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Solving a Boundary Value Problem via Fixed-Point Theorem on Ⓡ-Metric Space

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Abstract: In this paper, we prove the fixed-point theorem for rational contractive mapping on Ⓡ-metric space. Additionally, an Euclidean metric space with a binary relation example and an application to the first-order boundary value problem are given. Moreover, the obtained results generalize and extend some of the well-known results in the literature.

Keywords: rational contractive mapping; Ⓡ-complete metric space; fixed point

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1. Introduction

In 1922, the classical Banach contraction principle [1] made many inferences including the existence of fixed points for contractive mapping. The Banach contraction principle has extended and established in various metric space settings. Among these extended versions, Alam and Imdad [2,3] formulated a relation of continuity contraction and completeness on the theoretical analogues of the standard metric space notion. Furthermore, Ahmadullah et al. [4] and Boyd-Wong [5] extended their fixed-point theorems on nonlinear contraction mappings. On the other hand, Senapati and Dey [6] and many other authors have improved the notion of w-distance in relational metric space with an arbitrary binary relation. Ali, Imdad and Sessa [7] proved fixed-point theorems on Ⓡ-complete regular symmetric spaces. Alam, George, Imdad and Hasanuzzaman [8] proved fixed-point theorems for nonexpansive mappings under binary relations. Javed, Arshad, Baazeem and Nabil [9] proved fixed-point theorems on Ⓡ-complete metric spaces. Faruk, Ahmad Khan, Haq Khan and Alam [10] proved fixed-point theorems for generalized nonlinear contractions involving a new pair of auxiliary functions in a metric space endowed with a locally finitely T-transitive binary relation. Samet et al. [11] introduced the notion of α-admissible mappings and reported metric fixed-point results in Kannan contraction mappings. Hereby, many authors have extended and unified most of the results in metric fixed points in these mappings (as can be seen, e.g., in [12–15]). Several generalizations of the contraction mapping principle have been established since then by various mathematicians, resulting in an abundance of fixed-point theorems in metric spaces, which has continued until today. The fixed-point theorem relates to an arbitrary mapping from 1975 and 1976. Gopi Prasad [16] discussed the fixed points of Kannan contractive mappings in relational metric space. Fixed-point theorems in relational metric spaces with an application to boundary value problems was discussed by Gopi Prasad et al. [17]. Numerous researchers have conducted research on metric spaces for a number of years in an effort to obtain new extensions of the well-known boundary value problem. Many other researchers are focusing on several metric spaces during several years (as can be seen in [18–25]). In this paper, we prove the fixed-point theorem for rational contractive mapping on Ⓡ-metric space.

2. Preliminaries

Let us begin this section with some basic definitions, propositions and related theorems on metrical notions. Here, Ⓡ denotes a non-void binary relation (briefly, $\mathcal{BR}$), $\mathbb{N}$ represents a set of natural numbers and $\mathbb{N}_0$ indicates a set of whole numbers (that is $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Definition 1 ([2])**.** *Let Ψ be a non-void set under Ⓡ on Ψ and be defined as a subset of $\Psi \times \Psi$. Then, we mean that ρ relates ϱ if and only if $(\rho, \varrho) \in$ Ⓡ under Ⓡ.*

Definition 2 ([2])**.** *Let Ψ be a non-void set under Ⓡ on Ψ and ρ and $\varrho \in \Psi$. Then, ρ and ϱ are said to be Ⓡ-comparative if $(\rho, \varrho) \in$ Ⓡ (or) $(\varrho, \rho) \in$ Ⓡ. Let us denote that $[\rho, \varrho] \in$ Ⓡ.*

Definition 3 ([2])**.** *Let Ψ be a non-void set under Ⓡ on Ψ.*

(1) The dual relation or transpose or inverse of Ⓡ is $Ⓡ^{-1}$ and is defined as $Ⓡ^{-1} = \{(\rho, \varrho) \in \Psi^2 : (\varrho, \rho) \in Ⓡ\}$.
(2) The $\mathcal{BR}$ Ⓡ of the symmetric closure of $Ⓡ^{\mathfrak{s}}$ is defined as the set $Ⓡ \cup Ⓡ^{-1}$ (that is $Ⓡ^{\mathfrak{s}} : = Ⓡ \cup Ⓡ^{-1}$). In other words, $Ⓡ^{\mathfrak{s}}$ is the smallest symmetric relation on Ψ.

Definition 4 ([2])**.** *Let Ψ be a non-void set under Ⓡ on Ψ. If a sequence $\{\rho_\sigma\} \subset \Psi$ is said to be Ⓡ-preserving, and if $(\rho_\sigma, \rho_{\sigma+1}) \in$ Ⓡ for all $\sigma \in \mathbb{N}_0$.*

Definition 5 ([2])**.** *Let Ψ be a non-void set under Ⓡ on Ψ and Λ be a self-mapping on Ψ, which is said to be Λ-closed, if $(\rho, \varrho) \in Ⓡ \Rightarrow (\Lambda\rho, \Lambda\varrho) \in$ Ⓡ for all $\rho, \varrho \in \Psi$.*

Theorem 1 ([3])**.** *Let Ψ be a non-void set under Ⓡ on Ψ and Λ be a self-mapping on Ψ. If Ⓡ is Λ-closed, then Ⓡ is also Λ^σ-closed for all $\sigma \in \mathbb{N}_0$, where Λ^σ denotes the n-th iterate of Λ.*

Definition 6 ([3])**.** *Let $(\Psi, \psi, Ⓡ)$ be a metric space under a Ⓡ. Then, (Ψ, ψ) is said to be Ⓡ-complete, if every Ⓡ-preserving Cauchy sequence in Ψ converges to a point in Ψ.*

Definition 7 ([3])**.** *Let (Ψ, ψ) be a metric space and Ⓡ is a $\mathcal{BR}$ on Ψ and $\rho \in \Psi$. Let Λ be a self-mapping on Ψ which is said to be Ⓡ-continuous at ρ; if any Ⓡ-preserving sequence $\{\rho_\sigma\}$ such that $\rho_\sigma \xrightarrow{\psi} \rho$, then $\Lambda(\rho_\sigma) \xrightarrow{\psi} \Lambda(\rho)$. Moreover, Λ is called Ⓡ-continuous if it is Ⓡ-continuous at each point of Ψ.*

Definition 8 ([3])**.** *Let $(\Psi, \psi, Ⓡ)$ be a metric space under Ⓡ. A subset $\mathcal{E}$ of Ψ is said to be Ⓡ-connected, then there exists a path from ρ to ϱ in Ⓡ for all $\rho, \varrho \in \mathcal{E}$.*

Definition 9 ([3])**.** *Let Ψ be a non-void set with a binary relation Ⓡ on Ψ which is said to be transitive, if $(\rho, \varrho), (\varrho, \phi) \in Ⓡ \Rightarrow (\rho, \phi) \in Ⓡ, \rho, \varrho, \phi \in \Psi$.*

Definition 10 ([26])**.** *Let Ψ be a non-void set under Ⓡ on Ψ with a pair of points $\rho, \varrho \in \Psi$. If there is a finite sequence $\{\phi_0, \phi_1, \phi_2, ..., \phi_{\mathfrak{k}}\} \subset \Psi$ such that $(\phi_{\mathfrak{i}}, \phi_{\mathfrak{i}+1}) \in$ Ⓡ and $\phi_0 = \rho, \phi_{\mathfrak{k}} = \varrho$ for each $\mathfrak{i}(0 \leq \mathfrak{i} \leq \mathfrak{k} - 1)$, then this finite sequence is said to be a path of length $\mathfrak{k}$ from ρ to ϱ in Ⓡ.*

Let Ⓡ be a $\mathcal{BR}$ and Λ be a self-mapping which is a non-void set on Ψ,

(i) $\mathcal{F}(\Lambda) : =$ the set of all fixed points of Λ;
(ii) $\Psi(\Lambda; Ⓡ) : = \{\rho \in \Psi : (\rho, \Lambda\rho) \in Ⓡ\}$.

In 1968, Kannan [27] proved the fixed-point theorem on metric space as follows:

Theorem 2. *Let (Ψ, ψ) be a complete metric space and Λ be a self-mapping on Ψ. If Λ is Kannan contraction, that is, there exists $\jmath \in [0, \frac{1}{2})$ such that*

$$\psi(\Lambda\rho, \Lambda\varrho) \leq \jmath[\psi(\rho, \Lambda\rho) + \psi(\varrho, \Lambda\varrho)]$$

then Λ has a unique fixed point $\theta \in \Psi$ and for each $\rho \in \Psi$, the sequence of iterates $\{\Lambda^n\rho\}$ converges to θ.

Motivated by the above work, here we prove fixed-point theorems on Ⓡ-metric space under rational-type contraction mapping with an application.

3. Main Results

In this section, we first prove the existence of rational contractive mapping on Ⓡ-metric spaces. Here, we denote the complete metric space by $\mathcal{CM}$ space.

Theorem 3. *Let the mapping $\Lambda \colon \Psi \to \Psi$ and (Ψ, ψ) be a Ⓡ-$\mathcal{CM}$ space such that*

- *(a)* *$\Psi(\Lambda, Ⓡ)$ is non-void set;*
- *(b)* *Ⓡ is Λ-closed;*
- *(c)* *Λ is Ⓡ-continuous;*
- *(d)* *There exists $\imath, \jmath \in [0, \frac{1}{2})$ such that*

$$\psi(\Lambda\rho, \Lambda\varrho) \leq \imath \frac{\psi(\rho, \Lambda\rho) \cdot \psi(\varrho, \Lambda\varrho)}{1 + \psi(\varrho, \Lambda\varrho)} + \jmath\psi(\rho, \varrho),$$

for all $\rho, \varrho \in \Psi$ with $(\rho, \varrho) \in$ Ⓡ and $\imath + \jmath < 1$. Then, there exists $\rho \in \Psi$ such that $\rho \in \Lambda\rho$.

Proof. Let us assume (a), and choose ρ_0 as arbitrary element of $\Psi(\Lambda, Ⓡ)$. Construct a sequence $\{\rho_\sigma\}$ that is

$$\rho_\sigma = \Lambda^\sigma(\rho_0) \quad \text{for all} \quad \sigma \in \mathbb{N}_0. \tag{1}$$

Since $(\rho_0, \Lambda\rho_0) \in$ Ⓡ, using Λ-closedness of Ⓡ and Theorem 1, we have

$$(\Lambda^1\rho_0, \Lambda^2\rho_0), (\Lambda^2\rho_0, \Lambda^2\rho_0), \cdots, (\Lambda^\sigma\rho_0, \Lambda^{\sigma+1}\rho_0) \in Ⓡ.$$

So that

$$(\rho_\sigma, \rho_{\sigma+1}) \in Ⓡ \quad \text{for all} \quad \sigma \in \mathbb{N}_0. \tag{2}$$

Then, the sequence $\{\rho_\sigma\}$ is Ⓡ-preserving.

Let us apply contractive condition (d), we have

$$\begin{aligned}
\psi(\rho_\sigma, \rho_{\sigma+1}) =& \psi(\Lambda\rho_{\sigma-1}, \Lambda\rho_\sigma) \\
\leq& \imath \frac{\psi(\rho_{\sigma-1}, \Lambda\rho_{\sigma-1}) \cdot \psi(\rho_\sigma, \Lambda\rho_\sigma)}{1 + \psi(\rho_\sigma, \Lambda\rho_\sigma)} + \jmath\psi(\rho_{\sigma-1}, \rho_\sigma) \\
\leq& \imath \frac{\psi(\rho_{\sigma-1}, \rho_\sigma) \cdot \psi(\rho_\sigma, \rho_{\sigma+1})}{1 + \psi(\rho_\sigma, \rho_{\sigma+1})} + \jmath\psi(\rho_{\sigma-1}, \rho_\sigma) \\
\psi(\rho_\sigma, \rho_{\sigma+1}) \leq& \imath \frac{\psi(\rho_{\sigma-1}, \rho_\sigma) \cdot \psi(\rho_\sigma, \rho_{\sigma+1})}{1 + \psi(\rho_\sigma, \rho_{\sigma+1})} + \jmath\psi(\rho_{\sigma-1}, \rho_\sigma) \\
\psi(\rho_\sigma, \rho_{\sigma+1}) \leq& \imath \frac{\psi(\rho_{\sigma-1}, \rho_\sigma) \cdot \psi(\rho_\sigma, \rho_{\sigma+1})}{1 + \psi(\rho_\sigma, \rho_{\sigma+1})} + \jmath\psi(\rho_{\sigma-1}, \rho_\sigma) \\
\psi(\rho_\sigma, \rho_{\sigma+1}) \leq& (\imath + \jmath)\psi(\rho_{\sigma-1}, \rho_\sigma) \quad \text{for all} \quad \sigma \in \mathbb{N}_0.
\end{aligned}$$

By the inductive process, we obtain

$$\psi(\rho_\sigma, \rho_{\sigma+1}) \leq \left(\imath + \jmath\right)^\sigma \psi(\rho_0, \rho_1). \tag{3}$$

For any positive integers ς, σ with $\varsigma > \sigma$, we have

$$\begin{aligned}\psi(\rho_\sigma, \rho_\varsigma) \leq& \psi(\rho_\sigma, \rho_{\sigma+1}) + \cdots + \psi(\rho_{\varsigma-1}, \rho_\varsigma)\\ \leq& (\mathfrak{r}^\sigma + \cdots + \mathfrak{r}^{\varsigma-1})\psi(\rho_0, \rho_1), \quad \text{where} \quad \mathfrak{r} = \imath + \jmath\\ \leq& \frac{\mathfrak{r}^\sigma}{1-\mathfrak{r}}\psi(\rho_0, \rho_1),\end{aligned}$$

such that $\{\rho_\sigma\}$ is a Cauchy sequence. Since (Ψ, ψ) is Ⓡ-$\mathcal{CM}$space, there exists $\phi \in \Psi$; then,

$$\lim_{\sigma\to\infty} \rho_\sigma = \phi. \tag{4}$$

Since Λ is Ⓡ-continuous, then $\rho_{\sigma+1} = \Lambda\rho_\sigma \xrightarrow{\psi} \Lambda\rho$. Therefore,

$$\Lambda\rho = \rho.$$

Hence, ρ is a fixed point of Λ.

Suppose that ρ, ϱ is any two fixed points of Λ. Thus, we have $(\rho, \varrho) \in$ Ⓡ (or) $(\varrho, \rho) \in$ Ⓡ. For $(\rho, \varrho) \in$ Ⓡ, we have

$$\begin{aligned}\psi(\rho, \varrho) =& \psi(\Lambda(\rho), \Lambda(\varrho))\\ \leq& \imath\frac{\psi(\rho, \Lambda\rho)\cdot\psi(\varrho, \Lambda\varrho)}{1+\psi(\varrho, \Lambda\varrho)} + \jmath\psi(\rho, \varrho)\\ \leq& \jmath\psi(\rho, \varrho)\\ <& \psi(\rho, \varrho),\end{aligned}$$

which is a contradiction. Hence, we must have $\rho = \varrho$. Similarly, for $(\varrho, \rho) \in$ Ⓡ, we have $\rho = \varrho$. Hence, Λ has a unique fixed point. □

Example 1. *Let $\Psi = [0,3]$ equipped with a binary relation Ⓡ $= \{(0,0), (0, \frac{1}{2}), (\frac{1}{2}, 1), (1,1), (1, \frac{3}{2}), (\frac{3}{2}, 2), (2,2), (2, \frac{5}{2}), (\frac{5}{2}, 3)\}$ and Euclidean metric ψ_2; defined by*

$$\psi((\rho_1, \rho_2), (\varrho_1, \varrho_2)) = \sqrt{(\rho_1 - \varrho_1)^2 + (\rho_2 - \varrho_2)^2}$$

then Ψ is a Ⓡ-complete metric space. Define a function $\Lambda\colon \Psi \to \Psi$ such that

$$\Lambda(\rho_1, \rho_2) = \begin{cases}(\rho_1, 0) & \text{if} \quad \rho_1 \geq \rho_2\\ (0, \rho_2) & \text{if} \quad \rho_1 < \rho_2.\end{cases}$$

We notice that $\psi(\Lambda\rho, \Lambda\varrho) \leq \imath\frac{\psi(\rho,\Lambda\rho)\cdot\psi(\varrho,\Lambda\varrho)}{1+\psi(\varrho,\Lambda\varrho)} + \jmath\psi(\rho, \varrho)$ is not valid if (ρ, ϱ) or $(\varrho, \rho) \in \{(1,0), (0,2)\}$. As any given $\imath, \jmath \in [0,1]$, we have

$$\psi(\Lambda(1,1),\Lambda(\frac{3}{2},2)) <\imath\frac{\psi((1,1)\Lambda(1,1))\cdot\psi((\frac{3}{2},2),\Lambda(\frac{3}{2},2))}{1+\psi((\frac{3}{2},2),\Lambda(\frac{3}{2},2))}+\jmath(\psi((1,1),(\frac{3}{2},2))$$
$$\psi((1,0),(0,2)) <\imath\frac{\psi((1,1),(1,0))\cdot\psi((\frac{3}{2},2),(0,2))}{1+\psi((\frac{3}{2},2),(0,2))}+\jmath(\psi((1,1),(\frac{3}{2},2))$$
$$\sqrt{5} <\imath\frac{\frac{3}{2}}{1+\frac{3}{2}}+\jmath\sqrt{5}$$
$$\sqrt{5} <\imath\frac{3}{5}+\jmath\sqrt{5}.$$

Thus, Λ does not satisfy the fixed point. Then, our contractive condition holds in $(\rho,\varrho)\in$ Ⓡ for all $(\rho,\varrho)\in$ Ⓡ. Similarly, it can be easily verified that Λ is Ⓡ-continuous.

Thus, Λ is satisfied by all the conditions of the above Theorem 3. Hence, $(0,0)$ is the fixed point of Λ and has a unique fixed point.

Theorem 4. *Let the mapping $\Lambda\colon\Psi\to\Psi$ and (Ψ,ψ) be a Ⓡ-$\mathcal{CM}$ space such that:*

(a) $\Psi(\Lambda,$ Ⓡ$)$ *is non-void set;*
(b) Ⓡ *is Λ-closed;*
(c) Λ *is Ⓡ-continuous;*
(d) *There exits $\imath,\jmath\in[0,\frac{1}{2})$ such that*

$$\psi(\Lambda\rho,\Lambda\varrho)\le\imath\psi(\rho,\varrho)+\jmath[\psi(\rho,\Lambda\rho)+\psi(\varrho,\Lambda\varrho)]$$

for all $\rho,\varrho\in\Psi$ with $(\rho,\varrho)\in$ Ⓡ and $\imath+2\jmath<1$. Then, Λ has a fixed point.

Proof. Let us assume (a), and choose ρ_0 as arbitrary element of $\Psi(\Lambda,$ Ⓡ$)$. Construct a sequence $\{\rho_\sigma\}$ that is

$$\rho_\sigma=\Lambda^\sigma(\rho_0)\quad\text{for all}\quad\sigma\in\mathbb{N}_0. \tag{5}$$

Since $(\rho_0,\Lambda\rho_0)\in$ Ⓡ, using Λ-closedness of Ⓡ and Theorem 1, we have

$$(\Lambda^\sigma(\rho_0),\Lambda^{\sigma+1}(\rho_0))\in \text{Ⓡ}.$$

So that

$$(\rho_\sigma,\rho_{\sigma+1})\in \text{Ⓡ},\quad\text{for all}\quad\sigma\in\mathbb{N}_0. \tag{6}$$

Then, the sequence $\{\rho_\sigma\}$ is Ⓡ-preserving. Let us apply the contractive condition

$$\begin{aligned}\psi(\rho_\sigma,\rho_{\sigma+1}) &=\psi(\Lambda\rho_{\sigma-1},\Lambda\rho_\sigma)\\ &\le\imath\psi(\rho_{\sigma-1},\rho_\sigma)+\jmath[\psi(\rho_{\sigma-1},\Lambda\rho_{\sigma-1})+\psi(\rho_\sigma,\Lambda\rho_\sigma)]\\ &\le\imath\psi(\rho_{\sigma-1},\rho_\sigma)+\jmath[\psi(\rho_{\sigma-1},\rho_\sigma)+\psi(\rho_\sigma,\rho_{\sigma+1})]\\ &\le\imath\psi(\rho_{\sigma-1},\rho_\sigma)+\jmath\psi(\rho_{\sigma-1},\rho_\sigma)+\jmath\psi(\rho_\sigma,\rho_{\sigma+1})\\ \psi(\rho_\sigma,\rho_{\sigma+1})-\jmath\psi(\rho_\sigma,\rho_{\sigma+1}) &\le(\imath+\jmath)\psi(\rho_{\sigma-1},\rho_\sigma)\\ \psi(\rho_\sigma,\rho_{\sigma+1}) &\le\frac{\imath+\jmath}{1-\jmath}\psi(\rho_{\sigma-1},\rho_\sigma)\quad\text{for all}\quad\sigma\in\mathbb{N}_0.\end{aligned}$$

By the inductive process, we obtain

$$\psi(\rho_\sigma,\rho_{\sigma+1})\le\left(\frac{\imath+\jmath}{1-\jmath}\right)^\sigma\psi(\rho_0,\rho_1),\quad\text{for all}\quad\sigma\in\mathbb{N}_0. \tag{7}$$

For any positive integers ς, σ with $\varsigma > \sigma$, we have

$$\begin{aligned}\psi(\rho_\sigma, \rho_\varsigma) \leq& \psi(\rho_\sigma, \rho_{\sigma+1}) + \cdots + \psi(\rho_{\varsigma-1}, \rho_\varsigma)\\ \leq& (\mathfrak{r}^\sigma + \cdots + \mathfrak{r}^{\varsigma-1})\psi(\rho_0, \rho_1) \quad \text{where} \quad \mathfrak{r} = \frac{\imath + \jmath}{1 - \jmath}\\ \psi(\rho_\sigma, \rho_\varsigma) \leq& \frac{\mathfrak{r}^\sigma}{1 - \mathfrak{r}}\psi(\rho_0, \rho_1),\end{aligned}$$

such that $\{\rho_\sigma\}$ is a Cauchy sequence. Since (Ψ, ψ) is Ⓡ-$\mathcal{CM}$space, there exists $\phi \in \Psi$ then

$$\lim_{\sigma \to \infty} \rho_\sigma = \phi. \tag{8}$$

Since Λ is Ⓡ-continuous. Then, $\rho_{\sigma+1} = \Lambda\rho_\sigma \xrightarrow{\psi} \Lambda\rho$. Therefore,

$$\Lambda\rho = \rho,$$

Hence, ρ is a fixed point of Λ. □

Theorem 5. *In addition to the hypothesis of the above Theorem 4, if the following condition holds:*

$$\Lambda(\Psi) \textit{ is } Ⓡ^{\mathfrak{s}} - \textit{connected. Then } \Lambda \textit{ has a unique fixed point.}$$

Proof. Let ρ and ϱ be two fixed points of Λ, that is $\mathcal{F}(\Lambda) \neq \phi$ and $\rho, \varrho \in \mathcal{F}(\Lambda)$ then for all $\sigma \in \mathbb{N}_0$, we have

$$\Lambda^\sigma\rho = \rho, \Lambda^\sigma\varrho = \varrho. \tag{9}$$

By our assumption, there exists a path (say $\mathfrak{u}_0, \mathfrak{u}_1, \cdots \mathfrak{u}_{\mathfrak{k}}$) of some finite length $\mathfrak{k}$ in $Ⓡ^{\mathfrak{s}}$ from ρ to ϱ.

$$\mathfrak{u}_0 = \rho, \mathfrak{u}_{\mathfrak{k}} = \varrho \quad \text{and} \quad [\mathfrak{u}_{\mathfrak{i}}, \mathfrak{u}_{\mathfrak{i}+1}] \in Ⓡ. \tag{10}$$

Since the mapping is Λ-closed, using Ⓡ-complete and Ⓡ-continuous

$$(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1}) \in Ⓡ, \quad \text{for each} \quad \mathfrak{i}(0 \leq \mathfrak{i} \leq \mathfrak{k} - 1). \tag{11}$$

Let us apply the contractive condition

$$\begin{aligned}\psi(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1}) \leq& \imath\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1})\\ &+ \jmath[\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}) + \psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1})]\\ \leq& \imath\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1})\\ &+ \jmath[\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}) + \psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}) + \psi(^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1})]\\ =& \imath\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1})\\ &+ \jmath[\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}) + \psi(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}) + \psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1})]\\ \leq& \imath\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1})\\ &+ \jmath[\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}) + \psi(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1})]\\ \leq& \imath\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1})\\ &+ \jmath\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1}) + \jmath\psi(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1})\\ \psi(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1}) - \jmath\psi(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1}) \leq& (\imath + \jmath)\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1})\\ \psi(\Lambda^\sigma\mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma\mathfrak{u}_{\mathfrak{i}+1}) \leq& \left(\frac{\imath + \jmath}{1 - \jmath}\right)\psi(\Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}}, \Lambda^{\sigma-1}\mathfrak{u}_{\mathfrak{i}+1})\end{aligned}$$

For our convenience, we put $\psi_\sigma^{\mathfrak{i}} = \psi(\Lambda^\sigma \mathfrak{u}_{\mathfrak{i}}, \Lambda^\sigma \mathfrak{u}_{\mathfrak{i}+1})$. Therefore, we have

$$\psi_\sigma^{\mathfrak{i}} \le \left(\frac{\imath + \jmath}{1-\jmath}\right)\psi_{\sigma-1}^{\mathfrak{i}}, \quad \text{for each} \quad \mathfrak{i}(0 \le \mathfrak{i} \le \mathfrak{k}-1). \tag{12}$$

Using the inductive process,

$$\psi_\sigma^{\mathfrak{i}} \le \left(\frac{\imath + \jmath}{1-\jmath}\right)\psi_{\sigma-1}^{\mathfrak{i}} \le \left(\frac{\imath + \jmath}{1-\jmath}\right)^2\psi_{\sigma-2}^{\mathfrak{i}} \le \left(\frac{\imath + \jmath}{1-\jmath}\right)^\sigma\psi_0^{\mathfrak{i}}.$$

so that

$$\psi_\sigma^{\mathfrak{i}} \le \left(\frac{\imath + \jmath}{1-\jmath}\right)^\sigma\psi_0^{\mathfrak{i}}.$$

Taking the limit $\sigma \to \infty$ in the above inequality, we have

$$\lim_{\sigma\to\infty} \psi_\sigma^{\mathfrak{i}} = 0 \quad \text{for each} \quad \mathfrak{i}, (0 \le \mathfrak{i} \le \mathfrak{k}-1). \tag{13}$$

By the definition of triangular inequality in (13), we obtain

$$\psi(\rho, \varrho) = \psi(\Lambda^\sigma \mathfrak{u}_0, \Lambda^\sigma \mathfrak{u}_{\mathfrak{k}}) \le \psi_\sigma^0 + \psi_\sigma^1 + \cdots + \psi_\sigma^{\mathfrak{k}-1} \to 0 \text{ as } \sigma \to \infty.$$

Hence, Λ has a unique fixed point. □

Remark 1. *In the above Theorem 5, we put $\imath = 0$, then it can be reduced to Theorem 3.2 in [16].*

Example 2. *Let $\Psi = \{(0,0),(1,0),(1,1),(1,2),(1,3),(0,2),(2,1),(2,2),(2,3)\}$ equipped with binary relation $\circledR = \{((0,0),(0,0)),((1,0),(1,1)),((1,2),(1,3)),$ $((2,0),(0,0)),((0,0),(2,0)),((2,0),(0,2)),((2,2),(2,3))\}$ and Euclidean metric ψ_2; defined by*

$$\psi((\rho_1,\rho_2),(\varrho_1,\varrho_2)) = \sqrt{(\rho_1-\varrho_1)^2 + (\rho_2-\varrho_2)^2}$$

then Ψ is a $\circledR$-complete metric space. Define a function $\Lambda\colon \Psi \to \Psi$ such that

$$\Lambda(\rho_1,\rho_2) = \begin{cases} (\rho_1, 0) & \text{if} \quad \rho_1 \le \rho_2 \\ (0, \rho_2) & \text{if} \quad \rho_1 > \rho_2. \end{cases}$$

We notice that $\psi(\Lambda\rho, \Lambda\varrho) \le \imath\psi(\rho,\varrho) + \jmath[\psi(\rho,\Lambda\rho) + \psi(\varrho,\Lambda\varrho)]$ is not valid if (ρ,ϱ) or $(\varrho,\rho) \in \{(1,3),(2,1)\}$. As given any $\imath, \jmath \in [0,1]$, we have

$$\begin{aligned}
\psi(\Lambda(1,3),\Lambda(2,1)) &< \imath\psi((1,3),(2,1)) + \jmath[\psi((1,3),\Lambda(1,3)) + \psi((2,1),\Lambda(2,1))] \\
\psi((1,0),(0,1)) &< \imath\psi((1,3),(2,1)) + \jmath[\psi((1,3),(1,0)) + \psi((2,1),(0,1))] \\
\sqrt{2} &< \imath\sqrt{5} + \jmath(\sqrt{9}+2) \\
\sqrt{2} &< \imath(\sqrt{5}) + \jmath(5).
\end{aligned}$$

Thus, Λ does not satisfy the fixed point. Then, our contractive condition holds in $(\rho,\varrho) \in \circledR$ for all $(\rho,\varrho) \in \circledR$. Similarly, it can be easily verified that Λ is $\circledR$-continuous and $\Lambda(\Psi)$ is $\circledR^{\mathfrak{s}}$-connected.

Thus, Λ is satisfied by all the conditions of the above Theorems 4 and 5. Hence, $(0,0)$ is the fixed point of Λ and has a unique fixed point.

4. An Application

We give an application for the first-order periodic boundary value problem of a unique solution with a binary relation which is applicable in our main results. Let us consider the first-order periodic boundary value problem as follows:

$$\rho'(\theta) = \mathfrak{f}(\theta, \rho(\theta)); \quad \theta \in \mathcal{I} = [0, \Lambda]; \quad \rho(0) = \rho(\Lambda), \tag{14}$$

the map $\mathfrak{f}\colon \Delta \times Ⓡ \to Ⓡ$ is a continuous function and $\Lambda > 0$.

Let us denote the space of continuous function be $\Pi(\Delta)$ and defined on Δ. Let us recall some basic definition as follows:

Definition 11 ([13]). *If a function $\chi \in \Pi^1(\Delta)$ is said to be a lower solution of* (14), *if*

$$\begin{aligned} &\chi'(\theta) \le \mathfrak{f}(\theta, \chi(\theta)), \quad \theta \in \Delta, \\ &\rho(0) \le \rho(\Lambda). \end{aligned}$$

Definition 12 ([13]). *If a function $\chi \in \Pi^1(\Delta)$ is said to be a upper solution of* (14), *if*

$$\begin{aligned} &\chi'(\theta) \ge \mathfrak{f}(\theta, \chi(\theta)), \quad \theta \in \Delta, \\ &\rho(0) \ge \rho(\Lambda). \end{aligned}$$

Theorem 6. *In addition to the boundary value problem of* (14), *then there exists a $\rho > 0$ for all $\rho, \varrho \in Ⓡ$ with $\rho \le \varrho$,*

$$0 \le \jmath\mathfrak{f}(\theta, \varrho) + \imath\varrho - [\jmath\mathfrak{f}(\theta, \rho) + \imath\rho] \le \imath\mathfrak{f}(\varrho - \rho) + \jmath[\mathfrak{f}(\Lambda\rho - \rho) + \mathfrak{f}(\Lambda\varrho - \varrho)]. \tag{15}$$

Then, Λ has a unique solution.

Proof. From a boundary value problem, the equation can be considered as

$$\begin{aligned} &\jmath\rho'(\theta) + \imath\rho(\theta) = \jmath\mathfrak{f}(\rho(\theta), \rho^{\sigma}(\theta)) + \imath\rho(\theta); \quad \theta \in \mathcal{I} = [0, \Lambda] \\ &\rho(0) = \rho(\Lambda) \end{aligned}$$

From the above problem, the equation is equivalent to the integral equation

$$\rho(\theta) = \int_0^{\Lambda} \mathcal{G}(\theta, \mathfrak{s})[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^{\sigma}(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\psi\mathfrak{s}.$$

where

$$\mathcal{G}(\theta, \mathfrak{s}) = \begin{cases} \frac{\mathfrak{e}^{\imath(\Lambda + \mathfrak{s} - \theta)}}{\mathfrak{e}^{-\imath\theta} - 1}, & 0 \le \mathfrak{s} < \theta \le \Lambda, \\ \frac{\mathfrak{e}^{\jmath(\mathfrak{s} - \theta)}}{\mathfrak{e}^{-\jmath\theta} - 1}, & 0 \le \theta < \mathfrak{s} \le \Lambda. \end{cases}$$

A mapping from $\Lambda\colon \Pi(\Delta) \to \Pi(\Delta)$ and binary relation by

$$\begin{aligned} &(\rho)(\theta) = \int_0 \mathcal{G}(\theta, \mathfrak{s})[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^{\sigma}(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\psi\mathfrak{s}, \\ &Ⓡ = \{(\rho, \varrho) \in \Pi(\Delta) \times \Pi(\Delta) \colon \rho(\theta) \le \varrho(\theta) \text{for all} \theta \in \Delta\}. \end{aligned}$$

(i) $\psi(\rho, \varrho) = \sup|\rho(\theta) - \varrho(\theta)|$ is the sup-metric with $\Pi(\Delta)$ for $\theta \in \Delta$ and the complete metric space is $\rho, \varrho \in \Pi(\Delta)$ and hence $(\Pi(\Delta), \psi)$ is Ⓡ-complete.

(ii) Let us choose Ⓡ-preserving, sequence $\{\rho_{\sigma}\}$ such that $\rho_{\sigma} \xrightarrow{\psi} \phi$, for all $\theta \in \Delta$, then

$$\rho_0(\theta) \le \rho_1(\theta) \le \cdots \le \rho_{\sigma}(\theta) \le \rho_{\sigma+1} \le \cdots$$

and convergent to $\rho(\theta)$ which implies $\rho_\sigma(\theta) \leq \phi(\theta)$for all$\theta \in \Delta, \sigma \in \mathbb{N}_0$, which implies $[\rho_\sigma, \phi] \in \circledR$ for all $\sigma \in \mathbb{N}_0$.

Hence, $\circledR$-continuous.

(iii) Let $\alpha \in \Pi^1(\Delta)$ be a lower solution of (14), then

$$\jmath\rho'(\theta) + \imath\rho(\theta) \leq \jmath\mathfrak{f}(\rho(\theta), \rho^\sigma(\theta)) + \imath\rho(\theta), \text{for all}\theta \in \Delta.$$

Multiplying by $\mathfrak{e}^{\imath\theta+\jmath\theta}$, we have

$$(\rho(\theta)\mathfrak{e}^{(\imath+\jmath)\theta})' \leq [\jmath\mathfrak{f}(\rho(\theta), \rho^\sigma(\theta)) + \imath\rho(\theta)]\mathfrak{e}^{(\imath+\jmath)\theta}, \text{for all}\theta \in \Delta,$$

which implies

$$\rho(\theta)\mathfrak{e}^{(\imath+\jmath)\theta} \leq \rho(0) + \int_0^\theta [\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\mathfrak{e}^{(\imath+\jmath)\mathfrak{s}}\psi\mathfrak{s}, \tag{16}$$

As $\rho(0) \leq \rho(\Lambda)$,

$$\rho(0)\mathfrak{e}^{(\imath+\jmath)} \leq \rho(\Lambda)\mathfrak{e}^{(\imath+\jmath)} \leq \rho(0) + \int_0^\Lambda [\jmath\mathfrak{f}(\rho(\mathfrak{s}) + \rho^\sigma(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\mathfrak{e}^{(\imath+\jmath)\mathfrak{s}}\psi\mathfrak{s},$$

thus,

$$\rho(0) \leq \int_0^\Lambda \frac{\mathfrak{e}^{(\imath+\jmath)\mathfrak{s}}}{\mathfrak{e}^{(\imath+\jmath)} - 1}[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\psi\mathfrak{s}, \tag{17}$$

From (16) and (17),

$$\begin{aligned}
\rho(\theta)\mathfrak{e}^{(\imath+\jmath)} \leq& \int_0^\Lambda \frac{\mathfrak{e}^{(\imath+\jmath)\mathfrak{s}}}{\mathfrak{e}^{(\imath+\jmath)} - 1}[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\psi\mathfrak{s} \\
&+ \int_0^\theta [\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\mathfrak{e}^{(\imath+\jmath)\mathfrak{s}}\psi\mathfrak{s} \\
&= \int_0^\Lambda \frac{\mathfrak{e}^{\imath(+\mathfrak{s})}}{\mathfrak{e}^{-\imath\theta} - 1}[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s}))] + \imath\rho(\mathfrak{s})\psi\mathfrak{s} \\
&+ \int_0^\Lambda \frac{\mathfrak{e}^{\jmath(\mathfrak{s})}}{\mathfrak{e}^{-\jmath\theta} - 1}[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s}))] + \imath\rho(\mathfrak{s})\psi\mathfrak{s},
\end{aligned}$$

so that,

$$\begin{aligned}
\rho(\theta) \leq& \int_0^\Lambda \frac{\mathfrak{e}^{\imath(+\mathfrak{s}-\theta)}}{\mathfrak{e}^{-\imath\theta} - 1}[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s}))] + \imath\rho(\mathfrak{s})\psi\mathfrak{s} \\
&+ \int_\theta^\Lambda \frac{\mathfrak{e}^{\jmath(\mathfrak{s}-\theta)}}{\mathfrak{e}^{-\jmath\theta} - 1}[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s}))] + \imath\rho(\mathfrak{s})\psi\mathfrak{s}, \\
&= \int_0^\Lambda \mathcal{G}(\theta, \mathfrak{s})[\jmath\mathfrak{f}(\rho(\mathfrak{s}), \rho^\sigma(\mathfrak{s}))]\psi\mathfrak{s} \\
&= (\Lambda\rho)(\theta)
\end{aligned}$$

that is $(\rho(\theta), \Lambda\rho(\theta)) \in \circledR$ for all $\theta \in \Delta$, which implies that $\mathcal{X}(\Lambda, \circledR) \neq \phi$.

(iv) For any $(\rho, \varrho) \in \circledR$, that is $\rho(\theta) \leq \varrho(\theta)$

$$\jmath\mathfrak{f}(\rho(\theta), \rho^\sigma(\theta)) + \imath\rho(\theta) \leq \jmath\mathfrak{f}(\varrho(\theta), \varrho^\sigma(\theta)) + \imath\varrho(\theta), \quad \text{for all}\theta \in \Delta$$

and $\mathcal{G}(\theta,\mathfrak{s}) > 0$ for $(\theta,\mathfrak{s}) \in \Delta \times \Delta$,

$$\begin{aligned}(\Lambda\rho)(\theta) &= \int_0^\Lambda \mathcal{G}(\theta,\mathfrak{s})[\jmath\mathfrak{f}(\rho(\mathfrak{s}),\rho^\sigma(\mathfrak{s})) + \imath\rho(\mathfrak{s})]\psi\mathfrak{s},\\ &\leq \mathcal{G}(\theta,\mathfrak{s})[\jmath\mathfrak{f}(\varrho(\mathfrak{s}),\varrho^\sigma(\mathfrak{s})) + \imath\varrho(\mathfrak{s})]\psi\mathfrak{s},\\ &= (\Lambda\varrho)(\theta) \quad \text{for all} \theta \in \Delta\end{aligned}$$

which implies that $(\Lambda\rho,\Lambda\varrho) \in Ⓡ$, that is Ⓡ is Λ-closed.

(v) For all $(\rho,\varrho) \in Ⓡ$,

$$\begin{aligned}\psi(\Lambda\rho,\Lambda\varrho) &= \sup_{\theta\in\Delta}|(\Lambda\rho)(\theta) - (\Lambda\varrho)(\theta)| = \sup_{\theta\in\Delta}((\Lambda\varrho)(\theta) - (\Lambda\rho)(\theta))\\ &\leq \sup_{\theta\in\Delta}\int_0^\Lambda \mathcal{G}(\theta,\mathfrak{s})[\jmath\mathfrak{f}(\varrho(\mathfrak{s}),\varrho^\sigma(\mathfrak{s})) + \imath\varrho(\mathfrak{s}) - \jmath\mathfrak{f}(\rho(\mathfrak{s}),\rho^\sigma(\mathfrak{s})) - \imath\rho(\mathfrak{s})]\psi\mathfrak{s}\\ &\leq \sup_{\theta\in\Delta}\int_0^\Lambda \mathcal{G}(\theta,\mathfrak{s})\imath^2(\varrho(\mathfrak{s}) - \rho(\mathfrak{s}))\psi\mathfrak{s}\\ &+ \sup_{\theta\in\Delta}\int_0^\Lambda \mathcal{G}(\theta,\mathfrak{s})\jmath^2[\mathfrak{f}(\varrho(\mathfrak{s}),\varrho^\sigma(\mathfrak{s})) - \mathfrak{f}(\rho(\mathfrak{s}),\rho^\sigma(\mathfrak{s}))]\psi\mathfrak{s}\\ &= \imath^2\psi(\rho,\varrho)\int_0^\Lambda \mathcal{G}(\theta,\mathfrak{s})\psi\mathfrak{s} + \jmath^2[\psi(\rho,\Lambda\rho) + \psi(\varrho,\Lambda\varrho)]\int_0^\Lambda \mathcal{G}(\theta,\mathfrak{s})\psi\mathfrak{s}\\ &= \imath^2\psi(\rho,\varrho)\sup_{\theta\in\Delta}\frac{1}{\mathfrak{e}^{-\imath\theta} - 1}\left(\frac{1}{\imath}\mathfrak{e}^{\imath(\Lambda+\mathfrak{s}-\theta)}\Big|_0^\theta + \frac{1}{\imath}\mathfrak{e}^{\imath(\mathfrak{s}-\theta)}\Big|_\theta^\Lambda\right)\\ &+ \jmath^2[\psi(\rho,\Lambda\rho) + \psi(\varrho,\Lambda\varrho)]\sup_{\theta\in\Delta}\frac{1}{\mathfrak{e}^{-\jmath\theta} - 1}\left(\frac{1}{\jmath}\mathfrak{e}^{\jmath(+\mathfrak{s}-\theta)}\Big|_0^\theta + \frac{1}{\jmath}\mathfrak{e}^{\jmath(\mathfrak{s}-\theta)}\Big|_\theta^\Lambda\right)\\ &= \imath^2\psi(\rho,\varrho)\frac{1}{\imath\mathfrak{e}^{-\imath\theta} - 1}(\mathfrak{e}^{-\imath\theta} - 1) + \jmath^2[\psi(\rho,\Lambda\rho) + \psi(\varrho,\Lambda\varrho)]\frac{(\mathfrak{e}^{-\jmath\theta} - 1)}{\jmath\mathfrak{e}^{-\jmath\theta} - 1}\\ &= \imath\psi(\rho,\varrho) + \jmath[\psi(\rho,\Lambda\rho) + \psi(\varrho,\Lambda\varrho)]\\ \psi(\Lambda\rho,\Lambda\varrho) &\leq \imath\psi(\rho,\varrho) + \jmath[\psi(\rho,\Lambda\rho) + \psi(\varrho,\Lambda\varrho)], \quad \text{for all } \rho,\varrho \in Ⓡ.\end{aligned}$$

Hence, from the above Theorem 4, all conditions are satisfied. Thus, Λ has a unique fixed point. □

5. Conclusions

In this paper, we proved a unique fixed point theorem using the concept of rational contractive mappings in Ⓡ-metric space. A concrete illustration is given to demonstrate the validity of the concept and the degree of applicability of our findings. Özgür and Taş [28] proved the fixed-circle theorem on metric spaces. It remains an intriguing open problem to investigate the fixed circle on Ⓡ-metric space instead of fixed-point theorems on Ⓡ-metric space.

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