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Analysis of the Fractional Differential Equations Using Two Different Methods

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Abstract: Numerical methods play an important role in modern mathematical research, especially studying the symmetry analysis and obtaining the numerical solutions of fractional differential equation. In the current work, we use two numerical schemes to deal with fractional differential equations. In the first case, a combination of the group preserving scheme and fictitious time integration method (FTIM) is considered to solve the problem. Firstly, we applied the FTIM role, and then the GPS came to integrate the obtained new system using initial conditions. Figure and tables containing the solutions are provided. The tabulated numerical simulations are compared with the reproducing kernel Hilbert space method (RKHSM) as well as the exact solution. The methodology of RKHSM mainly relies on the right choice of the reproducing kernel functions. The results confirm that the FTIM finds the true solution. Additionally, these numerical results indicate the effectiveness of the proposed methods.

Keywords: fictitious time integration method; time-fractional heat equation; fractional differential equations; reproducing kernel Hilbert space method; group-preserving scheme

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1. Introduction

The fractional calculus' sense (FC) is presented after classical calculus, but after identifying the limitations of the classical one, many researchers weighed the notions of fractional calculus to comprehend the character systematically. Plenty of mathematicians promoted the vital establishment with the aid of new attributes and corresponding outcomes for FC [1–6]. Specifically, the particular functions are proposed to create novel non-integer integral and differential operators. These latter are presented by many people to investigate and symbolize different equations linked with phenomena [7–15]. The symmetric and anti-symmetric solitons of the fractional Schrödinger equation have been studied in [16]. In [17], the authors extended the Lie symmetry analysis to the time fractional generalized KdV equations. The Adomian decomposition technique for investigating the fractional KdV–Burgers equation was applied in [18]. Since the Chebyshev collocation technique is implemented for investigating the time-fractional nonlinear Klein–Gordon equation [19], a geometric method is applied for the Korteweg–de Vries equation [20]. A combination of FTIM and geometric method is applied for the fractional Burger–Huxley equation [21]. Additionally, in [22], a lie group approach is implemented for solving the fractional equation. Some other applications can be seen in[23–31].

Consider the following equation:

$$cq^{C}\mathcal{D}^{\alpha}_{0^+,t}u = \lambda u_{xx} + g(x,t), \quad (x,t) \in \Omega, \tag{1}$$

$$u(x,t) = H(x,t), \quad on \quad \Gamma, \tag{2}$$

where c denotes the specific heat, c is the density, λ describes the thermal conductivity coefficient, and ${}^{C}\mathcal{D}^{\alpha}_{0^+,t}$ shows the Caputo fractional derivative. Equations of this type are used to describe the transport processes with a long memory.The fractional heat equation is one of the most well-known fractional partial differential equations that describe the physical phenomenon. In recent years, solving the fractional heat equation magnetized the attention of mathematicians because of its importance. Many methods have been worked to solve this problem. The higher-order numerical method is used for investigating the fractional heat equation [32]. Additionally, the Laplace homotopy technique is worked to solve this equation [33]. In [34], authors used one-step backward-forward algorithms for multi-dimensional backward heat conduction problems.

Motivated by the above works, in this paper, two numerical methods are worked to solve this equation. One is a combination of a specific type of fictitious time integration technique and the Runge–Kutta method. The other is the RKHSM. The main paper's contributions are as follows:

- We present new results on the numerical simulation for the considered equation.
- We apply two effective numerical methods to obtain these new accurate results.
- The convergence analysis that confirms the theoretical parts of both methods is discussed.

Variable transformation of a time integration method, namely FTIM, was suggested by Atluri and Liu. Researchers used it for solving linear or nonlinear algebraic equations by defining the fictitious time and using it to derive a system of nonautonomous first-order ordinary differential equations that is equivalent to the original algebraic equations in an n-dimensional space. Some applications of this technique can be seen in [35–39].

In another aspect of this paper, as we mentioned before, we applied the RKHSM for solving the proposed equation.

Recently, the RKHSM has achieved great popularity and success. It became a powerful tool in treating different types of FPDEs, such as the fractional Bloch–Torrey equations [40] and fractional differential equations, including the ABC derivative [41], to name a few. See also [42–47] for more research about this method. The RKHSM has many advantages, such as its simplicity and flexibility in treating many fractional differential systems and the fact that it is a mesh-free method. The rest of the paper consists of the following: Section 2 recalls some essential concepts about fractional calculus and reproducing kernel theories. Sections 3 and 4 are where we see the main theory of the FTIM and RKHSM, respectively, to build a numerical solution for the considered problem. Before finishing with the conclusion part, we validate the proposed methods through two examples.

2. Basic Definitions

Definition 1. *The left-sided Riemannian–Liouville fractional integral of order* $\mu \in \mathbb{R}_+$ *of* $f \in C_\alpha, \alpha \geq -1$, *is*

$$I^{\mu}f(x) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(\eta)}{(x-\eta)^{1-\mu}} \mathrm{d}\eta, \;\; 0 < \mu, \;\; x > 0, \;\; I^0 f(x) = f(x). \tag{3}$$

Definition 2. *We write*

$$f(x) \in C^m_\alpha, \quad x > 0, m \in N \cup 0,$$

provided

$$f^{(m)} \in C_\alpha.$$

Definition 3. *Suppose a real function f (with* $0 < x$*) is in the space* C_α*,* $\alpha \in \mathcal{R}$*. By* $p(> \alpha)$ *such that* $f(x) = x^p f_1(x) \in C[0, \infty]$*. Obviously,* $C_\alpha \in C_\beta$ *if* $\beta \subset \alpha$*.*

Definition 4. *Suppose* $f \in C_{-1}^m, m \in N$*, the Caputo derivative of f is*

$$D^\mu f(x) = \begin{cases} [I^{m-\mu]} f(x), \;\; \mu \in (m-1, m], \\ \\ \dfrac{d^m}{dt^m} f(x), \;\; \mu = m. \end{cases} \tag{4}$$

$$I^\mu I^\nu f = I^{\mu+\nu}, \;\; 0 \le \mu, \; 0 \le \nu, \; f \in C_\alpha, \;\; 0 \le \alpha. \tag{5}$$

$$I^\mu x^\xi f = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} x^{\mu+\xi}, \;\; 0 < \mu, \;\; -1 < \xi, \;\; 0 < x. \tag{6}$$

Lemma 1. *Assume* $\alpha \in (m-1, m]$ *and* $f \in L_1[a, b]$*. Then,*

$$J_a^\mu f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{1-\mu} f(t) \mathrm{d}t, \;\; D_a^\alpha J_a^\alpha f(x) = f(x), \tag{7}$$

and

$$D_a^\alpha J_a^\alpha f(x) = f(x) - \sum_{k=1}^{m-1} f^{(k)}(0) \frac{(x-a)^k}{k!}, \;\; x > 0. \tag{8}$$

Definition 5. *The fractional derivative of f in the Caputo sense is*

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t) \mathrm{d}t. \tag{9}$$

Definition 6. *The Caputo time-fractional derivative operator is*

$${}^C\mathcal{D}_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-z)^{m-\alpha-1} \frac{\partial^m u(x,z)}{\partial z^m} \mathrm{d}z, \; \alpha \in (m-1, m), \\ \\ \frac{\partial^m u(x,t)}{\partial t^m}, \qquad \alpha = m, \end{cases} \tag{10}$$

for m to be the smallest integer that exceeds α*. The space-fractional derivative with* $\beta > 0$ *is described by*

$${}^C\mathcal{D}_x^\beta u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^x (x-s)^{m-\beta-1} \frac{\partial^m u(s,t)}{\partial s^m} \mathrm{d}s, m-1 < \beta < m, \\ \\ \frac{\partial^m u(x,t)}{\partial t^m}, \qquad \beta = m. \end{cases} \tag{11}$$

Notations

(i) We will write

$$AC[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is absolutely continuous on } [a, b]\}$$

to denote the collection of all absolutely continuous functions on $[a,b]$.

(ii) We write CCF to mean a completely continuous function.

Definition 7. *Define the function space* $\mathfrak{W}_2^3[a,b]$ *by*

$$\mathfrak{W}_2^3[a,b] = \{f(x) \mid f^{(\mathrm{j})} \in AC[a,b],\ \mathrm{j} = 0,1,2,\ f^{(3)} \in L^2[a,b],\ \text{and } f(a) = f(b) = 0\}.$$

Definition 8. *If* $f,\ g \in \mathfrak{W}_2^3[a,b]$, *the inner product and norm of this space are described to be*

$$\langle f,g\rangle_{\mathfrak{W}_2^3} = \sum_{\mathrm{i}=0}^{2} f^{(\mathrm{i})}(a) g^{(\mathrm{i})}(a) + \int_a^b f^{(3)}(x) g^{(3)}(x) \mathrm{d}x, \tag{12}$$

and

$$\|f\|_{\mathfrak{W}_2^3} = \sqrt{\langle f,f\rangle_{\mathfrak{W}_2^3}}. \tag{13}$$

Theorem 1. *The RK function of* $\mathfrak{W}_2^3[0,1]$ *is the function* $\mathtt{R}_\zeta(x)$ *described as*

$$\mathtt{R}_\zeta(x) = \begin{cases} \mathtt{r}(x,\zeta), & x \le \zeta, \\ \mathtt{r}(\zeta,x), & x > \zeta, \end{cases} \tag{14}$$

with

$$\begin{aligned} \mathtt{r}(x,\zeta) = {} & \frac{3\,x\,\zeta}{13} - \frac{x\zeta^5}{156} + \frac{5\,x\zeta^4}{156} - \frac{5\,x\zeta^3}{78} - \frac{5\,x\zeta^2}{26} + \frac{21\,x^2\zeta^2}{104} - \frac{x^2\zeta^5}{624} + \frac{5\,x^2\zeta^4}{624} - \frac{5\,x^2\zeta^3}{312} - \frac{5\,x^2\,\zeta}{26} + \frac{7\,x^3\zeta^2}{104} - \frac{x^3\zeta^5}{1872} \\ & + \frac{5\,x^3\zeta^4}{1872} - \frac{5\,x^3\zeta^3}{936} - \frac{5\,x^3(\zeta)}{78} - \frac{x^4(\zeta)}{104} + \frac{x^4\zeta^5}{3744} - \frac{5\,x^4\zeta^4}{3744} + \frac{5\,x^4\zeta^3}{1872} + \frac{5\,x^4\zeta^2}{624} - \frac{x^5\zeta^5}{18720} + \frac{x^5\zeta^4}{3744} - \frac{x^5\zeta^3}{1872} \\ & - \frac{x^5\zeta^2}{624} - \frac{x^5(\zeta)}{156} + \frac{x^5}{120}. \end{aligned} \tag{15}$$

For the proof, see [43].

Definition 9. *Define the function space* $\mathfrak{W}_2^2[c,d]$ *by*

$$\mathfrak{W}_2^2[c,d] = \{f(t) \mid f^{(\mathrm{j})} \in AC[c,d],\ \mathrm{j} = 0,1,\ f'' \in L^2[c,d],\ \text{and } f(c) = 0\}.$$

Definition 10. *If* $f,\ g \in \mathfrak{W}_2^2[c,d]$, *the inner product and norm are*

$$\langle f,g\rangle_{\mathfrak{W}_2^2} = \sum_{\mathrm{i}=0}^{1} f^{(\mathrm{i})}(c) g^{(\mathrm{i})}(c) + \int_c^d f''(t) g''(t) \mathrm{d}t, \tag{16}$$

$$\|f\|_{\mathfrak{W}_2^2} = \sqrt{\langle f,f\rangle_{\mathfrak{W}_2^2}}. \tag{17}$$

Theorem 2. *The RK function of* $\mathfrak{W}_2^2[c,d]$ *is the function* $\mathtt{K}_\eta(t)$ *defined by*

$$\mathtt{K}_\eta(t) = \begin{cases} -\frac{1}{3}\,c^3 + \frac{1}{2}\,c^2\eta + c^2 - \eta\,c + \left(\frac{1}{2}\,c^2 - \eta\,c - c + \eta\right)t + \frac{1}{2}\,\eta\,t^2 - \frac{1}{6}\,t^3, & t \le \eta, \\ -\frac{1}{3}\,c^3 + \frac{1}{2}\,c^2\eta - \frac{1}{6}\,\eta^3 + c^2 - \eta\,c + \left(\frac{1}{2}\,c^2 - \eta\,c + \frac{1}{2}\,\eta^2 - c + \eta\right)t, & t > \eta. \end{cases} \tag{18}$$

For the proof, see [43].

Definition 11. *If* $f,\ g \in \mathfrak{W}_2^1[a,b]$, *the inner product and norm are*

$$\langle f,g\rangle_{\mathfrak{W}_2^1} = f(a)\,g(a) + \int_a^b f'(x) g'(x) \mathrm{d}x, \tag{19}$$

and

$$\|f\|_{\mathfrak{W}_2^1} = \sqrt{\langle f, f\rangle_{\mathfrak{W}_2^1}}. \tag{20}$$

Theorem 3. *The RK function of* $\mathfrak{W}_2^1[a,b]$ *is the function* $\mathtt{F}_\zeta(x)$ *defined by*

$$\mathtt{F}_\zeta(x) = \begin{cases} -a+1+t, & t \le \zeta, \\ \zeta - a + 1, & t > \zeta. \end{cases} \tag{21}$$

For the proof, see [43].
Throughout $\Omega = [a,b] \times [c,d]$.

Definition 12. *Define the binary function space* $\mathfrak{W}_2^{(3,2)}(\Omega)$ *by*

$$\mathfrak{W}_2^{(3,2)}(\Omega) = \{u | \ \frac{\partial^3}{\partial x^2 \partial t} u \text{ is CCF on } \Omega,\ \frac{\partial^5}{\partial x^3 \partial t^2} u \in L^2(\Omega), \\ u(x,c) = u(a,t) = u(b,t) = 0\}.$$

Definition 13. *If* $u,\ v \in \mathfrak{W}_2^{(3,2)}(\Omega)$, *the inner product and norm of this space are*

$$\begin{aligned} \langle u, v\rangle_{\mathfrak{W}_2^{(3,2)}} = & \sum_{\mathrm{j}=0}^{1} \int_c^d \left[\frac{\partial^2}{\partial t^2}\frac{\partial^{\mathrm{j}}}{\partial x^{\mathrm{j}}} u(a,t) \frac{\partial^2}{\partial t^2}\frac{\partial^{\mathrm{j}}}{\partial x^{\mathrm{j}}} v(a,t)\right] \mathrm{d}t + \int_c^d \left[\frac{\partial^2}{\partial t^2} u(b,t) \frac{\partial^2}{\partial t^2} v(b,t)\right] \mathrm{d}t \\ & + \sum_{\mathrm{j}=0}^{1} \left\langle \frac{\partial^{\mathrm{j}}}{\partial t^{\mathrm{j}}} u(x,c), \frac{\partial^{\mathrm{j}}}{\partial t^{\mathrm{j}}} v(x,c)\right\rangle_{\mathfrak{W}_2^3} + \int_c^d \int_a^b \frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2} u(x,t) \frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2} v(x,t) \mathrm{d}x\, \mathrm{d}t, \end{aligned} \tag{22}$$

and

$$\|u\|_{\mathfrak{W}_2^{(3,2)}} = \sqrt{\langle u, u\rangle_{\mathfrak{W}_2^{(3,2)}}}. \tag{23}$$

Theorem 4. *The RK function of* $\mathfrak{W}_2^{(3,2)}(\Omega)$ *is the function*

$$\mathtt{Y}_{(\zeta,\eta)}(x,t) = \mathtt{R}_\zeta(x)\, \mathtt{K}_\eta(t). \tag{24}$$

Definition 14. *Define the binary function space* $\mathfrak{W}_2^{(1,1)}(\Omega)$ *by*

$$\mathfrak{W}_2^{(1,1)}(\Omega) = \left\{u(x,t) |\ u(x,t) \text{ is CCF in } \Omega,\ \frac{\partial^2}{\partial x \partial t} u(x,t) \in L^2(\Omega)\right\}.$$

Definition 15. *If* $u,\ v \in \mathfrak{W}_2^{(1,1)}(\Omega)$, *the inner product and norm of this space are*

$$\begin{aligned} \langle u, v\rangle_{\mathfrak{W}_2^{(1,1)}} = & \int_c^d \left[\frac{\partial}{\partial t} u(a,t) \frac{\partial}{\partial t} v(a,t)\right] \mathrm{d}t + \langle u(x,c), v(x,c)\rangle_{\mathfrak{W}_2^1} \\ & + \int_c^d \int_a^b \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t} u(x,t) \frac{\partial}{\partial x}\frac{\partial}{\partial t} v(x,t)\right] \mathrm{d}x\, \mathrm{d}t, \end{aligned} \tag{25}$$

and

$$\|u\|_{\mathfrak{W}_2^{(1,1)}} = \sqrt{\langle u, u\rangle_{\mathfrak{W}_2^{(1,1)}}}. \tag{26}$$

Theorem 5. *The RK function of* $\mathfrak{W}_2^{(1,1)}(\Omega)$ *is the function*

$$\mathtt{T}_{(\zeta,\eta)}(x,t) = \mathtt{F}_\zeta(x)\, \mathtt{F}_\eta(t). \tag{27}$$

3. The Fictitious Time Integration Method (FTIM)

Here, the FTIM is implemented. Consider the following equation:

$$c\, q\, {}^{C}\mathcal{D}_{0^+,t}^{\alpha} u + b(x) u_x + u_{xx} + g(x,t) = 0, \quad (x,t) \in \Omega. \tag{28}$$

Using (10), $0 < \alpha < 1$ and $0 < \beta \leq 1$ for Equation (28), we have

$$\frac{c\,q}{\Gamma(1-\alpha)} \int_0^t \frac{u_\sigma(x,\sigma)}{(t-\sigma)^\alpha} \mathrm{d}\sigma - b(x)u_x + u_{xx} + g(x,t) = 0. \tag{29}$$

To increase the stability of the technique, we propose a fictitious damping coefficient μ in Equation (29) by

$$\frac{\mu\,c\,q}{\Gamma(1-\alpha)} \int_0^t \frac{u_\sigma(x,\sigma)}{(t-\sigma)^\alpha} \mathrm{d}\sigma - \mu b(x)u_x + \mu u_{xx} + \mu g(x,t) = 0. \tag{30}$$

Imposing, in Equation (30), the transformation

$$\omega(x,t,\xi) = (1+\theta)^\lambda u(x,t),\ 0 < \lambda \leq 1, \tag{31}$$

yields

$$\frac{\mu}{(1+\theta)^\lambda}\left(\frac{c\,q}{\Gamma(1-\alpha)} \int_0^t \frac{\omega_\sigma(x,\sigma,\theta)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + b(x)\omega_x(x,t,\theta) + \omega_{xx}(x,t,\theta)\right) + \mu g(x,t) = 0. \tag{32}$$

We consider

$$\frac{\partial \omega}{\partial \theta} = \lambda(1+\theta)^{\lambda-1}u(x,t). \tag{33}$$

Equation (32) will be

$$\frac{\partial \omega}{\partial \theta} = \frac{\mu}{(1+\theta)^\lambda}\left(\frac{c\,q}{\Gamma(1-\alpha)} \int_0^t \frac{\omega_\sigma(x,\sigma,\theta)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + b(x)\omega_x(x,t,\theta) + \omega_{xx}(x,t,\theta)\right) + \mu g(x,t) + \lambda(1+\theta)^{\lambda-1}u. \tag{34}$$

Equation (34) can be converted to a new class of PDE for ω by choosing $u = \omega/(1+\theta)^\lambda$:

$$\frac{\partial \omega}{\partial \theta} = \frac{\mu}{(1+\theta)^\lambda}\left(\frac{c\,q}{\Gamma(1-\alpha)} \int_0^t \frac{\omega_\sigma(x,\sigma,\theta)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + b(x)\omega_x(x,t,\theta) + \omega_{xx}(x,t,\theta)\right) + \mu g(x,t) + \frac{\lambda\omega(x,t,\theta)}{1+\theta}. \tag{35}$$

Using

$$\frac{\partial}{\partial \theta}\left(\frac{\omega}{(1+\theta)^\lambda}\right) = \frac{\omega_\theta}{(1+\theta)^\lambda} - \frac{\lambda\omega}{(1+\theta)^{\lambda+1}}, \tag{36}$$

Implementing $1/(1+\theta)^\lambda$ for Equation (35), one obtains

$$\frac{\partial}{\partial \theta}\left(\frac{\omega}{(1+\theta)^\lambda}\right) = \frac{\mu}{(1+\theta)^\lambda}\left(\frac{c\,q}{\Gamma(1-\alpha)} \int_0^t \frac{\omega_\sigma(x,\sigma,\theta)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + b(x)\omega_x(x,t,\theta) + \omega_{xx}(x,t,\theta)\right) + \mu g(x,t). \tag{37}$$

By $u = \frac{\omega}{(1+\theta)^{\lambda}}$, we obtain

$$u_{\theta} = \frac{\mu}{(1+\theta)^{\lambda}}\left(\frac{c\,q}{\Gamma(1-\alpha)}\int_0^t \frac{u_{\sigma}(x,\sigma,\theta)}{(t-\sigma)^{\alpha}}\mathrm{d}\sigma + b(x)u_x(x,t,\theta) + u_{xx}(x,t,\theta)\right) + \mu g(x,t). \tag{38}$$

Suppose that $u_i^j(\theta) := u(x_i, t_j, \theta)$ as the discrete values of u at a grid point (x_i, t_j), and Equation (38) converts to

$$\frac{d}{d\xi}u_i^j(\theta) = \frac{\mu}{(1+\theta)^{\lambda}}\left(\frac{c\,q}{\Gamma(1-\alpha)}\int_0^{t_j} \frac{u_{\sigma}(x_i,\sigma,\theta)}{(t_j-\sigma)^{\alpha}}\mathrm{d}\sigma + b(x_i)u_x(x_i,t_j,\theta) + u_{xx}(x_i,t_j,\theta)\right) + \mu g(x_i,t_j), \tag{39}$$

where

$$\int_0^{t_j} \frac{u_{\sigma}(x_i,\sigma,\theta)}{(t_j-\sigma)^{\alpha}}\mathrm{d}\sigma \approx \sum_{l=1}^{j-1} \frac{u(x_i,t_{l+1},\theta) - u(x_i,t_l,\theta)}{\Delta t(t_j - t_l)^{\alpha}}, \tag{40}$$

where $t_j = j\Delta t$, $x_i = a + i\Delta x$, and $\Delta t = \frac{T}{n}$.

$\mathbf{u} = (u_1^1, u_1^2, \ldots, u_m^n)^T$, Equation (39) can be abstracted by

$$\mathbf{u}' = \mathbf{Q}(\mathbf{u}, \xi),\ \mathbf{u} \in \mathbb{R}^{m\times n}, \xi \in \mathbb{R},\ \ M = m \times n, \tag{41}$$

where $\mathbf{Q} \in \mathbb{R}^M$ is a vector-valued function of $\mathbf{u}$ and θ and $\mathbf{u}$ is an M-dimensional vector. Now, we implement the group-preserving scheme (GPS) [48] to solve Equation (39) as

$$\mathbf{u}_{s+1} = \mathbf{u}_s + \frac{\left[\cosh\left(\frac{\Delta\theta\|\mathbf{Q}_s\|}{\|\mathbf{u}_s\|}\right) - 1\right]\mathbf{Q}_s.\mathbf{u}_s + \sinh\left(\frac{\Delta\theta\|\mathbf{Q}_s\|}{\|\mathbf{u}_s\|}\right)\|\mathbf{u}_s\|\|\mathbf{Q}_s\|}{\|\mathbf{Q}_s\|^2}\mathbf{Q}_s = \mathbf{u}_s + \Xi_s\mathbf{Q}_s. \tag{42}$$

Now, we employ the GPS by taking the initial value of $u_i^j(0)$ to solve Equation (39) from the initial fictitious time $\theta = 0$ to a selected final fictitious time θ_f. Additionally, the terminating criterion for this method is

$$\sqrt{\sum_{i,j=1}^{m,n}[u_i^j(s+1) - u_i^j(s)]^2} \leq \varepsilon, \tag{43}$$

where ε is a picked convergence criterion. The solution of u will be obtained by

$$u_i^j = \frac{u_i^j(\theta_0)}{(1+\theta_0)^{\lambda}}, \tag{44}$$

where $\theta_0(\leq \theta_f)$ satisfies the above criterion.

4. The Application of RKHSM

4.1. Methodology for RKHSM

In this part, the RKHSM is used to solve Problems (1) and (2), using the following way:

Step 1: Considering the following transformation

$$v = u - P, \tag{45}$$

where $P(x,t) = -(f_2(0) - f_2(t))\frac{x}{b} - (f_1(0) - f_1(t))(1 - \frac{x}{b}) + u_0(x)$ for which $u(0,t) = f_1(t), u(b,t) = f_2(t)$, and $u(x,0) = u_0(x)$.

Consequently, the new form of (1) and (2) is as follows:

$$cq^C\mathcal{D}^{\alpha}_{0^+,t}v - \lambda v_{xx} = \hbar(x,t), \quad 0 \le x \le b,\ 0 \le t \le d, \tag{46}$$

With

$$v(x,t) = 0 \quad on \quad \Gamma, \tag{47}$$

where $\hbar(x,t) = \lambda P_{xx}(x,t) - cq^C\mathcal{D}^{\alpha}_{0^+,t}P(x,t) + g(x,t)$.

Step 2: Defining a linear operator $\mathfrak{T} : \mathfrak{W}_2^{(3,2)}(\Omega) \to \mathfrak{W}_2^{(1,1)}(\Omega)$ as follows

$$\begin{aligned} \mathfrak{T} : \mathfrak{W}_2^{(3,2)}(\Omega) &\to \mathfrak{W}_2^{(1,1)}(\Omega) \\ v &\to cq^C\mathcal{D}^{\alpha}_{0^+,t}v - \lambda v_{xx}. \end{aligned} \tag{48}$$

Lemma 2. *The operator $\mathfrak{T}$ is a bounded linear.*

Proof. We begin by checking directly that $\mathfrak{T}$ is bounded. So, we must show that

$$\|\mathfrak{T}v\|_{\mathfrak{W}_2^{(1,1)}} \le \mathtt{C}\|v\|_{\mathfrak{W}_2^{(3,2)}}, \ \text{with } \mathtt{C} > 0. \tag{49}$$

We have

$$\begin{aligned} \|\mathfrak{T}v(x,t)\|^2_{\mathfrak{W}_2^{(1,1)}} &= \langle \mathfrak{T}v(x,t), \mathfrak{T}v(x,t)\rangle_{\mathfrak{W}_2^{(1,1)}} \\ &= \int_c^d \left[\frac{\partial}{\partial t}\mathfrak{T}v(a,t)\right]^2 dt + \langle \mathfrak{T}v(x,c), \mathfrak{T}v(x,c)\rangle_{\mathfrak{W}_2^1} + \int_c^d \int_a^b \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t}\mathfrak{T}v(x,t)\right]^2 dx\, dt \\ &= [\mathfrak{T}v(a,c)]^2 + \int_a^b \left[\frac{\partial}{\partial x}\mathfrak{T}v(x,c)\right]^2 dx + \int_c^d \left[\frac{\partial}{\partial t}\mathfrak{T}v(a,t)\right]^2 dt \\ &\quad + \int_c^d \int_a^b \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t}\mathfrak{T}v(x,t)\right]^2 dxdt. \end{aligned} \tag{50}$$

In view of reproducing the property,

$$v(x,t) = \left\langle v(\diamond,*), \Upsilon_{(x,t)}(\diamond,*)\right\rangle_{\mathfrak{W}_2^{(3,2)}}. \tag{51}$$

In a similar way, we deduce

$$\frac{\partial^{\mathtt{i}}}{\partial x^{\mathtt{i}}}\frac{\partial^{\mathtt{j}}}{\partial t^{\mathtt{j}}}\mathfrak{T}v(x,t) = \left\langle v(\diamond,*), \frac{\partial^{\mathtt{i}}}{\partial x^{\mathtt{i}}}\frac{\partial^{\mathtt{j}}}{\partial t^{\mathtt{j}}}\mathfrak{T}\Upsilon_{(x,t)}(\diamond,*)\right\rangle_{\mathfrak{W}_2^{(3,2)}}, \quad \mathtt{i},\mathtt{j} \in \{0,1\}. \tag{52}$$

Applying the Schwarz inequality, we discover

$$\left|\frac{\partial^{\mathtt{i}}}{\partial x^{\mathtt{i}}}\frac{\partial^{\mathtt{j}}}{\partial t^{\mathtt{j}}}\mathfrak{T}v(x,t)\right| = \left|\left\langle v(\diamond,*), \frac{\partial^{\mathtt{i}}}{\partial x^{\mathtt{i}}}\frac{\partial^{\mathtt{j}}}{\partial t^{\mathtt{j}}}\mathfrak{T}\Upsilon_{(x,t)}(\diamond,*)\right\rangle_{\mathfrak{W}_2^{(3,2)}}\right| \le \|v\|_{\mathfrak{W}_2^{(3,2)}}\left\|\frac{\partial^{\mathtt{i}}}{\partial x^{\mathtt{i}}}\frac{\partial^{\mathtt{j}}}{\partial t^{\mathtt{j}}}\mathfrak{T}\Upsilon_{(x,t)}(\diamond,*)\right\|_{\mathfrak{W}_2^{(3,2)}}. \tag{53}$$

Since $\Upsilon_{(x,t)}(\diamond,*)$ is continuous, we consequently have

$$\left|\frac{\partial^{\mathtt{i}}}{\partial x^{\mathtt{i}}}\frac{\partial^{\mathtt{j}}}{\partial t^{\mathtt{j}}}\mathfrak{T}v(x,t)\right| \le \mathtt{C}_{\mathtt{i},\mathtt{j}}\|v\|_{\mathfrak{W}_2^{(3,2)}}, \quad \mathtt{i},\mathtt{j} \in \{0,1\}. \tag{54}$$

Hence

$$
\begin{aligned}
\|\mathfrak{T}v(x,t)\|^2_{\mathfrak{W}_2^{(1,1)}} \le & \mathtt{C}^2_{0,0}\|v\|^2_{\mathfrak{W}_2^{(3,2)}} + \int_a^b \mathtt{C}^2_{1,0}\|v\|^2_{\mathfrak{W}_2^{(3,2)}}\mathrm{d}\xi + \int_c^d \mathtt{C}^2_{0,1}\|v\|^2_{\mathfrak{W}_2^{(3,2)}}\mathrm{d}\tau \\
& + \int_c^d\int_a^b \mathtt{C}^2_{1,1}\|v\|^2_{\mathfrak{W}_2^{(3,2)}}\mathrm{d}x\,\mathrm{d}t \\
\le & \Big[\mathtt{C}^2_{0,0} + \mathtt{C}^2_{1,0}\,(b-a) + \mathtt{C}^2_{0,1}(d-c) + \mathtt{C}^2_{1,1}(b-a)\,(d-c)\Big]\|v\|^2_{\mathfrak{W}_2^{(3,2)}}.
\end{aligned} \tag{55}
$$

Therefore,

$$
\|\mathfrak{T}v(x,t)\|^2_{\mathfrak{W}_2^{(1,1)}} \le \mathtt{C}\|v\|^2_{\mathfrak{W}_2^{(3,2)}}, \tag{56}
$$

where $\mathfrak{C} = \mathtt{C}^2_{0,0} + \mathtt{C}^2_{1,0}\,(b-a) + \mathtt{C}^2_{0,1}(d-c) + \mathtt{C}^2_{1,1}(b-a)\,(d-c)$. □

We apply, next, the operator $\mathfrak{T}$ in order to reformulate the problem (46) and (47) to be

$$
\begin{cases} \mathfrak{T}v(x,t) = \hbar(x,t), & (x,t)\in\Omega, \\ v(x,t) = 0 & \text{on} \quad \Gamma, \end{cases} \tag{57}
$$

where $\hbar(x,t) = \lambda P_{xx}(x,t) - cq\,{}^{C}\mathcal{D}^{\alpha}_{0^+,t}P(x,t) + g(x,t)$.

Step 3: Construct the $\{\bar{\Theta}_{\mathrm{i}}\}_{\mathrm{i}=1}^{\infty}$ on $\mathfrak{W}_2^{(3,2)}(\Omega)$, providing this by using the Gram–Schmidt process:

$$
\bar{\Theta}_{\mathrm{i}}(x,t) = \sum_{\mathrm{k}=1}^{\mathrm{i}} \aleph_{\mathrm{ik}}\Theta_{\mathrm{k}}(x,t), \quad 0 < \aleph_{\mathrm{ii}},\ \mathrm{i} = 1,2,\ldots, \tag{58}
$$

where

- $\Theta_{\mathrm{i}}(x,t) = \mathfrak{T}^*\varrho_{\mathrm{i}}(x,t)$, in which $\mathfrak{T}^*$ denotes the adjoint of $\mathfrak{T}$ and $\varrho_{\mathrm{i}}(x,t) = \mathtt{T}_{(x_{\mathrm{i}},t_{\mathrm{i}})}(x,t)$ where $\mathtt{T}_{(x_{\mathrm{i}},t_{\mathrm{i}})}(x,t)$ is given by (27).
- The countable set $\{(x_{\mathrm{i}},t_{\mathrm{i}})\}_{\mathrm{i}=1}^{\infty}$ is dense in Ω.
- $\{\Theta_{\mathrm{i}}\}_{\mathrm{i}=1}^{\infty}$ is a function system in $\mathfrak{W}_2^{(3,2)}(\Omega)$ and the following shows the way that we can construct it:

$$
\begin{aligned}
\Theta_{\mathrm{i}}(x,t) = \mathfrak{T}^*\varrho_{\mathrm{i}}(x,t) &= \Big\langle \mathfrak{T}^*\varrho_{\mathrm{i}}(\zeta,\eta), \Upsilon_{(x,t)}(\zeta,\eta)\Big\rangle_{\mathfrak{W}_2^{(3,2)}} \\
&= \Big\langle \varrho_{\mathrm{i}}(\zeta,\eta), \mathfrak{T}_{(\zeta,\eta)}\Upsilon_{(x,t)}(\zeta,\eta)\Big\rangle_{\mathfrak{W}_2^{(1,1)}} \\
&= \Big\langle \mathtt{T}_{(\zeta_{\mathrm{i}},\eta_{\mathrm{i}})}(\zeta,\eta), \mathfrak{T}_{(\zeta,\eta)}\Upsilon_{(x,t)}(\zeta,\eta)\Big\rangle_{\mathfrak{W}_2^{(1,1)}} \\
&= \mathfrak{T}_{(\zeta,\eta)}\Upsilon_{(x,t)}(\zeta,\eta)|_{(\zeta,\eta)=(x_{\mathrm{i}},t_{\mathrm{i}})} \\
&= \mathfrak{T}_{(\zeta,\eta)}\Upsilon_{(\zeta,\eta)}(x,t)|_{(\zeta,\eta)=(x_{\mathrm{i}},t_{\mathrm{i}})} \\
&= \Big\{cq\,{}^{C}\mathcal{D}^{\alpha}_{0^+,\eta}\Upsilon_{(\zeta,\eta)}(x,t) - \lambda\partial^2_{\zeta^2}\Upsilon_{(\zeta,\eta)}(x,t)\Big\}\Big|_{(\zeta,\eta)=(x_{\mathrm{i}},t_{\mathrm{i}})}.
\end{aligned} \tag{59}
$$

- $\aleph_{\mathrm{ik}}$ is the orthogonalization coefficients which are defined by

$$
\aleph_{\mathrm{ij}} = \begin{cases} \frac{1}{\|\Theta_1\|}, & \text{for} \quad \mathrm{i}=\mathrm{j}=1, \\ \frac{1}{\varsigma_{\mathrm{i}}}, & \text{for} \quad \mathrm{i}=\mathrm{j}\neq 1, \\ -\frac{1}{\varsigma_{\mathrm{i}}}\sum_{\mathrm{k}=\mathrm{j}}^{\mathrm{i}-1} C_{\mathrm{ik}}\aleph_{\mathrm{kj}}, & \text{for} \quad \mathrm{i}>\mathrm{j}, \end{cases} \tag{60}
$$

where $\varsigma_{\mathrm{i}} = \sqrt{\|\Theta_{\mathrm{i}}\|^2 - \sum_{\mathrm{k}=1}^{\mathrm{i}-1} C^2_{\mathrm{ik}}}$, $C_{\mathrm{ik}} = \langle\Theta_{\mathrm{i}},\bar{\Theta}_{\mathrm{k}}\rangle_{\mathfrak{W}_2^{(3,2)}}$.

Theorem 6. *Assume $\{(x_{\mathrm{i}},t_{\mathrm{i}})\}_{\mathrm{i}=1}^{\infty}$ is dense; therefore, $\{\Theta_{\mathrm{i}}\}_{\mathrm{i}=1}^{\infty}$ is the complete system of $\mathfrak{W}_2^{(3,2)}(\Omega)$.*

Proof. Clearly, $\Theta_i(x,t) \in \mathfrak{W}_2^{(3,2)}(\Omega)$. Thus, for $v(x,t) \in \mathfrak{W}_2^{(3,2)}(\Omega)$,

$$\langle v(x,t), \Theta_i(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} = 0, \quad i = 1, 2, \ldots. \tag{61}$$

as

$$\langle v, \Theta_i \rangle_{\mathfrak{W}_2^{(3,2)}} = \langle v(x,t), \mathfrak{T}^* \varrho_i(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} = \langle \mathfrak{T} v(x,t), \varrho_i(x,t) \rangle_{\mathfrak{W}_2^{(1,1)}} = \mathfrak{T} v(x_i, t_i) = 0, \tag{62}$$

and due to the density of $\{(x_i, t_i)\}_{i=1}^{\infty}$ in Ω :

$$\mathfrak{T} v(x,t) = 0. \tag{63}$$

by applying $\mathfrak{T}^{-1}$,

$$v(x,t) = 0. \tag{64}$$

□

Step 4: The solution's representation is given by

Theorem 7. *Assume $\{(x_i, t_i)\}_{i=1}^{\infty}$ is a dense set on Ω and (57) has a unique solution on $\mathfrak{W}_2^{(3,2)}(\Omega)$, then*

$$v(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \hbar(x_k, t_k) \bar{\Theta}_i(x,t) \tag{65}$$

is the solution of (57), and the solution of (1) and (2) is

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \hbar(x_k, t_k) \bar{\Theta}_i(x,t) - P(x,t). \tag{66}$$

Proof. We know that the basis $\{\bar{\Theta}_i(x,t)\}_{i=1}^{\infty}$ is a complete orthonormal system in the space $\mathfrak{W}_2^{(3,2)}(\Omega)$, then

$$\begin{aligned} v(x,t) &= \textstyle\sum_{i=1}^{\infty} \langle v(x,t), \bar{\Theta}_i(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} \bar{\Theta}_i(x,t) \\ &= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \langle v(x,t), \Theta_k(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} \bar{\Theta}_i(x,t) \\ &= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \langle v(x,t), \mathfrak{T}^* \rho_k(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} \bar{\Theta}_i(x,t) \\ &= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \langle \mathfrak{T} v(x,t), \varrho_k(x,t) \rangle_{\mathfrak{W}_2^{(1,1)}} \bar{\Theta}_i(x,t) \\ &= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \left\langle \mathfrak{T} v(x,t), \mathrm{T}_{(x_k,t_k)}(x,t) \right\rangle_{\mathfrak{W}_2^{(1,1)}} \bar{\Theta}_i(x,t) \\ &= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \hbar(x_k, t_k) \bar{\Theta}_i(\xi, \tau), \end{aligned} \tag{67}$$

with $\hbar(x_k, t_k) = \mathfrak{T} v(x_k, t_k)$.

On the other hand, (66) follows directly from $u = v - P(x,t)$. □

Remarks

1. We have

$$v_n(x,t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \aleph_{ik} \hbar(x_k, t_k) \bar{\Theta}_i(x,t). \tag{68}$$

2. $\mathfrak{W}_2^{(3,2)}(\Omega)$ is a Hilbert space. Then, we deduce

$$\sum_{i=1}^{\infty} \sum_{k=1}^{i} \aleph_{ik} \hbar(x_k, t_k) \bar{\Theta}_i(x,t) < \infty. \tag{69}$$

4.2. Convergence Analysis

The approximate solution of (65) takes the form

$$v_n(x,t) = \sum_{i=1}^{n} \Lambda_i \bar{\Theta}_i(x,t), \tag{70}$$

where

$$\Lambda_i = \sum_{k=1}^{i} \aleph_{ik} \hbar(x_k, t_k). \tag{71}$$

Here, by letting $(x_1, t_1) = (a, c)$, it is possible to know the values of $v(x_1, t_1)$ from the initial and boundary conditions. In addition, $v_0(x_1, t_1) = v(x_1, t_1)$.

Theorem 8. *Suppose that $\|v_n\|_{\mathfrak{W}_2^{(3,2)}}$ is bounded in (57), $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense on Ω, and the solution of (70) is unique. Then, v_n converges to v and*

$$v_n(x,t) = \sum_{i=1}^{n} \Lambda_i \bar{\Theta}_i(x,t). \tag{72}$$

Proof. (i) From (70), we know

$$v_{n+1} = v_n + \Lambda_{n+1} \bar{\Theta}_{n+1}. \tag{73}$$

then,

$$\begin{aligned}
\|v_{n+1}\|^2_{\mathfrak{W}_2^{(3,2)}} &= \|v_n + \Lambda_{n+1}\bar{\Theta}_{n+1}\|^2_{\mathfrak{W}_2^{(3,2)}} \\
&= \langle v_n + \Lambda_{n+1}\bar{\Theta}_{n+1}, v_n + \Lambda_{n+1}\bar{\Theta}_{n+1} \rangle_{\mathfrak{W}_2^{(3,2)}} \\
&= \langle v_n, v_n \rangle_{\mathfrak{W}_2^{(3,2)}} + \langle v_n, \Lambda_{n+1}\bar{\Theta}_{n+1} \rangle_{\mathfrak{W}_2^{(3,2)}} + \langle \Lambda_{n+1}\bar{\Theta}_{n+1}, v_n \rangle_{\mathfrak{W}_2^{(3,2)}} + \langle \Lambda_{n+1}\bar{\Theta}_{n+1}, \Lambda_{n+1}\bar{\Theta}_{n+1} \rangle_{\mathfrak{W}_2^{(3,2)}} \\
&= \|v_n\|^2_{\mathfrak{W}_2^{(3,2)}} + \langle v_n, \Lambda_{n+1}\bar{\Theta}_{n+1} \rangle_{\mathfrak{W}_2^{(3,2)}} + \langle \Lambda_{n+1}\bar{\Theta}_{n+1}, v_n \rangle_{\mathfrak{W}_2^{(3,2)}} +, \Lambda^2_{n+1} \langle \bar{\Theta}_{n+1}, \bar{\Theta}_{n+1} \rangle_{\mathfrak{W}_2^{(3,2)}}.
\end{aligned} \tag{74}$$

Now, the orthogonality of $\{\bar{\Theta}_i(x,t)\}_{i=1}^{\infty}$ implies

$$\begin{aligned}
\|v_{n+1}\|^2_{\mathfrak{W}_2^{(3,2)}} &= \|v_n\|^2_{\mathfrak{W}_2^{(3,2)}} + \Lambda^2_{n+1} \\
&= \|v_{n-1}\|^2_{\mathfrak{W}_2^{(3,2)}} + \Lambda^2_n + \Lambda^2_{n+1} \\
&= \|v_{n-2}\|^2_{\mathfrak{W}_2^{(3,2)}} + \Lambda^2_{n-1} + \Lambda^2_n + \Lambda^2_{n+1} \\
&\vdots \\
&= \|v_1\|^2_{\mathfrak{W}_2^{(3,2)}} + \Lambda^2_2 + \Lambda^2_3 + \cdots + \Lambda^2_n + \Lambda^2_{n+1} \\
&= \|v_0\|^2_{\mathfrak{W}_2^{(3,2)}} + \textstyle\sum_{i=1}^{n+1} \Lambda^2_i,
\end{aligned} \tag{75}$$

Hence,

$$\|v_n\|_{\mathfrak{W}_2^{(3,2)}} \leq \|v_{n+1}\|_{\mathfrak{W}_2^{(3,2)}}. \tag{76}$$

The convergence of $\|v_n\|_{\mathfrak{W}_2^{(3,2)}}$ follows directly from the boundedness of $\|v_n\|_{\mathfrak{W}_2^{(3,2)}}$. So, there exists $\digamma$ such that

$$\sum_{i=1}^{\infty} \Lambda^2_i = \digamma, \tag{77}$$

where the constant $\digamma$ is positive.

As a result,

$$\left\{\Lambda^2_i\right\}_{i=1}^{\infty} \in \ell^2.$$

As $(v_m - v_{m-1}) \perp \cdots \perp (v_{n+1} - v_n)$ and for $n < m$, we write

$$\begin{aligned} \|v_m - v_n\|^2_{\mathfrak{W}_2^{(3,2)}} &= \|v_m - v_{m-1} + v_{m-1} - \cdots + v_{n+1} - v_n\|^2_{\mathfrak{W}_2^{(3,2)}} \\ &= \|v_m - v_{m-1}\|^2_{\mathfrak{W}_2^{(3,2)}} + \|v_{m-1} - v_{m-2}\|^2_{\mathfrak{W}_2^{(3,2)}} + \cdots + \|v_{n+1} - v_n\|^2_{\mathfrak{W}_2^{(3,2)}}. \end{aligned} \tag{78}$$

Furthermore,

$$\|v_m - v_{m-1}\|^2_{\mathfrak{W}_2^{(3,2)}} = \Lambda_m^2. \tag{79}$$

Thus,

$$\|v_m - v_n\|^2_{\mathfrak{W}_2^{(3,2)}} = \sum_{p=n+1}^{m} \Lambda_p^2 \to 0, \quad \text{as} \quad n,\ m \to \infty. \tag{80}$$

The completeness of $\mathfrak{W}_2^{(3,2)}(\Omega)$ allows us to deduce that $v_n \to \tilde{v}$ as $n \to \infty$.

(ii) To prove this, let us take the limits in (70)

$$\tilde{v}(x,t) = \sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \bar{\Theta}_{\mathrm{i}}(x,t). \tag{81}$$

We apply the linear operator $\mathfrak{T}$ to (81)

$$\mathfrak{T}\tilde{v}(x,t) = \sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \mathfrak{T}\bar{\Theta}_{\mathrm{i}}(x,t), \tag{82}$$

Hence,

$$\begin{aligned} \mathfrak{T}\tilde{v}(x_p,t_p) &= \textstyle\sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \langle \mathfrak{T}\bar{\Theta}_{\mathrm{i}}(x,t), \varrho_p(x,t) \rangle_{\mathfrak{W}_2^{(1,1)}} \\ &= \textstyle\sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \langle \bar{\Theta}_{\mathrm{i}}(x,t), \mathfrak{T}^*\varrho_p(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} \\ &= \textstyle\sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \langle \bar{\Theta}_{\mathrm{i}}(x,t), \Theta_p(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}}. \end{aligned} \tag{83}$$

Thus,

$$\aleph_{\mathrm{j}p} \mathfrak{T}\tilde{v}(x_p,t_p) = \aleph_{\mathrm{j}p} \left[\sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \langle \bar{\Theta}_{\mathrm{i}}(x,t), \Theta_p(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} \right], \tag{84}$$

and we take the summation $\sum_{p=1}^{\mathrm{j}}$ to deduce

$$\begin{aligned} \textstyle\sum_{p=1}^{\mathrm{j}} \aleph_{\mathrm{j}p} \mathfrak{T}\tilde{v}(x_p,t_p) = & \textstyle\sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \left\langle \bar{\Theta}_{\mathrm{i}}(x,t), \sum_{p=1}^{\mathrm{j}} \aleph_{\mathrm{j}p} \Theta_p(x,t) \right\rangle_{\mathfrak{W}_2^{(3,2)}} \\ & = \textstyle\sum_{\mathrm{i}=1}^{\infty} \Lambda_{\mathrm{i}} \langle \bar{\Theta}_{\mathrm{i}}(x,t), \bar{\Theta}_{\mathrm{j}}(x,t) \rangle_{\mathfrak{W}_2^{(3,2)}} \\ & = \Lambda_{\mathrm{j}}. \end{aligned} \tag{85}$$

Observe then from (71) that

$$\mathfrak{T}\tilde{v}(x_p,t_p) = \hbar(x_p,t_p). \tag{86}$$

For all $(\zeta,\eta) \in \Omega$, it exists $\{(x_{q\mathrm{j}},t_{q\mathrm{j}})\}_{\mathrm{j}=1}^{\infty}$ such that $(x_{q\mathrm{j}},t_{q\mathrm{j}}) \to (\zeta,\eta)$, as $\mathrm{j} \to \infty$. It is well-known to us that

$$\mathfrak{T}\tilde{v}(x_{q\mathrm{j}},t_{q\mathrm{j}}) = \hbar(x_{q\mathrm{j}},t_{q\mathrm{j}}). \tag{87}$$

Using the continuity of $\hbar$ and letting $\mathrm{j} \to \infty$ allows us to

$$\mathfrak{T}\tilde{v}(\zeta,\eta) = \hbar(\zeta,\eta). \tag{88}$$

□

5. Numerical Experiments

We apply the proposed methods to solve some problems. In **Example 1**, we use RKHSM to solve the considered equation, and the GPS is considered for **Example 2** to deal with the fractional convection–diffusion equation. Now, how to apply the RKHSM can be summarized in the following procedure:
Step 1: Setting $n = p \times q$;
Step 2: Setting $\Theta_i(x_i, t_i) = \mathfrak{T}_{(\zeta,\eta)}\Upsilon_{(x,t)}(\zeta,\eta)|_{(\zeta,\eta)=(x_i,t_i)}$;
Step 3: Calculating the orthogonalization coefficients $\aleph_{ij}$ using (60);
Step 4: Setting $\bar{\Theta}_i(x_i, t_i) = \sum_{k=1}^{i} \aleph_{ik}\Theta_k(x_i, t_i), \quad i = 1, 2, \ldots, n$;
Step 5: Choosing an initial guess $u_0(x_1, t_1)$;
Step 6: Setting $i = 1$;
Step 7: Setting $\Lambda_i = \sum_{k=1}^{i} \aleph_{ik}\hbar(x_k, t_k)$;
Step 8: $u_i(x_i, t_i) = \sum_{\ell=1}^{i} \Lambda_\ell \bar{\Theta}_\ell(x_\ell, t_\ell)$;
Step 9: If $i < n$, set $i = i + 1$. Go to step 7. Else stop, Where $x_i = \frac{i}{p}$, $i = 1, 2, \ldots, p$ and $t_j = \frac{j}{q}$, $j = 1, 2, \ldots, q$. n is the grid points' number.

Example 1. *Considering the following problem with the fractional order $\alpha = 0.5$:*

$$^{C}\mathcal{D}^{\alpha}_{0^+,t}u = u_{xx} + g(x,t), \quad (x,t) \in [0,1]^2, \tag{89}$$

where

$$u(x,t) = exp(x)x^2(1-x)^2t^\alpha,$$

and

$$g(x,t) = e^x\left(\left(x\left(x^3 + 6x^2 + x - 8\right) + 2\right)(-t^\alpha) - \frac{\pi(x-1)^2x^2\csc(\pi\alpha)}{\Gamma(-\alpha)}\right).$$

In this example, the RKHSM is tested with the standard grid points $x_i = \frac{i}{p}$, $i = 1, \ldots, p$ and $t_j = \frac{j}{q}$, $j = 1, \ldots, q$ with $p \times q = n = 100$. The comparison of (89) with (1) and (2) shows $\Omega = [0,1] \times [0,1]$, $cq = 1$, $\lambda = 1$, and $u(0,t) = u(1,t) = u(x,0) = 0$. Therefore, as we see in Section 4, the approximate solution of (89) takes the form

$$v_n(x,t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \aleph_{ik}g(x_k, t_k)\bar{\Theta}_i(x,t). \tag{90}$$

In Table 1, a numerical comparison between the obtained results via RKHSM with the exact solution, for α = 0.9, 0.8, 0.75, is given. These results clearly show that the approximate solution (using the RKHSM) converges to the exact solution. The results are in good agreement with each other, and this confirms the effectiveness of the RKHSM to solve this type of equation.

Table 1. Absolute errors of the RKHSM solution for Example 1.

	RKHSM-Absolute Error		
(x, t)	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.75$
(0.1, 0.1)	1.3016×10^{-3}	1.3446×10^{-3}	1.9199×10^{-3}
(0.3, 0.3)	1.0876×10^{-4}	3.8615×10^{-4}	8.1880×10^{-4}
(0.5, 0.5)	5.9252×10^{-5}	8.7214×10^{-4}	1.3792×10^{-3}
(0.7, 0.7)	5.9929×10^{-4}	1.0365×10^{-3}	1.6629×10^{-3}
(0.9, 0.9)	9.5379×10^{-4}	1.6982×10^{-4}	4.2782×10^{-4}

Example 2. *Consider the problem (28) with $b(x) = x$ and fractional order α = 0.01 where*

$$u(x,t) = x^2t^{2\alpha}\frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)},$$

and

$$g(x,t) = 2t^2 + 2x^2 + 2.$$

We solve this problem by using the GPS with taking $\Delta\theta = 1e - 10$, $m = n = 20$, and $\lambda = 1$. Additionally, an initial guess of $u_i^j(0) = 1e - 1$ is taken. Figure 1 shows the numerical solution, exact solution, absolute error, and absolute error's contour. Indeed, we present in Table 2 the values of absolute errors between the numerical solution (using the GPS) and the exact solution for Example 2. From this table's results, it is clear that the error estimate confirms the accuracy of this new method, and Figure 1 shows that both graphs are very similar in their behavior.

Table 2. Comparison between the exact solution and FITM solution for Example 2.

(x, t)	**Approximate**	**Exact**	**Absolute Error**
(0.1, 0.1)	0.0215	0.0213	1.5308×10^{-4}
(0.2, 0.2)	0.0893	0.0864	2.9000×10^{-3}
(0.3, 0.3)	0.2016	0.1960	5.6000×10^{-3}
(0.4, 0.4)	0.3133	0.3076	5.7000×10^{-3}
(0.5, 0.5)	0.4958	0.4958	3.9629×10^{-8}

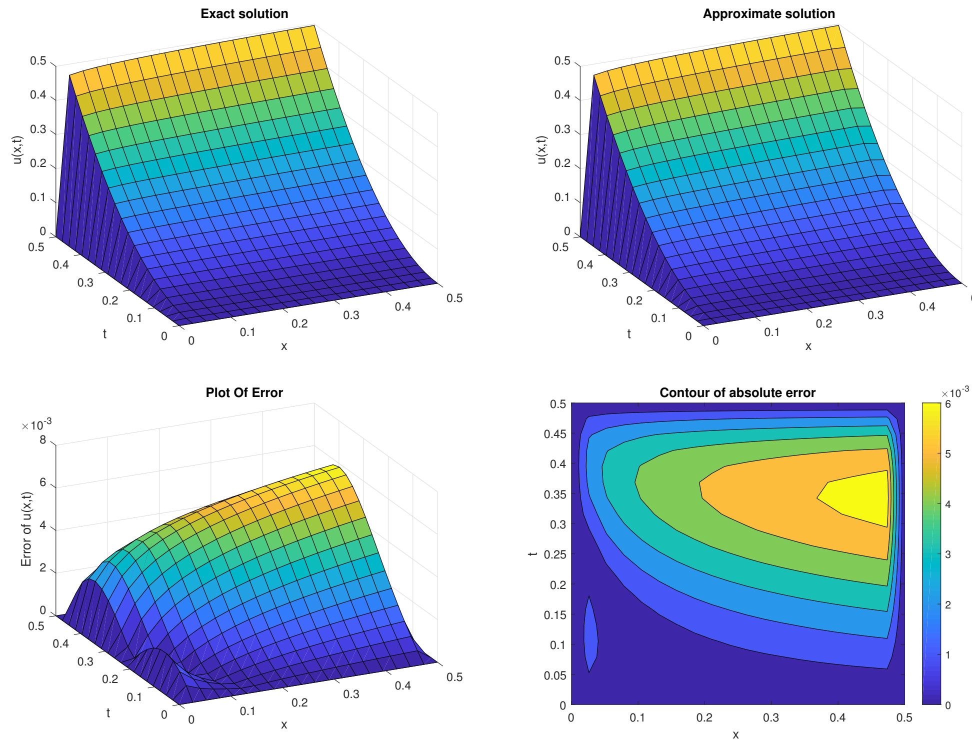

Figure 1. Solution under applying GPS for the fractional convection-diffusion equation in Example 2.

6. Conclusions

In the current work, we successfully implemented two numerical schemes to gain approximate solutions to the considered problems. One is the FITM, which converted the original problem into a new one with one extra dimension. After that, we used GPS to solve the problem. The other is the RKHSM, which was used for the mentioned problem. The main steps for applying this method are defining an appropriate bounded linear operator and constructing an orthonormal function system of the appropriate RKHS. Indeed, the both methods are shown to have good convergence. Two examples were employed to show the capacity and reliability of the FITM and RKHSM. Our obtained results are compared with exact results and they are found to be in good agreement with each other. From the numerical results, it can be observed the suitability, ease, and effectiveness of the proposed approaches for solving such types of fractional partial differential equation. This research opens the way for the use of the two proposed methods to study the mentioned problem for various new fractional derivatives. As part of our purpose, we plan to apply the FTIM and RKHSM to multidimensional fractional partial differential equations, which will be new in the literature.

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References

1. Liouville, J. Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions. *JEC Polytech.* **1832**, *13*, 1–69.
2. Riemann, G.F.B. *Versuch Einer Allgemeinen Auffassung der Integration und Differentiation*; Gesammelte Mathematische Werke: Leipzig, Germany, 1896.
3. Caputo, M. *Elasticita e Dissipazione*; Zanichelli: Bologna, Italy, 1969.
4. Miller, K.S.; Ross, B. *An Introduction to Fractional Calculus and Fractional Diffrential Equations*; Wiley: New York, NY, USA, 1993.
5. Podlubny, I. *Fractional Diffrential Equations*; Academic Press: New York, NY, USA, 1999.
6. Singh, H.; Kumar, D.; Baleanu, D. *Methods of Mathematical Modelling: Fractional Diffrential Equations*; CRC Press Taylor & Francis Group: New York, NY, USA, 2019.
7. Baleanu D.; Guvenc, Z.B.; Tenreiro Machado, J.A. *New Trends in Nanotechnology and Fractional Calculus Applications*; Springer: Dordrecht, The Netherlands, 2010.
8. Atangana, A.; Alkahtani, B.T. Analysis of non-homogenous heat model with new trend of derivative with fractional order. *Chaos Solit. Fract.* **2016**, *89*, 566–571.
9. Kumar, D.; Singh, J.; Tanwar, K.; Baleanu, D. A new fractional exothermic reactions model having constant heat source in porous media with power, exponential and Mittag-Leffler laws. *Int. Heat Mass Transf.* **2019**, *138*, 1222–1227. [CrossRef]
10. Baleanu, D.; Wu, G.C.; Zeng, S.D. Chaos analysis and asymptotic stability of generalized Caputo fractional diffrential equations. *Chaos Solit. Fract.* **2017**, *102*, 99–105. [CrossRef]
11. Veeresha, P.; Prakasha, D.G.; Baskonus, H.M. New numerical surfaces to the mathematical model of cancer chemotherapy effct in Caputo fractional derivatives. *Chaos* **2019**, *29*, 013119. [CrossRef]
12. Gao, W.; Veeresha, P.; Baskonus, H.M.; Prakasha, D.G.; Kumar, P. A new study of unreported cases of 2019-nCOV epidemic outbreaks. *Chaos Solit. Fract.* **2020**, *138*, 109929. [CrossRef]
13. Kumar, D.; Singh, J.; Baleanu, D. On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law. *Math. Meth. Appl. Sci.* **2020**, *43*, 443–457. [CrossRef]
14. Veeresha, P.; Prakasha, D.G.; Baleanu, D. Analysis of fractional Swift-Hohenberg equation using a novel computational technique. *Math. Meth. Appl. Sci.* **2020**, *43*, 1970–1987. [CrossRef]
15. Caputo, M.; Fabrizio, M. A new defiition of fractional derivative without singular kernel. *Prog. Fract. Diffr. Appl.* **2015**, *1*, 73–85.
16. Cao, Q.-H.; Dai, C.-Q. Symmetric and anti-symmetric solitons of the fractional second- and third-order nonlinear Schrödinger equation. *Chin. Phys. Lett.* **2021**, *38*, 090501. [CrossRef]
17. Chen, C.; Jiang Y.-L.; Wang X.-T. Lie symmetry analysis of the time fractional generalized KdV equations with variable coefficients. *Symmetry* **2019**, *11*, 1281. [CrossRef]
18. Wang, Q. Numerical solution for fractional KdV-Burgers equation by Adomian decomposition method *Appl. Math. Comput.* **2006**, *182*, 1048–1055.
19. Nagy, A.M. Numerical solution of time fractional nonlinear Klein-Gordon equation using Sinc Chebyshev collocation method. *Appl. Math. Comput.* **2017**, *310*, 139–148. [CrossRef]
20. Parto-Haghighi, M.; Akinlar, M.A.; Chu, Y. New numerical solutions of fractional-order Korteweg-de Vries equation. *Results Phys.* **2020**, *19*, 103326.

21. Inc, M.; Partohaghighi, M.; Akinlar, M.A.; Agarwal, P.; Chu, Y.-M. New solutions of fractional-order Burger-Huxley equation. *Results Phys.* **2020**, *18*, 103290. [CrossRef]
22. Hashemi, M.S.; Baleanu, D.; Parto-Haghighi, M. A lie group approach to solve the fractional Poisson equation. *Rom. J. Phys.* **2015**, *60*, 1289–1297.
23. Akgül, A.; Ahmad, H. Reproducing kernel method for Fangzhu's oscillator for water collection from air. *Math. Meth. Appl. Sci.* **2020**, 1–10. [CrossRef]
24. Ahmad, H.; Akgül, A.; Khan, T.A.; Stanimirović, P.S.; Chu, Y.-M. New perspective on the conventional solutions of the nonlinear time-fractional partial differential equations. *Complexity* **2020**, *2020*, 8829017. [CrossRef]
25. Inc, M.; Khan, M.N.; Ahmad, I.; Yao, S.W.; Ahmad, H.; Thounthong, P. Analysing time-fractional exotic options via efficient local meshless method. *Results Phys.* **2020**, *19*, 103385. [CrossRef]
26. Ahmad, I.; Khan, M.N.; Inc, M.; Ahmad, H.; Nisar, K.S. Numerical simulation of simulate an anomalous solute transport model via local meshless method. *Alex. Eng. J.* **2020**, *59*, 2827–2838. [CrossRef]
27. Hussain, M.; Haq, S. Weighted meshless spectral method for the solutions of multi-term time fractional advection-diffusion problems arising in heat and mass transfer. *Int. J. Heat Mass Transf.* **2019**, *129*, 1305–1316. [CrossRef]
28. Ahmad, I.; Ahmad, H.; Inc, M.; Yao, S.-W.; Almohsen, B. Application of local meshless method for the solution of two term time fractional-order multi-dimensional PDE arising in heat and mass transfer. *Therm. Sci.* **2020**, *24*, 95–105. [CrossRef]
29. Pitolli, F.; Sorgentone, C.; Pellegrino, E. Approximation of the Riesz-Caputo derivative by cubic splines. *Algorithms* **2022**, *15*, 69. Available online: https://www.mdpi.com/1999-4893/15/2/69 (accessed on 15 February 2022). [CrossRef]
30. Izadi, M.; Srivastava, H.M. A discretization approach for the nonlinear fractional logistic equation. *Entropy* **2020**, *22*, 1328. Available online: https://www.mdpi.com/1099-4300/22/11/1328 (accessed on 16 November 2020). [CrossRef]
31. Shymanskyi, V.; Sokolovskyy, Y. Finite element calculation of the linear elasticity problem for biomaterials with fractal structure. *Open Bioinform. J.* **2021**, *14*, 114–122. [CrossRef]
32. Priya, G.S.; Prakash, P.; Nieto, J.J.; Kayar, Z. Higher-order numerical scheme for the fractional heat equation with Dirichlet and Neumann boundary conditions. *Numer. Heat Transf.* **2013**, *63*, 540–559. [CrossRef]
33. Mehmet, Y.; Necati, O. Numerical inverse Laplace homotopy technique for fractional heat equations. *Therm. Sci.* **2018**, *22*, 185–194.
34. Chen, Y.-W. A highly accurate backward-forward algorithm for multi-dimensional backward heat conduction problems in fictitious time domains. *Int. J. Heat Mass Transf.* **2018**, *120*, 499–514. [CrossRef]
35. Partohaghighi, M.; Akinlar, M.A.; Weber, G.W. New solutions of hyperbolic telegraph equation. *J. Dyn. Games* **2021**, *8*, 129–138.
36. Partohaghighi, M.; Bayram, M.; Baleanu, D. On numerical solution of the time fractional advection-diffusion equation involving Atangana-Baleanu-Caputo derivative. *Open Phys.* **2019**, *17*, 816–822. [CrossRef]
37. Chen, Y.-W.; Liu, C.-S.; Chang, Y.-S.; Chang, J.-R. Constraint-type fictitious time integration method for solving non-linear multi-dimensional elliptic partial differential equations. *J. Mar. Sci. Technol.* **2020**, *28*, 168–178.
38. Chen, Y.-W. High order implicit and explicit Lie-group schemes for solving backward heat conduction problems. *Int. J. Heat Mass Transf.* **2016**, *101*, 1016–1029. [CrossRef]
39. Liu, C.S. Cone of non-linear dynamical system and group preserving schemes. *Int. J. Non Linear Mech.* **2021**, *36*, 1047–1068. [CrossRef]
40. Akgül, E.K. A novel method for the space and time fractional Bloch-torrey equations. *Therm. Sci.* **2018**, *22*, S253–S258. [CrossRef]
41. Akgül, A. A novel method for a fractional derivative with non-local and non-singular kernel. *Chaos Solit. Fract.* **2018**, *114*, 478–482. [CrossRef]
42. Hemati, F.; Ghasemi, M.; Khoshsiar Ghaziani, R. Numerical solution of the multiterm time-fractional diffusion equation based on reproducing kernel theory. *Numer. Methods Partial Differ. Equ.* **2021**, *37*, 44–68. [CrossRef]
43. Attia, N.; Akgül, A.; Seba, D.; Nour, A. Reproducing kernel Hilbert space method for the numerical solutions of fractional cancer tumor models. *Math. Meth. Appl. Sci.* **2020**, 1–22. [CrossRef]
44. Abu Arqub, O.; Shawagfeh, N. Solving optimal control problems of Fredholm constraint optimality via the reproducing kernel Hilbert space method with error estimates and convergence analysis. *Math. Methods Appl. Sci.* **2021**, *44*, 7915–7932. [CrossRef]
45. Abu Arqub, O.; Shawagfeh, N. Application of reproducing kernel algorithm for solving Dirichlet time-fractional diffusion-Gordon types equations in porous media. *J. Porous Media* **2019**, *22*, 411–434. [CrossRef]
46. Djennadi, S.; Shawagfeh, N.; Abu Arqub, O. A numerical algorithm in reproducing kernel-based approach for solving the inverse source problem of the time-space fractional diffusion equation. *Partial Differ. Equ. Appl. Math.* **2021**, *4*, 100164. [CrossRef]
47. Abu Arqub, O. Reproducing kernel algorithm for the analytical-numerical solutions of nonlinear systems of singular periodic boundary value problems. *Math. Prob. Eng.* **2015**, *2015*, 518406.. [CrossRef]
48. Cui, M.; Lin, Y. *Nonlinear Numerical Analysis in the Reproducing Kernel Space*; Nova Science Publishers Inc.: New York, NY, USA, 2009.