values of a and b:

$$a = -\frac{1}{2}(q-1) \qquad\qquad b = -\frac{1}{4}(q+1)^2 \tag{109}$$

For this election of a and b, equation (107) turns into the diffusion equation [4]:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{110}$$

With all these changes, the optimal exercise price becomes $S_f(t) \longrightarrow x_f(\tau)$. Let us see what happens with the payoff function:

$$\max\{K - S, 0\} = K\max\{1 - e^{\ln\frac{S}{K}}, 0\} = K\max\{1 - e^x, 0\} \tag{111}$$

We have to add to this expression the factor $e^{\frac{1}{2}(q-1)x+\frac{1}{4}(q+1)^2\tau}/K$ given by (98) and (103) to complete the transformation. As a result, the payoff function is:

$$f(x,\tau) = e^{\frac{1}{4}(q+1)^2\tau}\max\{e^{\frac{1}{2}(q-1)x} - e^{\frac{1}{2}(q+1)x}, 0\} \tag{112}$$

Once we have the diffusion equation (110) and the payoff function (112), we can formulate the free boundary problem after the transformations [4]:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{when } x > x_f(\tau) \tag{113}$$

$$u(x,\tau) = f(x,\tau) \qquad \text{when } x \le x_f(\tau) \tag{114}$$

The final condition (57) is now an initial condition in τ framework:

$$u(x,0) = f(x,0) = \max\{e^{\frac{1}{2}(q-1)x} - e^{\frac{1}{2}(q+1)x}, 0\} \tag{115}$$

We must also remember that the property

$$u(x,\tau) \ge f(x,\tau) = e^{\frac{1}{4}(q+1)^2\tau}\max\{e^{\frac{1}{2}(q-1)x} - e^{\frac{1}{2}(q+1)x}, 0\} \tag{116}$$

has to be verified, as well as the continuity conditions of u and $\partial u/\partial x$ when $x = x_f(\tau)$.

The asymptotic behaviour of $u(x,\tau)$ is:

$$\lim_{x\to\infty} u(x,\tau) = 0 \tag{117}$$

$$\lim_{x\to-\infty} u(x,\tau) = f(x,\tau) \tag{118}$$

For the first asymptotic behaviour we have to remember what happened when $S \to \infty$ in (59) and for the second one, we known from (114) that when $x \to -\infty$, then $u = f$.

We are going to restrict the problem to a finite interval to have boundary conditions for the numerical resolution. We consider the problem for values of x within $-x^- < x < x^+$, being $-x^-$ and x^+ large enough . Now we have the boundary conditions

$$u(x^+, \tau) = 0 \qquad u(-x^-, \tau) = f(-x^-, \tau) \tag{119}$$

This second boundary condition is different from (58). That is why the Black-Scholes formula is not valid for American put options.

With all this information, we can already write the problem as a **linear complementarity problem** with an initial and boundary conditions [4]:

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \cdot \Big(u(x,\tau) - f(x,\tau)\Big) = 0 \tag{120}$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geq 0 \qquad \Big(u(x,\tau) - f(x,\tau)\Big) \geq 0 \tag{121}$$

The initial condition is

$$u(x,0) = f(x,0) \tag{122}$$

and the boundary conditions

$$u(x^+, \tau) = f(x^+, \tau) = 0 \qquad u(-x^-, \tau) = f(-x^-, \tau) \tag{123}$$

Moreover, u and $\partial u / \partial x$ must also be continuous. As we can see, writing the problem this way, we have eliminated the explicit dependence on the free boundary. The problem can now be solved without the free boundary condition and later retake it from the solution. This will make it easier for us to solve the problem using numerical methods, which is our following step.

8.4 Finite-difference formulation: The Crank-Nicolson method

We are going to apply the finite-difference formulation to the linear complementarity problem, particularly the one of the **Crank-Nicolson method**. The Crank-Nicolson method is a combination of the explicit (or Forward Time Central Space, FTCS) method and the implicit (or Backward Time Central Space, BTCS) method. It is a widely used method to solve partial differential equations, as it is the case of the diffusion equation, and it is interesting due to its unconditional stability. Let us see now how is the method derived.

First of all, we need to replace the partial derivatives of the diffusion equation by approximations based on Taylor series. In the explicit method, approximations for the partial derivatives at a point (x, τ) are forward in time and central in space:

$$\frac{\partial u}{\partial \tau}(x,\tau) \approx \frac{u(x, \tau + \delta\tau) - u(x,\tau)}{\delta\tau} + \mathcal{O}\left((\delta\tau)^2\right) \tag{124}$$

$$\frac{\partial^2 u}{\partial x^2}(x,\tau) \approx \frac{u(x+\delta x, \tau) - 2u(x,\tau) + u(x-\delta x, \tau)}{(\delta x)^2} + \mathcal{O}\left((\delta x)^2\right) \tag{125}$$

In the implicit method, approximations at point (x,τ) are backward in time and central in space. In this case, however, we need to write the approximations at point $(x,\tau+\delta\tau)$ instead of (x,τ):

$$\frac{\partial u}{\partial \tau}(x,\tau+\delta\tau) \approx \frac{u(x,\tau+\delta\tau)-u(x,\tau)}{\delta\tau} + \mathcal{O}\left((\delta\tau)^2\right) \tag{126}$$

$$\frac{\partial^2 u}{\partial x^2}(x,\tau+\delta\tau) \approx \frac{u(x+\delta x,\tau+\delta\tau)-2u(x,\tau+\delta\tau)+u(x-\delta x,\tau+\delta\tau)}{(\delta x)^2} + \mathcal{O}\left((\delta x)^2\right) \tag{127}$$

The approximations for the Crank-Nicolson method at point $(x,\tau+\delta\tau/2)$ can be obtained averaging the approximations for the explicit method at point (x,τ) and the approximations for the implicit method at point $(x,\tau+\delta\tau)$ [16]:

$$\frac{\partial u}{\partial \tau}\left(x,\tau+\frac{\delta\tau}{2}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial \tau}(x,\tau)+\frac{\partial u}{\partial \tau}(x,\tau+\delta\tau)\right) \approx \frac{u(x,\tau+\delta\tau)-u(x,\tau)}{\delta\tau} + \mathcal{O}\left((\delta\tau)^2\right) \tag{128}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}\left(x,\tau+\frac{\delta\tau}{2}\right) &= \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}(x,\tau)+\frac{\partial^2 u}{\partial x^2}(x,\tau+\delta\tau)\right)\\ &\approx \frac{1}{2}\left(\frac{u(x+\delta x,\tau)-2u(x,\tau)+u(x-\delta x,\tau)}{(\delta x)^2}\right.\\ &+\left.\frac{u(x+\delta x,\tau+\delta\tau)-2u(x,\tau+\delta\tau)+u(x-\delta x,\tau+\delta\tau)}{(\delta x)^2}\right) + \mathcal{O}\left((\delta x)^2\right)\end{aligned} \tag{129}$$

We can now define the finite-difference mesh. We divide the x-axis into points which are separated by an equal space δx. We do the same with the τ-axis, but this time the space separation is $\delta\tau$. Thus we have turned the (x,τ)-plane into a mesh where the mesh points have positions $(n\delta x, m\delta\tau)$. We will only consider the values $u(x,\tau)$ at the mesh points. This can be written as [4]:

$$u_n^m = u(n\delta x, m\delta\tau) \tag{130}$$

As we have seen earlier, the values of x need to be truncated between a $-x^-$ and a x^+. For the case of the finite-difference mesh, this can be done using two integers N^- (negative) and N^+ (positive) so that

$$N^-\delta x \le n\delta x \le N^+\delta x \tag{131}$$

In the case of τ, we just need an integer M that verifies

$$0 \le m\delta\tau \le M\delta\tau \tag{132}$$

If we apply the finite-difference discretization to approximations (128) and (129), we get the necessary terms for the Crank-Nicolson scheme:

$$\frac{\partial u}{\partial \tau}\left(x,\tau+\frac{\delta\tau}{2}\right) \approx \frac{u_n^{m+1}-u_n^m}{\delta\tau} + \mathcal{O}\left((\delta\tau)^2\right) \tag{133}$$

$$\frac{\partial^2 u}{\partial x^2}\left(x, \tau + \frac{\delta\tau}{2}\right) \approx \frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} + \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2}\right) + \mathcal{O}\left((\delta x)^2\right) \tag{134}$$

We are now in position to write the linear complementarity problem using the finite-difference formulation and, more concretely, the Crank-Nicolson method. If we consider the terms $\mathcal{O}\left((\delta\tau)^2\right)$ and $\mathcal{O}\left((\delta x)^2\right)$ negligible, the inequality $\partial u/\partial\tau - \partial^2 u/\partial x^2 \geq 0$ (121) can be written as [4]:

$$u_n^{m+1} - u_n^m \geq \frac{\alpha}{2}\left(u_{n+1}^m - 2u_n^m + u_{n-1}^m + u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}\right) \tag{135}$$

where $\alpha = \delta\tau/(\delta x)^2$. The payoff function (112) can also be expressed using the finite-difference notation:

$$f_n^m = f(n\delta x, m\delta\tau) \tag{136}$$

so the inequality $u - f \geq 0$ (121) can be approximated by

$$u_n^m - f_n^m \geq 0 \tag{137}$$

The initial condition (122) is

$$u_n^0 = f_n^0 \tag{138}$$

and the boundary conditions (123)

$$u_{N^-}^m = f_{N^-}^m \qquad u_{N^+}^m = f_{N^+}^m \tag{139}$$

Regarding inequality (135), we can see that there are terms which correspond to time steps m and $m+1$. If we group the terms with m within a variable C_n^m

$$C_n^m = (1-\alpha)\, u_n^m + \frac{\alpha}{2}\left(u_{n+1}^m + u_{n-1}^m\right) \tag{140}$$

we can write the inequality as

$$(1+\alpha)\, u_n^{m+1} - \frac{\alpha}{2}\left(u_{n+1}^{m+1} + u_{n-1}^{m+1}\right) \geq C_n^m \tag{141}$$

With all this information, we can say that the linear complementarity equation (120) can be approximated by

$$\left((1+\alpha)\, u_n^{m+1} - \frac{\alpha}{2}\left(u_{n+1}^{m+1} + u_{n-1}^{m+1}\right) - C_n^m\right)\left(u_n^{m+1} - f_n^{m+1}\right) = 0 \tag{142}$$

Now we need to find a way to solve the problem, that is, to find all u_n^m. All we can say for now is that the algorithm will do a sweep across all the values of $n\delta x$ from N^- to N^+ for every time step $m\delta\tau$ from 0 to M. When we are calculating the terms at time step $m+1$, C_n^m can be explicitly obtained because we already know the terms at time step m.

8.5 Matrix formulation

Before we introduce the method of resolution, we are going to write equation (142) using matrices. First of all, we concentrate on the first factor. We denote by $\mathbf{u}^{m+1}$ the vector containing the values of u for all $n\delta x$ at a specific time step $(m+1)\delta\tau$. We also define a matrix $\mathbf{A}$ and a vector $\mathbf{b}^m$ so that the factor takes the form $(\mathbf{A}\mathbf{u}^{m+1} - \mathbf{b}^m)$ [4].

$$\mathbf{A} = \begin{pmatrix} 1+\alpha & -\frac{\alpha}{2} & 0 & \dots & 0 \\ -\frac{\alpha}{2} & 1+\alpha & -\frac{\alpha}{2} & & \vdots \\ 0 & -\frac{\alpha}{2} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{\alpha}{2} \\ 0 & \dots & 0 & -\frac{\alpha}{2} & 1+\alpha \end{pmatrix} \tag{143}$$

$$\mathbf{u}^{m+1} = \begin{pmatrix} u^{m+1}_{N^-+1} \\ \vdots \\ u^{m+1}_0 \\ \vdots \\ u^{m+1}_{N^+-1} \end{pmatrix} \qquad \mathbf{b}^m = \begin{pmatrix} b^m_{N^-+1} \\ \vdots \\ b^m_0 \\ \vdots \\ b^m_{N^+-1} \end{pmatrix} = \begin{pmatrix} C^m_{N^-+1} \\ \vdots \\ C^m_0 \\ \vdots \\ C^m_{N^+-1} \end{pmatrix} + \frac{\alpha}{2} \begin{pmatrix} u^{m+1}_{N^-} = f^{m+1}_{N^-} \\ 0 \\ \vdots \\ 0 \\ u^{m+1}_{N^+} = f^{m+1}_{N^+} \end{pmatrix} \tag{144}$$

At first instance, we exclude the terms N^- and N^+ both in $\mathbf{A}$ and $\mathbf{u}^{m+1}$ for the multiplication $\mathbf{A}\mathbf{u}^{m+1}$, since they are already known from the boundary conditions (139). However, we later retake the boundary terms in the second vector of $\mathbf{b}^m$ for them to appear in the equations (142) for the steps $n = N^- + 1$ and $n = N^+ - 1$. This vector has dimension $(N^+ - N^- - 1)$, as well as the first vector of $\mathbf{b}^m$ and $\mathbf{u}^{m+1}$. The matrix $\mathbf{A}$ is squared and tridiagonal and has dimensions $(N^+ - N^- - 1) \times (N^+ - N^- - 1)$.

For the second factor in equation (142), we already have the vector notation for $\mathbf{u}^{m+1}$ and we just need to define the same way $\mathbf{f}^{m+1}$ as the vector containing the values of the payoff function f for all $n\delta x$ at a specific time step $(m+1)\delta\tau$. The factor becomes $\left(\mathbf{u}^{m+1} - \mathbf{f}^{m+1}\right)$.

With this **matrix formulation**, the linear complementarity problem can be written as follows [4]:

$$\left(\mathbf{A}\mathbf{u}^{m+1} - \mathbf{b}^m\right)\left(\mathbf{u}^{m+1} - \mathbf{f}^{m+1}\right) = 0 \tag{145}$$

$$\left(\mathbf{A}\mathbf{u}^{m+1} - \mathbf{b}^m\right) \geq 0 \qquad \mathbf{u}^{m+1} \geq \mathbf{f}^{m+1} \tag{146}$$

8.6 The LU method

The method we are going to use to solve the problem is known as the **LU method**. Ref[4] solves the problem with the Projected SOR method, not the LU. It uses the LU method with the implicit scheme to price European options. We are going to apply the same procedure but for American options with the Crank-Nicolson scheme. This method is used to solve systems of linear equations as $(\mathbf{A}\mathbf{u}^{m+1} - \mathbf{b}^m) = 0$. It is based on the LU decomposition, which consists on factorizing a matrix, in this case $\mathbf{A}$, into a product of a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$, so that $\mathbf{A} = \mathbf{LU}$.

$$A = \begin{pmatrix} 1+\alpha & -\frac{\alpha}{2} & 0 & \dots & 0 \\ -\frac{\alpha}{2} & 1+\alpha & -\frac{\alpha}{2} & & \vdots \\ 0 & -\frac{\alpha}{2} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{\alpha}{2} \\ 0 & \dots & 0 & -\frac{\alpha}{2} & 1+\alpha \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{N^-+1} & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & l_{N^+-2} & 1 \end{pmatrix} \cdot \begin{pmatrix} y_{N^-+1} & z_{N^-+1} & 0 & \dots & 0 \\ 0 & y_{N^-+2} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & z_{N^+-2} \\ 0 & \dots & 0 & 0 & y_{N^+-1} \end{pmatrix} \tag{147}$$

To determine l_n, y_n and z_n we have to multiply the matrices

$$\mathbf{LU} = \begin{pmatrix} y_{N^-+1} & z_{N^-+1} & 0 & \dots & 0 \\ l_{N^-+1}y_{N^-+1} & l_{N^-+1}z_{N^-+1} + y_{N^-+2} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & z_{N^+-2} \\ 0 & \dots & 0 & l_{N^+-2}y_{N^+-2} & l_{N^+-2}z_{N^+-2} + y_{N^+-1} \end{pmatrix} \tag{148}$$

and equal the result to matrix $\mathbf{A}$. Here we find that

$$z_n = -\frac{\alpha}{2} \qquad l_n = -\frac{\alpha}{2y_n} \qquad \text{for } n = N^- + 1, \dots, N^+ - 2 \tag{149}$$

$$y_{N^-+1} = 1 + \alpha \tag{150}$$

$$y_n = (1+\alpha) - \frac{\alpha^2}{4y_{n-1}} \qquad \text{for } n = N^- + 2, \dots, N^+ - 1 \tag{151}$$

As we can see, we are just interested in the values of y_n for $n = N^- + 1, \dots, N^+ - 1$. We can now divide the problem $(\mathbf{A}\mathbf{u}^{m+1} - \mathbf{b}^m) = 0$ into two sub-problems:

$$\mathbf{L}\mathbf{w}^m = \mathbf{b}^m \qquad \mathbf{U}\mathbf{u}^{m+1} = \mathbf{w}^m \tag{152}$$

This is the same as doing $\mathbf{L}\left(\mathbf{U}\mathbf{u}^{m+1}\right) = \mathbf{b}^m$, but with an intermediate vector $\mathbf{w}^m$. Substituting expressions (149), (150) and (151) into the matrices $\mathbf{L}$ and $\mathbf{U}$, we have that the sub-problems are

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{\alpha}{2y_{N^-+1}} & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -\frac{\alpha}{2y_{N^+-2}} & 1 \end{pmatrix} \cdot \begin{pmatrix} w^m_{N^-+1} \\ \vdots \\ w^m_0 \\ \vdots \\ w^m_{N^+-1} \end{pmatrix} = \begin{pmatrix} b^m_{N^-+1} \\ \vdots \\ b^m_0 \\ \vdots \\ b^m_{N^+-1} \end{pmatrix} \tag{153}$$

and

$$\begin{pmatrix} y_{N^-+1} & -\frac{\alpha}{2} & 0 & \dots & 0 \\ 0 & y_{N^-+2} & -\frac{\alpha}{2} & & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{\alpha}{2} \\ 0 & \dots & \dots & 0 & y_{N^+-1} \end{pmatrix} \cdot \begin{pmatrix} u^{m+1}_{N^-+1} \\ \vdots \\ u^{m+1}_0 \\ \vdots \\ u^{m+1}_{N^+-1} \end{pmatrix} = \begin{pmatrix} w^m_{N^-+1} \\ \vdots \\ w^m_0 \\ \vdots \\ w^m_{N^+-1} \end{pmatrix} \tag{154}$$

We begin with the first system (153). We can directly derive the value of $w^m_{N^-+1}$ and then increasing n from $N^- + 1$, obtain the terms w^m_n knowing the previous one w^m_{n-1}.

$$w^m_{N^-+1} = b^m_{N^-+1} \tag{155}$$

$$w^m_n = b^m_n + \frac{\alpha w^m_{n-1}}{2y_{n-1}} \qquad \text{for } n = N^- + 2, \dots, N^+ - 1 \tag{156}$$

In the second system (154), we directly get $u^{m+1}_{N^+-1}$ and then decreasing n from $N^+ - 1$, we can obtain the terms u^{m+1}_n knowing u^{m+1}_{n+1}.

$$u^{m+1}_{N^+-1} = \frac{w^m_{N^+-1}}{y_{N^+-1}} \tag{157}$$

$$u^{m+1}_n = \frac{w^m_n + \frac{\alpha}{2} u^{m+1}_{n+1}}{y_n} \qquad \text{for } n = N^+ - 2, \dots, N^- + 1 \tag{158}$$

With all these expressions, we can build the algorithm for the LU method to solve our linear system ($\mathbf{A}\mathbf{u}^{m+1} = \mathbf{b}^m$) in (145). We suppose that $\mathbf{A}$ and $\mathbf{b}^m$ are already known. The algorithm has the following steps:

1. Find all the values y_n starting from y_{N^-+1} (150) and using (151).

2. Calculate the vector $\mathbf{w}^m$ starting with the component $w^m_{N^-+1}$ (155) and then using (156) to obtain the rest.

3. Obtain the vector $\mathbf{u}^{m+1}$ starting from $u^{m+1}_{N^+-1}$ (157) and calculating the other components with (158).

This is the algorithm that solves the linear system ($\mathbf{A}\mathbf{u}^{m+1} = \mathbf{b}^m$), but this is just one part of the resolution of the linear complementarity problem. The algorithm for the whole problem (145)(146) is the following:

1. In the beginning, we have the matrix $\mathbf{A}$ and the initial (138) and boundary conditions (139). With the initial condition, we can obtain $\mathbf{u}^0$ (first time step $m = 0$).

2. We calculate $\mathbf{b}^0$ and solve the linear system $\left(\mathbf{A}\mathbf{u}^1 = \mathbf{b}^0\right)$ using the LU method to obtain $\mathbf{u}^1$. The terms $u^1_{N^-}$ and $u^1_{N^+}$ do not appear in the solution of the linear system but are known from the boundary conditions. According to the expression $\mathbf{u}^{m+1} \geq \mathbf{f}^{m+1}$, every component of the vector $\mathbf{u}^1$ has to be greater than or equal to the correspondent component of the vector $\mathbf{f}^1$. This means that when we are calculating u^1_n using (157) and (158), we have to check whether it is greater than or equal to f^1_n. In case it was smaller, we would have to force $u^1_n = f^1_n$.

3. Once we have $\mathbf{u}^1$, we calculate $\mathbf{b}^1$ to be able to solve the system $\left(\mathbf{A}\mathbf{u}^2 = \mathbf{b}^1\right)$ and obtain $\mathbf{u}^2$. We have to check that $\mathbf{u}^2 \geq \mathbf{f}^2$ the same way as before and do the necessary readjustments.

4. The process is repeated until we calculate $\mathbf{u}^M$.

We have seen that in some cases we have to force the value of u^m_n to be equal to f^m_n. This suggests us that we have to make some changes in expressions (157) and (158):

$$u^{m+1}_{N^+-1} = \max\left(\frac{w^m_{N^+-1}}{y_{N^+-1}}, f^{m+1}_{N^+-1}\right) \tag{159}$$

$$u^{m+1}_n = \max\left(\frac{w^m_n + \frac{\alpha}{2}u^{m+1}_{n-1}}{y_n}, f^{m+1}_n\right) \qquad \text{for } n = N^+ - 2, \ldots, N^- + 1 \tag{160}$$

We are now going to build our own Python code for the whole algorithm of resolution. The algorithm gives us the values of u with respect to x and τ, but we are also going to calculate the values of $P(S,t)$ reversing the transformations. Moreover, we are going to try to find the optimal exercise price $x_f(\tau)$ at every time step and then convert it into $S_f(t)$. To obtain x_f at a time step $m\delta\tau$, we have to find the value of n for which $u^m_n = f^m_n$ and which verifies that for $n' > n$, $u^m_{n'} > f^m_{n'}$, excluding $n = N^-$ and $n = N^+$.

8.7 Python code

We are now going to write our Python code for the algorithm.

```
import math
import matplotlib.pyplot as plt

class AmericanPut:

    def __init__(self, r, sigma, T, K):
        self.__r = r
        self.__sigma = sigma
        self.__T = T
        self.__K = K
        self.__q = 2 * r / (sigma * sigma)
        self.__S = []
        self.__t = []
        self.__P = [[]]
        self.__Sf = []

    def get_t(self):
        return self.__t

    def get_S(self):
        return self.__S

    def get_P(self):
        return self.__P

    def get_Sf(self):
        return self.__Sf

    def payoff(self, x, tau):    #Payoff function f(x,tau)
        return math.exp(0.25 * (self.__q + 1) * (self.__q + 1) * tau) * \
               max(math.exp(0.5 * (self.__q - 1) * x) - math.exp(0.5 * (self.__q + 1) * x), 0.0)

    def from_u_to_P(self, u, x, tau):   #Transformation from u to P
        P = []
        for i in range(len(u)):
            P.append(self.__K * math.exp(-0.5 * (self.__q - 1) * x[i]
                                  - 0.25 * (self.__q + 1) * (self.__q + 1) * tau) * u[i])
        return P

    @staticmethod
    def find_y(alpha, N):    #Method that finds all the y_n
        y = [1 + alpha]
        for i in range(1, N):
            y.append(1 + alpha - alpha * alpha / (4 * y[i - 1]))
        return y

    def find_b(self, u, x, tau, alpha):   #Method that calculates b^m
        b = []
        for i in range(1, len(u) - 1):
            C = (1 - alpha) * u[i] + 0.5 * alpha * (u[i + 1] + u[i - 1])
            b.append(C)
        b[0] += 0.5 * alpha * self.payoff(x[0], tau)
        b[-1] += 0.5 * alpha * self.payoff(x[-1], tau)
        return b
```

```
    def LU(self, b, y, x, tau, alpha, length):  #LU solver
        w = [b[0]]
        check_sf = True
        for i in range(1, length):
            w.append(b[i] + 0.5 * alpha * w[i - 1] / y[i - 1])
        u = [0 for k in range(length)]
        u[length - 1] = max(w[length - 1] / y[length - 1], self.payoff(x[length], tau))
        for j in range(length - 2, -1, -1):
            u[j] = max((w[j] + 0.5 * alpha * u[j + 1]) / y[j], self.payoff(x[j + 1], tau))
            if check_sf and u[j] == self.payoff(x[j + 1], tau):
                xf = x[j + 1]
                check_sf = False
        self.__Sf.append(self.__K * math.exp(xf))
        return u

    def values(self, dtau, dx, Nmin, Nplus):  #Method that gives the values of P(S,t)
        alpha = dtau / (dx * dx)
        N = Nplus - Nmin + 1
        M = int(0.5 * self.__sigma * self.__sigma * self.__T / dtau)
        self.__P = [[0 for i in range(N)] for j in range(M)]
        self.__t = [self.__T]
        x = []
        u = []
        for l in range(N):
            x.append((Nmin + l) * dx)
            self.__S.append(self.__K * math.exp(x[l]))
            u.append(self.payoff(x[l], 0.0))  # Finding u^0
        self.__P[0][:] = self.from_u_to_P(u, x, 0.0)
        y = self.find_y(alpha, len(u) - 2)
        for i in range(1, M): #Finding u^m from m=1 up to m=M
            tau = i * dtau
            self.__t.append(self.__T - tau / (0.5 * self.__sigma * self.__sigma))
            b = self.find_b(u, x, tau, alpha)
            u[1:N - 1] = self.LU(b, y, x, tau, alpha, len(u) - 2)
            u[0] = self.payoff(x[0], tau)
            u[N - 1] = self.payoff(x[N - 1], tau)
            self.__P[i][:] = self.from_u_to_P(u, x, tau)
        pass

if __name__ == '__main__':
    american_put = AmericanPut(r= , sigma= , T= , K= )
    american_put.values(dtau= , dx= , Nmin= , Nplus= )
    t = american_put.get_t()
    S = american_put.get_S()
    P = american_put.get_P()
    Sf = american_put.get_Sf()
```

To create an object that represents an American put option in this code, we need to set an interest rate r, a volatility σ, an expiration date T and a strike price K. Once the object is created, we have to call the method *values* if we want to price the option. The inputs for this method are the time interval $dtau$, the x interval dx and the minimum an maximum numbers for n, that is, $Nmin$ and $Nplus$. The method applies the algorithm of resolution and fills the lists t and S and the matrix P with the correspondent values. A row in P matrix represents a time step and contains the values of the option for all values of x at that time step. The method also fills the list S_f containing the optimal exercise price at each time step. The transformations from $u(x, \tau)$ to $P(S, t)$ are internally made.

8.8 Numerical example

We are now going to run the code with a numerical example. We consider an American put option and the following parameters:

- Risk-free interest rate r=0.03 (3%)
- Volatility σ=0.35 $\sqrt{year}^{-1}$ (35%)
- Expiration date $T = 0.5$ year (6 months)
- Strike price $K = 10\$$

Firstly, we calculate the values of the option with $dtau = 0.0005$, $dx = 0.00125$, $Nmin = -4000$ and $Nplus = 600$. The program fills the lists t, S, S_f and the matrix P. If we were interested in the value of the option for some specific S and t, we could take it from P. The same for S_f as a function of t.

We are now going to plot some results. On the one hand, we represent in 3D in Figure 4 the value of the option P as a function of the underlying asset price S and time t. When the asset price S tends to 0, the option value P goes to the strike price $K = 10\$$ the same way as the payoff function ($\max\{K - S, 0\}$), and as S increases significantly, P approaches 0. This means that the boundary conditions (123) are verified. The plot also contains the optimal exercise boundary, which is the red line, for the given the values of the option. The values of P above the line are cases for which the option would be executed and the values under the line, for which it would be hold.

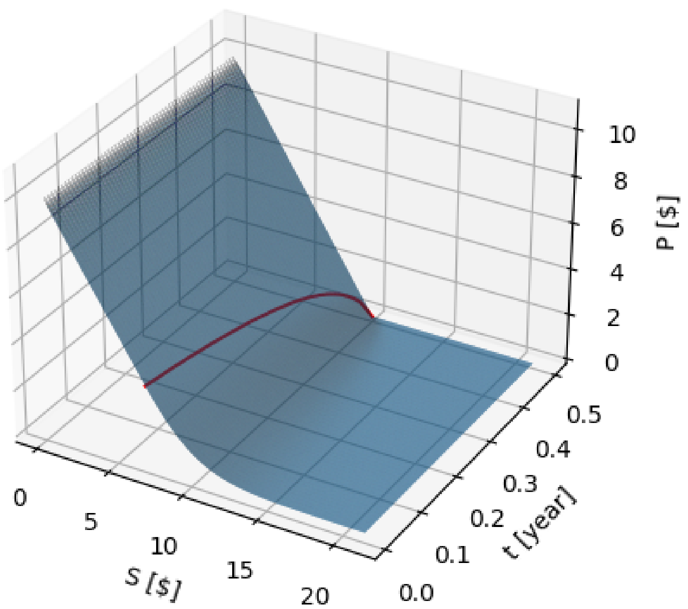
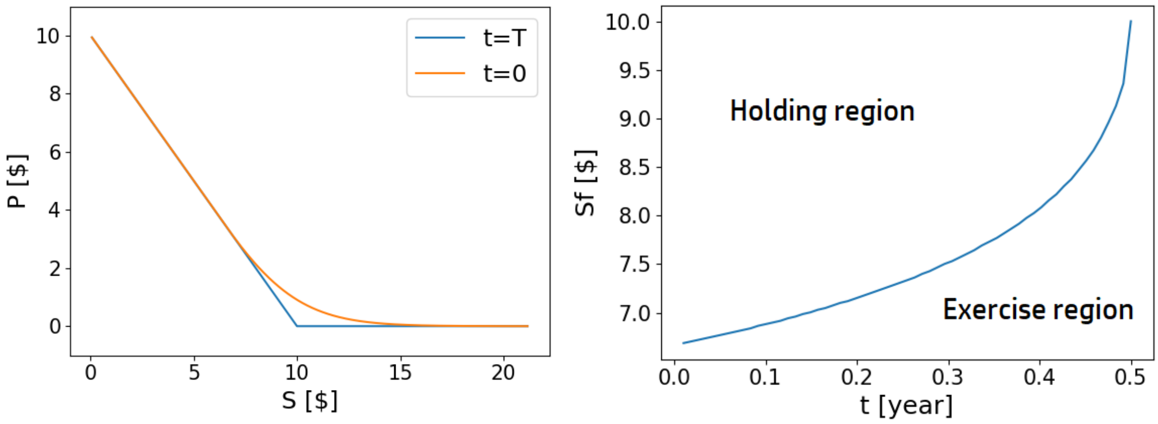


Figure 4: 3D plot for $P(S,t)$. The red line represents the optimal exercise boundary.

We are also going to plot the 2D function $P(S)$ for two different times $t = 0$ and $t = T$, as well as the optimal exercise price function $S_f(t)$. The resulting plots are the ones in Figure 5. In the first one, Figure 5.a, we observe that for $t = T$, the value of $P(S)$ is equal to the payoff function (Figure 1.b) as it has to be according to the final condition (86). When $t = 0$, the function $P(S)$ is greater than or equal to the payoff function, so condition (85) is satisfied. From Figures 4 and 5.a and the analysis of the boundary and final conditions we have made, we can infer that the behaviour of $P(S,t)$ is correct.

Regarding Figure 5.b, we deduce that when time approaches expiration date, the optimal exercise price increases and tends to the strike price $K = 10\$$. This is logical since at time $t = T$, $P(S,T) = \max\{K - S, 0\}$ and exercise should be done only if $S < K$. The region behind the line is the exercise region and the region above the line is the one to hold. As time evolves, we are more predisposed to execute the option and sell the asset despite obtaining a smaller payoff. The reason for this is that, at early times, the value of the option is greater than the payoff function for more values of S (it is visualized in Figure 5.a) and, in those cases, it is better to hold the option than to execute it.



(a) $P(S)$ for $t = 0$ and $t = T$

(b) Optimal exercise price function $S_f(t)$

Figure 5: Function $P(S)$ for $t = 0$ and $t = T$ and optimal exercise price function $S_f(t)$.

9 Conclusions

In this work we have established a connection between physics and finance. Concretely, we have analysed some physical concepts in the valuation of options.

First of all, we have learnt some basic financial concepts. It was crucial to understand what is an option for the rest of the project. We have also introduced some physical notions related to Brownian motion. We have seen that diffusion processes are governed by the Brownian motion and we have found different solutions for the diffusion equation based on Green's function. Furthermore, we have explained the mathematical formulation of the Brownian motion, that is, the Wiener process.

With the main financial and physical concepts already assimilated, we have analysed the behavior of the underlying asset price. We have seen that it can modelled with a random walk based on a Brownian motion. We have found that in the random walk model, the asset price follows a Geometric Brownian motion and that the probability density function of its logarithm suffered a diffusion process. To visualize this diffusion process, we have plotted some functions.

Our next step has been to pose the Black-Scholes model based on the random walk model for the underlying asset. We have derived the Black-Scholes partial differential equation used to price options. Once we had the equation, we have looked for an explicit solution for European put options. This was the Black-Scholes formula. We have solved the problem as a diffusion problem with the payoff function playing the role of the initial distribution. However, first we have had to transform the Black-Scholes equation into a diffusion equation. Once we had the Black-Scholes formulas for both European call and put options, we have used them with a numerical example. We have found that the behaviour of the options was the expected.

In the last section, we have studied the problem of pricing American options. We have seen that we cannot find an explicit solution and that for American call options, early exercised is never recommended. We have analysed the free boundary problem for American put options and transformed it into a linear complementarity problem to obtain a solution. Once more. we have worked with the Black-Scholes equation transformed into a diffusion equation. Writing the problem as a linear complementarity one, eliminated the dependence on the free boundary and made resolution easier. We have used the finite difference formulation and the Crank-Nicolson scheme with matrices to solve the problem. We have also built an algorithm that uses the LU method to solve linear systems of equations and translated it into our own Python code. The last step has been to run this code with a numerical example and analyse the results. We have seen that the function of the option value had the right behaviour and we have also obtained the optimal exercise price function. The value of the optimal exercise price increased and converged to the strike price as time approached expiration date.

Overall, we have found that there is a relation between physics and finance. Particularly, we have found that the Brownian motion and diffusion equation play an important role in the valuation of options. The influence of physics in finance is a fact and that is why many financial lines of investigation involve the use of physics.

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