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An Efficient Approach for Solving Differential Equations in the Frame of a New Fractional Derivative Operator

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Abstract: Recently, a new fractional derivative operator has been introduced so that it presents the combination of the Riemann–Liouville integral and Caputo derivative. This paper aims to enhance the reproducing kernel Hilbert space method (RKHSM, for short) for solving certain fractional differential equations involving this new derivative. This is the first time that the application of the RKHSM is employed for solving some differential equations with the new operator. We illustrate the convergence analysis of the applicability and reliability of the suggested approaches. The results confirm that the RKHSM finds the true solution. Additionally, these numerical results indicate the effectiveness of the proposed method.

Keywords: fractional differential equations; proportional-Caputo hybrid operator; constant proportional-Caputo operator; reproducing kernel Hilbert space method

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1. Introduction

The subject of the derivative concept has become considerably important and popular due to its many applications in the broad disciplines of chemistry, biology, engineering, applied physics, and many others. Fractional calculus has an important role in modeling various fascinating complex phenomena in the form of ordinary or partial differential equations: to mention a few, time-fractional Schrödinger equations [1,2], the time-fractional Benjamin–Bona-Mahony equation [2], time-fractional Burgers equation [3], time-fractional Korteweg–de Vries equation [4], and time-fractional Kuramoto–Sivashinsky equation [5]. The symmetric and anti-symmetric solutions of the fractional Schrödinger equation have been studied in [6]. In [7], the authors extended the Lie symmetry analysis to the time fractional generalized KdV equations. However, using the classical concept of derivative does not fill the gap that exists in different fields. This latter need modernized the classical concept of the derivative to the rich concept of fractional derivative [8,9]. For the most part, the non-local behaviors of many processes can be formulated as fractional differential equations (FDEs, for short) [10]. Here, the exact solutions of the FDEs are too complicated to be found. From this point, some well-known numerical techniques have been needed to find an approximate approach to these equations [11–13].

Many researchers worked on developing the fractional calculus concept and investigating new ways to define fractional derivatives which range from Riemann–Liouville to new hybrid proportional-Caputo fractional derivatives. The hybrid proportional-Caputo is the novel suggested fractional operator [14]. In their paper, they used an elementary FDE to discover that their new operator is deeply connected with the bivariate Mittag–Leffler function, which arises naturally from the modeling of certain systems of the real world. In addition, their new hybrid fractional operator may be useful as a tool to study the anomalous behavior of dynamical systems in diverse real data, such as chaotic systems, visco-elasticity, electro-chemistry, and physics.

In this research, motivated by the work of Baleanu et al. [14], we apply the RKHSM to fractional differential equations with respect to the new hybrid fractional operator.

The RKHSM is a widely used numerical method for solving non-linear systems. This method was proposed in 1908 [15] and is an effective numerical method for complex non-linear problems without discretization. Many researchers applied it to solve several types of equations [16–21]. Its principal advantages are the feature that it is easy to be applied, especially because it is meshfree, and its capability to deal with diverse complex differential equations. The highlights of the manuscript can be summarized as follows: (i) an efficient numerical technique is employed for solving some differential equations with the new operator; (ii) the effect of the new fractional derivative is shown in the obtained outcomes; (iii) the superior performance of the used method is confirmed via comparing the numerical solutions with the true ones.

In Section 2, we will discuss some basic tools to apply the RKHSM. After doing the preparations we need, we will describe how to apply the RKHSM in Section 3. In Section 4, some applications are presented. Finally, the conclusion is given.

2. Mathematical Concepts

2.1. The New Fractional Derivative Operator

Definition 1. *The Caputo derivative of order of $\hbar(\tau)$ is described by [9]*

$$ {}_0^C D_\tau^\gamma \hbar(\tau) = {}_0^{RL} I_\tau^{1-\gamma} \hbar'(\tau) = \frac{1}{\Gamma(1-\gamma)} \int_0^\tau \hbar'(\eta)(\tau-\eta)^{-\gamma} d\eta, \tag{1} $$

where ${}_0^{RL} I_\tau^\gamma$ is the Riemann–Liouville integral which is given by the definition below.

Definition 2. *Let $\hbar$ be an integrable function. The Riemann–Liouville integral of order $\gamma > 0$ of $\hbar$ is given by [9]*

$$ {}_0^{RL} I_\tau^\gamma \hbar(\tau) = \frac{1}{\Gamma(\gamma)} \int_0^\tau \hbar(\eta)(\tau-\eta)^{\gamma-1} d\eta. \tag{2} $$

The more essential properties related to the above operators can be obtained from [22].

Definition 3. *Let $0 \le \gamma \le 1$ and $\mathtt{K}_0, \mathtt{K}_1 \in C([0,1] \times \mathbb{R}, \mathbb{R}^+)$. The proportional derivative operator of order γ of a differentiable function $\hbar$ is given by [23]*

$$ {}^P D_\gamma \hbar(\tau) = \mathtt{K}_1(\gamma,\tau)\hbar(\tau) + \mathtt{K}_0(\gamma,\tau)\hbar'(\tau), \tag{3} $$

where the functions $\mathtt{K}_0$ and $\mathtt{K}_1$ satisfy the following conditions:

$$ \lim_{\gamma \to 0^+} \mathtt{K}_0(\gamma,\tau) = 0; \quad \lim_{\gamma \to 1^-} \mathtt{K}_0(\gamma,\tau) = 1; \quad \mathtt{K}_0(\gamma,\tau) \neq 0,\ 0 < \gamma \le 1; \quad \forall \tau \in \mathbb{R}, \tag{4} $$

$$ \lim_{\gamma \to 0^+} \mathtt{K}_1(\gamma,\tau) = 1; \quad \lim_{\gamma \to 1^-} \mathtt{K}_1(\gamma,\tau) = 0; \quad \mathtt{K}_1(\gamma,\tau) \neq 0,\ 0 \le \gamma < 1; \quad \forall \tau \in \mathbb{R}. \tag{5} $$

Remark 1. *For the special cases of γ, we can obtain from (3)–(5) two different cases. The first one, if $\gamma = 0$, then (3) reduces to the function itself, i.e., ${}^P D_0 \hbar(\tau) = \hbar(\tau)$. In the case $\gamma = 1$, (3) reduces to the standard differentiation operator, i.e., ${}^P D_1 \hbar(\tau) = \frac{d}{d\tau}\hbar(\tau) = \hbar'(\tau)$.*

Remark 2. *The proportional (conformable) derivative goes back to Khalil et al. [24]. In [25], some properties of the conformable derivative were investigated. In the same year, a modified proportional derivative was explored in [26] with more properties. Definition 3 represents a precise definition of the proportional derivative. For more details related to the proportional operator see, for instance, [27].*

It is interesting to note that the proportional derivative operator (3) of order γ can be expressed as a special case where $\mathtt{K}_0$ and $\mathtt{K}_1$ depend only on γ, they are constant functions with respect to τ. So as a consequence, this particular case can be defined as follows.

Definition 4. *Let $0 \le \gamma \le 1$ and $\mathtt{K}_0, \mathtt{K}_1 \in C([0,1], \mathbb{R}^+)$. The constant proportional (CP, for short) derivative operator of order γ of a differentiable function $\hbar$ is given by [14]*

$$^{CP}D_{\gamma}\hbar(\tau) = \mathtt{K}_1(\gamma)\hbar(\tau) + \mathtt{K}_0(\gamma)\hbar'(\tau), \tag{6}$$

where the functions $\mathtt{K}_0$ and $\mathtt{K}_1$ satisfy the following conditions:

$$\lim_{\gamma\to 0^+} \mathtt{K}_0(\gamma) = 0; \quad \lim_{\gamma\to 1^-} \mathtt{K}_0(\gamma) = 1; \quad \mathtt{K}_0(\gamma) \neq 0,\ 0 < \gamma \le 1; \tag{7}$$

$$\lim_{\gamma\to 0^+} \mathtt{K}_1(\gamma) = 1; \quad \lim_{\gamma\to 1^-} \mathtt{K}_1(\gamma) = 0; \quad \mathtt{K}_1(\gamma) \neq 0,\ 0 \le \gamma < 1. \tag{8}$$

Recently, Baleanu et al. [14] introduced the concept of a new hybrid fractional operator which can be defined in two possible ways. One is to combine both proportional and Caputo definitions. The other is to combine both constant proportional and Caputo definitions. The concept of these operators is formalized as follows.

Definition 5. *Let the function $\hbar$ be differentiable and let $\hbar$ with its derivative $\hbar'$ be locally L^1 functions on $\mathbb{R}^+$ [14].*

1. *The proportional-Caputo (PC, for short) hybrid operator of order γ, for $\hbar$ is given by*

$$\begin{aligned} {}^{PC}_{\ 0}D^{\gamma}_{\tau}\hbar(\tau) &= {}^{RL}_{\ 0}I^{1-\gamma}_{\tau}\left[{}^{P}D_{\gamma}\hbar(\tau)\right] \\ &= \begin{cases} \frac{1}{\Gamma(1-\gamma)}\int_0^{\tau}(\mathtt{K}_1(\gamma,\eta)\hbar(\eta) + \mathtt{K}_0(\gamma,\eta)\hbar'(\eta))(\tau-\eta)^{-\gamma}d\eta, & 0<\gamma<1, \\ \int_0^{\tau}\hbar(\eta)d\eta, & \gamma = 0, \\ \hbar'(\tau), & \gamma = 1. \end{cases} \end{aligned} \tag{9}$$

2. *The constant proportional-Caputo (CPC, for short) hybrid operator of order γ, for $\hbar$ is given by*

$$\begin{aligned} {}^{CPC}_{\ \ 0}D^{\gamma}_{\tau}\hbar(\tau) &= {}^{RL}_{\ 0}I^{1-\gamma}_{\tau}\left[{}^{CP}D_{\gamma}\hbar(\tau)\right] \\ &= \begin{cases} \frac{1}{\Gamma(1-\gamma)}\int_0^{\tau}(\mathtt{K}_1(\gamma)\hbar(\eta) + \mathtt{K}_0(\gamma)\hbar'(\eta))(\tau-\eta)^{-\gamma}d\eta, & 0<\gamma<1, \\ \int_0^{\tau}\hbar(\eta)d\eta, & \gamma = 0, \\ \hbar'(\tau), & \gamma = 1. \end{cases} \end{aligned} \tag{10}$$

Proposition 1. *The hybrid CPC and PC fractional operators are non-local and singular [14].*

Theorem 1. *The Laplace transform of the hybrid CPC operator ${}^{CPC}_{\ \ 0}D^{\gamma}_{\tau}$ is represented by [14]*

$$L\left[{}^{CPC}_{\ \ 0}D^{\gamma}_{\tau}\hbar(\tau)\right] = \left[\frac{\mathtt{K}_1(\gamma)}{s} + \mathtt{K}_0(\gamma)\right]s^{\gamma}\hat{\hbar}(s) - \mathtt{K}_0(\gamma)s^{\gamma-1}\hbar(0), \tag{11}$$

where the function $\hbar(\tau)$ is differentiable. $\hbar$, with its derivative $\hbar'$, are locally L^1 functions on $\mathbb{R}^+$, and $\hat{\hbar}(s)$ exists.

Among the main properties of the PC and CPC fractional operators, we mention the following ones, namely, the inversion relations:

- $\left({}^{PC}_{0}I^{\gamma}_{\tau}\, {}^{PC}_{0}D^{\gamma}_{\tau}\hbar\right)(\tau) = \hbar(\tau) - \exp\left(-\int_0^{\tau} \frac{K_1(\gamma,s)}{K_0(\gamma,s)}ds\right)\hbar(0),$
- $\left({}^{CPC}{}_{0}I^{\gamma}_{\tau}\, {}^{CPC}{}_{0}D^{\gamma}_{\tau}\hbar\right)(\tau) = \hbar(\tau) - \exp\left(-\frac{K_1(\gamma)}{K_0(\gamma)}\tau\right)\hbar(0),$
- $\left({}^{PC}_{0}D^{\gamma}_{\tau}\, {}^{PC}_{0}I^{\gamma}_{\tau}\hbar\right)(\tau) = \left({}^{CPC}_{0}D^{\gamma}_{\tau}\, {}^{CPC}{}_{0}I^{\gamma}_{\tau}\hbar\right)(\tau) = \hbar(\tau) - \frac{\tau^{-\gamma}}{\Gamma(1-\gamma)} \lim_{\tau\to 0} {}^{RL}_{0}I^{\gamma}_{\tau}\hbar(\tau),$

where ${}^{PC}_{0}I^{\gamma}_{\tau}$ and ${}^{CPC}{}_{0}I^{\gamma}_{\tau}$ denote the inverse of the operators ${}^{PC}_{0}D^{\gamma}_{\tau}$ and ${}^{CPC}{}_{0}D^{\gamma}_{\tau}$, respectively. They were constructed in [14] as follows.

Proposition 2. *The inverse operators to the CPC and PC fractional derivatives are defined, respectively, as [14]*

$$ {}^{PC}_{0}I^{\gamma}_{\tau}\hbar(\tau) = \int_0^{\tau} \exp\left(-\int_{\eta}^{\tau} \frac{K_1(\gamma,s)}{K_0(\gamma,s)}ds\right) \frac{{}^{RL}_{0}D^{1-\gamma}_{\eta}\hbar(\eta)}{K_0(\gamma,\eta)}d\eta, \tag{12} $$

$$ {}^{CPC}{}_{0}I^{\gamma}_{\tau}\hbar(\tau) = \frac{1}{K_0(\gamma)} \int_0^{\tau} \exp\left(-\frac{K_1(\gamma)}{K_0(\gamma)}(\tau-\eta)\right) {}^{RL}_{0}D^{1-\gamma}_{\eta}\hbar(\eta)d\eta. \tag{13} $$

These satisfy the inversion relations which are mentioned just above. The ${}^{RL}_{0}D^{\gamma}_{\tau}$ *denotes the Riemann–Liouville derivative which is defined as follows:*

$$ {}^{RL}_{0}D^{\gamma}_{\tau}\hbar(\tau) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{d\tau} \int_0^{\tau} \hbar(\eta)(\tau-\eta)^{-\gamma}d\eta. \tag{14} $$

For more details on CPC and PC hybrid fractional operators see [14].

Remark 3. *Of special interest, Baleanu et al. [14] realized that the specific case*

$$ K_0(\gamma,\tau) = \gamma\tau^{1-\gamma}, \qquad K_1(\gamma,\tau) = (1-\gamma)\tau^{\gamma}. \tag{15} $$

will not be useful in applications because of the lack of dimensional agreement in (3) while this is important for physical consistency.

Remark 4. *We shall pay special attention in this paper to two specific cases when*

1. *For any $\sigma \in (0,+\infty)$, we take*

$$ K_0(\gamma) = \gamma\sigma^{1-\gamma}, \qquad K_1(\gamma) = (1-\gamma)\sigma^{\gamma} \tag{16} $$

2. *For any $\sigma \in (0,+\infty)$, we take*

$$ K_0(\gamma) = \gamma\sigma^{1+\gamma}, \qquad K_1(\gamma) = (1-\gamma)\sigma^{\gamma} \tag{17} $$

2.2. The Reproducing Kernel Theory

The following are some fundamental definitions and theorems required from the reproducing kernel theory.

Definition 6. *We say that a function $K : S \times S \to \mathbb{C}$ is a reproducing kernel of the space H provided [28]:*

1. *$K(\cdot,t) \in H$, $\forall\, t \in S$.*
2. *$\langle \mathfrak{f}, K(\cdot,t)\rangle = \mathfrak{f}(t)$, $\forall \mathfrak{f} \in H$ and $\forall\, t \in S$.*

where H is a Hilbert space over S and $S \neq \varnothing$.

Remark 5. *Note that the assertion (2) is called the reproducing property (RP).*

Remark 6. *An RKHS "H" is a Hilbert space endowed with an RK "K".*

Definition 7 ([28])**.** *The function space $W_2^2[0,T]$ consists of all functions $\mathfrak{f}$ for which $\mathfrak{f}$ and $\mathfrak{f}'$ are absolutely continuous functions on $[0,T]$, $\mathfrak{f}'' \in L^2[0,T]$, and $\mathfrak{f}(0)=0$.*

Definition 8 ([28])**.** *If $\mathfrak{f},\mathfrak{g} \in W_2^2[0,T]$, then the inner product and norm are*

$$\langle \mathfrak{f},\mathfrak{g}\rangle_{W_2^2} = \sum_{\imath=0}^{1} \mathfrak{f}^{(\imath)}(0)\mathfrak{g}^{(\imath)}(0) + \int_0^T \mathfrak{f}^{(\mathrm{m})}(t)\mathfrak{g}^{(\mathrm{m})}(t)\mathrm{d}t,$$

and

$$\|\mathfrak{f}\|_{W_2^2} = \sqrt{\langle \mathfrak{f},\mathfrak{f}\rangle_{W_2^2}}.$$

Theorem 2. *We obtain the RK function $\mathcal{S}_\eta(t)$ of $W_2^2[0,T]$ as:*

$$\mathcal{S}_\eta(t) = \begin{cases} t\,\eta + 1/2\,\eta\,t^2 - 1/6\,t^3 & ,t \le \eta, \\ -1/6\,\eta^3 + 1/2\,t\,\eta^2 + t\,\eta & ,t > \eta. \end{cases} \tag{18}$$

Proof. We must prove

$$\langle \mathfrak{f},\mathcal{S}_\eta\rangle_{W_2^2} = \mathfrak{f}(\eta).$$

We have

$$\langle \mathfrak{f},\mathcal{S}_\eta\rangle_{W_2^2} = \mathfrak{f}(0)\mathcal{S}_\eta(0) + \mathfrak{f}'(0)\mathcal{S}_\eta'(0) + \mathfrak{f}'(0)\mathcal{S}_\eta'(0) + \int_0^T \mathfrak{f}''(t)\mathcal{S}_\eta''(t)\,\mathrm{d}t.$$

Applying integration by parts, we obtain:

$$\langle \mathfrak{f},\mathcal{S}_\eta\rangle_{W_2^2} = \mathfrak{f}(0)\mathcal{S}_\eta(0) + \mathfrak{f}'(0)\mathcal{S}_\eta'(0) + \mathfrak{f}'(T)\mathcal{S}_\eta''(T) - \mathfrak{f}'(0)\mathcal{S}_\eta''(0) - \int_0^T \mathfrak{f}'(t)\mathcal{S}_\eta^{(3)}(t)\mathrm{d}t.$$

Since $\mathfrak{f}(t) \in W_2^2[0,T]$, we have

$$\mathfrak{f}(0) = 0. \tag{19}$$

Then

$$\langle \mathfrak{f},\mathcal{S}_\eta\rangle_{W_2^2} = \mathfrak{f}'(0)\mathcal{S}_\eta'(0) + \mathfrak{f}'(T)\mathcal{S}_\eta''(T) - \mathfrak{f}'(0)\mathcal{S}_\eta''(0) - \int_0^T \mathfrak{f}'(t)\mathcal{S}_\eta^{(3)}(t)\mathrm{d}t.$$

We need to compute $\mathcal{S}_\eta'(0)$, $\mathcal{S}_\eta''(0)$, and $\mathcal{S}_\eta''(T)$:

$$\mathcal{S}_\eta'(0) = \eta,$$

$$\mathcal{S}_\eta''(0) = \eta,$$

$$\mathcal{S}_\eta''(T) = 0.$$

By using the above equations, we obtain:

$$\langle \mathfrak{f},\mathcal{S}_\eta\rangle_{W_2^2} = -\int_0^T \mathfrak{f}'(t)\mathcal{S}_\eta^{(3)}(t)\mathrm{d}t. \tag{20}$$

We have

$$\mathcal{S}_\eta^{(3)}(t) = \begin{cases} -1 & ,t \le \eta, \\ 0 & ,t > \eta. \end{cases}$$

Thus

$$\langle \mathfrak{f}, \mathcal{S}_\eta \rangle_{W_2^2} = -\int_0^\eta \mathfrak{f}'(t)\mathcal{S}_\eta^{(3)}(t)\mathrm{d}t - \int_\eta^T \mathfrak{f}'(t)\mathcal{S}_\eta^{(3)}(t)\mathrm{d}t$$
$$= \int_0^\eta \mathfrak{f}'(t)\,\mathrm{d}t$$
$$= \mathfrak{f}(\eta) - \mathfrak{f}(0),$$

and, since $\mathfrak{f}(t) \in W_2^2[0,T]$, we deduce

$$\langle \mathfrak{f}, \mathcal{S}_\eta \rangle_{W_2^2} = \mathfrak{f}(\eta).$$

□

Definition 9 ([28]). *The function space $W_2^1[0,T]$ consists of all functions $\mathfrak{f}$ for which $\mathfrak{f}$ is absolutely continuous function on $[0,T]$ and $\mathfrak{f}' \in L^2[0,T]$.*

Definition 10 ([28]). *If $\mathfrak{f}, \mathfrak{g} \in W_2^1[0,T]$, then the inner product and norm are*

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{W_2^1} = \mathfrak{f}(0)\mathfrak{g}(0) + \int_0^T \mathfrak{f}'(t)\mathfrak{g}'(t)\mathrm{d}t,$$

and

$$\|\mathfrak{f}\|_{W_2^1} = \sqrt{\langle \mathfrak{f}, \mathfrak{f} \rangle_{W_2^1}}.$$

Theorem 3. *We obtain the RK function $\mathcal{R}_\eta(t)$ of $W_2^1[0,T]$ as:*

$$\mathcal{R}_\eta(t) = \begin{cases} 1+t & ,t \le \eta, \\ \eta+1 & ,t > \eta. \end{cases} \tag{21}$$

Proof. We must prove

$$\langle \mathfrak{f}, \mathcal{R}_\eta \rangle_{W_2^1} = \mathfrak{f}(\eta).$$

We have

$$\langle \mathfrak{f}, \mathcal{R}_\eta \rangle_{W_2^1} = \mathfrak{f}(0)\mathcal{R}_\eta(0) + \int_0^T \mathfrak{f}'(x)\mathcal{R}_\eta'(x)\,\mathrm{d}x.$$

We need to compute $\mathcal{R}_\eta(0)$:

$$\mathcal{R}_\eta(0) = 1,$$

By using the above equation, we obtain:

$$\langle \mathfrak{f}, \mathcal{R}_\eta \rangle_{W_2^1} = \mathfrak{f}(0) + \int_0^T \mathfrak{f}'(t)\mathcal{R}_\eta'(t)\,\mathrm{d}t.$$

We have

$$\mathcal{R}_\eta'(t) = \begin{cases} 1 & ,t \le \eta, \\ 0 & ,t > \eta. \end{cases}$$

Thus

$$\langle \mathfrak{f}, \mathcal{R}_\eta \rangle_{W_2^1} = \mathfrak{f}(0) + \int_0^\eta \mathfrak{f}'(t)\mathcal{R}_\eta'(t)\,\mathrm{d}t + \int_\eta^T \mathfrak{f}'(t)\mathcal{R}_\eta'(t)\,\mathrm{d}t$$
$$= \mathfrak{f}(0) + \int_0^\eta \mathfrak{f}'(t)\,\mathrm{d}t$$
$$= \mathfrak{f}(0) + \mathfrak{f}(\eta) - \mathfrak{f}(0),$$

so, we deduce

$$\langle \mathfrak{f}, \mathcal{R}_\eta \rangle_{W_2^1} = \mathfrak{f}(\eta).$$

□

3. The RKHS Approach

Considering the following fractional initial value problem:

$$\begin{cases} {}^{CPC}_{0}D_t^{\gamma}\mathfrak{f}(t) = \phi(t, \mathfrak{f}(t)), \quad t \in [0, T], \\ \mathfrak{f}(0) = \mu. \end{cases} \tag{22}$$

where ${}^{CPC}_{0}D_t^{\gamma}\mathfrak{f}(t)$ is given in (10).

One skillful way to investigate the considered problem by using RKHSM is to homogenize the initial condition $\mathfrak{f}(0) = \mu$. To do so, the transformation has the form:

$$\mathfrak{g}(t) = \mathfrak{f}(t) - \mu.$$

Then, (22) becomes

$$\begin{cases} {}^{CPC}_{0}D_t^{\gamma}\mathfrak{g}(t) = \Lambda(t, \mathfrak{g}(t)), \quad t \in [0, T], \\ \mathfrak{g}(0) = 0. \end{cases} \tag{23}$$

where $\Lambda(t, \mathfrak{g}(t)) = \phi(t, \mathfrak{g}(t) + \mu) - \left(\mu t^{1-\gamma} \mathrm{K}_1(\gamma)\right)/\Gamma(2-\gamma)$.

The first step is to define a linear operator $\mathfrak{O} : W_2^2[0, T] \to W_2^1[0, T]$ defined by

$$\mathfrak{O}\mathfrak{g}(t) = {}^{CPC}_{0}D_t^{\gamma}\mathfrak{g}(t). \tag{24}$$

Theorem 4. *The operator $\mathfrak{O} : W_2^2[0, T] \to W_2^1[0, T]$ is bounded and linear.*

Proof. For checking the linearity, let $\mathfrak{g}(t), \mathfrak{m}(t) \in W_2^2[0, T]$. Then,

$$\begin{aligned} \mathfrak{O}(\mathfrak{g} + \mathfrak{m})(t) &= {}^{CPC}_{0}D_t^{\gamma}(\mathfrak{g} + \mathfrak{m})(t), \\ &= \frac{1}{\Gamma(1-\gamma)} \int_0^t \Big(\mathrm{K}_1(\gamma)(\mathfrak{g} + \mathfrak{m})(\tau) + \mathrm{K}_0(\gamma)(\mathfrak{g} + \mathfrak{m})'(\tau)\Big)(t-\tau)^{-\gamma} d\tau, \\ &= {}^{CPC}_{0}D_t^{\gamma}\mathfrak{g}(t) + {}^{CPC}_{0}D_t^{\gamma}\mathfrak{m}(t), \\ &= \mathfrak{O}\mathfrak{g}(t) + \mathfrak{O}\mathfrak{m}(t). \end{aligned}$$

Additionally, let $\mathfrak{g}(t) \in W_2^2[0, T]$ and $\xi \in \mathbb{R}$. Then

$$\begin{aligned} \mathfrak{O}(\xi\mathfrak{g})(t) &= {}^{CPC}_{0}D_t^{\gamma}(\xi\mathfrak{g})(t), \\ &= \frac{1}{\Gamma(1-\gamma)} \int_0^t \Big(\mathrm{K}_1(\gamma)(\xi\mathfrak{g})(\tau) + \mathrm{K}_0(\gamma)(\xi\mathfrak{g})'(\tau)\Big)(t-\tau)^{-\gamma} d\tau, \\ &= \xi {}^{CPC}_{0}D_t^{\gamma}\mathfrak{g}(t), \\ &= \xi\mathfrak{O}\mathfrak{g}(t). \end{aligned}$$

We can now prove that $\mathfrak{O}$ is bounded. This shows that

$$\|\mathfrak{O}\mathfrak{g}\|_{W_2^1} \leq \Xi \|\mathfrak{g}\|_{W_2^2}, \text{ with } \Xi > 0.$$

From Definition 10, we have

$$\|\mathfrak{O}\mathfrak{g}(t)\|_{W_2^1}^2 = \langle \mathfrak{O}\mathfrak{g}(t), \mathfrak{O}\mathfrak{g}(t) \rangle_{W_2^1} = [\mathfrak{O}\mathfrak{g}(0)]^2 + \int_0^T [\mathfrak{O}\mathfrak{g}'(t)]^2 dt.$$

By virtue of the RP, we have

$$\mathfrak{g}(t) = \langle \mathfrak{g}(\centerdot), \mathcal{S}_t(\centerdot) \rangle_{W_2^2}.$$

In addition,

$$\mathfrak{D}\mathfrak{g}(t) = \langle \mathfrak{g}(\centerdot), \mathfrak{D}\mathcal{S}_t(\centerdot) \rangle_{W_2^2},$$
$$\mathfrak{D}\mathfrak{g}'(t) = \langle \mathfrak{g}(\centerdot), \partial_t(\mathfrak{D}\mathcal{S}_t(\centerdot)) \rangle_{W_2^2}.$$

Using the Schwarz inequality and the continuity of $\mathcal{S}_t(\centerdot)$ to obtain

$$|\mathfrak{D}\mathfrak{g}(t)| = \left| \langle \mathfrak{g}(\centerdot), \mathfrak{D}\mathcal{S}_t(\centerdot) \rangle_{W_2^2} \right| \le \|\mathfrak{g}\|_{W_2^2} \|\mathfrak{D}\mathcal{S}_t(\centerdot)\|_{W_2^2} \le \Xi_1 \|\mathfrak{g}\|_{W_2^2}. \tag{25}$$

In the same way,

$$\left|\mathfrak{D}\mathfrak{g}'(t)\right| \le \Xi_2 \|\mathfrak{g}\|_{W_2^2}.$$

Hence

$$\begin{aligned} \|\mathfrak{D}\mathfrak{g}(t)\|^2_{W_2^2} &\le \Xi_1^2 \|\mathfrak{g}\|^2_{W_2^2} + \int_0^T \Xi_2^2 \|\mathfrak{g}\|^2_{W_2^2} dt, \\ &= \left(\Xi_1^2 + T\Xi_2^2\right) \|\mathfrak{g}\|^2_{W_2^2}. \end{aligned} \tag{26}$$

From (26), we conclude that $\|\mathfrak{D}\mathfrak{g}(\tau)\|_{\mathscr{W}_2^1} \le \Xi \|\mathfrak{g}\|_{\mathscr{W}_2^2}$, where $\Xi = \Xi_1^2 + T\Xi_2^2$. □

By applying (24), we can rewrite (23) as follows

$$\begin{cases} \mathfrak{D}\mathfrak{g}(t) = \Lambda(t, \mathfrak{g}(t)), \quad t \in [0, T], \\ \mathfrak{g}(0) = 0. \end{cases} \tag{27}$$

where $\Lambda(t, \mathfrak{g}(t)) = \phi(t, \mathfrak{g}(t) + \mu) - \left(\mu t^{1-\gamma} \mathrm{K}_1(\gamma)\right) / \Gamma(2-\gamma)$.

Prior to constructing the numerical solution of (27), we should first construct the orthogonal function system of $W_2^2[0, T]$. For this, let us introduce the useful functions:

$$\kappa_\imath(t) = \mathcal{R}_{t_\imath}(t) \quad \text{and} \quad \psi_\imath(t) = \mathfrak{D}^* \kappa_\imath(t),$$

where

- $\mathcal{R}_{t_\imath}(t)$ represents the RKF associated with $W_2^1[0, T]$.
- $\mathfrak{D}^*$ is the formal adjoint of $\mathfrak{D}$.
- $\{t_\imath\}_{\imath=1}^{\infty}$ is a dense countable set in $[0, T]$.

The Gram–Schmidt process gives the following orthonormal system $\{\bar{\psi}_\imath\}_{\imath=1}^{\infty}$:

$$\bar{\psi}_\imath(t) = \sum_{k=1}^{\imath} \varpi_{\imath k} \psi_k(t), \quad \varpi_{\imath\imath} > 0, \ \imath = 1, 2, \ldots. \tag{28}$$

Here, $\{\psi_\imath\}_{\imath=1}^{\infty}$ denotes a functioning system in $W_2^2[0, T]$ where its expression can be determined by the following way:

$$\varpi_{\imath\jmath} = \begin{cases} \frac{1}{\|\psi_1\|} & \text{for} \quad \imath = \jmath = 1, \\ \frac{1}{e_\imath} & \text{for} \quad \imath = \jmath \ne 1, \\ -\frac{1}{e_\imath} \sum_{k=\jmath}^{\imath-1} C_{\imath k} \varpi_{k\jmath} & \text{for} \quad \imath > \jmath, \end{cases} \tag{29}$$

where $e_\imath = \sqrt{\|\psi_\imath\|^2 - \sum_{k=1}^{\imath-1} C_{\imath k}^2}, C_{\imath k} = \langle \psi_\imath, \bar{\psi}_k \rangle_{\mathscr{W}_2^2}$.

Remark 7. *Not that the formula for $\psi_\imath(t)$ is given by*

$$\psi_\imath(t) = \mathfrak{O}^*\kappa_\imath(t) = \langle \mathfrak{O}^*\kappa_\imath(\eta), \mathcal{S}_t(\eta)\rangle_{W_2^2} = \langle \kappa_\imath(\eta), \mathfrak{O}\mathcal{S}_t(\eta)\rangle_{W_2^1} = \langle \mathcal{R}_{t_\imath}(\eta), \mathfrak{O}\mathcal{S}_t(\eta)\rangle_{W_2^1} = \mathfrak{O}_\eta \mathcal{S}_t(\eta)|_{\eta=t_\imath}.$$

The operator $\mathfrak{O}_\eta$ means that $\mathfrak{O}$ is applied to η variables.

Theorem 5. *Assume $\{t_\imath\}_{\imath=1}^{\infty}$ is dense on $[0,T]$, then $\{\psi_\imath\}_{\imath=1}^{\infty}$ is the complete system of the space $W_2^2[0,T]$.*

Proof. We see easily that $\psi_\imath(t) \in W_2^2[0,T]$. Therefore, for each fixed $\mathfrak{g}(t) \in W_2^2[0,T]$,

$$\langle \mathfrak{g}(t), \psi_\imath(t)\rangle_{W_2^2} = 0, \ \imath = 1,2,\ldots,$$

Since

$$\langle \mathfrak{g}(t), \psi_\imath(t)\rangle_{W_2^2} = \langle \mathfrak{g}(t), \mathfrak{O}^*\kappa_\imath(t)\rangle_{W_2^2} = \langle \mathfrak{O}\mathfrak{g}(t), \kappa_\imath(t)\rangle_{W_2^1} = \mathfrak{O}\mathfrak{g}(t_\imath) = 0.$$

Due to the density of $\{t_\imath\}_{\imath=1}^{\infty}$ in $[0,T]$, one can get

$$\mathfrak{O}\mathfrak{g}(t) = 0.$$

Additionally, the existence of $\mathfrak{O}^{-1}$ implies.

$$\mathfrak{g}(t) = 0.$$

□

Lemma 1. *Assume $\mathfrak{g}(t) \in W_2^2[0,T]$, then*

$$\left\|\mathfrak{g}^{(\imath)}(t)\right\|_C \leq \digamma \|\mathfrak{g}(t)\|_{W_2^2}, \quad \imath = 0,1.$$

$\digamma$ is non-negative and $\|\mathfrak{g}(t)\|_C = \max_{t\in[0,T]} |\mathfrak{g}(t)|$.

Proof. $\forall t \in [0,T]$ we have

$$\mathfrak{g}^{(\imath)}(t) = \left\langle \mathfrak{g}(\cdot), \partial_t^{(\imath)} \mathcal{S}_t(\cdot)\right\rangle_{W_2^2}, \quad \imath = 0,1,$$

and

$$\left\|\partial_t^{(\imath)}\mathcal{S}_t\right\|_{W_2^2} \leq \digamma_\imath, \quad \imath = 0,1.$$

As a result,

$$\left|\mathfrak{g}^{(\imath)}(t)\right| = \left|\left\langle \mathfrak{g}(\cdot), \partial_t^{(\imath)}\mathcal{S}_t(\cdot)\right\rangle_{W_2^2}\right| \leq \left\|\partial_t^{(\imath)}\mathcal{S}_t\right\|_{W_2^2}\|\mathfrak{g}\|_{W_2^2} \leq \digamma_\imath \|\mathfrak{g}\|_{W_2^2}, \ \imath = 0,1. \tag{30}$$

Here $\digamma = \max_{\imath=0,1}\{\digamma_\imath\}$. □

Theorem 6. *Assume $\{t_\imath\}_{\imath=1}^{\infty}$ is dense in $[0,T]$ and there exists a unique solution $\mathfrak{g}(t)$ for (27) in $W_2^2[0,T]$, then the solution's representation of (27) is given by*

$$\mathfrak{g}(t) = \sum_{\imath=1}^{\infty}\sum_{k=1}^{\imath} \varpi_{\imath k}\Lambda(t_k, \mathfrak{g}(t_k))\bar{\psi}_\imath(t), \tag{31}$$

and the solution of (22) will be as follows

$$\mathfrak{f}(t) = \left(\sum_{\imath=1}^{\infty} \sum_{k=1}^{\imath} \varpi_{\imath k} \Lambda(t_k, \mathfrak{g}(t_k)) \bar{\psi}_\imath(t) \right) + \mu. \tag{32}$$

Proof. On account of the completeness of $\{\bar{\psi}_\imath(t)\}_{\imath=1}^{\infty}$ in the space $W_2^2[0,T]$, we compute

$$\begin{aligned}
\mathfrak{g}(t) &= \sum_{\imath=1}^{\infty} \langle \mathfrak{g}(t), \bar{\psi}_\imath(t) \rangle_{W_2^2} \bar{\psi}_\imath(t) \\
&= \sum_{\imath=1}^{\infty} \left\langle \mathfrak{g}(t), \sum_{k=1}^{\imath} \varpi_{\imath k} \psi_k(t) \right\rangle_{W_2^2} \bar{\psi}_\imath(t) \\
&= \sum_{\imath=1}^{\infty} \sum_{k=1}^{\imath} \varpi_{\imath k} \langle \mathfrak{g}(t), \psi_k(t) \rangle_{W_2^2} \bar{\psi}_\imath(t) \\
&= \sum_{\imath=1}^{\infty} \sum_{k=1}^{\imath} \varpi_{\imath k} \langle \mathfrak{g}(t), \mathfrak{O}^* \kappa_k(t) \rangle_{W_2^2} \bar{\psi}_\imath(t) \\
&= \sum_{\imath=1}^{\infty} \sum_{k=1}^{\imath} \varpi_{\imath k} \langle \mathfrak{O}\mathfrak{g}(t), \kappa_k(t) \rangle_{W_2^1} \bar{\psi}_\imath(t) \\
&= \sum_{\imath=1}^{\infty} \sum_{k=1}^{\imath} \varpi_{\imath k} \langle \mathfrak{O}\mathfrak{g}(t), \mathcal{R}_t(t_k) \rangle_{W_2^1} \bar{\psi}_\imath(t) \\
&= \sum_{\imath=1}^{\infty} \sum_{k=1}^{\imath} \varpi_{\imath k} \Lambda(t_k, \mathfrak{g}(t_k)) \bar{\psi}_\imath(t).
\end{aligned}$$

with $\Lambda(t_k, \mathfrak{g}(t_k)) = \mathfrak{O}\mathfrak{g}(t_k)$. (32) following very easily from the considered change of variables $\mathfrak{f}(t) = \mathfrak{g}(t) + \mu$. □

Remark 8. *The n-term approximate solution of (31) is given as*

$$\mathfrak{g}_n(t) = \sum_{\imath=1}^{n} \sum_{k=1}^{\imath} \varpi_{\imath k} \Lambda(t_k, \mathfrak{g}(t_k)) \bar{\psi}_\imath(t).$$

Remark 9. *As $W_2^2[0,T]$ is a Hilbert space, we deduce*

$$\sum_{\imath=1}^{\infty} \sum_{k=1}^{\imath} \varpi_{\imath k} \Lambda(t_k, \mathfrak{g}(t_k)) \bar{\psi}_\imath(t) < \infty.$$

This means that the approximate solution $\mathfrak{g}_n(t)$ is convergent in the norm.

Theorem 7. *Lets g_n be an approximate solution in the reproducing kernel Hilbert space $W_2^2[0,T]$. Then,*

1. *$\mathfrak{g}_n(t)$ converges uniformly to $\mathfrak{g}(t)$.*
2. *$\mathfrak{g}_n'(t)$ converges uniformly to $\mathfrak{g}'(t)$.*

Proof. For the first result, we need to estimate the term on the left below:
$\forall t \in [0,T]$,

$$\begin{aligned}
|\mathfrak{g}_n(t) - \mathfrak{g}(t)| &= \left| \langle \mathfrak{g}_n(\cdot) - \mathfrak{g}(\cdot), \mathcal{S}_t(\cdot) \rangle_{W_2^2} \right| \\
&\le \|\mathcal{S}_t\|_{W_2^2} \|\mathfrak{g}_n - \mathfrak{g}\|_{W_2^2} \\
&\le \mathcal{C}_0 \|\mathfrak{g}_n - \mathfrak{g}\|_{W_2^2}
\end{aligned}$$

where $\mathcal{C}_0$ is a constant.

In the same way, we get

$$\left|\mathfrak{g}_n'(t)-\mathfrak{g}'(t)\right| \le \|\partial_t \mathcal{S}_t\|_{W_2^2}\|\mathfrak{g}_n'-\mathfrak{g}'\|_{W_2^2},$$

due to the uniform boundedness of $\partial_t \mathcal{S}_t(\bullet)$, we have

$$\|\partial_t \mathcal{S}_t\|_{W_2^2} \le \mathcal{C}_1,$$

where $\mathcal{C}_1$ is a positive constant.

Hence

$$\left|\mathfrak{g}_n'(t)-\mathfrak{g}'(t)\right| \le \mathcal{C}_1\|\mathfrak{g}_n'-\mathfrak{g}'\|_{W_2^2}.$$

□

4. Numerical Experiments

In the following, some applications of differential equations based on the new hybrid CPC derivative are tested to confirm the theoretical part.

Example 1. *Considering the following simple problem:*

$$\begin{cases} {}^{CPC}{}_0D_t^{\gamma}\mathfrak{f}(t)=0, \quad 0<t\le 1, \\ \mathfrak{f}(0)=\frac{1}{2}. \end{cases} \tag{33}$$

The exact solution to the above problem takes the following form [14]

$$\mathfrak{f}(t)=\frac{1}{2}\exp\left(-\frac{\mathrm{K}_1(\gamma)}{\mathrm{K}_0(\gamma)}t\right).$$

1. *First situation (FS, for short):* $\mathrm{K}_0(\gamma)=\gamma\sigma^{1-\gamma}, \mathrm{K}_1(\gamma)=(1-\gamma)\sigma^{\gamma}, \sigma=\frac{1}{2}$.
2. *Second situation (SS, for short):* $\mathrm{K}_0(\gamma)=\gamma\sigma^{1+\gamma}, \mathrm{K}_1(\gamma)=(1-\gamma)\sigma^{\gamma}, \sigma=\frac{1}{2}$.

After homogenizing the initial conditions in (33) by using the transformation:

$$\mathfrak{g}(t)=\mathfrak{f}(t)-\frac{1}{2},$$

we obtain

$$\begin{cases} {}^{CPC}{}_0D_t^{\gamma}\mathfrak{g}(t)=-\frac{t^{1-\gamma}\mathrm{K}_1(\gamma)}{\Gamma(2-\gamma)}, \quad 0<t\le 1, \\ \mathfrak{g}(0)=0. \end{cases} \tag{34}$$

This RKHSM is tested with the grid points $t_\imath=\frac{\imath}{n}$, $\imath=1,2,\dots,n$. The exact solution (ES), approximate solution (AS), absolute error (AE), and relative error RE) of (33) are shown in Tables 1–4 when $\gamma\in\{0.25,0.5,0.75,0.9\}$ and $t\in[0,1]$ for both FS and SS. Additionally, the comparison between the ES and RKHSM's solution for $\gamma=0.25,0.5,0.75,0.9$ are depicted together in Figures 1–3 for FS and in Figures 4–6 for SS. From the results, we can see that the numerical solutions are found to be in excellent agreement with the exact solutions.

Table 1. First and second situations: results for Example 1 when $\gamma = 0.25$.

	First Situation				Second Situation			
t	**ES**	**AS**	**AE**	**RE**	**ES**	**AS**	**AE**	**RE**
0	0.5000000000	0.5000000000	0	0	0.5000000000	0.5000000000	0	0
0.1	0.3271255460	0.3271255449	1.10×10^{-9}	$3.362623352 \times 10^{-9}$	0.2744058180	0.2744058158	2.20×10^{-9}	$8.017322723 \times 10^{-9}$
0.2	0.2140222456	0.2140222429	2.70×10^{-9}	$1.261551103 \times 10^{-8}$	0.1505971060	0.1505971036	2.40×10^{-9}	$1.593656122 \times 10^{-8}$
0.3	0.1400242879	0.1400242852	2.70×10^{-9}	$1.928236908 \times 10^{-8}$	0.8264944410	0.0826494408	3.30×10^{-9}	$3.992767327 \times 10^{-8}$
0.4	0.9161104330	0.0916110390	4.30×10^{-9}	$4.693757265 \times 10^{-8}$	0.0453589766	0.4535896840	8.24×10^{-9}	$1.816619468 \times 10^{-7}$
0.5	0.0599366251	0.0599366190	6.10×10^{-9}	$1.017741655 \times 10^{-7}$	0.0248935342	0.0248935222	1.20×10^{-8}	$4.812494648 \times 10^{-7}$
0.6	0.0392136024	0.0392135880	1.44×10^{-8}	$3.672195100 \times 10^{-7}$	0.0136618612	0.0136618364	2.50×10^{-8}	$1.816736358 \times 10^{-6}$
0.7	0.0256555422	0.0256555220	2.02×10^{-8}	$7.873542427 \times 10^{-7}$	0.0074977884	0.0074977619	2.65×10^{-8}	$3.535709272 \times 10^{-6}$
0.8	0.0167851665	0.0167851500	1.65×10^{-8}	$9.824150395 \times 10^{-7}$	0.4114873524	0.0041148555	1.80×10^{-8}	$4.380207531 \times 10^{-6}$
0.9	0.0109817135	0.0109816940	1.95×10^{-8}	$1.776589781 \times 10^{-6}$	0.0022582905	0.0022582600	3.05×10^{-8}	$1.349339262 \times 10^{-5}$
1	0.0071847981	0.0071847830	1.51×10^{-8}	$2.095396403 \times 10^{-6}$	0.0012393761	0.0012393490	2.71×10^{-8}	$2.185615832 \times 10^{-5}$

Table 2. First and second situations: results for Example 1 when $\gamma = 0.5$.

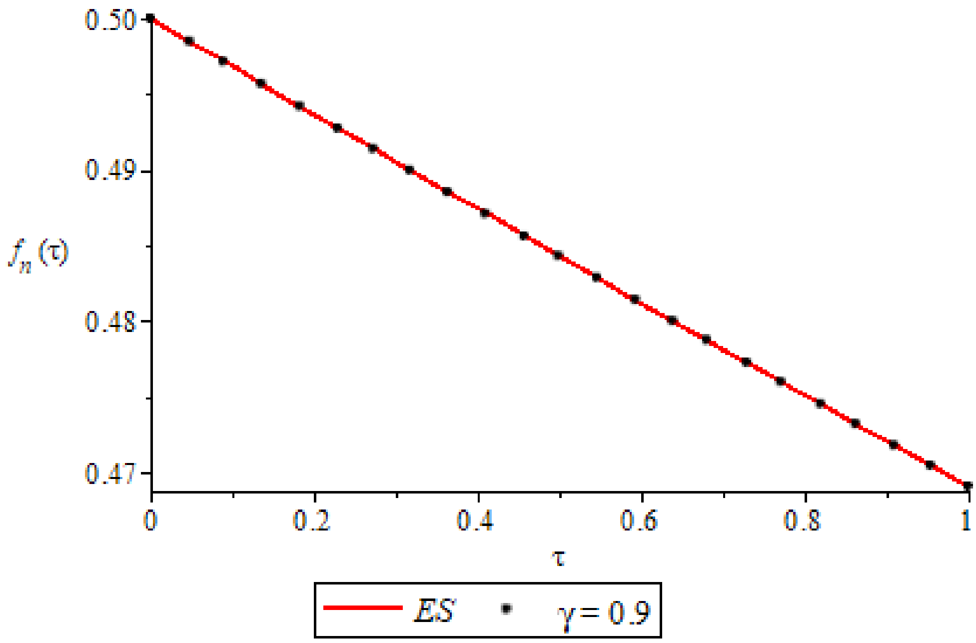
	First Situation				Second Situation			
t	**ES**	**AS**	**AE**	**RE**	**ES**	**AS**	**AE**	**RE**
0	0.5000000000	0.5000000000	0	0	0.5000000000	0.5000000000	0	0
0.1	0.4524187090	0.4524187086	4.0×10^{-10}	$8.84136735 \times 10^{-10}$	0.4093653766	0.4093653764	2.0×10^{-10}	$4.8856110 \times 10^{-10}$
0.2	0.4093653766	0.4093653757	9.0×10^{-10}	$2.19852496 \times 10^{-9}$	0.3351600230	0.3351600235	5.0×10^{-10}	1.4918247×10^{-9}
0.3	0.3704091104	0.3704091095	9.0×10^{-10}	$2.42974585 \times 10^{-9}$	0.2744058180	0.2744058200	2.0×10^{-9}	7.2884752×10^{-9}
0.4	0.3351600230	0.3351600200	3.0×10^{-9}	$8.95094819 \times 10^{-9}$	0.2246644820	0.2246644820	0	0
0.5	0.3032653298	0.3032653250	4.8×10^{-9}	$1.58277242 \times 10^{-8}$	0.1839397206	0.1839397200	6.0×10^{-10}	3.2619382×10^{-9}
0.6	0.2744058180	0.2744058110	7.0×10^{-9}	$2.55096632 \times 10^{-8}$	0.1505971060	0.1505970990	7.0×10^{-9}	4.6481637×10^{-8}
0.7	0.2482926519	0.2482926420	9.9×10^{-9}	$3.98723036 \times 10^{-8}$	0.1232984820	0.1232984720	1.0×10^{-8}	8.1103999×10^{-8}
0.8	0.2246644820	0.2246644730	9.0×10^{-9}	$4.00597367 \times 10^{-8}$	0.1009482590	0.1009482500	9.0×10^{-9}	8.9154584×10^{-8}
0.9	0.2032848298	0.2032848130	1.7×10^{-8}	$8.26426646 \times 10^{-8}$	0.0826494441	0.0826494270	1.7×10^{-8}	2.0689794×10^{-7}
1	0.1839397206	0.1839397270	6.4×10^{-9}	$3.47940074 \times 10^{-8}$	0.0676676416	0.0676676600	1.8×10^{-8}	2.7191727×10^{-7}

Table 3. First and second situations: results for Example 1 when $\gamma = 0.75$.

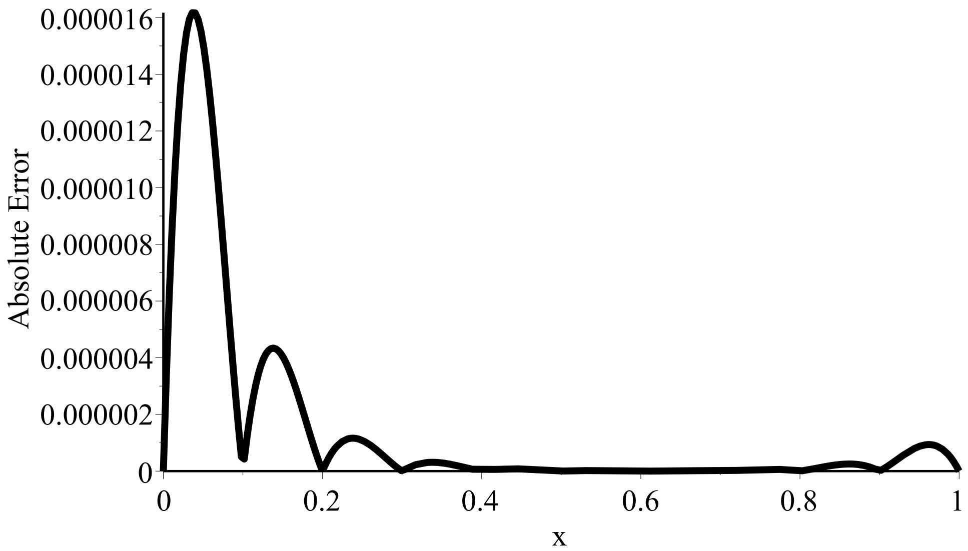
	First Situation				Second Situation			
t	ES	AS	AE	RE	ES	AS	AE	RE
0	0.5000000000	0.5000000000	0	0	0.5000000000	0.5000000000	0	0
0.1	0.4883526910	0.4524187086	1.0×10^{-10}	$2.0477004 \times 10^{-10}$	0.4677534925	0.4677534923	2.0×10^{-10}	$4.27575642 \times 10^{-10}$
0.2	0.4769767018	0.4769767016	2.0×10^{-10}	$4.19307692 \times 10^{-10}$	0.4375866596	0.4375866591	5.0×10^{-10}	$1.14263081 \times 10^{-9}$
0.3	0.4658657117	0.4658657115	2.0×10^{-10}	$4.29308264 \times 10^{-10}$	0.4093653766	0.4093653762	4.0×10^{-10}	$9.77122206 \times 10^{-10}$
0.4	0.4550135480	0.4550135476	4.0×10^{-10}	$8.79094703 \times 10^{-10}$	0.3829641692	0.3829641680	1.2×10^{-9}	$3.13345241 \times 10^{-9}$
0.5	0.4444141812	0.4444141806	6.0×10^{-10}	$1.350091931 \times 10^{-9}$	0.3582656552	0.3582656525	2.7×10^{-9}	7.5363071×10^{-9}
0.6	0.4340617227	0.4340617217	1.0×10^{-9}	$2.303819820 \times 10^{-9}$	0.3351600230	0.3351600190	4.0×10^{-9}	$1.19345976 \times 10^{-8}$
0.7	0.4239504207	0.4239504194	1.3×10^{-9}	$3.066396297 \times 10^{-9}$	0.3135445426	0.3135445370	5.6×10^{-9}	$1.78603013 \times 10^{-8}$
0.8	0.4140746577	0.4140746565	1.2×10^{-9}	$2.898028116 \times 10^{-9}$	0.2933231098	0.2933231050	4.8×10^{-9}	$1.63642067 \times 10^{-8}$
0.9	0.4044289468	0.4044289448	2.0×10^{-9}	$4.945244439 \times 10^{-9}$	0.2744058180	0.2744058080	1.0×10^{-8}	$3.64423760 \times 10^{-8}$
1	0.3950079290	0.3950079298	8.0×10^{-10}	$2.025275801 \times 10^{-9}$	0.2567085596	0.2567085640	4.4×10^{-9}	$1.71400596 \times 10^{-8}$

Table 4. First and second situations: results for Example 1 when $\gamma = 0.9$.

	First Situation				Second Situation			
t	ES	AS	AE	RE	ES	AS	AE	RE
0	0.5000000000	0.5000000000	0	0	0.5000000000	0.5000000000	0	0
0.1	0.4968193310	0.4968193310	0	0	0.4890114362	0.4890114361	1.0×10^{-10}	$2.04494195 \times 10^{-10}$
0.2	0.4936588953	0.4936588953	0	0	0.4782643696	0.4782643695	1.0×10^{-10}	$2.09089379 \times 10^{-10}$
0.3	0.4905185642	0.4905185642	0	0	0.4677534925	0.4677534924	1.0×10^{-10}	$2.13787821 \times 10^{-10}$
0.4	0.4873982098	0.4873982098	0	0	0.4574736144	0.4574736142	2.0×10^{-10}	$4.37183684 \times 10^{-10}$
0.5	0.4842977051	0.4842977051	0	0	0.4474196584	0.4474196581	3.0×10^{-10}	$6.70511441 \times 10^{-10}$
0.6	0.4812169237	0.4812169236	1.0×10^{-10}	$2.07806490 \times 10^{-10}$	0.4375866596	0.4375866588	8.0×10^{-10}	$1.82820930 \times 10^{-9}$
0.7	0.4781557402	0.4781557400	2.0×10^{-10}	$4.18273761 \times 10^{-10}$	0.4279697617	0.4279697607	1.0×10^{-9}	$2.33661368 \times 10^{-9}$
0.8	0.4751140299	0.4751140298	1.0×10^{-10}	$2.10475789 \times 10^{-10}$	0.4185642156	0.4185642147	9.0×10^{-10}	$2.15020770 \times 10^{-9}$
0.9	0.4720916690	0.4720916688	2.0×10^{-10}	$4.23646536 \times 10^{-10}$	0.4093653766	0.4093653750	1.6×10^{-9}	$3.90848883 \times 10^{-9}$
1	0.4690885343	0.4690885343	0	0	0.4003687014	0.4003687022	8.0×10^{-10}	$1.99815819 \times 10^{-9}$

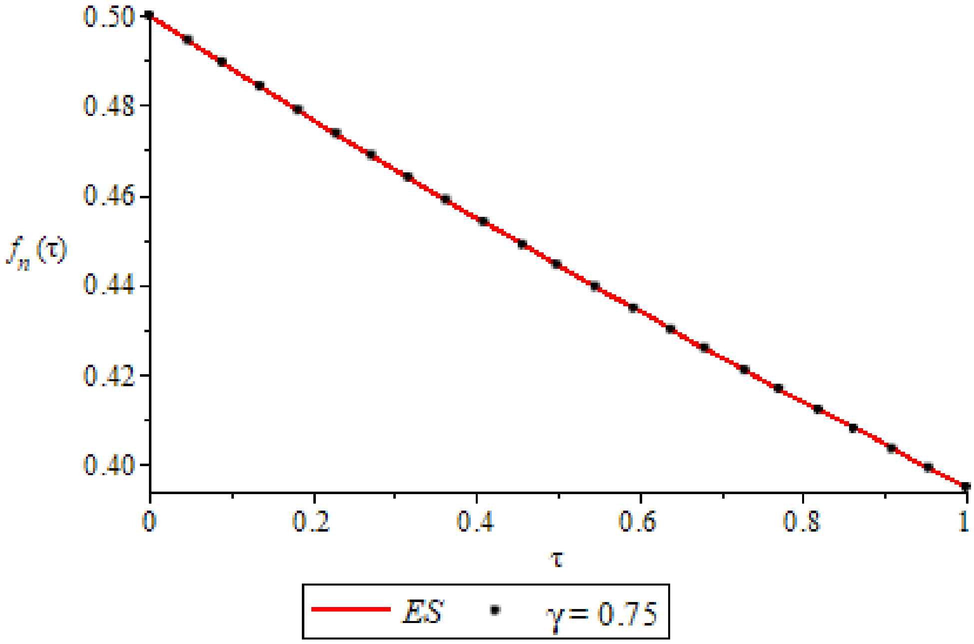


(**a**) Exact and the RKHSM's solutions.

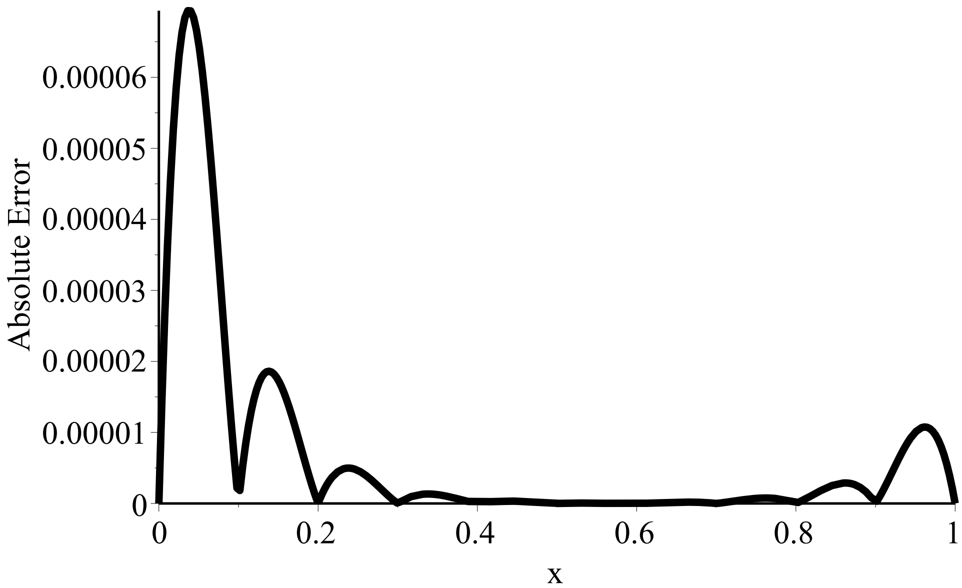


(**b**) Absolute error of the RKHSM.

Figure 1. Comparison of numerical solutions of the RKHSM by the ES for the FS of Example 1.

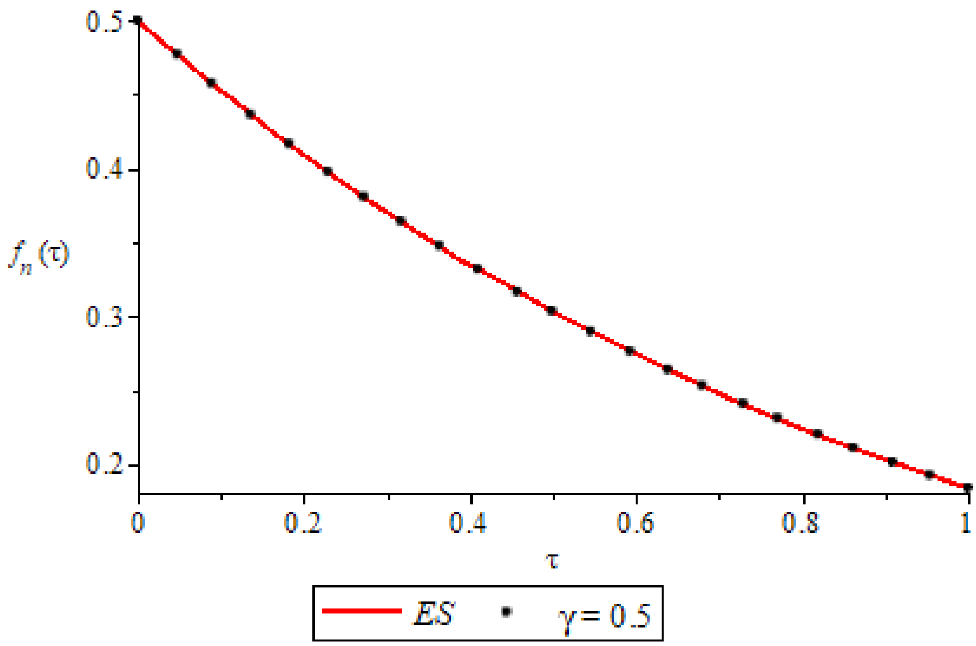


(**a**) Exact and the RKHSM's solutions.



(**b**) Absolute error of the RKHSM.

Figure 2. Comparison of numerical solutions of the RKHSM by the ES for the FS of Example 1.



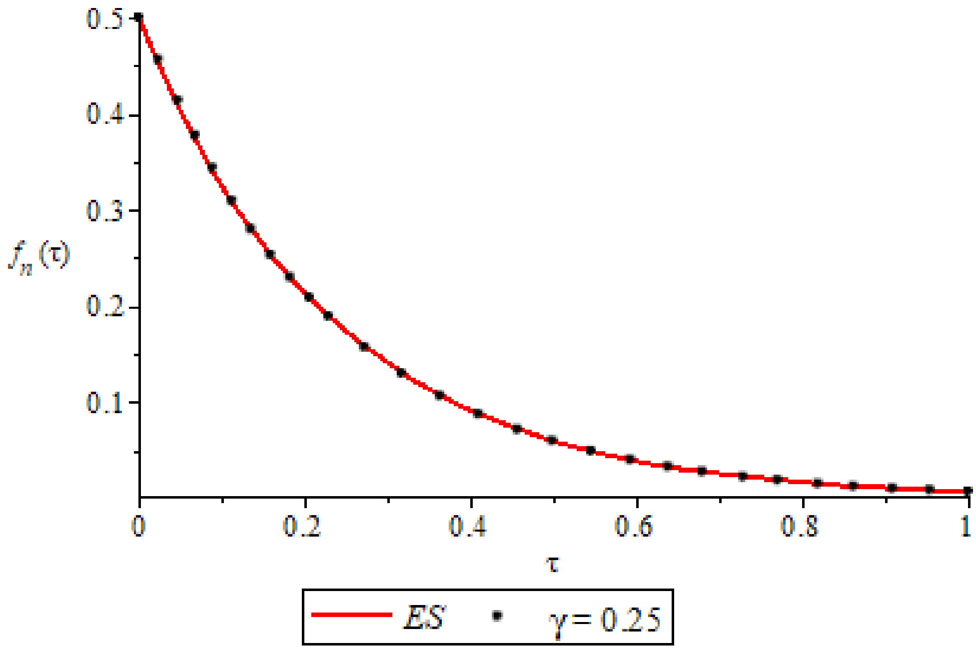
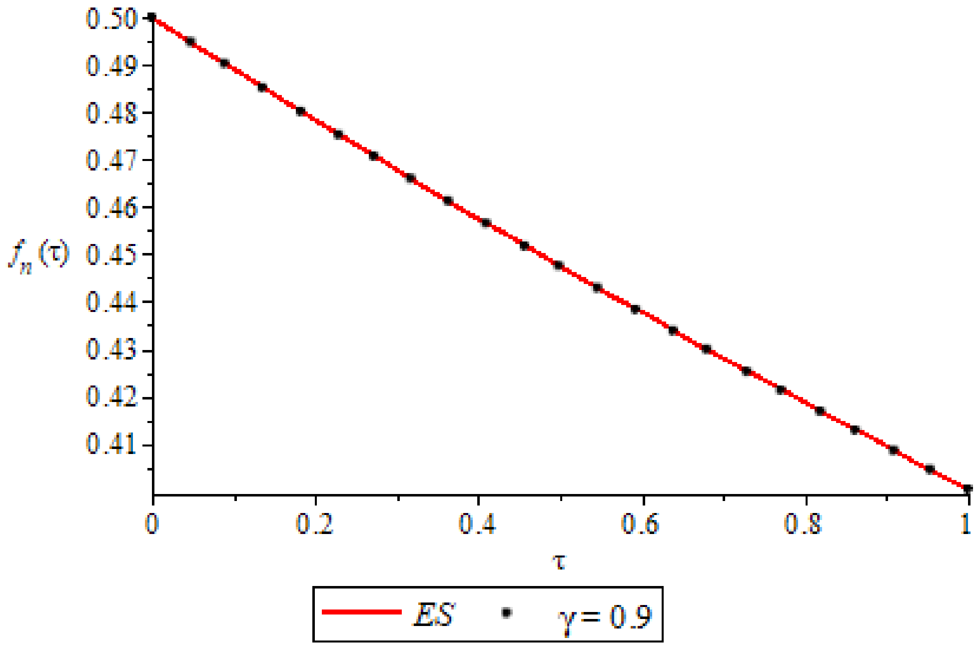
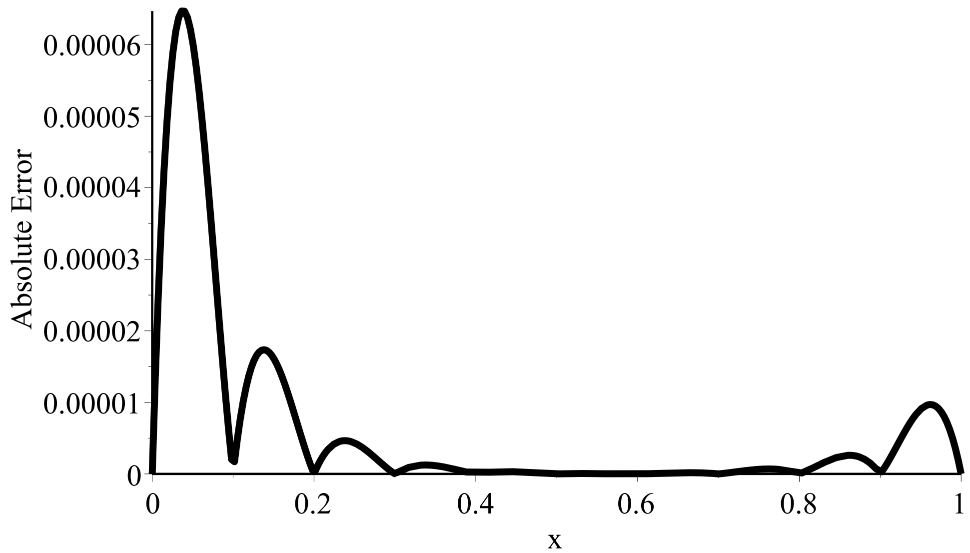


Figure 3. Comparison of numerical solutions of the RKHSM by the ES for the FS of Example 1.

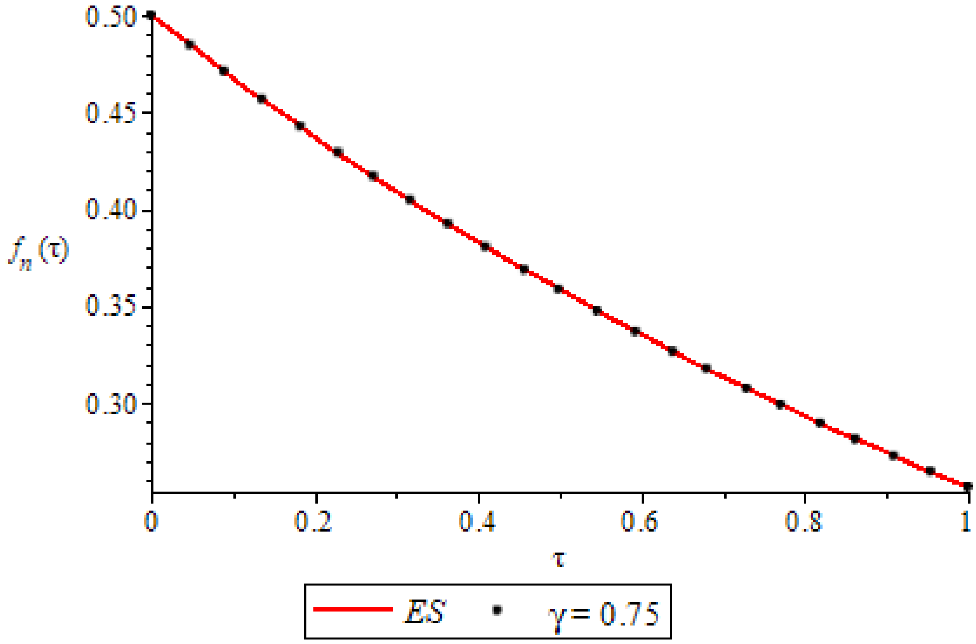


(**a**) Exact and the RKHSM's solutions.

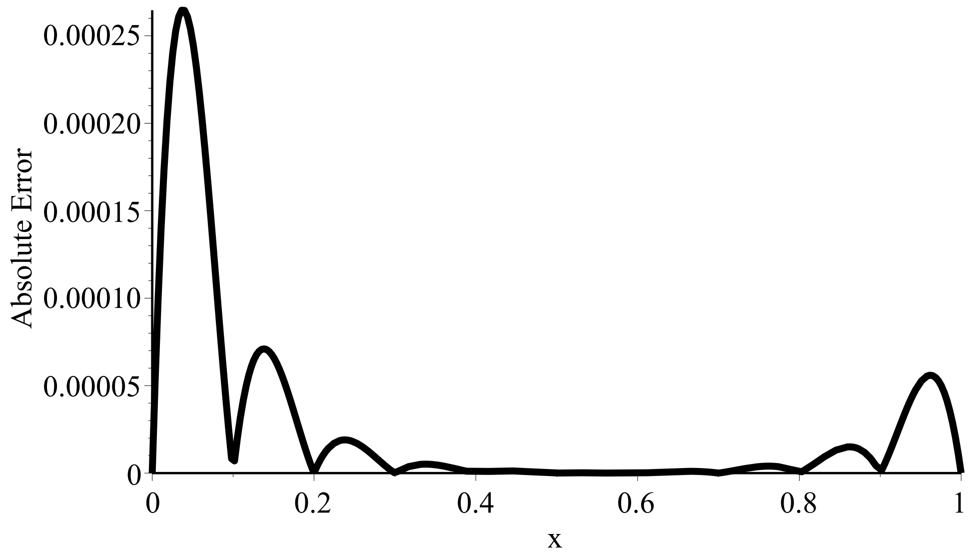


(**b**) Absolute error of the RKHSM.

Figure 4. Comparison of numerical solutions of the RKHSM by the ES for the SS of Example 1.



(**a**) Exact and the RKHSM's solutions.



(**b**) Absolute error of the RKHSM.

Figure 5. Comparison of numerical solutions of the RKHSM by the ES for the SS of Example 1.

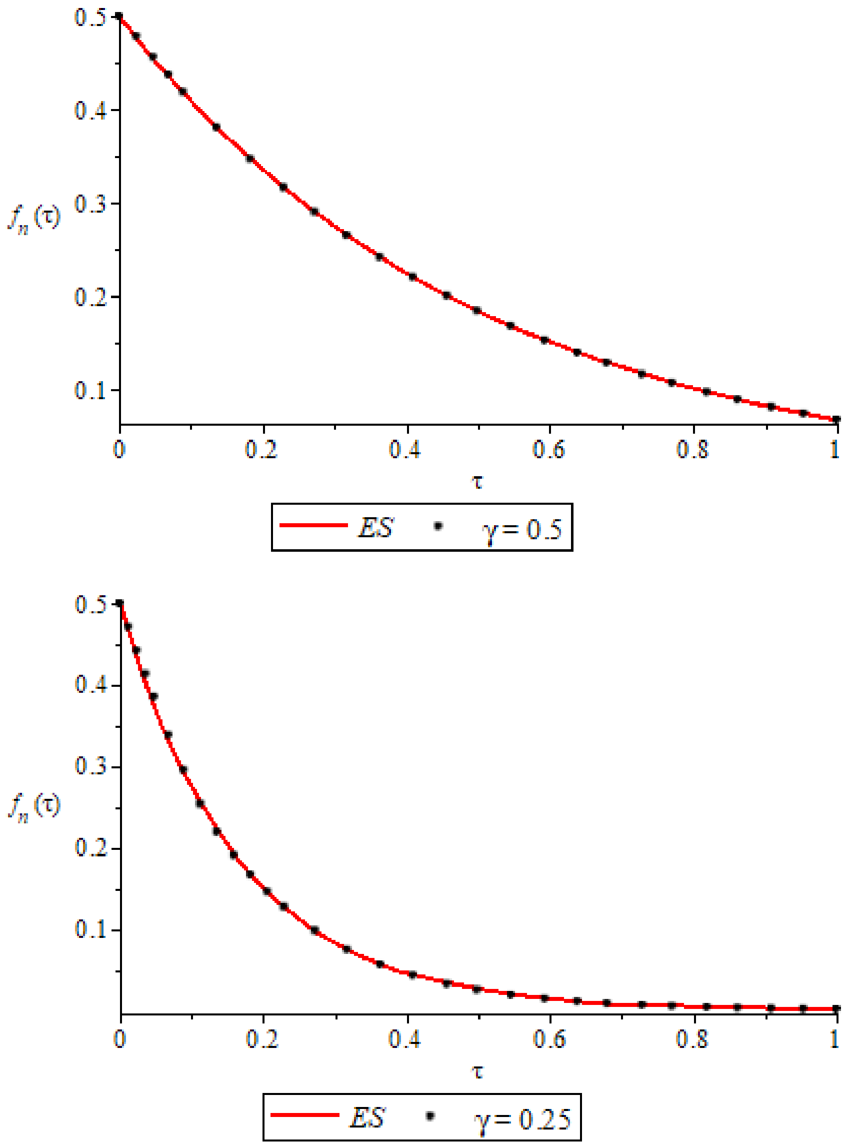


Figure 6. Comparison of numerical solutions of the RKHSM by the ES for the SS of Example 1.

Example 2. *Considering the following simple problem:*

$$\begin{cases} {}^{CPC}{}_0D_t^{\gamma}\mathfrak{f}(t) = t, \quad 0 < t \leq 1, \\ \mathfrak{f}(0) = 0. \end{cases} \tag{35}$$

The exact solution to the above problem takes the following form

$$\mathfrak{f}(t) = \frac{\exp\left(-\frac{t\mathrm{K}_1(\gamma)}{\mathrm{K}_0(\gamma)}\right)t^{\gamma}\left(-\Gamma(1+\gamma) + \Gamma\left(1+\gamma, -\frac{t\mathrm{K}_1(\gamma)}{\mathrm{K}_0(\gamma)}\right)\right)\left(-\frac{t\mathrm{K}_1(\gamma)}{\mathrm{K}_0(\gamma)}\right)^{-\gamma}}{\Gamma(1+\gamma)\mathrm{K}_1(\gamma)}.$$

1. *First situation (FS, for short):* $\mathrm{K}_0(\gamma) = \gamma\sigma^{1-\gamma}, \mathrm{K}_1(\gamma) = (1-\gamma)\sigma^{\gamma}, \sigma = \frac{1}{2}$.
2. *Second situation (SS, for short):* $\mathrm{K}_0(\gamma) = \gamma\sigma^{1+\gamma}, \mathrm{K}_1(\gamma) = (1-\gamma)\sigma^{\gamma}, \sigma = \frac{1}{2}$.

This RKHSM is tested with the grid points $t_\imath = \frac{\imath}{n}$, $\imath = 1, 2, \ldots, n$. *The ES, AS, AE, and RE of* (35) *are shown in Tables 5–8 when* $\gamma \in \{0.25, 0.5, 0.75, 0.9\}$ *and* $t \in [0, 1]$ *for both the FS and SS. Additionally, the comparisons between the ES and RKHSM's solution for both cases are depicted in Figure 7 for the FS and Figure 8 for the SS, when* $\gamma = 0.25, 0.5, 0.75, 0.9$. *From the results, we can see that the numerical solutions are found to be in excellent agreement with the exact solutions.*

Table 5. First and second situations: results for Example 2 when $\gamma = 0.25$.

	First Situation				Second Situation			
t	**ES**	**AS**	**AE**	**RE**	**ES**	**AS**	**AE**	**RE**
0.1	0.278390226	0.278390225	8.0×10^{-10}	$2.873664108 \times 10^{-9}$	0.366578617	0.366578616	1.7×10^{-9}	$4.637477257 \times 10^{-9}$
0.2	0.559438322	0.559438324	2.5×10^{-9}	$4.468767876 \times 10^{-9}$	0.694736216	0.694736218	1.3×10^{-9}	$1.871213807 \times 10^{-9}$
0.3	0.794667664	0.794667667	3.2×10^{-9}	$4.026840585 \times 10^{-9}$	0.941462921	0.941462918	2.9×10^{-9}	$3.080312495 \times 10^{-9}$
0.4	0.986000956	0.986000987	3.1×10^{-8}	$3.184581091 \times 10^{-8}$	1.125538939	1.125538968	2.9×10^{-8}	$2.576543467 \times 10^{-8}$
0.5	1.141251776	1.141251787	1.1×10^{-8}	$9.638539217 \times 10^{-9}$	1.265695498	1.265695503	5.0×10^{-9}	$3.950397238 \times 10^{-9}$
0.6	1.268231856	1.268231894	3.8×10^{-8}	$2.996297548 \times 10^{-8}$	1.375697794	1.375697830	3.6×10^{-8}	$2.616853800 \times 10^{-8}$
0.7	1.373458393	1.373458428	3.5×10^{-8}	$2.548311633 \times 10^{-8}$	1.464920275	1.464920304	2.9×10^{-8}	$1.979629915 \times 10^{-8}$
0.8	1.462030835	1.462030879	4.4×10^{-8}	$3.009512450 \times 10^{-8}$	1.539590658	1.539590698	4.0×10^{-8}	$2.598093187 \times 10^{-8}$
0.9	1.537826398	1.537826474	7.6×10^{-8}	$4.942040278 \times 10^{-8}$	1.603828534	1.603828600	6.6×10^{-8}	$4.115153123 \times 10^{-8}$
1	1.603753900	1.603753849	5.1×10^{-8}	$3.180039032 \times 10^{-8}$	0.001239376	1.660377912	3.0×10^{-8}	$1.806817547 \times 10^{-8}$

Table 6. First and second situations: results for Example 2 when $\gamma = 0.5$.

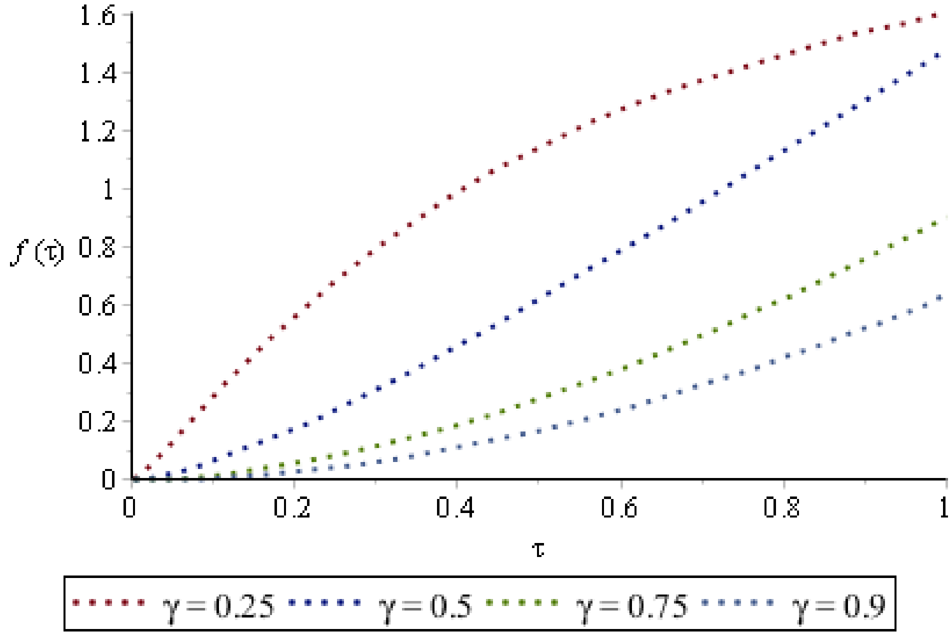
	First Situation				Second Situation			
t	**ES**	**AS**	**AE**	**RE**	**ES**	**AS**	**AE**	**RE**
0.1	0.064667410	0.064667411	9.9×10^{-10}	$1.530910242 \times 10^{-8}$	0.124390490	0.124390493	2.9×10^{-9}	$2.331367938 \times 10^{-8}$
0.2	0.175914718	0.175914719	1.5×10^{-9}	$8.526859025 \times 10^{-9}$	0.326098542	0.326098549	6.5×10^{-9}	$1.993262514 \times 10^{-8}$
0.3	0.311030443	0.311030447	4.4×10^{-9}	$1.414652521 \times 10^{-8}$	0.556730844	0.556730856	1.2×10^{-8}	$2.227288130 \times 10^{-8}$
0.4	0.461172981	0.461172979	1.7×10^{-9}	$3.686252383 \times 10^{-9}$	0.798597356	0.798597370	1.4×10^{-8}	$1.803161493 \times 10^{-8}$
0.5	0.621108506	0.621108511	4.7×10^{-9}	$7.567115817 \times 10^{-9}$	1.042442942	1.265695503	1.9×10^{-8}	$1.822641756 \times 10^{-8}$
0.6	0.787336310	0.787336312	2.1×10^{-9}	$2.667221077 \times 10^{-9}$	1.283029637	1.375697830	3.6×10^{-8}	$2.805858881 \times 10^{-8}$
0.7	0.957370780	0.957370788	8.2×10^{-9}	$8.565124582 \times 10^{-9}$	1.517362373	1.464920304	4.2×10^{-8}	$2.767961161 \times 10^{-8}$
0.8	1.129387212	1.129387215	3.0×10^{-9}	$2.656307746 \times 10^{-9}$	1.743794486	1.539590698	4.6×10^{-8}	$2.637925603 \times 10^{-8}$
0.9	1.302020083	1.302020093	1.0×10^{-8}	$7.680373084 \times 10^{-9}$	1.961520086	1.603828600	8.3×10^{-8}	$4.231412368 \times 10^{-8}$
1	1.474236919	1.474236919	0	0	2.170265034	2.170264989	4.5×10^{-8}	$2.073479473 \times 10^{-8}$

Table 7. First and second situations: results for Example 2 when $\gamma = 0.75$.

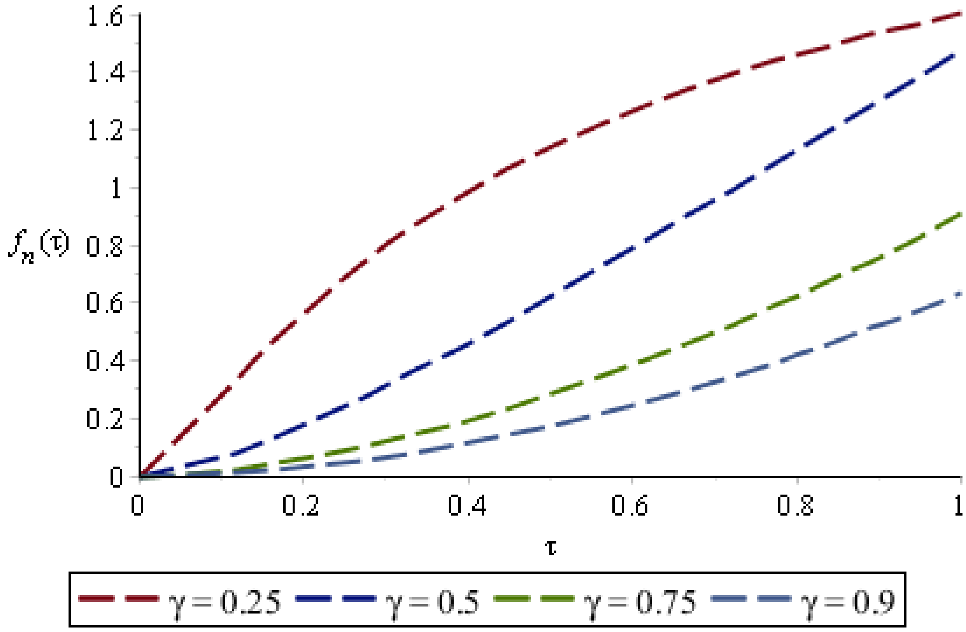
	First Situation				Second Situation			
t	ES	AS	AE	RE	ES	AS	AE	RE
0.1	0.01738193388	0.01738193250	1.38×10^{-9}	$7.939277698 \times 10^{-8}$	0.048404880	0.048404879	1.1×10^{-9}	$2.355134427 \times 10^{-8}$
0.2	0.05796964429	0.05796964023	4.06×10^{-9}	$7.003665539 \times 10^{-8}$	0.158979414	0.158979408	5.5×10^{-9}	$3.459567419 \times 10^{-8}$
0.3	0.1168624847	0.1168624762	8.5×10^{-9}	$7.273506140 \times 10^{-8}$	0.315694535	0.315694524	1.1×10^{-8}	$3.579409442 \times 10^{-8}$
0.4	0.1917111802	0.1917111584	2.2×10^{-8}	$1.137127213 \times 10^{-7}$	0.510261243	0.510261205	3.8×10^{-8}	$7.427567849 \times 10^{-8}$
0.5	0.2809210058	0.2809209865	1.9×10^{-8}	$6.870258757 \times 10^{-8}$	0.736856811	0.736856784	2.7×10^{-8}	$3.596356798 \times 10^{-8}$
0.6	0.3832748550	0.3832748188	3.6×10^{-8}	$9.444919104 \times 10^{-8}$	0.990974626	0.990974570	5.6×10^{-8}	$5.600547032 \times 10^{-8}$
0.7	0.4977838273	0.4977837878	4.0×10^{-8}	$7.935171421 \times 10^{-8}$	1.268951107	1.268951049	5.8×10^{-8}	$4.570704078 \times 10^{-8}$
0.8	0.6236125669	0.6236125196	4.7×10^{-8}	$7.584837527 \times 10^{-8}$	1.567719773	1.567719703	7.0×10^{-8}	$4.465083697 \times 10^{-8}$
0.9	0.7600365805	0.7600365012	7.9×10^{-8}	$1.043370833 \times 10^{-7}$	1.884664677	1.884664559	1.2×10^{-7}	$6.261060731 \times 10^{-8}$
1	0.9064155258	0.9064155820	5.6×10^{-8}	$6.200246840 \times 10^{-8}$	2.217524640	2.217524727	8.7×10^{-8}	$3.923293497 \times 10^{-8}$

Table 8. First and second situations: results for Example 2 when $\gamma = 0.9$.

	First Situation				Second Situation			
t	ES	AS	AE	RE	ES	AS	AE	RE
0.1	0.0081861790	0.0081861771	1.9×10^{-9}	$2.313655737 \times 10^{-7}$	0.028351089	0.028351084	5.2×10^{-9}	$1.844726329 \times 10^{-7}$
0.2	0.0304849118	0.0304849070	4.8×10^{-9}	$1.567988792 \times 10^{-7}$	0.105006807	0.105006793	1.4×10^{-8}	$1.314200522 \times 10^{-7}$
0.3	0.0657211749	0.0657211649	1.0×10^{-8}	$1.526144354 \times 10^{-7}$	0.225161367	0.225161338	2.9×10^{-8}	$1.279082660 \times 10^{-7}$
0.4	0.1132757197	0.1132756975	2.2×10^{-8}	$1.959819815 \times 10^{-7}$	0.386004901	0.386004835	6.7×10^{-8}	$1.725366693 \times 10^{-7}$
0.5	0.1727089417	0.1727089198	2.2×10^{-8}	$1.268029309 \times 10^{-7}$	0.585395620	0.585395557	6.4×10^{-8}	$1.088152999 \times 10^{-7}$
0.6	0.2436739618	0.2436739235	3.8×10^{-8}	$1.571772368 \times 10^{-7}$	0.821549439	0.821549325	1.1×10^{-7}	$1.380318635 \times 10^{-7}$
0.7	0.3258809599	0.3258809169	4.3×10^{-8}	$1.319500225 \times 10^{-7}$	1.092910710	1.092910583	1.3×10^{-7}	$1.162034545 \times 10^{-7}$
0.8	0.4190787537	0.4190787040	5.0×10^{-8}	$1.185934614 \times 10^{-7}$	1.398085171	1.398085022	1.5×10^{-7}	$1.065743369 \times 10^{-7}$
0.9	0.5230442467	0.5230441624	8.4×10^{-8}	$1.611718330 \times 10^{-7}$	1.735800499	1.735800248	2.5×10^{-7}	$1.446018711 \times 10^{-7}$
1	0.6375755634	0.6375756254	6.2×10^{-8}	$9.724337562 \times 10^{-8}$	2.104880901	2.104881081	1.8×10^{-7}	$8.551552723 \times 10^{-8}$

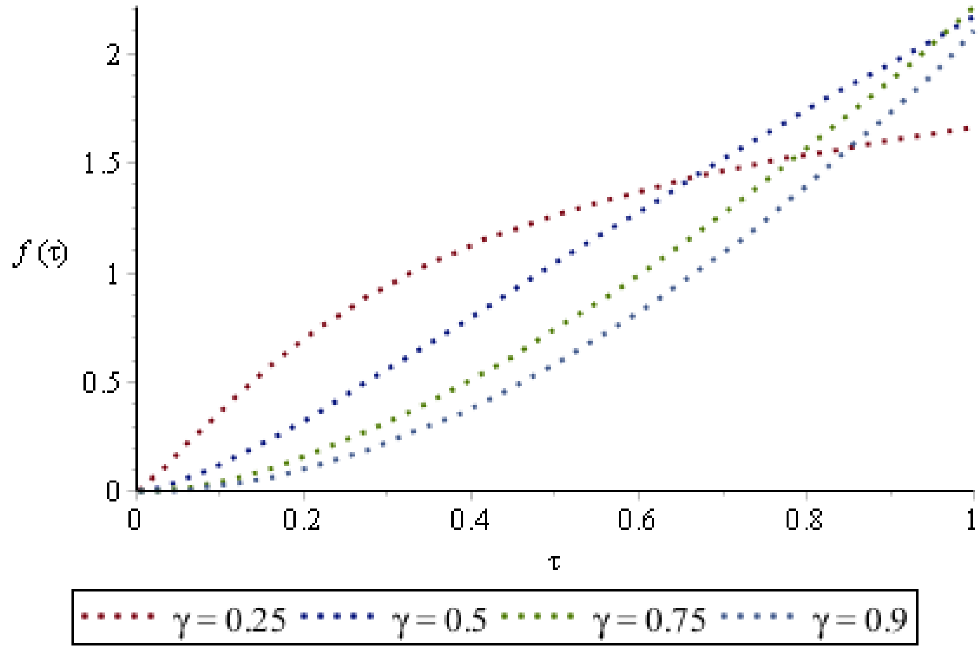


(**a**) Exact solutions for different values of γ.

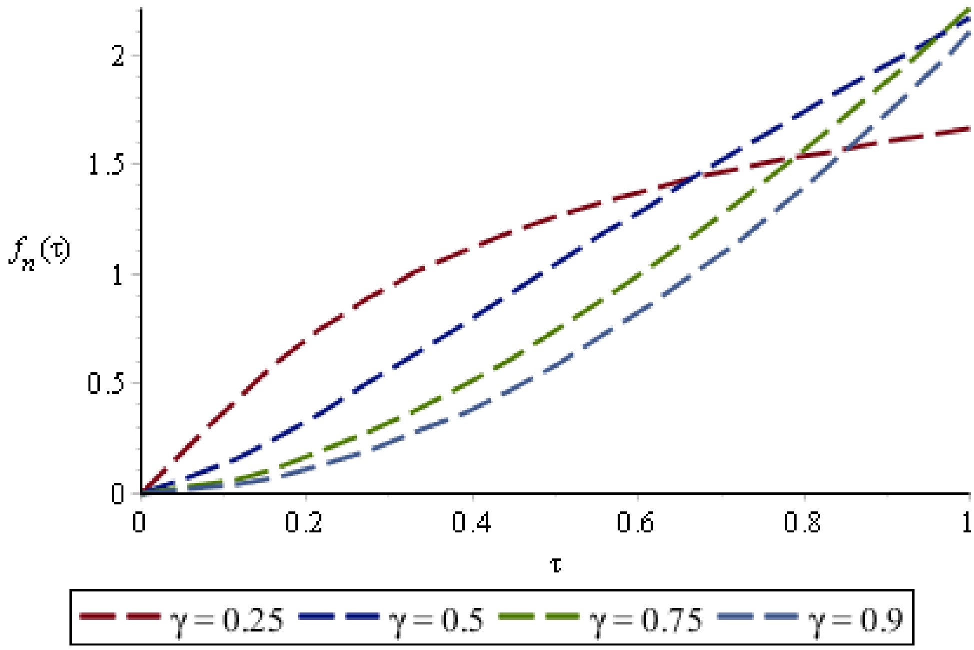


(**b**) Approximate solutions for different values of γ.

Figure 7. Comparison of numerical solutions of the RKHSM by the ES for the FS of Example 2.



(**a**) Exact solutions for different values of γ.



(**b**) Approximate solutions for different values of γ.

Figure 8. Comparison of numerical solutions of the RKHSM by the ES for the SS of Example 2.

5. Conclusions

In this paper, an efficient method has been applied successfully for solving FDEs. The approximate solution $\mathfrak{g}_n(t)$ and its derivative both converge uniformly. The accuracy and applicability of the proposed method are validated by computing the numerical solutions at many grid points. The boundedness of the linear operator is demonstrated. The results show that the proposed method is a powerful tool to approximate many other non-linear problems which are described by the new hybrid fractional derivative operator. This research opens the way for the use of the RKHSM to study the mentioned problem for various new fractional derivatives. As part of our purpose, we plan to apply the RKHSM to multidimensional fractional partial differential equations that are described with CPC derivative, which will be new in the literature.

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