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A Study on the Modified Form of Riemann-Type Fractional Inequalities via Convex Functions and Related Applications

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Abstract: In this article, we provide constraints for the sum by employing a generalized modified form of fractional integrals of Riemann-type via convex functions. The mean fractional inequalities for functions with convex absolute value derivatives are discovered. Hermite–Hadamard-type fractional inequalities for a symmetric convex function are explored. These results are achieved using a fresh and innovative methodology for the modified form of generalized fractional integrals. Some applications for the results explored in the paper are briefly reviewed.

Keywords: modified form of fractional integral; convex function; inequality; increasing functions

MSC: 26D15; 26D10; 26A33; 34B27



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1. Introduction and Preliminaries

The area of mathematics known as fractional calculus focuses on the applications of derivatives and integrals under fractional orders, i.e., the order that is not a natural number or integer. The theory of fractional calculus has undergone rapid growth and has attracted the interest of several academics from a wide range of fields. For instance, mathematical modeling with these operators has suddenly become widespread, and the simulation power of these operators has made it possible to use such models in various disciplines. Some instances of these applications can be seen in many newly published papers such as a fractional model of Syphilis [1], a hybrid fractional model of a thermostat [2], an existence study on two models of epidemic strains [3], a fractional Caputo–Fabrizio version of a Mumps virus model [4], two models of Q fever [5], a fractional HIV-1 model with uncertainty [6], a fractal–fractional model of AH1N1/09 virus and CD4⁺ T-cells [7,8], and hybrid fractional proportional problems [9]. In the framework of fractional calculus equipped with Mittag–Leffler kernels, Samraiz et al. introduced generalized fractional operators with applications in mathematical physics, and Fernandez et al. [10] introduced Hermite–Hadamard integral inequalities. For a novel generalized harmonic convexity, Yang et al. [11] presented two generalized fractional extensions of both the Fejer–Hadamard and Hadamard inequalities. For convex functions that cover the aforementioned conclusion, such as Riemann–Liouville fractional integrals, Mohammad et al. [12] established several inequalities of the Hermite–Hadamard type. Jia et al. [13] turned to Hadamard-type inequalities under the RL-fractional operators of generalized convex functions, together with their weighted equivalents known as Fejer–Hadamard-type inequalities. By using an

h -convex function and the Caputo–Fabrizio fractional operator, Chen et al. [14] provided an inequality of the Hermite–Hadamard type. They also gave several new results for the h -convex function’s class. Recently, it has been discovered that fractional calculus is incredibly helpful for simulating challenging issues in engineering, mechanics, medicine, chemistry, physics and many other fields [15–17].

Convex functions have a long and distinguished history in science for more than a century and have been a hot topic of research. Various convex function estimations, extensions, and modifications have been put out by researchers. In addition to introducing the idea of generalized harmonically ψ -convex functions in the fractal domain, Jile et al. [18] also developed a new identity that has been linked to several original and well-known inequalities of the mid-point-type, Ostrowski-type, and trapezoid-type. By utilizing Hölder’s, Minkowski, and power mean inequalities through quantum calculus, Khan et al. [19] provided some new bounds for Ostrowski-type functionals. With the help of a new category of n -polynomial s -type convex functions, Rashid et al. [20] created a variety of new extensions of the Hermite–Hadamard- and Ostrowski-type inequalities and produced integral identities for differentiable functions of the first and second order. By utilizing the methods of fractal analysis and (s, m) -convexity, Abdeljawad et al. [21] generated some acceptable theorems in relation to generalized Hermite–Hadamard-type inequalities and local fractional Simpson-like inequalities. Kashuri et al. [22] conducted research to examine the fundamental algebraic characteristics of the new general category of functions known as the (n, m) -generalized convex. Ma et al. [23] proposed and analyzed a more broadly defined class of convex functions and also developed different inequalities for this class of convex function.

In addition to ensuring a convex function’s integrability, this gives an estimate from both sides of the mean value. The Hermite–Hadamard–Fejer integral inequalities were proposed by Muhammad et al. [24] under the existing notions in fractional calculus along with positive symmetric weighted kernels. On fractal sets of real line numbers, Sun [25] presented the idea of generalized harmonically convex functions and established generalized Hermite–Hadamard inequalities for such functions. Aljaaidi et al. [26] introduced a fresh fractional Hermite–Hadamard-inequality by applying the proportional fractional operators of integrable functions to another continuous and strictly growing function. Bounoua et al. [27] developed innovative fractional integral inequalities of the Volterra–Fredholm and Hermite–Hadamard type as a useful tool for identifying the boundaries of solutions to fractional differential equations and fractional integral equations.

For the flow of our work, we first need the following definitions. In 1905, Jensen [28] introduced a convex function by the following definition.

Definition 1. A function $\psi : [a, b] \rightarrow \mathbb{R}$ is called convex if

$$\psi(vx + (1 - v)y) \leq v\psi(x) + (1 - v)\psi(y)$$

for all $x, y \in [a, b]$ and all $v \in [0, 1]$.

Definition 2. Let $\psi \in L[c, d]$. Then the left-sided and right-sided Riemann–Liouville fractional integrals of order $\xi > 0$ with $c \geq 0$ are given by

$$J_{c^+}^{\xi} \Psi(z) = \frac{1}{\Gamma(\xi)} \int_c^z (z - t)^{\xi-1} \psi(t) dt, \quad z > c,$$

$$J_{d^-}^{\xi} \Psi(z) = \frac{1}{\Gamma(\xi)} \int_z^d (t - z)^{\xi-1} \psi(t) dt, \quad z < d.$$

Remember that the space of functions that can be integrated across the interval $[c, d]$ is represented by the notation $L[c, d]$.

The k -Riemann–Liouville fractional integral is presented in [29] as follows:

Definition 3. Let $\psi \in L[c, d]$. The k -fractional integral (right and left) of order $\xi, k > 0$ with $c \geq 0$ is formulated as

$$J_{c^+}^{\xi, k} \Psi(z) = \frac{1}{k\Gamma_k(\xi)} \int_c^z (z-t)^{\frac{\xi}{k}-1} \psi(t) dt, \quad z > c,$$

$$J_{d^-}^{\xi, k} \Psi(z) = \frac{1}{k\Gamma_k(\xi)} \int_z^d (t-z)^{\frac{\xi}{k}-1} \psi(t) dt, \quad z < d,$$

where $\Gamma_k(\cdot)$ is the k -Gamma function (see [30]).

The Riemann–Liouville fractional integral with respect to an increasing function is another extension of the standard Riemann–Liouville fractional integral (see [31]).

Definition 4 ([31]). Let Ω be a positive and increasing map on $(c, d]$, including a continuous derivative Ω' on (c, d) . Additionally, let $\psi : [c, d] \rightarrow \mathfrak{R}$ be an integrable function. The fractional integrals (left-sided and right-sided) of ψ with respect to another function Ω on $[c, d]$ of order $\xi > 0$ are

$$J_{\Omega, c^+}^{\xi} \Psi(z) = \frac{1}{\Gamma(\xi)} \int_c^z (\Omega(z) - \Omega(t))^{\xi-1} \Omega'(t) \psi(t) dt, \quad z > c,$$

$$J_{\Omega, d^-}^{\xi} \Psi(z) = \frac{1}{\Gamma(\xi)} \int_z^d (\Omega(t) - \Omega(z))^{\xi-1} \Omega'(t) \psi(t) dt, \quad z < d.$$

The next definition defines a k -analogue of the previous definition.

Definition 5 ([32]). Let Ω be a positive and increasing map on $(c, d]$, including a continuous derivative Ω' on (c, d) . Additionally, let $\psi : [c, d] \rightarrow \mathfrak{R}$ be an integrable function. The fractional integrals (left-sided and right-sided) of ψ with respect to another function Ω on $[c, d]$ of order $\xi > 0$ are

$$J_{\Omega, c^+}^{\xi, k} \Psi(z) = \frac{1}{k\Gamma_k(\xi)} \int_c^z (\Omega(z) - \Omega(t))^{\frac{\xi}{k}-1} \Omega'(t) \psi(t) dt, \quad z > c,$$

$$J_{\Omega, d^-}^{\xi, k} \Psi(z) = \frac{1}{k\Gamma_k(\xi)} \int_z^d (\Omega(t) - \Omega(z))^{\frac{\xi}{k}-1} \Omega'(t) \psi(t) dt, \quad z < d.$$

Definition 6 ([33]). Let s be a real number excluding 1, k be a positive number, $\xi, \rho, \omega, \gamma \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\xi) > 0$, and $n \in \mathbb{N}$. Let $\psi > 0$ be an increasing map on $(0, q]$, including continuous derivative ψ' on $(0, q)$ and $\psi \in L[0, q]$. Then the modified form of the (k, s) -fractional integral with order ξ is defined as follows

$${}_k J_{c^+; \rho, \xi}^{\omega, \gamma} \Psi(z) = \frac{(s+1)^{\frac{\xi}{k}-1}}{k} \int_c^z \left(\Omega^{s+1}(z) - \Omega^{s+1}(t) \right)^{\frac{\xi}{k}-1}$$

$$\times E_{k; \rho, \xi}^{\gamma} \left(w(\Omega^{s+1}(z) - \Omega^{s+1}(t)) \right)^{\frac{\rho}{k}} \Omega^s(t) \Omega'(t) \psi(t) dt,$$

where $\Omega^{s+1}(t) = (\Omega(t))^{s+1}$ and $E_{k; \rho, \xi}^{\gamma}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n, k} \theta^n}{\Gamma_k(\rho n + \xi) n!}$ is the Mittag–Leffler function.

Remark 1. Case 1: Corresponding to the choice $s = 0$ and $\gamma = 0$, we get the k -integral operator given in Definition 5.

Case 2: Corresponding to the choice $s = 0$, $\gamma = 0$, and $k = 1$, we get the integral operator given in Definition 4.

Case 3: Corresponding to the choice $s = 0$, $\gamma = 0$, and $\Omega(z) = z$, we get the k -integral operator given in Definition 3.

Case 4: Corresponding to the choice $s = 0$, $\gamma = 0$, $\Omega(z) = z$, and $k = 1$, we get the integral operator given in Definition 2.

Now, in the next section, we establish our main results.

2. Main Results

We start this section by stating the following theorem.

Theorem 1. Let $\Psi, \Omega : [c, d] \rightarrow \mathfrak{R}$ be two functions with Ψ as a positive convex and Ω as a strictly increasing and differentiable function with $(\Omega^{s+1})'(z) \in L[c, d]$. Then for $\xi, \zeta \geq k$, we have the following inequality:

$$\begin{aligned}
 & k(s+1) {}_{c^+}^{\xi, s} J_{k, \rho, \zeta}^{\omega, \gamma} \Psi(z) + k(s+1) {}_{d^-}^{\xi, s} J_{k, \rho, \zeta}^{\omega, \gamma} \Psi(z) \\
 & \leq \sum_{n=0}^{\infty} \frac{(\gamma)_{n, k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) n!} \left[\frac{(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}-1}}{(z-c)\Gamma_k(\rho n + \zeta)} \right. \\
 & \times \left((z-c)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} \Psi(c) \right. \\
 & \left. - (\Psi(c) - \Psi(z)) \int_c^z (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho n}{k}+1} dt \right) \\
 & \left. + \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\xi}{k}-1}}{(d-z)\Gamma_k(\rho n + \zeta)} \right. \\
 & \times \left((d-z)(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho n}{k}+1} \Psi(d) \right. \\
 & \left. - (\Psi(d) - \Psi(z)) \int_z^d (\Omega^{s+1}(t) - \Omega^{s+1}(z))^{\frac{\rho n}{k}+1} dt \right) \Big]. \tag{1}
 \end{aligned}$$

Proof. Based on the fact that Ω is strictly increasing and differentiable, $(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}-1} \leq (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}-1}$, where $z \in [c, d]$ and $t \in [c, z]$, $\xi \geq k$, and $(\Omega^{s+1})'(z) > 0$. Hence, the following inequality is valid

$$(\Omega^{s+1})'(t)(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}-1} \leq (\Omega^{s+1})'(t)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}-1}. \tag{2}$$

By using the convexity of Ψ , we have

$$\Psi(t) \leq \frac{z-t}{z-c} \Psi(c) + \frac{t-c}{z-c} \Psi(z). \tag{3}$$

From (2) and (3), one can have

$$\begin{aligned}
 & (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}-1} (s+1) \Omega^s(t) \Omega'(t) \Psi(t) \\
 & \leq \frac{(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}-1}}{z-c} \left[(s+1) \Omega^s(t) \Omega'(t) \left[(z-t) \Psi(c) + (t-c) \Psi(z) \right] \right].
 \end{aligned}$$

Multiplying both sides by $E_{k;\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right)$ and integrating over $[c, z]$, we have

$$\begin{aligned} & (s+1) \int_c^z (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\zeta}{k}-1} E_{k;\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right) \Omega^s(t) \Omega'(t) \Psi(t) dt \\ & \leq \frac{(s+1)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\zeta}{k}-1}}{z-c} \\ & \times \left[\Psi(c) \int_c^z (z-t) E_{k;\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right) \Omega^s(t) \Omega'(t) dt \right. \\ & \left. + \Psi(z) \int_c^z (t-c) E_{k;\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right) \Omega^s(t) \Omega'(t) dt \right]. \end{aligned}$$

By using Definition 6, we have

$$\begin{aligned} & k(s+1) \frac{\zeta}{k} J_{c^+; \rho, \zeta}^{\omega, \gamma} \Psi(z) \\ & \leq \sum_{n=0}^\infty \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) n!} \left[\frac{(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\zeta}{k}-1}}{(z-c) \Gamma_k(\rho n + \zeta)} \right. \\ & \times \left((z-c)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} \Psi(c) \right. \\ & \left. \left. - (\Psi(c) - \Psi(z)) \int_c^z (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho n}{k}+1} dt \right) \right]. \end{aligned} \tag{4}$$

Now for $z \in [c, d]$ and $t \in [z, d]$, $\zeta \geq k$, the following inequality is valid:

$$(\Omega^{s+1})'(t) (\Omega^{s+1}(t) - \Omega^{s+1}(z))^{\frac{\zeta}{k}-1} \leq (\Omega^{s+1})'(t) (\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\zeta}{k}-1}. \tag{5}$$

Again from the convexity of Ψ , we have

$$\Psi(t) \leq \frac{t-z}{d-z} \Psi(d) + \frac{d-t}{d-z} \Psi(z). \tag{6}$$

The next inequality can be obtained from (5) and (6) by using the same procedure as for (2) and (3), i.e.,

$$\begin{aligned} & k(s+1) \frac{\zeta}{k} J_{d^-; \rho, \zeta}^{\omega, \gamma} \Psi(z) \\ & \leq \sum_{n=0}^\infty \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) n!} \left[\frac{(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\zeta}{k}-1}}{(d-z) \Gamma_k(\rho n + \zeta)} \right. \\ & \times \left((d-z)(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho n}{k}+1} \Psi(d) \right. \\ & \left. \left. - (\Psi(d) - \Psi(z)) \int_z^d (\Omega^{s+1}(t) - \Omega^{s+1}(z))^{\frac{\rho n}{k}+1} dt \right) \right]. \end{aligned} \tag{7}$$

By adding inequalities (4) and (7), we get the desired inequality. \square

In the following remarks, we show the generality of our findings.

Remark 2. Corresponding to the choice $s = \gamma = 0$ in Theorem 1, we get ([34] Theorem 6).

Remark 3. By choosing $s = \gamma = 0$ and $k = 1$ in Theorem 1, we get ([35] Theorem 1).

Remark 4. If we take $s = \gamma = 0, k = 1,$ and $\Omega(z) = z$ in Theorem 1, then we get ([36] Theorem 1).

Theorem 2. Consider two functions $\Psi, \Omega : [c, d] \rightarrow \mathfrak{R}$. Additionally, let Ψ be differentiable, $|\Psi'|$ be convex, and Ω be a strictly increasing and differentiable function with $(\Omega^{s+1})'(z) \in L[c, d]$. Then for $\xi, \zeta \geq 0$ and $k > 0$, we have

$$\begin{aligned} & \left| \left(\frac{\xi}{k} + \frac{\rho n}{k} \right) k(s+1) \frac{\xi}{k} J_{c^+; \rho, \xi}^{\omega, \gamma} \Psi(z) + \left(\frac{\zeta}{k} + \frac{\rho n}{k} \right) k(s+1) \frac{\zeta}{k} J_{d^-; \rho, \zeta}^{\omega, \gamma} \Psi(z) \right. \\ & - \left(\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \xi) n!} (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k} + \frac{\rho n}{k}} \Psi(c) \right. \\ & \left. \left. + \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\zeta}{k} + \frac{\rho n}{k}} \Psi(d) \right) \right| \\ & \leq \frac{(z-c)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} E_{k; \rho, \xi}^{\gamma} \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) |\Psi'(c)|}{2} \\ & + \frac{(d-z)(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\zeta}{k}} E_{k; \rho, \zeta}^{\gamma} \left(\omega (\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho}{k}} \right) |\Psi'(d)|}{2} \\ & + |\Psi'(z)| \left(\frac{(z-c)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} E_{k; \rho, \xi}^{\gamma} \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right)}{2} \right. \\ & \left. + \frac{(d-z)(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\zeta}{k}} E_{k; \rho, \zeta}^{\gamma} \left(\omega (\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho}{k}} \right)}{2} \right). \end{aligned} \tag{8}$$

Proof. By utilizing the convexity of $|\Psi'|$, we get

$$|\Psi'(t)| \leq \frac{z-t}{z-c} |\Psi'(c)| + \frac{t-c}{z-c} |\Psi'(z)|,$$

and

$$\Psi'(t) \leq \frac{z-t}{z-c} |\Psi'(c)| + \frac{t-c}{z-c} |\Psi'(z)|. \tag{9}$$

Since the function Ω is strictly increasing,

$$(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}} \leq (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}}, \tag{10}$$

where $z \in [c, d]$ and $t \in [c, z], \xi \geq 0, k > 0$.

By combining (9) and (10), we can write

$$\begin{aligned} & (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}} \Psi'(t) \\ & \leq \frac{(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}}}{z-c} \left[(z-t) |\Psi'(c)| + (t-c) |\Psi'(z)| \right]. \end{aligned}$$

Multiplying both sides by $E_{k; \rho, \xi}^{\gamma} \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right)$ and integrating over $[c, z]$, we have

$$\begin{aligned}
 & \int_c^z (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}} E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right) \Psi'(t) dt \\
 & \leq \frac{(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}}}{z - c} \left[|\Psi'(c)| \int_c^z (z - t) E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right) dt \right. \\
 & \quad \left. + |\Psi'(z)| \int_c^z (t - c) E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right) dt \right] \\
 & \leq \frac{(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}}}{z - c} \left[|\Psi'(c)| \int_c^z (z - t) E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) dt \right. \\
 & \quad \left. + |\Psi'(z)| \int_c^z (t - c) E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) dt \right] \\
 & = (z - c) (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) \left[\frac{|\Psi'(c)| + |\Psi'(z)|}{2} \right]. \tag{11}
 \end{aligned}$$

Consider

$$\begin{aligned}
 & \int_c^z (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}} E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\rho}{k}} \right) \Psi'(t) dt \\
 & = - \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \xi) n!} (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k} + \frac{\rho n}{k}} \Psi(c) \\
 & \quad + \left(\frac{\xi}{k} + \frac{\rho n}{k} \right) k(s + 1) \frac{\xi}{k} J_{c^+; \rho, \xi}^{\omega, \gamma} \Psi(z).
 \end{aligned}$$

Therefore, (11) becomes

$$\begin{aligned}
 & \left(\frac{\xi}{k} + \frac{\rho n}{k} \right) k(s + 1) \frac{\xi}{k} J_{c^+; \rho, \xi}^{\omega, \gamma} \Psi(z) \\
 & - \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \xi) n!} (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k} + \frac{\rho n}{k}} \Psi(c) \\
 & \leq (z - c) (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) \left[\frac{|\Psi'(c)| + |\Psi'(z)|}{2} \right]. \tag{12}
 \end{aligned}$$

Again, from the convexity of $|\Psi'|$, we have

$$\Psi'(t) \geq - \left(\frac{z - t}{z - c} |\Psi'(c)| + \frac{t - c}{z - c} |\Psi'(z)| \right).$$

The next inequality is obtained by using the same method that we used to get (12); i.e.,

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \xi) n!} (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k} + \frac{\rho n}{k}} \Psi(c) \\
 & - \left(\frac{\xi}{k} + \frac{\rho n}{k} \right) k(s + 1) \frac{\xi}{k} J_{c^+; \rho, \xi}^{\omega, \gamma} \Psi(z) \\
 & \leq (z - c) (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} E_{k;\rho,\xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) \left[\frac{|\Psi'(c)| + |\Psi'(z)|}{2} \right]. \tag{13}
 \end{aligned}$$

By using the modulus property on inequalities (12) and (13), we obtain

$$\begin{aligned} & \left| \left(\frac{\zeta}{k} + \frac{\rho n}{k} \right) k \Gamma_k(\zeta)(s+1) {}^{\zeta} J_{c^+; \rho, \zeta}^{\omega, \gamma} \Psi(z) \right. \\ & \left. - \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(z) - \Omega^{s+1}(c)) {}^{\zeta} J_{c^+; \rho, \zeta}^{\omega, \gamma} \Psi(c) \right| \\ & \leq (z-c) (\Omega^{s+1}(z) - \Omega^{s+1}(c)) {}^{\zeta} E_{k; \rho, \zeta}^{\gamma} \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c)) {}^{\rho/k} \right) \left[\frac{|\Psi'(c)| + |\Psi'(z)|}{2} \right]. \end{aligned} \tag{14}$$

Now by utilizing the convexity of $|\Psi'|$ on $[d, z]$, i.e.,

$$|\Psi'(t)| \leq \frac{t-z}{d-z} |\Psi'(d)| + \frac{d-t}{d-z} |\Psi'(z)|,$$

and

$$((\Omega^{s+1}(t) - \Omega^{s+1}(z)) {}^{\zeta/k} \leq (\Omega^{s+1}(d) - \Omega^{s+1}(z)) {}^{\zeta/k}.$$

and applying same procedure adopted to obtain (14), we get

$$\begin{aligned} & \left| \left(\frac{\zeta}{k} + \frac{\rho n}{k} \right) k (s+1) {}^{\zeta} J_{d^-; \rho, \zeta}^{\omega, \gamma} \Psi(z) \right. \\ & \left. - \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(z)) {}^{\zeta} J_{d^-; \rho, \zeta}^{\omega, \gamma} \Psi(d) \right| \\ & \leq (d-z) (\Omega^{s+1}(d) - \Omega^{s+1}(z)) {}^{\zeta} E_{k; \rho, \zeta}^{\gamma} \left(\omega (\Omega^{s+1}(d) - \Omega^{s+1}(z)) {}^{\rho/k} \right) \left[\frac{|\Psi'(d)| + |\Psi'(z)|}{2} \right]. \end{aligned} \tag{15}$$

From inequalities (14) and (15) via triangular inequality, we get the required result. \square

Again, the following remarks show the generality of our results.

Remark 5. Corresponding to the choice $s = \gamma = 0$ in Theorem 2, we get ([34] Theorem 8).

Remark 6. By choosing $s = \gamma = 0$ and $k = 1$ in Theorem 2, we get ([35] Theorem 2).

Remark 7. If we take $s = \gamma = 0, k = 1$, and $\Omega(z) = z$ in Theorem 2, then we get ([36] Theorem 2).

We need the following useful lemma for our next main results:

Lemma 1. Let $\Psi : [c, d] \rightarrow \mathfrak{R}$ be a convex mapping. If Ψ is symmetric about $\frac{c+d}{2}$, then we have

$$\Psi\left(\frac{c+d}{2}\right) \leq \Psi(z), \quad z \in [c, d]. \tag{16}$$

Theorem 3. Let $\Psi : [c, d] \rightarrow \mathfrak{R}$ be convex and symmetric about $\frac{c+d}{2}$ and Ω be a strictly increasing differentiable function with $(\Omega^{s+1})'(z) \in L[c, d]$. Then for $\xi, \zeta \geq 0$ and $k > 0$, we have

$$\begin{aligned} & \Psi\left(\frac{c+d}{2}\right) \left[\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n + \zeta}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(c)) {}^{\zeta/k + \frac{\rho n}{k} + 1} \right. \\ & \left. + \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n + \zeta}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(c)) {}^{\zeta/k + \frac{\rho n}{k} + 1} \right] \end{aligned}$$

$$\begin{aligned}
 &\leq k(s+1)^{1+\frac{\xi}{k}} J_{c^+; \rho, \xi+1}^{\omega, \gamma} \Psi(d) + k(s+1)^{1+\frac{\xi}{k}} J_{d^-; \rho, \xi+1}^{\omega, \gamma} \Psi(c) \\
 &\leq \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho^n}{k} + 1\right) n!} \left[\frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}}}{(d-c)\Gamma_k(\rho n + \xi)} \right. \\
 &\quad \times \left((d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k} + 1} \Psi(c) \right. \\
 &\quad \left. - (\Psi(c) - \Psi(d)) \int_c^d (\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho n}{k} + 1} dz \right) \\
 &\quad + \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}}}{(d-c)\Gamma_k(\rho n + \xi)} \\
 &\quad \times \left((d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k} + 1} \Psi(d) \right. \\
 &\quad \left. - (\Psi(d) - \Psi(c)) \int_c^d (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho n}{k} + 1} dz \right) \Big]. \tag{17}
 \end{aligned}$$

Proof. Based on the fact that Ω is strictly increasing and differentiable, $(\Omega^{s+1}(z) - \Omega^{s+1}(t))^{\frac{\xi}{k}-1} \leq (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}-1}$, where $z \in [c, d]$ and $t \in [c, z]$, $\xi \geq k$, and $(\Omega^{s+1})'(z) > 0$. Hence, the following inequality is valid:

$$(\Omega^{s+1})'(z)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} \leq (\Omega^{s+1})'(z)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}}. \tag{18}$$

From the convexity of Ψ , we have

$$\Psi(z) \leq \frac{z-c}{d-c} \Psi(d) + \frac{d-z}{d-c} \Psi(c). \tag{19}$$

From (18) and (19), one can have

$$\begin{aligned}
 &(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} (s+1) \Omega^s(z) \Omega'(z) \Psi(z) \\
 &\leq \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}}}{d-c} \left[(s+1) \Omega^s(z) \Omega'(z) \left[(z-c) \Psi(d) + (d-z) \Psi(c) \right] \right].
 \end{aligned}$$

Multiplying both sides by $E_{k, \rho, \xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} \right)$ and integrating over $[c, d]$, we have

$$\begin{aligned}
 &(s+1) \int_c^d (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} E_{k, \rho, \xi}^\gamma \left(\omega (\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}} \right) \Omega^s(z) \Omega'(z) \Psi(z) dz \\
 &\leq \frac{(s+1)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}}}{d-c} \\
 &\quad \times \left[\Psi(d) \int_c^d (z-c) E_{k, \rho, \xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} \right) \Omega^s(z) \Omega'(z) dz \right. \\
 &\quad \left. + \Psi(c) \int_c^d (d-z) E_{k, \rho, \xi}^\gamma \left(\omega (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\xi}{k}} \right) \Omega^s(z) \Omega'(z) dz \right].
 \end{aligned}$$

By using Definition 6, we have

$$\begin{aligned}
 & k(s+1)^{1+\frac{\zeta}{k}} {}_k J_{d^-; \rho, \zeta+1}^{\omega, \gamma} \Psi(c) \\
 & \leq \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k}}}{(d-c)} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} \\
 & \times \left[(d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} \Psi(d) \right. \\
 & \left. - (\Psi(d) - \Psi(c)) \int_c^d (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} dz \right]. \tag{20}
 \end{aligned}$$

Now for $z \in [c, d]$ and $t \in [z, d], \zeta \geq 0, k > 0$, the following inequality is valid:

$$(\Omega^{s+1})'(z)(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\zeta}{k}} \leq (\Omega^{s+1})'(z)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k}}. \tag{21}$$

By utilizing (19) and (21) and the same strategy as for (20), we obtain

$$\begin{aligned}
 & k(s+1)^{1+\frac{\zeta}{k}} {}_k J_{c^+; \rho, \zeta+1}^{\omega, \gamma} \Psi(d) \\
 & \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k}}}{(d-c)} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} \\
 & \times \left[(d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} \Psi(c) \right. \\
 & \left. - (\Psi(c) - \Psi(d)) \int_c^d (\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho n}{k}+1} dz \right]. \tag{22}
 \end{aligned}$$

By adding (20) and (22), we have

$$\begin{aligned}
 & k(s+1)^{1+\frac{\zeta}{k}} {}_k J_{c^+; \rho, \zeta+1}^{\omega, \gamma} \Psi(d) + k(s+1)^{1+\frac{\zeta}{k}} {}_k J_{d^-; \rho, \zeta+1}^{\omega, \gamma} \Psi(c) \\
 & \leq \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k}}}{(d-c)} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} \\
 & \times \left[(d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} \Psi(c) \right. \\
 & \left. - (\Psi(c) - \Psi(d)) \int_c^d (\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho n}{k}+1} dz \right] \\
 & + \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k}}}{(d-c)} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} \\
 & \times \left[(d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} \Psi(d) \right. \\
 & \left. - (\Psi(d) - \Psi(c)) \int_c^d (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho n}{k}+1} dz \right]. \tag{23}
 \end{aligned}$$

Multiplying (16) by $(\Omega^{s+1})'(z)(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\zeta}{k}} E_{k,\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right)$, and then by integrating over $[c, d]$, we have

$$\begin{aligned} & \Psi\left(\frac{c+d}{2}\right) \left[(s+1) \int_c^d (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\zeta}{k}} \right. \\ & \left. E_{k,\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) \Omega^s(z) \Omega'(z) dz \right] \\ & \leq (s+1) \int_c^d (\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\zeta}{k}} \\ & E_{k,\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(z) - \Omega^{s+1}(c))^{\frac{\rho}{k}} \right) \Omega^s(z) \Omega'(z) \Psi(z) dz. \end{aligned}$$

By using Definition 6, we have

$$\begin{aligned} & \Psi\left(\frac{c+d}{2}\right) \sum_{n=0}^\infty \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n + \zeta}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k} + \frac{\rho n}{k} + 1} \\ & \leq k(s+1)^{1 + \frac{\zeta}{k} s} J_{d^-; \rho, \zeta + 1}^{\omega, \gamma} \Psi(c). \end{aligned} \tag{24}$$

Similarly, using Lemma 1 and multiplying (16) by

$$(\Omega^{s+1})'(z)(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\zeta}{k}} E_{k,\rho,\zeta}^\gamma \left(\omega(\Omega^{s+1}(d) - \Omega^{s+1}(z))^{\frac{\rho}{k}} \right),$$

and then by integrating over $[c, d]$, we have

$$\begin{aligned} & \Psi\left(\frac{c+d}{2}\right) \sum_{n=0}^\infty \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n + \zeta}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k} + \frac{\rho n}{k} + 1} \\ & \leq k(s+1)^{1 + \frac{\zeta}{k} s} J_{c^+; \rho, \zeta + 1}^{\omega, \gamma} \Psi(d). \end{aligned} \tag{25}$$

The following inequality holds when (24) and (25) are added:

$$\begin{aligned} & \Psi\left(\frac{c+d}{2}\right) \left[\sum_{n=0}^\infty \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n + \zeta}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k} + \frac{\rho n}{k} + 1} \right. \\ & \left. + \sum_{n=0}^\infty \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n + \zeta}{k} + 1\right) \Gamma_k(\rho n + \zeta) n!} (\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\zeta}{k} + \frac{\rho n}{k} + 1} \right] \\ & \leq k(s+1)^{1 + \frac{\zeta}{k} s} J_{c^+; \rho, \zeta + 1}^{\omega, \gamma} \Psi(d) + k(s+1)^{1 + \frac{\zeta}{k} s} J_{d^-; \rho, \zeta + 1}^{\omega, \gamma} \Psi(c). \end{aligned} \tag{26}$$

From (23) and (26), we get (17), and the proof is completed. \square

The following remarks show the generality of our theorem.

Remark 8. Corresponding to the choice $s = \gamma = 0$ in Theorem 3, we get ([34] Theorem 11).

Remark 9. By choosing $s = \gamma = 0$ and $k = 1$ in Theorem 3, we get ([35] Theorem 3).

Remark 10. If we take $s = \gamma = 0$, $k = 1$, and $\Omega(z) = z$ in Theorem 3, then we get ([36] Theorem 3).

3. Applications

We present a few applications of the findings from the previous part in this section. First, we use Theorem 1 to get the outcome shown below.

Theorem 4. According to the assumptions of Theorem 1, we have

$$\begin{aligned}
 & k(s+1)^{\frac{\xi}{k}} J_{c^+; \rho, \xi}^{\omega, \gamma} \Psi(d) + k(s+1)^{\frac{\xi}{k}} J_{d^-; \rho, \xi}^{\omega, \gamma} \Psi(c) \\
 & \leq \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\left(\frac{\rho n}{k} + 1\right) n!} \left[\frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}-1}}{(d-c) \Gamma_k(\rho n + \xi)} \right. \\
 & \times \left((d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k} + 1} \Psi(c) \right. \\
 & \left. - (\Psi(c) - \Psi(d)) \int_c^d (\Omega^{s+1}(d) - \Omega^{s+1}(t))^{\frac{\rho n}{k} + 1} dt \right) \\
 & + \frac{(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\xi}{k}-1}}{(d-c) \Gamma_k(\rho n + \xi)} \\
 & \times \left((d-c)(\Omega^{s+1}(d) - \Omega^{s+1}(c))^{\frac{\rho n}{k} + 1} \Psi(d) \right. \\
 & \left. - (\Psi(d) - \Psi(c)) \int_c^d (\Omega^{s+1}(t) - \Omega^{s+1}(c))^{\frac{\rho n}{k} + 1} dt \right) \Big]. \tag{27}
 \end{aligned}$$

Proof. If we take $z = c$ and $z = d$ in inequality (1), then we get inequality (27). \square

Remark 11. Corresponding to the choice $s = \gamma = 0$ in Theorem 4, we get ([34] Theorem 13).

Remark 12. Corresponding to the choice $s = \gamma = 0, \xi = \zeta = k = 1$ and $\Omega(z) = z$ in Theorem 4, we get ([36] Corollary 2).

Theorem 2 is then used to achieve the required conclusions.

Theorem 5. According to the assumptions of Theorem 2, we get

$$\begin{aligned}
 & \left| \left(\frac{\xi}{k} + \frac{\rho n}{k} \right) k(s+1)^{\frac{\xi}{k}} J_{c^+; \rho, \xi}^{\omega, \gamma} \Psi\left(\frac{c+d}{2}\right) + \left(\frac{\xi}{k} + \frac{\rho n}{k} \right) k(s+1)^{\frac{\xi}{k}} J_{d^-; \rho, \xi}^{\omega, \gamma} \Psi\left(\frac{c+d}{2}\right) \right. \\
 & \left. - \left(\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \xi) n!} \left(\Omega^{s+1}\left(\frac{c+d}{2}\right) - \Omega^{s+1}(c) \right)^{\frac{\xi}{k} + \frac{\rho n}{k}} \Psi(c) \right. \right. \\
 & \left. \left. + \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} \omega^n}{\Gamma_k(\rho n + \xi) n!} \left(\Omega^{s+1}(d) - \Omega^{s+1}\left(\frac{c+d}{2}\right) \right)^{\frac{\xi}{k} + \frac{\rho n}{k}} \Psi(d) \right) \right| \\
 & \leq \frac{(d-c) \left(\Omega^{s+1}\left(\frac{c+d}{2}\right) - \Omega^{s+1}(c) \right)^{\frac{\xi}{k}} E_{k; \rho, \xi}^{\gamma} \left(\omega \left(\Omega^{s+1}\left(\frac{c+d}{2}\right) - \Omega^{s+1}(c) \right)^{\frac{\rho}{k}} \right) |\Psi'(c)|}{4} \\
 & + \frac{(d-c) \left(\Omega^{s+1}(d) - \Omega^{s+1}\left(\frac{c+d}{2}\right) \right)^{\frac{\xi}{k}} E_{k; \rho, \xi}^{\gamma} \left(\omega \left(\Omega^{s+1}(d) - \Omega^{s+1}\left(\frac{c+d}{2}\right) \right)^{\frac{\rho}{k}} \right) |\Psi'(d)|}{4} \\
 & + \left| \Psi'\left(\frac{c+d}{2}\right) \right| \left(\frac{(d-c) \left(\Omega^{s+1}\left(\frac{c+d}{2}\right) - \Omega^{s+1}(c) \right)^{\frac{\xi}{k}} E_{k; \rho, \xi}^{\gamma} \left(\omega \left(\Omega^{s+1}\left(\frac{c+d}{2}\right) - \Omega^{s+1}(c) \right)^{\frac{\rho}{k}} \right)}{4} \right. \\
 & \left. + \frac{(d-c) \left(\Omega^{s+1}(d) - \Omega^{s+1}\left(\frac{c+d}{2}\right) \right)^{\frac{\xi}{k}} E_{k; \rho, \xi}^{\gamma} \left(\omega \left(\Omega^{s+1}(d) - \Omega^{s+1}\left(\frac{c+d}{2}\right) \right)^{\frac{\rho}{k}} \right)}{4} \right). \tag{28}
 \end{aligned}$$

Proof. If we take $z = \left(\frac{c+d}{2}\right)$ in inequality (8), then we have inequality (28). \square

Remark 13. Corresponding to the choice $s = \gamma = 0$ in Theorem 5, we get ([34] Theorem 16).

Example 1. By choosing $s = \gamma = 0$, $\xi = \zeta = k = 1$ and $\Omega(z) = z$ in Theorem 5, we get

$$\left| \frac{1}{d-c} \int_c^d \Psi(t) dt - \frac{\Psi(c) + \Psi(d)}{2} \right| \leq \frac{d-c}{8} \left[|\Psi'(c)| + |\Psi'(d)| + 2 \left| \Psi' \left(\frac{c+d}{2} \right) \right| \right]. \quad (29)$$

Example 2 ([37] Theorem 2.2). If we take $\Psi' \left(\frac{c+d}{2} \right) = 0$ in inequality (29), then we have

$$\left| \frac{1}{d-c} \int_c^d \Psi(t) dt - \frac{\Psi(c) + \Psi(d)}{2} \right| \leq \frac{d-c}{8} \left[|\Psi'(c)| + |\Psi'(d)| \right].$$

4. Conclusions

In this study, we looked into convex functions based on fractional integral inequalities for the modified form of the fractional integral of Riemann-type integrals. For differentiable functions with convex absolute value derivatives, several related fractional inequalities are also discussed. Furthermore, Hermite–Hadamard-type fractional inequalities for a symmetric and convex function are explored. The special cases obtained against the main results are the indicator that this article’s implications are more widespread than those existing in the literature. Using suitable fractional integral operators, this method can be used to produce additional conclusions for different kinds of convex functions and other fractional integral operators.

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