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CW complexes and Cellular Homology

Final Degree Dissertation
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Ander Arriola Corpion

Supervisors:
Oihana Garayalde Ocaña
Jon González Sánchez

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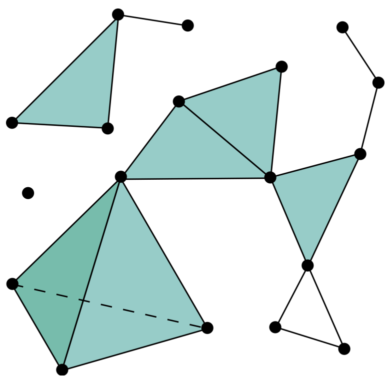
Introduction

A fundamental problem in topology is to determine if two spaces are homeomorphic or not. Generally, the way to prove that two spaces are homeomorphic is to give a homeomorphism between them. In most of the cases, though, this is a very difficult task. Thus, we could try a different approach and try to determine which spaces are not homeomorphic. At first sight, one can think that this is an even more difficult task, because, by definition, we would have to prove that no homeomorphism exists between the two spaces. However, there are some properties that are invariant under homeomorphisms, called **topological properties**, and so if we prove that a space X satisfies some topological property that the space Y does not, we can conclude that they are not homeomorphic.

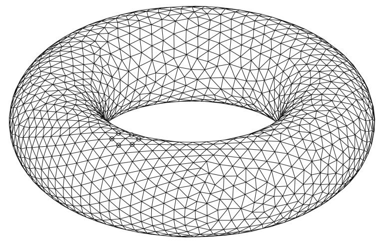
One such topological property is compactness. The circle $\mathbb{S}^1$ is not homeomorphic to $\mathbb{R}$ because $\mathbb{S}^1$ is compact and $\mathbb{R}$ is not. Another such topological property is connectedness. Indeed, $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$ because removing a point from $\mathbb{R}$ disconnects the space, while $\mathbb{R}^2$ without a point is still a connected space. These two properties are enough to distinguish some basic topological spaces, but a more complicated machinery is needed to distinguish, for example, most of the compact connected surfaces.

In the course *Ampliación de topología* we studied the **Fundamental group**. This group was introduced by Henri Poincaré (1854-1912). The idea is to give a group structure to the loops in a topological space in such a way that two homotopy equivalent topological spaces have isomorphic fundamental groups. Thus, we saw in the course that the fundamental group can be used, for example, to prove that $\mathbb{S}^1$ is not contractible or to prove that $\mathbb{S}^2$ is not homeomorphic to the 2-torus and to the Klein bottle. However, this group only involves 1-dimensional loops and fails to distinguish, for example, the spheres $\mathbb{S}^n$ with $n \geq 2$.

Homology was originally related with triangulation of manifolds. A simplicial complex is a set composed of points, line segments, triangles and their n-dimensional counterparts (see figure 1a). If a simplicial complex is homeomorphic to a manifold we say that it is a **triangulation** of the manifold



(a) A simplicial complex. Picture taken from **here**.



(b) A triangulation of the 2-torus. Picture taken from **here**.

Figure 1: Examples of simplicial complexes.

(see Figure 1b). The problem is that not all manifolds can be triangulated. Topological manifolds of dimensions 2 and 3 can always be triangulated, but there are compact 4-manifolds that admit no triangulation*.

The first work related to what we now know as homology was ”Sopra gli spazi di un numero qualunque di dimensioni”, published in 1871 by Enrico Betti (1823-1892). There he described some topologically invariant numbers that could be associated to a triangulated manifold describing the number of “n-dimensional holes” for each $n \geq 0$. These numbers were later known as **Betti numbers**, and they were obtained in a purely combinatorial way from the simplicial complex homeomorphic to the manifold. The first recognisable theory of homology was published by Henri Poincaré in 1895. He developed the **simplicial homology** of a triangulated manifold and create what is now called a **chain complex**. In the early XX century, Emmy Noether and, independently, Leopold Vietoris and Walther Mayer further developed the general theory of homology groups, and introduced the modern algebraic approach to the matter.

However, it is not so easy to prove that simplicial homology groups are homotopy-invariant. To prove this, a more general theory is used, called **singular homology**. In this more general setting some properties of homology groups are more easily proven, and one can show that it is equivalent to simplicial homology for triangulated manifolds. The price we pay for this is that we get rid of the combinatorial aspect of homology and the computation of homology groups becomes a very hard task.

In 1949, J. H. C. Whitehead introduced a special kind of topological

*`https://en.wikipedia.org/wiki/E8_manifold`

spaces, called **CW complexes**, to meet the needs of homology and homotopy theory. CW complexes are a generalization of simplicial complexes and are more flexible, while retaining a combinatorial nature that allows for computation. These spaces are constructed by starting with a discrete set of points and successively attaching n-cells (spaces homeomorphic to Euclidean n-balls) of increasing dimensions. It turns out that many interesting spaces can be constructed in this way, and that many topological properties can be deduced for these spaces just from their construction. In fact, every compact manifold is homotopy-equivalent to a CW complex (see [1, Corollary A.12]).

There are two standard ways to define CW complexes. The first way is to define a CW complex as a space formed inductively, starting from a discrete set of points X_0, attaching some 1-cells to it to form a space X_1, attaching some 2-cells to it to form a space X_2 and so forth. Many authors, such as Allen Hatcher in [1], define them in this way, and it is generally a good way to think about CW complexes to build intuition about their properties. The other way is to define what a CW decomposition of a space X is, and to define a CW complex as a topological space that admits such a decomposition. Then, one can prove that the inductively built space admits a CW decomposition making it a CW complex, so both definitions are equivalent. Other authors, such as John M. Lee in [2], choose this way. In this work the second way was chosen because it makes proving some topological properties of CW complexes easier.

The objective of this work is to give an introduction to singular homology groups, to present CW complexes and their basic properties and to discuss the advantages these spaces have when computing their homology groups. The work is structured as follows:

- **Chapter 1:** we introduce the general theory of singular homology and prove the homotopy invariance of homology groups, following the proof in [2, Theorem 13.8].

- **Chapter 2:** we define relative and reduced homology groups, and we build an exact sequence that relates the homology groups of a quotient X/A with the homology groups of X and A, mainly following [1].

- **Chapter 3:** we define CW complexes following [2] and we prove the equivalence of the two possible definitions of CW complexes discussed in the previous paragraph. We also introduce some examples of CW complexes.

- **Chapter 4:** we study a special homology theory for CW complexes, called cellular homology, and show that it is equivalent to singular homology for CW complexes. We use cellular homology to compute the homology groups of some CW complexes introduced in Chapter 3.

As for the Appendices, in Appendix A we collect some solved exercises. The computations of the homology groups of a point and the homology groups of the sphere $\mathbb{S}^n$ are included there as exercises. In order to make the work as self-contained as possible, we included in Appendix B and Appendix C most of the preliminary results used during the work, proofs included. The definition of singular homology groups requires some preliminaries in commutative algebra which are included in Appendix B. The first part of Appendix B is about free modules and the second part collects basic general results about chain complexes and exact sequences. Appendix C contains all the topological constructions that will be used to define CW complexes and cellular homology. The only exception is Section C.7, which contains part of the proof of Theorem 3.3.3. Moreover, Appendix D contains part of the proof of the homotopy invariance of singular homology groups, and Appendix E contains a complete proof of the Excision Theorem. These two proofs are not included in the work due to their technical nature, but are written there to make the work as self-contained as possible.

Notation Glossary

$\mathbb{R}^n$	The euclidean n-space. As a convention, if $n = 0$, we define $\mathbb{R}^0$ to be a singleton.
I	The unit interval $[0, 1]$.
$\mathbb{D}^n$	The unit disk in $\mathbb{R}^n$. All points of distance 1 or less from the origin. As a convention, if $n = 0$, we define $\mathbb{D}^0$ to be a singleton.
$\mathbb{B}^n$	The unit open ball in $\mathbb{R}^n$. All points of distance less than 1 from the origin. As a convention, if $n = 0$, we define $\mathbb{B}^0$ to be a singleton.
$\mathbb{S}^n$	The unit sphere in $\mathbb{R}^{n+1}$. All points of distance 1 from the origin.
$B_r(x_0)$	The open ball of radius r centered in x_0 in the euclidean space $\mathbb{R}^n$.
$\operatorname{int} A$	The topological interior of A.
$\operatorname{cl} A$	The topological closure of A.
$\operatorname{fr} A$	The topological boundary/frontier of A.
Id	The identity function.
$\bigsqcup_{i \in I} X_i$	Disjoint union of the family $\{X_i\}_{i \in} I$.
$\bigvee_{i \in I} X_i$	Wedge sum of the family $\{X_i\}_{i \in I}$.
$X \diagup A$	Topological quotient of X by A.
$\cong$	Isomorphism.
$\frac{M}{N}$	Quotient of R-modules $N \subseteq M$.
$\bigoplus$	Direct sum of modules/homomorphisms.
$\operatorname{Ker} f$	The kernel of the homeomorphism f.
$\operatorname{Im} f$	The image of the homeomorphism f.
$A - B$	Set theoretic difference of A and B.

Chapter 1

Singular Homology

The goal of this first chapter is to give the definition of singular homology groups and to prove that singular homology groups are homotopy invariant. In the last part of the chapter we show the relation between path-connectivity and homology groups.

1.1 Simplices

For every integer number $n \geq 0$ we define the n-dimensional analogue of the triangle: the n-simplex.

Definition 1.1.1. Let $\mathbb{R}^m$ be the euclidean space and $p_0, \dots, p_n \in \mathbb{R}^m$ be $n+1$ affinely independent points. The **n-simplex** generated by $p_0, \dots, p_n$ is the convex hull of $\{p_0, \dots, p_n\}$. That is,

$$[p_0, \dots, p_n] = \Big\{ \sum_{i=0}^{n} \lambda_i p_i \in \mathbb{R}^m \ \Big| \sum_{i=0}^{n} \lambda_i = 1, \ \ 0 \leq \lambda_i \leq 1, \ i = 0, \dots, n. \Big\}.$$

The points $p_0, \dots, p_n$ are called the **vertices** of the n-simplex. The ordering of the vertices determines the **orientation** of the simplex. The n-simplex generated by $\{e_0, \dots, e_n\} \subset \mathbb{R}^{n+1}$, $e_i = (0, \dots, \overset{i}{1}, \dots, 0)$, with ordering according to the increasing subscripts is called the **standard n-simplex**, and it is denoted by Δ^n. That is,

$$\begin{aligned} \Delta^n &= [e_0, \dots, e_n] \\ &= \Big\{ \sum_{i=0}^{n} \lambda_i e_i \in \mathbb{R}^{n+1} \ \Big| \sum_{i=0}^{n} \lambda_i = 1, \ \ 0 \leq \lambda_i \leq 1, \ i = 0, \dots, n. \Big\} \\ &= \Big\{ (\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} \ \Big| \sum_{i=0}^{n} \lambda_i = 1, \ \ 0 \leq \lambda_i \leq 1, \ i = 0, \dots, n. \Big\}. \end{aligned}$$

Specifying an ordering of the vertices determines the following canonical homeomorphism from the standard n-simplex Δ^n to the n-simplex $[p_0, \dots, p_n]$:

$$\begin{array}{cccc} \varphi_{[p_0,\ldots,p_n]}: & \Delta^n & \longrightarrow & [p_0,\ldots,p_n] \\ & (\lambda_0,\ldots,\lambda_n) & \longmapsto & \sum_{i=0}^n \lambda_i p_i. \end{array}$$

Definition 1.1.2. Let $[p_0,\ldots,p_n]$ be an n-simplex. A n-simplex generated by a subset of $\{p_0,\ldots,p_n\}$ is called a **face** of $[p_0,\ldots,p_n]$. If the vertex p_i is removed, we will write it as $[p_0,\ldots,\hat{p_i},\ldots,p_n]$.

Let $\Delta^n = [e_0,e_1,\ldots,e_n]$ be the standard n-simplex. The canonical homeomorphism from the standard $(n-1)$ simplex to the face $[e_0,\ldots,\hat{e_i},\ldots,e_n]$ of Δ^n will be denoted as $\varphi_{i,n} = \varphi_{[e_0,\ldots,\hat{e_i},\ldots,e_n]}$. That is,

$$\begin{array}{cccc} \varphi_{i,n}: & \Delta^{n-1} & \longrightarrow & \underset{i}{\Delta^n} \\ & (\lambda_0,\ldots,\lambda_{n-1}) & \mapsto & (\lambda_0,\ldots,\lambda_{i-1},0,\lambda_i,\ldots,\lambda_{n-1}). \end{array}$$

1.2 Singular Homology

Definition 1.2.1. Let X be a topological space, and let $n \geq 0$ be an integer. A **singular n-simplex** in a space X is a continuous map $\sigma : \Delta^n \longrightarrow X$. We will write

$$\Omega_n(X) = \{\ \sigma : \Delta^n \longrightarrow X \mid \sigma \text{ continuous}\}.$$

Definition 1.2.2. Let X be a topological space. The free $\mathbb{Z}$-module generated by $\Omega_n(X)$ is called the **n-dimensional singular chain group** of X and will be denoted as $C_n(X)$. Elements of $C_n(X)$ are called **singular n-chains**.

Remark 1.2.1. Observe that equivalently, $C_n(X)$ is the free abelian group generated by $\Omega_n(X)$. By the notation used in Section B.1, $C_n(X) = \mathbb{Z}^{\Omega_n(X)}$. For more detailed information about free modules, see Section B.1.

Let $\sigma : \Delta^n \longrightarrow X$ be a singular n-simplex. If we remove a vertex from Δ^n, the resulting face of Δ^n can be identified with Δ^{n-1} via the canonical homeomorphism. Thus, the restriction of σ to this face of Δ^n is a singular $(n-1)$-simplex. In order to define this restriction rigorously we will use the **face map**.

Definition 1.2.3. Let X be a topological space and $n \geq 0$. For each $i = 0,\ldots,n$ we define the **i-th face map** as:

$$\begin{array}{cccc} [\ \cdot\ ]_i: & \Omega_n(X) & \longrightarrow & \Omega_{n-1}(X) \\ & \sigma & \longmapsto & [\sigma]_i = \sigma \circ \varphi_{i,n}. \end{array}$$

The face map sends each singular n-simplex to a singular $(n-1)$-simplex.

Definition 1.2.4. Let $n \geq 1$ and let $\sigma : \Delta^n \longrightarrow X$ be a singular n-simplex. We define the **boundary** of σ as

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i [\sigma]_i \ \in\ C_{n-1}(X),$$

if $n \geq 1$. If $n = 0$, we define the boundary of any 0-singular simplex to be 0. That is, $\partial_0(\sigma) = 0$ for any $\sigma \in \Omega_0(X)$.

The extension of ∂_n from the basis $\Omega_n(X)$ to the whole $C_n(X)$ is the **boundary map**,

$$\begin{array}{rccc} \partial_n : & C_n(X) & \longrightarrow & C_{n-1}(X) \\ & \sum_{\sigma\in\Omega_n(X)} \lambda_\sigma \sigma & \mapsto & \sum_{\sigma\in\Omega_n(X)} \lambda_\sigma \partial_n(\sigma). \end{array}$$

Remark 1.2.2. For any $n \geq 0$, ∂_n is a $\mathbb{Z}$-module homomorphism by theorem B.1.4 in Appendix B. In order to simplify the notation we may simply write $\partial_n = \partial$ whenever it is clear which map it is.

Our goal now is to prove that the singular chain groups and the boundary maps form a chain complex, that is, $\partial_n \circ \partial_{n+1} = 0$ for any $n \geq 0$. A general definition of chain complexes is given in Section B.2.

Theorem 1.2.1. *For any $n \geq 0$, the boundary maps satisfy the following identity:*

$$\partial_n \circ \partial_{n+1} = 0.$$

Proof. If $n = 0$, as $\partial_0 = 0$, then $\partial_0 \circ \partial_1 = 0$.

For $n \geq 1$, by linearity, it suffices to show by that $\partial_n(\partial_{n+1}(\sigma)) = 0$ for any singular n-simplex σ.

$$\begin{aligned} \partial_n(\partial_{n+1}(\sigma)) &= \partial_n(\sum_{i=0}^{n+1}(-1)^i[\sigma]_i) = \sum_{i=0}^{n+1}(-1)^i\partial_n([\sigma]_i) \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n+1}(-1)^i(-1)^j \ [[\sigma]_i]_j \\ &= \sum_{0\leq j<i\leq n+1}(-1)^{i+j}[[\sigma]_i]_j + \sum_{0\leq i\leq j\leq n}(-1)^{i+j}[[\sigma]_i]_j, \end{aligned}$$

rearranging indices in the second summation ($i' = j+1, j' = i$) we get:

$$\partial_n(\partial_{n+1}(\sigma)) = \sum_{0\leq j<i\leq n+1}(-1)^{i+j}[[\sigma]_i]_j + \sum_{0\leq j'<i'\leq n+1}(-1)^{i'+j'-1}[[\sigma]_{j'}]_{i'-1}.$$

Notice that for $j < i$:

$$\varphi_{i,n+1} \circ \varphi_{j,n} = \varphi_{j,n+1} \circ \varphi_{i-1,n}.$$

Indeed, for any $(\lambda_0, \ldots, \lambda_{n-1}) \in \Delta^{n-1}$,

$$\varphi_{i,n+1}(\varphi_{j,n}(\lambda_0, \ldots, \lambda_{n-1})) = \varphi_{i,n+1}(\lambda_0, \ldots, \lambda_{j-1}, \overset{j}{0}, \lambda_j, \ldots, \lambda_{n-1})$$

$$= (\lambda_0, \dots, \lambda_{j-1}, \overset{j}{0}, \lambda_j, \dots, \lambda_{i-2}, \overset{i}{0}, \lambda_{i-1}, \dots, \lambda_{n-1}),$$

$$\varphi_{j,n+1}(\varphi_{i-1,n}(\lambda_0, \dots, \lambda_{n-1})) = \varphi_{i,n+1}(\lambda_0, \dots, \lambda_{i-2}, \overset{i-1}{0}, \lambda_{i-1}, \dots, \lambda_{n-1})$$

$$= (\lambda_0, \dots, \lambda_{j-1}, \overset{j}{0}, \lambda_j, \dots, \lambda_{i-2}, \overset{i}{0}, \lambda_{i-1}, \dots, \lambda_{n-1}).$$

Therefore, $[[\sigma]_i]_j = [[\sigma]_j]_{i-1}$ for $j < i$. Knowing this,

$$\partial_n(\partial_{n+1}(\sigma)) = \sum_{0 \le j < i \le n+1} (-1)^{i+j}[[\sigma]_i]_j + \sum_{0 \le j' < i' \le n+1} (-1)^{i'+j'-1}[[\sigma]_{j'}]_{i'-1} = 0,$$

as each term in the first summation is cancelled by one in the second.

□

By the previous proposition the groups $C_n(X)$ and the boundary map ∂ form a chain complex called **singular chain complex**. We will denote it as $(C_*(X), \partial_*)$. We can now define our main object to study.

Definition 1.2.5. Let X be a topological space. The homology groups of the chain complex $(C_*(X), \partial_*)$ are called **singular homology groups**. That is, the n-th singular homology group is

$$H_n(X) = \frac{\operatorname{Ker}(\partial_n)}{\operatorname{Im}(\partial_{n+1})}.$$

Elements of $\operatorname{Ker} \partial_n$ are called **cycles** and elements of $\operatorname{Im} \partial_{n+1}$ are called **boundaries**. The elements of $H_n(X)$ are cosets of $\operatorname{Im} \partial_{n+1}$ called **homology classes**. Two cycles with the same homology class are said to be **homologous**, which means that their difference is a boundary.

The homology groups of a point can be computed directly from the definition. This is done in Exercise 1 of Appendix A.

Example 1.2.1. Let $X = \{p\}$ be a point. The homology groups of X are the following:

$$H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

1.3 Homotopy invariance of homology groups

1.3.1 Chain maps and induced homomorphisms

In this section we will show that any continuous map between two topological spaces induces a chain map between their singular chain complexes, which induces a homomorphism between homology groups of their respective chain complexes. See Section B.2 of Appendix B for the general definition and properties of chain maps.

Let X, Y be topological spaces and $f : X \longrightarrow Y$ be a continuous map. Notice that if $\sigma \in \Omega_n(X)$, then $f \circ \sigma \in \Omega_n(Y)$ as the composition of continuous maps is continuous. Thus, we have a map

$$\begin{array}{ccc} \Omega_n(X) & \longrightarrow & \Omega_n(Y) \\ \sigma & \mapsto & f \circ \sigma, \end{array}$$

which can be extended to a homomorphism:

$$\begin{array}{cccc} f_\# : & C_n(X) & \longrightarrow & C_n(Y) \\ & \sum_{\sigma \in \Omega_n(X)} \lambda_\sigma \sigma & \mapsto & \sum_{\sigma \in \Omega_n(X)} \lambda_\sigma (f \circ \sigma) \end{array}$$

by Corollary B.1.5 in Appendix B. Observe that for any $\sigma \in \Omega_{n+1}(X)$,

$$\begin{aligned} f_\#(\partial(\sigma)) &= f_\#\Big(\sum_{i=0}^{n+1} (-1)^i [\sigma]_i \Big) = \sum_{i=0}^{n+1} (-1)^i f_\#([\sigma]_i) \\ &= \sum_{i=0}^{n+1} (-1)^i f \circ (\sigma \circ \varphi_{i,n+1}) = \sum_{i=0}^{n+1} (-1)^i (f \circ \sigma) \circ \varphi_{i,n+1} \\ &= \sum_{i=0}^{n+1} (-1)^i [f \circ \sigma]_i = \sum_{i=0}^{n+1} (-1)^i [f_\#(\sigma)]_i = \partial(f_\#(\sigma)). \end{aligned}$$

By linearity, the same holds for any sum in $C_{n+1}(X)$. Thus, this sequence of homomorphisms defines a chain map from the singular chain complex of X to that of Y, that is, the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \cdots \\ & & \downarrow f_\# & \circlearrowleft & \downarrow f_\# & \circlearrowleft & \downarrow f_\# & & \\ \cdots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \cdots \end{array}$$

This chain map induces a homomorphism $f_* : H_n(X) \longrightarrow H_n(Y)$ in homology groups called the **homomorphism induced by f**,

$$\begin{array}{cccc} f_* : & H_n(X) & \longrightarrow & H_n(Y) \\ & c + \operatorname{Im} \partial_{n+1} & \mapsto & f_\#(c) + \operatorname{Im} \partial_{n+1}. \end{array}$$

Proposition 1.3.1. *Let X, Y and Z be topological spaces and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be continuous maps. The following hold:*

(i) $(g \circ f)_* = g_* \circ f_*$.

(ii) $\mathbf{Id}_* = \mathbf{Id}$. *That is, the identity map in a topological space induces the identity map in the homology groups.*

Proof. Observe that for any $\sigma \in \Omega_n(X)$,

$$(g \circ f)_\#(\sigma) = (g \circ f) \circ \sigma = g \circ (f \circ \sigma) = g_\#(f_\#(\sigma)) = (g_\# \circ f_\#)(\sigma).$$

Thus, by linearity, the same holds for any n-chain of $C_n(X)$. By definition of f_* it is clear that (i) holds.

To show (ii), let $\mathbf{Id} : X \longrightarrow X$ be the identity map. Observe that for any singular n-simplex σ, $\mathbf{Id}_\#(\sigma) = \mathbf{Id} \circ \sigma = \sigma$ so it is clear that $\mathbf{Id}_\#$ is the identity map of $C_n(X)$. Therefore, (ii) holds. □

Remark 1.3.1. Observe that whenever $f : X \longrightarrow Y$ is a homeomorphism the map $\sigma \mapsto f \circ \sigma$ from $\Omega_n(X)$ to $\Omega_n(Y)$ is a bijection so the extension $f_\#$ is an isomorphism. Therefore, as a direct consequence of Lemma B.2.1 in Appendix B we get that $H_n(X)$ and $H_n(Y)$ are isomorphic for any $n \geq 0$. The goal of this chapter is to prove that homology groups are not only isomorphic when the spaces are homeomorphic, they are isomorphic when the spaces are homotopy equivalent too which is a weaker condition.

1.3.2 Homotopy invariance

We first start by giving some basic definitions.

Definition 1.3.1. Let X and Y be topological spaces and $f, g : X \longrightarrow Y$ be continuous functions. A **homotopy** from f to g is a continuous map

$$H : X \times I \longrightarrow Y,$$

where $I = [0, 1]$, such that for any $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In this case, we say that f and g are **homotopic** and write $f \simeq g$ or $f \overset{H}{\simeq} g$ if we want to specify that the homotopy is given by H. In particular, if g is a constant map we say that f is **null-homotopic**.

Definition 1.3.2. Let X and Y be topological spaces. A continuous map $f : X \longrightarrow Y$ is said to be a **homotopy equivalence** if there exists some continuous map $g : Y \longrightarrow X$ such that $g \circ f \simeq \mathbf{Id}_X$ and $f \circ g \simeq \mathbf{Id}_Y$. In this case we say that X and Y are **homotopy equivalent**.

Definition 1.3.3. Let X be a topological space. We say that X is **contractible** if it is homotopy equivalent to a point.

Definition 1.3.4. Let X be a toplogical space and let $A \subseteq X$ be a subspace. A continuous map $F : X \times I \to Y$ is a **deformation retraction** of X onto A if for every $x \in X$ and $a \in A$, $F(x, 0) = x$, $F(x, 1) \in A$ and $F(a, 1) = a$. In this case, we say that A is a **deformation retraction** of X.

If in the definition of a deformation retraction we add the requirement that $F(a, t) = a$ for every $t \in I$ and $a \in A$, then F is called a **strong deformation retract**, and A is a **strong deformation retraction** of X.

Our goal is to prove that homotopic maps induce the same homomorphism on homology. To prove that result we will use the following Lemma. The definition of a chain homotopy is given in Section B.2. Due to the length and technical nature of the proof of the Lemma, it is attached in Appendix D.

Lemma 1.3.2. *Let X be a topological space and $I = [0, 1]$. The chain maps induced by*

$$\begin{array}{rccc} \iota_0 : & X & \longrightarrow & X \times I \\ & x & \mapsto & (x, 0) \end{array} \quad \textit{and} \quad \begin{array}{rccc} \iota_1 : & X & \longrightarrow & X \times I \\ & x & \mapsto & (x, 1) \end{array}$$

are chain homotopic. In particular, they induce the same homomorphism in homology groups.

Now we go for the main theorem of this Chapter.

Theorem 1.3.3. *Let X and Y be topological spaces and let $f, g : X \longrightarrow Y$ be continuous maps. If f and g are homotopic, they induce the same homomorphism in homology groups. That is, $f_* = g_*$.*

Proof. We know by Lemma 1.3.2 that $(\iota_0)_* = (\iota_1)_*$. Let $f, g : X \longrightarrow Y$ be two continuous maps homotopic by a homotopy H. Observe that for any $x \in X$, $(H \circ \iota_0)(x) = H(x, 0) = f(x)$ and $(H \circ \iota_1)(x) = H(x, 1) = g(x)$, then

$$f_* = (H \circ \iota_0)_* = H_* \circ (\iota_0)_* = H_* \circ (\iota_1)_* = (H \circ \iota_1)_* = g_*,$$

and we get the general result. □

Corollary 1.3.4 (Homotopy invariance of homology groups)**.** *Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a homotopy equivalence. Then, $f_* : H_n(X) \longrightarrow H_n(Y)$ is an isomorphism for every $n \geq 0$.*

Proof. If f is a homotopy equivalence, there exists some continuous map $g : Y \longrightarrow X$ such that $g \circ f \simeq \mathbf{Id}_X$ and $f \circ g \simeq \mathbf{Id}_Y$. By Theorem 1.3.3, for every $n \geq 0$,

$$g_* \circ f_* = (g \circ f)_* = (\mathbf{Id}_X)_* = \mathbf{Id}_{H_n(X)},$$

and

$$f_* \circ g_* = (f \circ g)_* = (\mathbf{Id}_Y)_* = \mathbf{Id}_{H_n(Y)}.$$

Therefore, f_* is a bijection. □

We computed the homology groups of a point in Example 1.2.1. By the previous result we are able to compute the homology groups of any contractible space.

Corollary 1.3.5. *Let X be a contractible topological space. Then,*

$$H_n(X) = \begin{cases} \mathbb{Z} & \textit{if } n = 0, \\ 0 & \textit{if } n \geq 1. \end{cases}$$

1.4 Connectivity and homology groups

We are now ready to study the connection between connectivity and homology groups.

Proposition 1.4.1. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Then, the canonical injections $\imath_i : X_i \hookrightarrow \bigsqcup_{i\in I} X_i$ induce the isomorphism $\oplus_{i\in I}(\imath_i)_* : \bigoplus_{i\in I} H_n(X_i) \longrightarrow H_n(\bigsqcup_{i\in I} X_i)$ for any $n \geq 0$.*

Proof. Let $\partial_n^i : C_n(X_i) \longrightarrow C_{n-1}(X_i)$ be the boundary map of the chain $C_*(X_i)$. Observe that the sums $\oplus_{i\in I} C_*(X_i)$ form a chain complex with the boundary maps $\oplus_{i\in I}\partial_n^i$.

Consider the following map:

$$\begin{array}{cccc} \oplus_{i\in I}(\imath_i)_\# : & \bigoplus_{i\in I} C_n(X_i) & \longrightarrow & C_n(\bigsqcup_{i\in I} X_i) \\ & \sum_{i\in I}\sum_{\sigma\in\Omega_n(X_i)} \lambda_\sigma \sigma & \longmapsto & \sum_{i\in I}\sum_{\sigma\in\Omega_n(X_i)} \lambda_\sigma(\imath_i\circ\sigma). \end{array}$$

Notice that $\sum_{i\in I}\sum_{\sigma\in\Omega_n(X_i)} \lambda_\sigma(\imath_i\circ\sigma) = 0$ if and only if $\lambda_\sigma = 0$ for every $\sigma \in \Omega_n(X_i)$, $i \in I$. Thus, the map is injective.

The surjectivity of the map comes from the fact that each $\imath_i(X_i)$ is disconnected from the other components in the disjoint union. Take any $\sigma \in \Omega_n(\bigsqcup_{i\in I} X_i)$. Δ^n is connected so the image $\sigma(\Delta^n)$ is connected too by continuity. Thus, it is contained in some unique component $\imath_i(X_i)$. Therefore, as each restriction $\imath_{i|X_i}$ is a homeomorphism, $(\imath_{i\ |X_i})^{-1}\circ\sigma \in \Omega_n(X_i)$ and its image by $(\imath_i)_\#$ is obviously σ. Therefore each sum in $C_n(\bigsqcup_{i\in I} X_i)$ can be divided into sums in which all the singular simplices are images of simplices in a unique $C_n(X_i)$. This means that the map is surjective.

As each $(\imath_i)_\#$ is a chain map from $(C_*(X_i), \partial_*^i)$ to $(C_*(\sqcup_{i\in I} X_i), \partial_*)$, the sum $\oplus_{i\in I}(\imath_i)_\#$ is a chain map from $(\oplus_{i\in I} C_*(X_i), \oplus_{i\in I}\partial_n^i)$ to $(C_*(\sqcup_{i\in I} X_i), \partial_*)$. Thus it induces an isomorphism $(\oplus_{i\in I}(\imath_i)_\#)_* : H_n(\oplus_{i\in I} C_*(X_i)) \to H_n(\sqcup_{i\in I} X_i)$ for every $n \geq 0$. We also have an isomorphism

$$\bigoplus_{i\in I} H_n(X_i) = \bigoplus_{i\in I} \frac{\operatorname{Ker}\partial_n^i}{\operatorname{Im}\partial_n^i} \to \frac{\oplus_{i\in I}\operatorname{Ker}\partial_n^i}{\oplus_{i\in I}\operatorname{Im}\partial_n^i} = \frac{\operatorname{Ker}\oplus_{i\in I}\partial_n^i}{\operatorname{Im}\oplus_{i\in I}\partial_n^i} = H_n(\oplus_{i\in I} C_*(X_i)),$$

where the equality $\frac{\oplus_{i\in I}\operatorname{Ker}\partial_n^i}{\oplus_{i\in I}\operatorname{Im}\partial_n^i} = \frac{\operatorname{Ker}\oplus_{i\in I}\partial_n^i}{\operatorname{Im}\oplus_{i\in I}\partial_n^i}$ holds due to Proposition B.1.7 in Appendix B. Finally, observe that the composition of the two previous isomorphisms is exactly $\oplus_{i\in I}(\imath_i)_*$. Hence, $\oplus_{i\in I}(\imath_i)_*$ is also an isomorphism. □

Any topological space X is the disjoint union of its connected components with the inclusion maps as canonical injections. Thus, Proposition 1.4.1 directly implies the following result.

Proposition 1.4.2. *Let X be a topological space and $\{X_i\}_{i\in I}$ be its connected components. The inclusions $\imath_i : X_i \hookrightarrow X$ induce an isomorphism $\oplus_{i\in I}(\imath_i)_* : \bigoplus_{i\in I} H_n(X_i) \longrightarrow H_n(X)$ for every $n \geq 0$.*

Proposition 1.4.3. *Let X be a nonempty, path connected topological space. Then,*

$$H_0(X) \cong \mathbb{Z}.$$

Proof. By definition, we know that $H_0(X) = \frac{C_0(X)}{\operatorname{Im}\partial_1}$. We define the following homomorphism:

$$\begin{array}{cccc} \xi : & C_0(X) & \longrightarrow & \mathbb{Z} \\ & \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma & \mapsto & \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma. \end{array}$$

Since X is nonempty, $\Omega_0(X)$ is nonempty. Thus, for any $\lambda \in \mathbb{Z}$ there is $\lambda\sigma \in C_0(X)$ such that $\xi(\lambda\sigma) = \lambda$, so ξ is surjective.

We aim to prove that $\operatorname{Ker}\xi = \operatorname{Im}\partial_1$. For any singular 1-simplex $\sigma : \Delta^1 \to X$ we have $\xi(\partial_1(\sigma)) = \xi([\sigma]_0 - [\sigma]_1) = \xi([\sigma]_0) - \xi([\sigma]_1) = 1 - 1 = 0$. Thus, $\operatorname{Im}\partial_1 \subseteq \operatorname{Ker}\xi$.

For the reverse inclusion, let $\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma \in \operatorname{Ker}\xi$. This means that $\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma = 0$. The maps $\sigma \in \Omega_0(X)$ map the point $\Delta^0 = 1 \in \mathbb{R}$ to a point in X. Let $x_0 \in X$. For any 0-simplex σ, as X is path-connected, there is a path $\alpha_\sigma : [0,1] \longrightarrow X$ from $\sigma(1)$ to x_0. Let σ' be the singular 0-simplex such that $\sigma'(1) = x_0$. Consider the following map:

$$\begin{array}{cccc} \pi : & \Delta^1 & \longrightarrow & [0,1] \\ & (\lambda_0, \lambda_1) & \mapsto & \lambda_1. \end{array}$$

The map π is continuous and set $\tau_\sigma = \alpha_\sigma \circ \pi \in \Omega_1(X)$. Notice that

$$[\tau_\sigma]_0(1) = \alpha_\sigma\big(\pi(\ \varphi_{0,1}(1)\ )\big) = \alpha_\sigma\big(\pi(0,1)\big) = \alpha_\sigma(0) = \sigma(1) \Rightarrow [\tau_\sigma]_0 = \sigma,$$

$$[\tau_\sigma]_1(1) = \alpha_\sigma\big(\pi(\ \varphi_{1,1}(1)\ )\big) = \alpha_\sigma\big(\pi(1,0)\big) = \alpha_\sigma(1) = x_0 = \sigma'(1) \Rightarrow [\tau_\sigma]_1 = \sigma'.$$

Thus,

$$\partial_1(\tau_\sigma) = [\tau_\sigma]_0 - [\tau_\sigma]_1 = \sigma - \sigma',$$

and

$$\begin{aligned} \partial_1\big(\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \tau_\sigma\big) &= \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \partial_1(\tau_\sigma) = \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma - \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma' \\ &= \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma - \big(\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma\big)\sigma' = \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma. \end{aligned}$$

This means that $\sum_{\sigma\in\Omega_0(X)}\lambda_\sigma\sigma$ is the boundary of $\sum_{\sigma\in\Omega_0(X)}\lambda_\sigma\tau_\sigma \in C_1(X)$, so $\sum_{\sigma\in\Omega_0(X)}\lambda_\sigma\sigma \in \operatorname{Im}\partial_1$, which proves that $\operatorname{Ker}\xi \subseteq \operatorname{Im}\partial_1$. By the first isomorphism theorem,

$$H_0(X) = \frac{C_0(X)}{\operatorname{Im}\partial_1} = \frac{C_0(X)}{\operatorname{Ker}\xi} \cong \mathbb{Z}.$$

□

Corollary 1.4.4. *Let X be a topological space. If $\{X_i\}_{i\in I}$ are the path-connected components of X,*

$$H_0(X) \cong \bigoplus_{i\in I}\mathbb{Z}.$$

Proof. By Propositions 1.4.2 and 1.4.3,

$$H_0(X) \cong \bigoplus_{i\in I} H_0(X_k) \cong \bigoplus_{i\in I}\mathbb{Z}.$$

□

Remark 1.4.1. By Corollary 1.4.4, the zero homology group is completely determined by the path-connected components of the space. However, in general, it is not possible to compute the rest of the homology groups just from the definition. By Corollary 1.3.5, we know how homology groups of contractible spaces are, but it might not be possible to make computations in such a direct way for more complicated spaces. Singular homology groups are easy to define and we have proved that they are homotopy-invariant, but more work is required in order to compute the homology groups of more topological spaces.

Chapter 2

The exact sequence for good pairs

Let X be a topological space and let $A \subseteq X$ be a subspace. In this second Chapter we build an exact sequence that relates the homology groups of the quotient space ${}^{X}\!/\!_{A}$ with the homology groups of X and A. This exact sequence will be used to compute the homology groups of the sphere $\mathbb{S}^n$. The definition of the quotient ${}^{X}\!/\!_{A}$ is given in Section C.2, and the general definition and basic properties of exact sequences are given in Section B.2.

2.1 Relative homology groups

It sometimes happens in mathematics that by ignoring a certain amount of data one obtains a simpler theory that gives results that could not be obtained in the original setting. An example of this is arithmetic mod n, where one ignores multiples of n. At first, one could think that the simplest analogy of this in homology would be that if $A \subseteq X$, then $H_n(A)$ would be contained in $H_n(X)$ as a subgroup and the quotient group $\frac{H_n(X)}{H_n(A)}$ would be isomorphic to $H_n({}^{X}\!/\!_{A})$. While this does hold in some cases, if it held in general then homology theory would collapse since every space X can be embedded as a subspace of a contractible space, namely the cone $CX = (X \times I)/(X \times \{0\})$, which has trivial homology groups.

It turns out that if one ignores all singular chains in a subspace of a given space we obtain a better result.

Definition 2.1.1. Let X be a topological space and let $A \subseteq X$ be a subspace. As $C_n(A) \subseteq C_n(X)$, we define the **n-th relative chain group** to be the quotient

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}.$$

Elements in $C_n(X, A)$ are called **relative chains**.

The boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, so it induces a quotient boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$, called **relative boundary map**, that sends each relative chain $c + C_n(A)$ to the class $\partial(c) + C_{n-1}(A)$. Letting n vary, we obtain the following sequence:

$$\ldots \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_0(X, A) \to 0 \tag{2.1}$$

The relation $\partial^2 = 0$ holds for these boundary maps since it holds before passing to quotient groups. So (2.1) is a chain complex, called **relative chain complex**, and denoted by $(C_*(X, A), \partial_*)$.

Definition 2.1.2. The homology groups of the chain complex $(C_*(X, A), \partial_*)$ are called **relative homology groups** and are denoted by $H_n(X, A)$.

If $A = \{x_0\}$ is a point, we simply write $C_n(X, \{x_0\}) = C_n(X, x_0)$ and $H_n(X, \{x_0\}) = H_n(X, x_0)$.

Remark 2.1.1. From the definition of the relative boundary map we observe that:

- The elements in $H_n(X, A)$ are represented by **relative cycles**. That is, chains $c \in C_n(X)$ such that $\partial(c) \in C_{n-1}(A)$.
- A trivial relative cycle c is a **relative boundary**. That is, $c = \partial(b) + a$ for some $b \in C_{n+1}(X)$ and $a \in C_n(A)$.

These properties make precise the intuitive idea that $H_n(X, A)$ is "homology of X modulo A".

Let $\imath : C_n(A) \hookrightarrow C_n(X)$ be the inclusion map and $\pi : C_n(X) \longrightarrow C_n(X, A)$ be the quotient map. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(A) & \xrightarrow{\imath} & C_n(X) & \xrightarrow{\pi} & C_n(X, A) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{\imath} & C_{n-1}(X) & \xrightarrow{\pi} & C_{n-1}(X, A) & \longrightarrow & 0. \end{array}$$

Observe that by definition of the relative boundary map the diagram is commutative so $\imath$ and π are chain maps. Moreover $\imath$ is injective, π is surjective and $\operatorname{Im} \imath = \operatorname{Ker} \pi = C_n(A)$, so we have the following short exact sequence:

$$0 \to C_n(A) \xrightarrow{\imath} C_n(X) \xrightarrow{\pi} C_n(X, A) \to 0. \tag{2.2}$$

Short exact sequences of chain complexes induce a long exact sequence in their homology groups. This result is called **Zig-zag lemma**, and it is explained and proved in detail in Lemma B.2.3 in Appendix B. Thus, relative homology groups fit into the following long exact sequence:

$$\begin{aligned} \dots \to H_n(A) \xrightarrow{\imath_*} H_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \dots \\ \dots \to H_0(X, A) \to 0. \end{aligned} \tag{2.3}$$

The connecting homomorphism $\partial_* : H_n(X, A) \longrightarrow H_{n-1}(A)$ has a simple description. If a class in $H_n(X, A)$ is represented by a relative cycle $c \in C_n(X, A)$, the image is the class of $\partial(c)$ in $H_{n-1}(A)$.

There are induced homomorphisms for relative homology groups just as there are in the non-relative, or "absolute", case.

Definition 2.1.3. Let X, Y be topological spaces and let $A \subseteq X$, $B \subseteq Y$ be subspaces. A map $f : X \longrightarrow Y$ with $f(A) \subseteq B$ is called a **pair map** and it is denoted as $f : (X, A) \longrightarrow (Y, B)$.

Pair maps induce well defined homomorphisms $f_\# : C_n(X, A) \to C_n(Y, B)$ in the quotient groups since the chain map $f_\# : C_n(X) \to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$. This happens because for every $\sigma \in \Omega_n(A)$, $f \circ \sigma \in \Omega_n(B)$. The relation $f_\# \circ \partial = \partial \circ f_\#$ clearly holds as it holds before passing to the quotient. Therefore, these new chain maps induce homomorphisms $f_* : H_n(X, A) \longrightarrow H_n(Y, B)$. All the properties given in Proposition 1.3.1 also hold for the induced maps in relative homology.

We now want to give the homotopy invariance result for relative homology. For that, we need to give the notion of homotopy for a pair.

Definition 2.1.4. Two maps $f, g : (X, A) \longrightarrow (Y, B)$ are said to be **pair homotopic** if there exists a homotopy $H : X \times I \longrightarrow Y$ from f to g such that for any $a \in A$ and $t \in I$, $H(a, t) \in B$. In that case, we will say that H is a **homotopy of pairs**.

Definition 2.1.5. Let X, Y be topological spaces and $A \subseteq X$, $B \subseteq Y$. A map $f : (X, A) \longrightarrow (Y, B)$ is said to be a **homotopy equivalence** of the pairs (X, A) and (Y, B) if there is a map $g : (Y, B) \longrightarrow (X, A)$ such that $f \circ g$ and $g \circ f$ are pair homotopic to $\mathbf{Id}_Y : (Y, B) \longrightarrow (Y, B)$ and $\mathbf{Id}_X : (X, A) \longrightarrow (X, A)$ respectively.

If such a map f exists between the pairs (X, A) and (Y, B), the pairs are said to be **homotopy equivalent**.

The homotopy invariance of relative homology groups can be proven in the same way as in the non-relative case. Observe that the two maps defined in Lemma 1.3.2 are pair maps $(X, A) \to (X \times I, A \times I)$, and one can prove that the chain maps induced by them in the relative chain complex are chain homotopic in the same way.

Theorem 2.1.1. *If two pair maps $f, g : (X, A) \longrightarrow (Y, B)$ are pair homotopic, then $f_* = g_* : H_n(X, A) \longrightarrow H_n(Y, B)$.*

We also have the analogue of Proposition 1.4.1 for the relative case.

Proposition 2.1.2. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces and a subset $A_i \subseteq X_i$ for each $i \in I$. Then, the canonical injections $\imath_i : (X_i, A_i) \hookrightarrow (\sqcup_{i\in I} X_i, \sqcup_{i\in I} A_i)$ induce an isomorphism $\oplus_{i\in I}(\imath_i)_* : \bigoplus_{i\in I} H_n(X_i, A_i) \longrightarrow H_n(\sqcup_{i\in I} X_i, \sqcup_{i\in I} A_i)$.*

Proof. We observe that the isomorphism in Proposition 1.4.1

$$\begin{array}{ccc} \bigoplus_{i\in I} C_n(X_i) & \longrightarrow & C_n(\bigsqcup_{i\in I} X_i) \\ \sum_{i\in I} \sum_{\sigma\in\Omega_n(X_i)} \lambda_\sigma \sigma & \longmapsto & \sum_{i\in I} \sum_{\sigma\in\Omega_n(X_i)} \lambda_\sigma (\imath_i \circ \sigma), \end{array}$$

sends $\bigoplus_{i\in I} C_n(A_i)$ to $C_n(\bigsqcup_{i\in I} A_i)$. Therefore, this map induces the following isomorphism:

$$\frac{\oplus_{i\in I} C_n(X_i)}{\oplus_{i\in I} C_n(A_i)} \longrightarrow \frac{C_n(\sqcup_{i\in I} X_i)}{C_n(\sqcup_{i\in I} A_i)} = C_n(\sqcup_{i\in I} X_i, \sqcup_{i\in I} A_i).$$

On the other hand, we also have the following isomorphism:

$$\bigoplus_{i\in I} C_n(X_i, A_i) = \bigoplus_{i\in I} \frac{C_n(X_i)}{C_n(A_i)} \longrightarrow \frac{\oplus_{i\in I} C_n(X_i)}{\oplus_{i\in I} C_n(A_i)}.$$

The composition of both isomorphisms is exactly

$$\oplus_{i\in I}(\imath_i)_\# : \bigoplus_{i\in I} C_n(X_i, A_i) \longrightarrow C_n(\sqcup_{i\in I} X_i, \sqcup_{i\in I} A_i),$$

so it is also an isomorphism. The result for relative homology groups follows as in Proposition 1.4.1. □

An easy generalization of the long exact sequence of a pair (X, A) is the long exact sequence of a triple (X, A, B) where $B \subseteq A \subseteq X$. As $B \subseteq A$, we can consider the inclusion $\imath : C_n(A, B) \hookrightarrow C_n(X, B)$ and the projection $\pi : C_n(X, B) \to C_n(X, A)$. These are chain maps too and they form the following short exact sequence:

$$0 \to C_n(A, B) \xrightarrow{\imath} C_n(X, B) \xrightarrow{\pi} C_n(X, A) \to 0.$$

By Lemma B.2.3 the homology groups of each chain complex fit into the following long exact sequence:

$$\ldots \to H_n(A, B) \xrightarrow{\imath_*} H_n(X, B) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A, B) \to \ldots \quad (2.4)$$

A fundamental property of relative homology groups is the **Excision theorem**, describing when the relative homology groups $H_n(X, A)$ are unaffected by deleting (or excising) a subset $Z \subseteq A$.

Theorem 2.1.3 (Excision theorem)**.** *Let X be a topological space and $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A. Then, the inclusion $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X-Z, A-Z) \to H_n(X, A)$ for all $n \geq 0$.*

Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion $(B, A\cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A\cap B) \to H_n(X, A)$ for all $n \geq 0$.

The proof of this theorem involves some technical results and is included in Appendix E. The equivalence of the two assertions is also explained there.

2.2 Reduced homology groups and the exact sequence for good pairs

Let X be a topological space and $A \subseteq X$. We want to obtain a connection between the relative homology groups and the homology groups of the quotient X/A. To do this, it is convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, including zero. This is done by defining the **reduced homology groups**. Consider the map ξ defined in Proposition 1.4.3:

$$\begin{array}{rccc} \xi : & C_0(X) & \longrightarrow & \mathbb{Z} \\ & \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma & \mapsto & \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma . \end{array}$$

For $\sigma \in C_1(X)$, notice that $\partial(\sigma) = [\sigma]_0 - [\sigma]_1$, so $\xi(\partial(\sigma)) = 1 - 1 = 0$. This means that $\xi \circ \partial = 0$ and, as ξ is surjective, we can extend the usual chain complex to

$$\ldots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \ldots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\xi} \mathbb{Z} \to 0. \quad (2.5)$$

Definition 2.2.1. The homology groups of the chain complex (2.5) are called **reduced homology groups** and are denoted as $\widetilde{H}_n(X)$.

It is clear from the definition that $H_n(X) = \widetilde{H}_n(X)$ for any $n \geq 1$. Moreover, as $\operatorname{Im} \partial_1 \subseteq \operatorname{Ker} \xi$, the map

$$\begin{array}{ccc} H_0(X) & \longrightarrow & \mathbb{Z} \\ c + \operatorname{Im} \partial_1 & \mapsto & \xi(c), \end{array}$$

is a well defined surjective homomorphism. Its kernel is $\frac{\operatorname{Ker}\xi}{\operatorname{Im}\partial_1}$ which is $\widetilde{H}_0(X)$ by definition. By the first isomorphism theorem, we get that

$$\frac{H_0(X)}{\widetilde{H}_0(X)} \cong \mathbb{Z},$$

which is equivalent to saying that $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

Remark 2.2.1. Let X and Y be topological spaces. We will now show that continuous maps $f : X \longrightarrow Y$ also induce homomorphisms in the reduced homology groups. Consider the diagram:

$$\begin{array}{ccccccccccccc}
\dots & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & C_0(X) & \xrightarrow{\xi} & \mathbb{Z} & \xrightarrow{\partial} & 0 \\
 & & \downarrow f_\# & & \downarrow f_\# & & & & \downarrow f_\# & & \downarrow \mathbf{Id} & & \\
\dots & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & C_0(Y) & \xrightarrow{\xi} & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

We already know that $\partial \circ f_\# = f_\# \circ \partial$, so it suffices to show that $\xi = \xi \circ f_\#$. Let $\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma \sigma \in C_0(X)$. Indeed,

$$\xi(f_\#(\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma\sigma)) = \xi(\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma(f\circ\sigma)) = \sum_{\sigma\in\Omega_0(X)} \lambda_\sigma = \xi(\sum_{\sigma\in\Omega_0(X)} \lambda_\sigma\sigma).$$

Thus, we have an induced homomorphism $f_* : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(Y)$ for any $n \geq 0$. This also means that if we restrict an induced homomorphism between non-reduced homology groups $f_* : H_n(X) \longrightarrow H_n(Y)$ to $\widetilde{H}_n(X)$ we get the induced homomorphism between reduced homology groups $\widetilde{H}_n(X) \xrightarrow{f_*} \widetilde{H}_n(Y)$. Thus, Theorem 1.3.3 and Corollary 1.3.4 can be reformulated for the reduced case.

Theorem 2.2.1. *Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. If f and g are homotopic, they induce the same homomorphism in the reduced homology groups.*

In particular, if f is a homotopy equivalence then the induced homomorphism in the reduced homology groups is an isomorphism.

We are first going to relate reduced homology groups and relative homology groups by a long exact sequence. Let us extend each chain in (2.2) as we did in (2.5). In this way, obtain the following diagram:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & C_n(A) & \xrightarrow{\imath} & C_n(X) & \xrightarrow{\pi} & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{\imath} & C_{n-1}(X) & \xrightarrow{\pi} & C_{n-1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & C_0(A) & \xrightarrow{\imath} & C_0(X) & \xrightarrow{\pi} & C_0(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathbf{Id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

It is clear that the diagram is commutative, and the horizontal rows are short exact sequences. Therefore, by Lemma B.2.3 there is a long exact sequence:

$$
\begin{aligned}
\ldots \to \widetilde{H}_n(A) \xrightarrow{\imath_*} \widetilde{H}_n(X) \xrightarrow{\pi_*} H_n(X,A) \xrightarrow{\partial_*} \widetilde{H}_{n-1}(A) \to \ldots \\
\ldots \to H_0(X,A) \to 0.
\end{aligned} \tag{2.6}
$$

From this sequence we get the following result:

Proposition 2.2.2. *Let X be a topological space and $x_0 \in X$ be a point. Then, for every $n \geq 0$ the quotient map $\pi : C_n(X) \longrightarrow C_n(X,x_0)$ induces isomorphisms $\pi_* : \widetilde{H}_n(X) \longrightarrow H_n(X,x_0)$.*

Proof. Observe that if $A = \{x_0\}$, $\widetilde{H}_n(A) \cong 0$ for every n so from the long exact sequence (2.6) we get the following exact sequence:

$$0 \to \widetilde{H}_n(X) \xrightarrow{\pi_*} H_n(X,x_0) \to 0$$

which means that $\widetilde{H}_n(X) \xrightarrow{\pi_*} H_n(X,x_0)$ are isomorphisms. □

Our goal now is to study under which conditions the relative homology groups of a pair (X,A) and the reduced homology groups of the quotient X/A are isomorphic. Once we know this, we will be able to substitute $H_n(X,A)$ by $\widetilde{H}_n(X/A)$ in the sequence (2.6) obtaining the desired exact sequence.

Definition 2.2.2. Let X be a topological space and $A \subseteq X$ a nonempty closed subspace of X. The pair (X,A) is called a **good pair** if there exists some open subset $U \subseteq X$ such that $A \subseteq U$ and A is a strong deformation retract of U.

Proposition 2.2.3. *Let (X, A) be a good pair. The quotient map $q : (X, A) \longrightarrow (X/A, A/A)$ induces isomorphisms*

$$q_* : H_n(X, A) \longrightarrow H_n(X/A, A/A)$$

for all $n \geq 0$.

Proof. Let U be an open subset of X that strongly deformation retracts onto A. On the one hand we have the long exact sequence for the triple (X, U, A):

$$\ldots \to H_n(U, A) \xrightarrow{\imath_*} H_n(X, A) \xrightarrow{\pi_*} H_n(X, U) \xrightarrow{\partial_*} H_{n-1}(U, A) \to \ldots$$

Observe that as A is a strong deformation retract of U the pairs (U, A) and (A, A) are homotopy equivalent. Therefore, $H_n(U, A) \cong H_n(A, A) \cong 0$. Thus, for any n, we have the following exact sequence:

$$0 \to H_n(X, A) \longrightarrow H_n(X, U) \to 0,$$

and $H_n(X, A) \xrightarrow{\pi_*} H_n(X, U)$ is an isomorphism.

On the other hand, U is an strong deformation retract of A, so there is an strong deformation retraction $F : U \times I \to A$ such that $F(a, t) = a$ for any $t \in [0, 1]$ and $a \in A$. Then, we may define $\overline{F} : U/A \times [0, 1] \to A/A$ in the quotient as $\overline{F}(q(x), t) = q(F(x, t))$. It is clearly a strong deformation retract. Therefore, as before, $(U/A, A/A)$ and $(A/A, A/A)$ are homotopy equivalent and from the long exact sequence for the triple $(X/A, U/A, A/A)$ we get an isomorphism $H_n(X/A, A/A) \to H_n(X/A, U/A)$ for every $n \geq 0$.

Moreover, by Theorem 2.1.3 we have isomorphisms $H_n(X - A, U - A) \to H_n(X, U)$ and $H_n(X/A - A/A, U/A - A/A) \longrightarrow H_n(X/A, U/A)$ induced by inclusions $(X - A, U - A) \hookrightarrow (X, U)$ and $(X/A - A/A, U/A - A/A) \hookrightarrow (X/A, U/A)$ respectively. These maps fit into the following diagram:

$$\begin{array}{ccccc}
H_n(X, A) & \longrightarrow & H_n(X, U) & \longleftarrow & H_n(X - A, U - A) \\
\downarrow q_* & & \downarrow q_* & & \downarrow q_* \\
H_n(X/A, A/A) & \longrightarrow & H_n(X/A, U/A) & \longleftarrow & H_n(X/A - A/A, U/A - A/A)
\end{array}$$

The horizontal maps are all isomorphisms. The left isomorphisms are induced by quotient maps of groups and the right isomorphisms are induced by inclusions. Therefore, is clear that the diagram is commutative.

Finally, as A is closed, by Proposition C.2.1 $q_{|X-A}$ is a homeomorphism and thus, the vertical map q_* on the right hand side is an isomorphism. By commutatvity, we have that all the maps q_* are isomorphisms, getting the result.

□

Corollary 2.2.4. *Let* (X, A) *be a good pair. Then, for every* $n \geq 0$ *we have an isomorphism* $H_n(X, A) \to \widetilde{H}_n(X/A)$.

Proof. On the one hand, notice that as the space ${}^{A}\!/_{A}$ is a singleton contained in X/A we have an isomorphism $\widetilde{H}_n(X/A) \xrightarrow{\pi_*} H_n(X/A, A/A)$ for any $n \geq 0$ by Proposition 2.2.2. On the other hand, by Proposition 2.2.3, we have an isomorphism $H_n(X, A) \xrightarrow{q_*} H_n(X/A, A/A)$ for any $n \geq 0$. Thus, the composition

$$H_n(X, A) \xrightarrow{\pi_*^{-1} \circ q_*} \widetilde{H}_n(X/A)$$

is an isomorphism for any $n \geq 0$. □

By the previous corollary and Remark B.2.3 we can substitute $H_n(X, A)$ by $\widetilde{H}_n({}^{X}\!/_{A})$ in the exact sequence (2.6) and the next result follows.

Theorem 2.2.5 (The exact sequence for good pairs)**.** *Let* (X, A) *be a good pair. Then, there is an exact sequence*

$$\ldots \to \widetilde{H}_n(A) \xrightarrow{\imath_*} \widetilde{H}_n(X) \xrightarrow{q_*} \widetilde{H}_n(X/A) \xrightarrow{\partial_*} \widetilde{H}_{n-1}(A) \to \ldots$$
$$\ldots \to \widetilde{H}_0(X/A) \to \{0\}$$

where $\imath : A \hookrightarrow X$ *is the inclusion and* $q : X \longrightarrow X/A$ *the quotient map.*

The homology groups of the spheres $\mathbb{S}^n$ can be computed using Theorem 2.2.5. This is done in Exercise 2 of Appendix A.

Example 2.2.1. For any $n \geq 0$, the reduced homology groups of the sphere are the following:

$$\widetilde{H}_m(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

It is important to remark that Proposition 1.4.1 does not hold in the reduced case. For example, let $X = \{p, q\}$ be two distinct points. By Proposition 1.4.1, $H_0(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and then, $\widetilde{H}_0(X) \cong \mathbb{Z}$. But this is not the same as $\widetilde{H}_0(\{p\}) \oplus \widetilde{H}_0(\{q\}) = 0 \oplus 0 = 0$. The solution is to consider the wedge sum instead of the disjoint union, that is, a "one-point union" of a family of topological spaces. The definition of the wedge sum is given in Section C.3.

Corollary 2.2.6. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. If the wedge sum $\bigvee_{i\in I} X_i$ is formed at base points $x_i \in X_i$ so that the pairs (X_i, x_i) are good, the inclusions $\iota_i : X_i \longrightarrow \bigvee_{i\in I} X_i$ induce an isomorphism $\oplus_{i\in I}(\iota_i)_* : \bigoplus_{i\in I} \widetilde{H}_n(X_i) \longrightarrow \widetilde{H}_n(\bigvee_{i\in I} X_i)$.*

Proof. Let $X = \bigsqcup_{i\in I} X_i$ and $A = \bigsqcup_{i\in I}\{x_i\}$. By definition, $\bigvee_{i\in I} X_i = {}^{X}\!/_{A}$. Let $\jmath_i : X_i \hookrightarrow X$ be the canonical injection and let $q : X \to {}^{X}\!/_{A}$ be the quotient map. Then, $\iota_i = q \circ \jmath_i$ for any $i \in I$.

We first show that the pair (X, A) is good. To prove this, let $U_i \subseteq X_i$ be the open set that strongly deformation retracts to $\{x_i\}$ via $F_i : U_i \times I \to \{x_i\}$. On the one hand, $\imath_j^{-1}(A) = \{x_j\}$ is closed in X_j as (X_j, x_j) is a good pair for any $j \in I$. Thus, $\bigsqcup_{i\in I}\{x_i\}$ is a closed subset. On the other hand, strong deformation retract of X onto A is $F : X \times I \to A$ given by $F((x, i), t) = \imath_i(F_i(x, t))$.

Let $\pi_i : C_n(X_i) \to C_n(X_i, x_i)$ and $\pi : C_n(\bigvee_{i\in I} X_i) = C_n({}^{X}\!/_{A}) \longrightarrow C_n({}^{X}\!/_{A}, {}^{A}\!/_{A})$ be quotient maps of modules. Then, by Proposition 2.1.2, Proposition 2.2.3 and Corollary 2.2.4 we have the following sequence of isomorphisms:

$$\bigoplus_{i\in I} \widetilde{H}_n(X_i) \xrightarrow{\oplus_{i\in I}(\pi_i)_*} \bigoplus_{i\in I} H_n(X_i, x_i) \xrightarrow{\oplus_{i\in I}(\imath_i)_*} H_n(X, A) \xrightarrow{\pi_*^{-1} \circ q_*} \widetilde{H}_n(\bigvee_{i\in I} X_i).$$

This composition is precisely the map $\oplus_{i\in I}(\iota_i)_*$. □

Chapter 3

CW complexes

In this chapter we will follow [2] to provide the definition and basic properties of CW Complexes. We first define what a cell decomposition of a space is, and we define CW complexes as spaces that have a cell decomposition that meets two additional properties. In the last part of the chapter we prove that the spaces that have been built attaching cells of increasing dimensions are also CW complexes, showing that both definitions explained in the introduction are actually equivalent.

3.1 Cell decompositions

Definition 3.1.1. Let $n \geq 0$. An **open n-cell** is a topological space that is homeomorphic to the open unit ball $\mathbb{B}^n$ and a **closed n-cell** is a topological space that is homeomorphic to the closed disk $\mathbb{D}^n$. Points are considered both open and closed 0-cells, since we define $\mathbb{D}^0$ and $\mathbb{B}^0$ to be singletons.

For an open or closed n-cell we say that n is the **dimension** of the cell.

Remark 3.1.1. The fact that the dimension of a cell complex is well defined relies on the Theorem of Invariance of Dimension. This theorem states that no non empty open subset of $\mathbb{R}^n$ can be homeomorphic to any open subset of $\mathbb{R}^m$ if $m \neq n$. Thus, an n-cell can not be a m-cell for $m \neq n$. This theorem is proved in Exercise 4 in Appendix A.

Proposition 3.1.1. *Let $n \geq 1$ and let $D \subseteq \mathbb{R}^n$ be a compact convex subset with nonempty interior. Then, given any point $p \in \operatorname{int} D$, there exists a homeomorphism $F : \mathbb{D}^n \longrightarrow D$ that sends 0 to p, $\mathbb{B}^n$ to $\operatorname{int} D$ and $\mathbb{S}^{n-1}$ to $\operatorname{fr} D$. In particular, D is a closed n-cell and its interior is an open n-cell.*

Proof. Proof left as an exercise. See Appendix A, Exercise 6. □

Definition 3.1.2. Let $n \geq 1$, let D be a closed n-cell and let $f : \mathbb{D}^n \to D$ be a homeomorphism. We define the **boundary** of D as the set $\operatorname{fr} D = f(\mathbb{S}^{n-1})$, and the **interior** of D as $\operatorname{int} D = f(\mathbb{B}^n)$.

Remark 3.1.2. The boundary and the interior of a closed n-cell are well defined because any homeomorphism $\mathbb{D}^n \to \mathbb{D}^n$ maps $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$ and $\mathbb{B}^n$ to $\mathbb{B}^n$. This fact is proved in Excercise 3, Appendix A. Thus, if we have two homeomorphisms $f, g : \mathbb{D}^n \to D$, then, since $g^{-1} \circ f : \mathbb{D}^n \to \mathbb{D}^n$ is a homeomorphism,

$$f(\mathbb{S}^{n-1}) = g\big((g^{-1} \circ f)(\mathbb{S}^{n-1})\big) = g(\mathbb{S}^{n-1}),$$

$$f(\mathbb{B}^n) = g\big((g^{-1} \circ f)(\mathbb{B}^n)\big) = g(\mathbb{B}^n).$$

Let $D \subseteq \mathbb{R}^n$ be a compact convex subset with nonempty interior. Then, Proposition 3.1.1 shows that D is a closed n-cell and the interior and boundary of D defined in Definition 3.1.2 coincide with the topological interior and boundary of D respectively as a subset of the euclidean space.

Examples 1.

- Every closed interval in $\mathbb{R}$ is a closed 1-cell.
- Every compact region in the plane bounded by a regular polygon is a closed 2-cell. A solid tetrahedron and a solid cube are closed 3-cells.

Definition 3.1.3. Let X be a nonempty topological space. A **cell decomposition** of X is a partition $\mathcal{E}$ of X into open cells of various dimensions such that the following condition is satisfied: for each cell $e \in \mathcal{E}$ of dimension $n \geq 1$, there exists a continuous map Φ_e from some closed n-cell D into X that restricts to a homeomorphism from $\operatorname{int} D$ into e and maps $\operatorname{fr} D$ into the union of all cells of $\mathcal{E}$ of dimensions strictly less than n. This map Φ_e is called the **characteristic map for e**.

Definition 3.1.4. A **cell complex** is a Hausdorff space X together with a specific cell decomposition $\mathcal{E}$ of X. The open cells in $\mathcal{E}$ are typically just called the **cells** of X.

Remark 3.1.3. Let $(X, \mathcal{E})$ be a cell complex. Although each $e \in \mathcal{E}$ is an open cell it might not be an open subset of X.

Lemma 3.1.2. *Let $(X, \mathcal{E})$ be a cell complex. For $n \geq 1$, let $e \in \mathcal{E}$ be an n-cell of X and $\Phi : D \longrightarrow X$ be its characteristic map. Then,*

(i) $\Phi(D) = \operatorname{cl}(e)$.

(ii) $\Phi(\operatorname{fr} D) = \operatorname{cl}(e) - e$.

In particular, $\operatorname{cl}(e) - e$ is contained in a union of cells of strictly less dimension than n.

Proof. Since Φ is a continuous map between a compact and a Hausdorff space, it is a closed map. As it is closed and continuous $\Phi(\operatorname{cl}(A)) = \operatorname{cl}(\Phi(A))$ for any subset $A \subseteq D$. In particular,

$$\Phi(D) = \Phi(\operatorname{cl}(\operatorname{int} D)) = \operatorname{cl}(\Phi(\operatorname{int} D)) = \operatorname{cl}(e).$$

From here, $\text{cl}(e) - e = \Phi(D) - \Phi(\text{int}\, D) = \Phi(D - \text{int}\, D) = \Phi(\text{fr}\, D)$, and we deduce by definition of cell complexes that it is contained in a union of cells of strictly less dimension. □

Definition 3.1.5. Let $(X, \mathcal{E})$ be a cell complex. We say that $(X, \mathcal{E})$ is **finite** if $\mathcal{E}$ is a finite set. The cell complex is called **locally finite** if the collection of open cells $\mathcal{E}$ is locally finite.

For the general definition of local finiteness see Section C.5. We will later see that locally finite cell complexes (and thus, finite cell complexes) automatically satisfy the additional conditions to be a CW complex.

Remark 3.1.4. It is perfectly possible for a given space to have many different cell decompositions. Technically, the term *cell complex* refers to a space together with a specific cell decomposition.

3.2 CW complexes

For finite cell complexes the definitions given so far serve well, but for infinite complexes to have the desired properties, two additional restrictions must be added.

Definition 3.2.1. A **CW complex** is a cell complex $(X, \mathcal{E})$ satisfying the following additional conditions:

(C) The closure of each cell is contained in a union of finitely many cells.

(W) The topology of X is coherent with the family $\text{cl}(\mathcal{E}) = \{ \text{cl}(e) \mid e \in \mathcal{E} \}$. That is, a subset $C \subseteq X$ is closed in X if and only if $\text{cl}(e) \cap C$ is closed in $\text{cl}(e)$ for any $e \in \mathcal{E}$.

A cell decomposition of a space X satisfying (C) and (W)* is called a **CW decomposition of X**. If a space X admits a CW decomposition we usually say it is a CW complex and we omit writing $\mathcal{E}$ explicitly.

The general definition and properties of coherent topologies are given in Section C.6.

Remark 3.2.1. Observe that by Lemma 3.1.2 and by condition (C), for each cell e of a CW complex, $\text{cl}(e) - e$ is contained in a finite union of cells of strictly less dimension.

For locally finite complexes (and thus all finite ones) the two conditions are automatically satisfied as next proposition shows.

*The letters C and W come from the names originally J. H. C. Whitehead gave to these two conditions. Condition (C) was called **closure-finiteness**, and the coherent topology described in (W) was called **weak topology**.

Proposition 3.2.1. *Let X be a Hausdorff space and let $\mathcal{E}$ be a cell decomposition of X. If $\mathcal{E}$ is locally finite, then it is a CW decomposition of X.*

Proof. To prove the first condition observe that for each $e \in \mathcal{E}$, every point in $\mathrm{cl}(e)$ has a neighbourhood that intersects only finitely many cells of $\mathcal{E}$ by local finiteness. Since $\mathrm{cl}(e)$ is compact, it can be covered with finitely many of those neighbourhoods.

To prove the second condition, suppose that the intersection of $A \subseteq X$ with all $\mathrm{cl}(e) \in \mathrm{cl}(\mathcal{E})$ is closed in $\mathrm{cl}(e)$. By Proposition C.5.1 in Appendix C, $\mathcal{E}$ being locally finite is equivalent to $\mathrm{cl}(\mathcal{E})$ being locally finite. Thus, given $x \in X - A$, let W be a neighbourhood of x that intersects the closures of only finitely many cells, say $\mathrm{cl}(e_1), \dots, \mathrm{cl}(e_n)$. Since $A \cap \mathrm{cl}(e_i)$ is closed in $\mathrm{cl}(e_i)$, and each $\mathrm{cl}(e_i)$ is closed in X, it follows that each intersection $A \cap \mathrm{cl}(e_i)$ is closed in X. Thus, the set

$$W - A = W - \Big(\big(A \cap \mathrm{cl}(e_1)\big) \cup \ldots \cup \big(A \cap \mathrm{cl}(e_n)\big)\Big)$$

is an open neighbourhood of x contained in $X - A$. Hence, $X - A$ is open, and A is closed in X. □

Definition 3.2.2. Let X be a CW complex. If there is an integer n such that all the cells of X have dimension at most n, we say that X is **finite-dimensional**. Otherwise, we say it is **infinite-dimensional**.

If X is finite dimensional, the **dimension of X** is the largest integer n such that X contains at least one n-cell. We will write it by $\dim X$.

Remark 3.2.2. It is clear that finite complexes are always finite dimensional.

Here is a case in which open cells actually are open subsets of X.

Proposition 3.2.2. *Suppose that X is an n-dimensional CW complex. Then, every n-cell of X is an open subset of X.*

Proof. If $n = 0$, X is a discrete space. Let $n \geq 1$. Suppose that e_0 is an n-cell of X and let $\Phi : D \longrightarrow X$ be the characteristic map for e_0.

If we restrict the codomain of Φ and consider it as a map onto $\mathrm{cl}(e_0)$, $\Phi : D \longrightarrow \mathrm{cl}(e_0)$ is a continuous map between a compact and a Hausdorff space. Thus, it is a closed map. Since it is surjective, we have that it is an identification map†. Thus, since $\Phi^{-1}(e_0) = \mathrm{int}\, D$ is open in D, it follows

†A surjective map $f : X \to Y$ is an identification map (also called a quotient map, for example, in [2]) if it satisfies that $V \subseteq Y$ is open if and only if $f^{-1}(V)$ is open. Equivalently, f is an identification map if it satisfies that $C \subseteq Y$ is closed if and only if $f^{-1}(C)$ is closed. For more detailed information, check [2, Chapter 2, Quotient spaces].

that e_0 is open in $\mathrm{cl}(e_0)$.

On the other hand, if e is any other cell of X, $e \cap e_0 = \emptyset$ so $e_0 \cap \mathrm{cl}(e)$ is contained in $\mathrm{cl}(e) - e$. As e has dimension at most n (X is n-dimensional) $\mathrm{cl}(e) - e$ is contained in a union of cells of strictly less dimension than n. Since e_0 has dimension n, it follows that $e_0 \cap \mathrm{cl}(e) = \emptyset$. Thus, the intersection of e_0 with the closure of every cell is open and by (W) e_0 is open in X. □

Definition 3.2.3. Let X be a CW complex. A **subcomplex** of X is a subspace $Y \subseteq X$ that is a union of cells of X such that if Y contains a cell, it also contains its closure.

The following proposition follows inmediately from the definition.

Proposition 3.2.3. *The union and intersection of any collection of subcomplexes of a CW complex are themselves subcomplexes.*

Theorem 3.2.4. *Let X be a CW complex and $Y \subseteq X$ a subcomplex. Then,*

(i) *Y is a CW complex with the subspace topology and the cell decomposition that inherits from X.*

(ii) *Y is closed in X.*

Proof. We begin proving (i). Clearly Y is Hausdorff and by definition it is the disjoint union of its cells. In addition, for any cell $e \subseteq Y$ a characteristic map for e in X serves as characteristic map for e in Y, so Y is a cell complex.

To check condition (C), let $e \subseteq Y$ be a cell of Y and note that $\mathrm{cl}(e)$ is contained in a union of finitely many cells of X. Since $\mathrm{cl}(e) \subseteq Y$, these cells must also be cells of Y.

To check condition (W), let $S \subseteq Y$ be a subset such that $Y \cap \mathrm{cl}(e)$ is closed in $\mathrm{cl}(e)$ for any cell e contained in Y. We have to show that S is closed in Y. Let e_0 be a cell of X that is not contained in Y. We know that $\mathrm{cl}(e_0) - e_0$ is contained in the union of finitely many cells of X. Some of these, say $e_1, \ldots, e_k$, might be contained in Y. Then $\mathrm{cl}(e_1) \cup \ldots \cup \mathrm{cl}(e_k) \subseteq Y$ and

$$\begin{aligned} S \cap \mathrm{cl}(e_0) &= S \cap \Big(\mathrm{cl}(e_1) \ \cup \ \ldots \ \cup \ \mathrm{cl}(e_k) \Big) \cap \mathrm{cl}(e_0) \\ &= \Big(\big(S \cap \mathrm{cl}(e_1)\big) \cup \ldots \cup \big(S \cap \mathrm{cl}(e_k)\big) \Big) \cap \mathrm{cl}(e_0), \end{aligned}$$

which is closed in $\mathrm{cl}(e_0)$. As $\mathrm{cl}(e_0)$ is closed in X, S is closed in X and therefore in Y.

Finally to show (ii) just follow the preceding paragraph for $S = Y$. □

Definition 3.2.4. For each $n \geq 0$, the **n-skeleton of X** is the subspace $X_n \subseteq X$ consisting of the union of all cells of dimensions less than or equal to n.

Proposition 3.2.5. *Let X be a CW complex. For any $n \geq 0$, the n-skeleton X_n is a subcomplex of X and has dimension at most n.*

Proof. By definition each X_n is a union of cells of X of dimension at most n, and we know that the closure of each cell in X is contained in a union of cells of equal or strictly less dimension. Therefore, the closure is in X_n. □

Proposition 3.2.6. *Let X be a CW complex. The topology of X is coherent with the collection of n-skeletons $\{X_n\}_{n\geq 0}$.*

Proof. Let $A \subseteq X$ be a subset such that $A \cap X_n$ is closed for every $n \geq 0$. Let e be any cell of X of dimension n. Then, $\mathrm{cl}(e) \subseteq X_n$, and so $A \cap \mathrm{cl}(e) = (A \cap X_n) \cap \mathrm{cl}(e)$ is closed in $\mathrm{cl}(e)$. As this happens for any cell e of X, by condition (W) A is closed in X. □

Finally we address the question of compactness, which is easy to detect in CW complexes.

Lemma 3.2.7. *In any CW complex, the closure of each cell is contained in a finite subcomplex.*

Proof. Let X be a CW complex and let $e \subseteq X$ be an n-cell. We prove the result by induction on n. If $n = 0$, e is a point. Since X is Hausdorff, $\mathrm{cl}(e) = e$ so the result is true.

We assume the result for any cell of dimension less than n. By condition (C), $\mathrm{cl}(e) - e$ is contained in the union of finitely many cells of dimension lower than n, each of them contained in a finite subcomplex by induction hypothesis. The union of these finite subcomplexes together with e is a finite subcomplex containing $\mathrm{cl}(e)$. □

Lemma 3.2.8. *Let X be a CW complex. A subspace of X is closed and discrete if and only if its intersection with each cell is finite.*

Proof. Suppose $S \subseteq X$ is closed and discrete. For each cell e of X, since $S \cap \mathrm{cl}(e)$ is a closed subset of the compact set $\mathrm{cl}(e)$, $S \cap \mathrm{cl}(e)$ is compact too. Moreover, $S \cap \mathrm{cl}(e)$ is also a discrete space, and a compact, discrete space must be finite. Hence, the intersection $S \cap \mathrm{cl}(e)$ is finite and so is $S \cap e$.

Conversely, suppose that S is a subset whose intersection with each cell is finite. Let $E \subseteq S$. The intersection of each cell with E is also finite. By Lemma 3.2.7 the closure of each cell e is contained in a finite subcomplex of X, so the hypothesis implies that $E \cap \mathrm{cl}(e)$ is finite. Thus, as X is Hausdorff,

$E \cap \mathrm{cl}(e)$ is closed in $\mathrm{cl}(e)$ for any cell e and by (W), E is closed in X. We have proved that any subset of S (even S itself) is closed in X. Therefore S is a closed and discrete subspace of X. □

Theorem 3.2.9. *Let X be a CW complex. A subset of X is compact if and only if it is closed in X and contained in a finite subcomplex.*

Proof. Let $Y \subseteq X$ be a finite subcomplex and let $e_1, \dots, e_k$ be the family of cells of Y inherited from X. Then,

$$Y = \bigcup_{i=1}^{k} \mathrm{cl}(e_i)$$

is a finite union of compact sets, so it is compact. Thus, any closed subset $K \subseteq X$ contained in a finite subcomplex must be also compact.

Conversely, suppose that $K \subseteq X$ is compact. By contradiction, if K intersects infinitely many cells, by choosing one point in each intersection we obtain an infinite closed discrete subset of K which is impossible by Lemma 3.2.8. Therefore, K is contained in the union of finitely many cells, and the closure of each such cell is contained in a finite subcomplex by Lemma 3.2.7. Thus, K is contained in the union of all of those finite subcomplexes, which is a finite subcomplex of X. □

The following corollary immediately follows from the theorem.

Corollary 3.2.10. *A CW complex is compact if and only if it is a finite complex.*

3.3 Inductive construction of CW complexes

In this final subsection we describe how to construct CW complexes by attaching cells of succesively higher dimensions. In Section C.4 we define **adjunction spaces**, which formalize the notion of "attaching" a topological space to another.

Lemma 3.3.1. *Let X be a CW complex. Let $\{e_i\}_{i\in I}$ be the collection of cells of X and for each $i \in I$, let $\Phi_i : D_i \longrightarrow X$ be the characteristic map of the cell e_i. Then, the map $\Phi : \bigsqcup_{i\in I} D_i \longrightarrow X$ whose restriction to each D_i is Φ_i, is an identification map.*

Proof. The map is clearly surjective by definition of a CW complex. Moreover, the restriction of Φ to each D_i is Φ which is continuous, so Φ is continuous by Theorem C.1.3.

Let $C \subseteq \bigsqcup_{i\in I} D_i$ be a closed set. Observe that each Φ_i is a closed map as it is a continuous function between a compact and a Hausdorff space. Thus, for each $i \in I$,

$$\Phi(C) \cap \mathrm{cl}(e_i) = \Phi_i(C \cap D_i)$$

is closed in $\mathrm{cl}(e_i)$ and by (W) $\Phi(C)$ is closed in X. Therefore Φ is an identification. □

The next proposition shows that a topological space with a CW decomposition can be seen as a space constructed by inductively attaching its n-skeletons.

Proposition 3.3.2. *Let X be a CW complex. Each n-skeleton X_n is obtained from X_{n-1} by attaching a collection of n-cells.*

Proof. Let $\{e_i^n\}_{i\in I}$ be the collection of n-cells of X and for each n-cell e_n^i, let $\Phi_i^n : D_i^n \longrightarrow X$ be a characteristic map. We define $\varphi : \bigsqcup_{i\in I} \mathrm{fr}\, D_i^n \to X$ to be the map whose restriction to each $\mathrm{fr}\, D_i^n$ is equal to the restriction of Φ_i^n to $\mathrm{fr}\, D_i^n$. By definition of a cell complex φ takes its values in X_{n-1} so we can form the adjunction space $X_{n-1} \bigcup_\varphi \left(\bigsqcup_{i\in I} D_i^n \right)$.

Consider the map $\Phi : X_{n-1} \bigsqcup \left(\sqcup_{i\in I} D_i^n \right) \longrightarrow X_n$ that is equal to the inclusion on X_{n-1} and to $\Phi_{i\in I}^n$ on each D_i^n. Since it makes the same identifications as the quotient map of the space $X_{n-1} \bigcup_\varphi \left(\bigsqcup_{i\in I} D_i^n \right)$, if we show that Φ is an identification map, by uniqueness of quotient spaces‡ we get that X_n is homeomorphic to the adjunction space, as desired.

On the one hand, the restriction of Φ to X_{n-1} is the inclusion map $X_{n-1} \hookrightarrow X_n$, which is continuous from Proposition 3.2.6. On the other hand, the restriction of Φ to each D_i^n is Φ_i^n which is continuous by definition. Thus, Φ is continuous. It is also clear that it is surjective.

To conclude that it is an identification map it is left to show that if $\Phi^{-1}(B)$ is closed in $X_{n-1} \bigsqcup \left(\sqcup_{i\in I} D_i^n \right)$ for some $B \subseteq X_n$, then B is closed in X_n. Notice that $\Phi^{-1}(B)$ being closed means that $\Phi^{-1}(B) \cap X_{n-1}$ is closed in X_{n-1} and that each $\Phi^{-1}(B) \cap D_i^n$ is closed in D_i^n.

On the one hand, $\Phi_{|X_{n-1}}$ is the inclusion $X_{n-1} \hookrightarrow X_n$. Hence, $\Phi^{-1}(B) \cap X_{n-1} = (\Phi_{|X_{n-1}})^{-1}(B \cap X_{n-1}) = B \cap X_{n-1}$ and by the first assertion $B \cap X_{n-1}$ is closed in X_{n-1}, which means that $B \cap \mathrm{cl}(e)$ is closed in $\mathrm{cl}(e)$ for all cells of dimension strictly smaller than n. On the other hand, $\Phi_{|D_i^n} = \Phi_i^n$ and $\Phi^{-1}(B) \cap D_i^n = (\Phi_i^n)^{-1}(B \cap \mathrm{cl}(e_i^n))$ is closed in D_i^n by the second assertion.

‡Check [2, Theorem 3.75]

Characteristic maps are closed maps so $\Phi_i^n((\Phi_i^n)^{-1}(B \cap \mathrm{cl}(e_i^n))) = B \cap \mathrm{cl}(e_i^n)$ is closed in $\mathrm{cl}(e_i^n)$. Thus, the intersection of B with the closure of each n-cell of X_n is also closed and therefore, B is closed in X_n. $\square$

The next theorem shows the converse of the previous proposition: it shows that a space built by attaching cells of successively higher dimensions is a CW complex, which means that the two definitions of CW complexes are equivalent.

Theorem 3.3.3. *Suppose that $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ is a sequence of topological spaces satisfying the following conditions:*

(i) *X_0 is a non empty discrete space.*

(ii) *For each $n \geq 1$, X_n is obtained from X_{n-1} by attaching a (possibly empty) collection of n-cells.*

Then $X = \bigcup_{n\geq 0} X_n$ has a unique topology coherent with the family $\{X_n\}_{n\geq 0}$, and a unique cell decomposition making it into a CW complex whose n-skeleton is X_n for each n.

Proof. By hypothesis, X_0 is a discrete nonempty space and for each $n \geq 1$, we have attached a union of some closed n-cells $\bigsqcup_{i\in I_n} D_i^n$ to X_{n-1} by an attaching map $\phi_n : \bigsqcup_{i\in I_n} \mathrm{fr}\, D_i^n \to X_{n-1}$. Let $q_n : X_{n-1} \bigsqcup \left(\sqcup_{i\in I_n} D_i^n \right) \to X_n$ be the quotient map of the adjunction space. By Proposition C.4.1 in Appendix C, q_n embeds each X_{n-1} in X_n as a closed subspace and each $\bigsqcup_{i\in I_n} D_i^n - \bigsqcup_{i\in I_n} \mathrm{fr}\, D_i^n = \bigsqcup_{i\in I_n} \mathrm{int}\, D_i^n$ as an open subspace.

We give a topology on X by declaring a subset $C \subseteq X$ to be closed if and only if $C \cap X_n$ is closed for each $n \geq 0$. It is immediate that this is a topology: the unique topology on X coherent with $\{X_n\}_{n\geq 0}$.

With this topology each X_n is a subspace of X. If $C \subseteq X$ is closed in X, each $C \cap X_n$ is closed by definition of the topology. Conversely, if $C \subseteq X_n$ is closed in X_n, since each X_{m-1} is closed in X_m it follows that $C \cap X_m$ is closed in X_m for any $m \geq 0$ and thus C is closed in X.

Next we define the cell decomposition of X. Note that $X_n - X_{n-1}$ is an open subset of X_n homeomorphic to $\bigsqcup_{i\in I_n} \mathrm{int}\, D_i^n$, which is a disjoint union of open n-cells. For every $n \geq 1$, we define the n-cells of X to be the components $\{e_i^n\}_{i\in I_n} = \{q_n(\mathrm{int}\, D_i^n)\}_{i\in I_n}$ of $X_n - X_{n-1}$. These are subspaces of X_n, and hence of X, and X is the disjoint union of all of them and X_0. For each n-cell e_j^n, we define the characteristic map $\Phi_j^n : D_j^n \longrightarrow X$ as the composition

$$D_j^n \hookrightarrow X_{n-1} \bigsqcup \left(\sqcup_{i\in I_n} D_i^n \right) \xrightarrow{q_n} X_n \hookrightarrow X,$$

where the first and last maps are inclusions and the one in the middle is the quotient map. The first inclusion is continuous because inclusions in disjoint union spaces are continuous. The last inclusion is continuous by the definition of the topology. So it is clear that Φ_j^n is continuous. Moreover, we have built the attaching spaces so that q_n maps the boundary of each closed n-cell to X_{n-1} so it is clear that Φ_j^n maps $\operatorname{fr} D_j^n$ to X_{n-1}. Finally, the restriction of Φ_j^n to $\operatorname{int} D_j^n$ is equal to the inclusion of $\operatorname{int} D_j^n$ into the disjoint union, an embedding, followed by the restriction of q_n into $\operatorname{int} D_j^n$, which is a homeomorphism onto e_j^n, and finally the inclusion into X. Thus, $\Phi^n_{j|_{\operatorname{int} D_j^n}}$ is a homeomorphism onto e_j^n. This proves that X has a cell decomposition for which X_n is the n-skeleton for each $n \geq 0$. Since the n-cells of any such decomposition are the components of $X_n - X_{n-1}$, this is the unique such cell decomposition.

To show that X is a cell complex, it is left to prove that it is Hausdorff. This proof is quite technical and it is included in Section C.7.

To finish the proof we show that X satisfies conditions (C) and (W). If X contains only finitely many cells we can stop here because every finite cell complex is automatically a CW complex. For the general case, first we prove by induction on n that these conditions are satisfied by X_n for each n. They obviously hold for X_0 since it is a discrete space. Suppose they hold for X_k, $0 \leq k < n$, that is, suppose that X_k is a CW complex if $0 \leq k < n$.

To prove that X_n satisfies (C), notice that for any k-cell with $1 \leq k \leq n$, $\operatorname{cl}(e_i^k) - e_i^k = \Phi(e_i^k)$ is a compact subset of the CW complex X_{k-1}, and therefore by Theorem 3.2.9 it is contained in a finite subcomplex of X_{k-1}. Therefore $\operatorname{cl}(e_i^k)$ is contained in a union of finitely many cells.

To check (W), suppose that $B \subseteq X_n$ has a closed intersection with $\operatorname{cl}(e)$ for any cell e in X_n. Since $B \cap \operatorname{cl}(e_i^k)$ is closed in $\operatorname{cl}(e_i^k)$ for every k-cell e_i^k for $0 \leq k < n$ and X_{n-1} satisfies condition (W), $B \cap X_{n-1}$ is closed in X_{n-1}. Also, $B \cap \operatorname{cl}(e_i^n)$ is closed for any n-cell e_i^n. Then, $q_n^{-1}(B)$ is closed in $X_{n-1} \bigsqcup \big(\sqcup_{i \in I_n} D_i^n \big)$ because, on the one hand, $q_n^{-1}(B) \cap X_{n-1} = q^{-1}_{n|_{X_{n-1}}}(B \cap X_{n-1})$ is closed in X_{n-1}, and on the other hand $q_n^{-1}(B) \cap D_i^n = q^{-1}_{n|_{D_i^n}}(B \cap \operatorname{cl}(e_i^n))$ is closed in D_i^n. Therefore, B is closed in X_n by definition of the quotient topology on X_n.

Finally, we show that X itself satisfies conditions (C) and (W). Condition (C) follows because the closure of each cell lies in some X_n, and X_n is a CW complex. To prove (W), suppose $B \subseteq X$ has a closed intersection with $\operatorname{cl}(e)$ for every cell e in X. Then by the discussion in the preceding paragraph $B \cap X_n$ is closed of any $n \geq 0$, so B is closed in X by definition of the topology on X. □

We finish the section talking about quotients of CW complexes.

Theorem 3.3.4. *Let X be a CW complex and let $Y \subseteq X$ be a subcomplex. Then, the quotient ${}^{X}\!/\!_{Y}$ inherits a CW complex structure from X.*

Proof. Let $\{e_i^n\}_{i\in I_n}$ be the collection of n-cells of X and $\Phi_i^n : D_i^n \to X$ be the characteristic map of each e_i^n.

Let $q : X \to {}^{X}\!/\!_{Y}$ be the quotient map. By Theorem 3.2.4 Y is closed in X so $q_{|X-Y} : X-Y \to {}^{X}\!/\!_{Y} - {}^{Y}\!/\!_{Y}$ is a homeomorphism by Proposition C.2.1.

Let us give a cell decomposition of ${}^{X}\!/\!_{Y}$. Observe that ${}^{Y}\!/\!_{Y}$ is a point, so all cells contained in Y become 0-cells in the quotient.

The n-cells of $X - Y$ are embedded as n-cells in ${}^{X}\!/\!_{Y}$ by q. The characteristic maps of the cells can be taken to be the compositions

$$D_i^n \xrightarrow{\Phi_i^n} X - Y \xrightarrow{q} {}^{X}\!/\!_{Y} - {}^{Y}\!/\!_{Y}.$$

As $q_{|X-Y}$ is a homeomorphism it is clear that it is is a characteristic map. As X is Hausdorff, ${}^{X}\!/\!_{Y}$ is Hausdorff too so it is a cell complex.

Conditions (C) and (W) follow for this cell decomposition by definition of the quotient topology.

□

3.4 Examples of CW complexes

We finish the chapter giving some examples of Cell and CW decompositions.

3.4.1 Examples of cell decompositions that are not CW decompositions

These two examples have been taken from [2].

Example 3.4.1 (Failure of condition (W))**.** Let $X \subseteq \mathbb{R}^2$ be the union of the closed line segments from the origin to $(1, 0)$ and to the points $(1, \frac{1}{n})$ for $n \in \mathbb{N}$ with the subspace topology. Call ℓ_0 to the line segment to $(1, 0)$ and ℓ_n to the line segment to $(1, \frac{1}{n})$.

Define a cell decomposition as follows:

- The 0 cells are $(0, 0)$, $p_0 = (1, 0)$ and $p_n = (1, \frac{1}{n})$.
- The 1-cells are the line segments minus their endpoints: $e_n = \ell_n - \{(0, 0), p_n\}$, $n \in \mathbb{N} \cup \{0\}$.

It is clearly a cell decomposition. Moreover, the closure of each 0-cell is the 0-cell itself, and the closure of each 1-cell is $\mathrm{cl}(e_n) = e_n \cup \{p_n\} \cup \{(0,0)\} = \ell_n$ so condition (C) holds.

However, condition (W) does not hold. The intersections of $E = \{(\frac{1}{n}, \frac{1}{n^2})\}_{n\in\mathbb{N}}$ with the cells of X are closed but the set itself is not closed in X, because it has the origin as a limit point and $(0,0) \notin E$.

Example 3.4.2 (Failure of condition (C))**.** We define a cell decomposition of $\mathbb{D}^2$ as follows:

- Countably many 0-cells consisting of $\Big\{\big(\cos(\frac{2\pi}{n}), \sin(\frac{2\pi}{n})\big)\Big\}_{n\in\mathbb{N}}$.
- Countably many 1-cells consisting of the open arcs between the 0-cells.
- A single 2-cell consisting of the interior of the disk.

Condition (W) does hold because the closure of the 2-cell is the whole space $\mathbb{D}^2$ so if a subset has closed intersection with the closures of all the cells in particular the intersection with $\mathbb{D}^2$ is closed, which means that it is closed in $\mathbb{D}^2$. Condition (C) does not hold, though. For example, the closure of the 2-cell is not contained in a union of finitely many cells.

3.4.2 Examples of CW decompositions

Example 3.4.3 (Graphs)**.** A CW complex of dimension less than or equal to 1 is a **graph**. The 0-cells are the vertices and the 1-cells are the edges of the graph.

Example 3.4.4 (A CW decomposition of $\mathbb{R}$)**.** A cell decomposition of $\mathbb{R}$ is obtained by defining the 0-cells to be the integers, and the 1-cells to be the intervals $(n, n+1)$ for $n \in \mathbb{Z}$ with characteristic maps $[n, n+1] \to \mathbb{R}$ given by inclusion. The conditions (C) and (W) follow because it is a locally finite decomposition.

Example 3.4.5 (CW decomposition of $\mathbb{S}^n$)**.** We give a CW decompositon of $\mathbb{S}^n$ with only one 0-cell and one n-cell. The 0 cell is the north pole $N = (0, \dots, 0, 1)$ and the n-cell is $\mathbb{S}^n - N$. A characteristic map for the n-cell is

$$\begin{array}{rccc} \Phi: & \mathbb{D}^n & \longrightarrow & \mathbb{S}^n \\ & x & \mapsto & (2\sqrt{1-|x|^2}\, x,\ 2|x|^2 - 1), \end{array}$$

which collapses $\mathrm{fr}\,\mathbb{D}^n = \mathbb{S}^{n-1}$ to N.

Example 3.4.6 (The infinite dimensional sphere)**.** We give a CW decomposition of $\mathbb{S}^n$ with two cells of each dimension $0, \dots, n$. We will build it inductively:

- Start with $\mathbb{S}^0$, a discrete space of two points.
- For each $n \geq 1$, obtain $\mathbb{S}^n$ from $\mathbb{S}^{n-1}$ by attaching two n-cells $\mathbb{D}^n \sqcup \mathbb{D}^n$ with an attaching map

$$\phi_n : \operatorname{fr} \mathbb{D}^n \sqcup \operatorname{fr} \mathbb{D}^n = \mathbb{S}^{n-1} \sqcup \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$$

  whose restriction to each $\mathbb{S}^{n-1}$ is the identity on $\mathbb{S}^{n-1}$.

We obtain a CW decomposition of $\mathbb{S}^n$ which has $\mathbb{S}^k$ as the k-skeleton for each $k = 0, \dots, n$. Continuing this process, we obtain an infinite dimensional CW complex $\mathbb{S}^\infty = \bigcup_{n\geq 0} \mathbb{S}^n$ with two cells in every dimension. It contains every sphere $\mathbb{S}^n$ as a subcomplex.

Example 3.4.7 (Wedge sum of spheres)**.** Let $X = \bigvee_{i\in I} \mathbb{S}^n$ be a wedge sum of spheres formed by gluing the north poles $N \in \mathbb{S}^n$. A CW decomposition is given as follows:

- A 0-cell, the base point $[N] \in \bigvee_{i\in I} \mathbb{S}^n$.
- A n-cell for each sphere in the wedge sum. A characteristic map Φ_i for each cell is

$$\mathbb{D}^n \xrightarrow{\Phi} \mathbb{S}^n \xhookrightarrow{\imath_i} \bigsqcup_{i\in I} \mathbb{S}^n \xrightarrow{q} \bigvee_{i\in I} \mathbb{S}^n,$$

  where Φ is the map defined in Example 3.2.5, $\imath_i$ is the inclusion and q the quotient map.

Let X be a CW complex and consider the quotient of the n-skeleton by the $(n-1)$-skeleton X_n/X_{n-1}. This is a CW complex with an n-cell for each n-cell of X and a 0-cell. Consider the wedge sum of spheres $\bigvee_i \mathbb{S}^n$, joining as many spheres as n-cells of X. This is also a CW complex with the same number of n-cells as X and a 0-cell. There is really one way of adjoining those n-cells to a point (send all boundaries to the 0-cell) so this two spaces must be homeomorphic.

Example 3.4.8 (The real projective space)**.** The real projective space $\mathbb{RP}^n$ is defined as the quotient space of $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation $x \sim y$ if $y = \lambda x$ for some $\lambda \neq 0$. Restricting to vectors of length 1, one can also define $\mathbb{RP}^n$ to be the quotient space of $\mathbb{S}^n$ with antipodal points identified.

If $n = 0$, $\mathbb{RP}^n$ is just a point, so it is a 0-cell. If $n \geq 1$, let $\mathbb{D}^n/\sim$ be the quotient space of $\mathbb{D}^n$ identifying antipodal points of $\operatorname{fr} \mathbb{D}^n = \mathbb{S}^{n-1}$. Observe that $[x] \mapsto [(x, \sqrt{1 - ||x||^2})]$ defines a homeomorphism $\mathbb{D}^n/\sim \longrightarrow \mathbb{S}^n/\sim = \mathbb{RP}^n$. This map identifies $\operatorname{fr} \mathbb{D}^n/\sim = \operatorname{fr} \mathbb{S}^{n-1}/\sim$ with the points

$[(x_1, \ldots, x_{n-1}, 0)] \in \mathbb{RP}^n$. Since $\operatorname{fr} \mathbb{D}^n = \mathbb{S}^{n-1}$ with antipodal points identified is just $\mathbb{RP}^{n-1}$, we conclude that

$$\mathbb{RP}^n = \mathbb{D}^n \cup_q \mathbb{RP}^{n-1},$$

where $q : \mathbb{S}^{n-1} \to {}^{\mathbb{S}^{n-1}}\!/\!\sim\, = \mathbb{RP}^{n-1}$ is the quotient map.

It follows by induction that $\mathbb{RP}^n$ has a CW complex structure with one k-cell for each $k = 0, \ldots, n$. If we continue this process, the union $\mathbb{RP}^\infty = \cup_{n\geq 0}\mathbb{RP}^n$ is a cell complex with one cell in each dimension.

Example 3.4.9 (The complex projective space)**.** The complex projective space $\mathbb{CP}^n$ is defined as the quotient space of $\mathbb{C}^{n+1} - \{0\}$ under the equivalence relation $x \sim y$ if $y = \lambda x$ for some $\lambda \neq 0$. Equivalently, $\mathbb{CP}^n$ is the quotient of the unit sphere $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ with $x \sim y$ if $y = \lambda x$ with $|\lambda| = 1$.

If $n = 0$, $\mathbb{CP}^0$ is just a point, so it is a 0-cell. Let $n \geq 1$. The points in $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^n$ with last coordinate real and nonnegative are of the form $(z, \sqrt{1 - |z|}) \in \mathbb{C}^n \times \mathbb{C}$ with $|z| \leq 1$. Let D be the the set containing such points. Then, the map $z \mapsto (z, \sqrt{1 - |z|})$ from $\mathbb{D}^{2n} \subseteq \mathbb{C}^n$ to D is a homeomorphism. D contains a copy of the sphere $\mathbb{S}^{2n-1}$, consisisting of the points $(z, 0) \in D$ with $|z| = 1$. We identify the points $x, y \in \mathbb{S}^{2n-1} \subseteq D$ if $y = \lambda x$ with $|\lambda| = 1$, and we show that the inclusion $D \hookrightarrow \mathbb{S}^{2n+1}$ induces a homeomorphism ${}^{D}\!/\!\sim\, \to {}^{\mathbb{S}^{2n+1}}\!/\!\sim\, = \mathbb{CP}^n$ in the quotient.

To that aim, we distinguish two cases. On the one hand, if $(z_1, \ldots, z_n, z_{n+1}) \in \mathbb{S}^{2n+1}$ with $z_{n+1} \neq 0$,

$$\begin{aligned}(z_1, \ldots, z_n, z_{n+1}) &\sim \frac{\overline{z_{n+1}}}{|z_{n+1}|}(z_1, \ldots, z_n, z_{n+1}) \\ &= \Big(\frac{\overline{z_{n+1}}z_1}{|z_{n+1}|}, \ldots, \frac{\overline{z_{n+1}}z_n}{|z_{n+1}|}, |z_{n+1}|\Big) \in D,\end{aligned}$$

and this is the unique representative of its class in D. On the other hand, any point $(z_1, \ldots, z_n, 0) \in \mathbb{S}^{2n+1}$ is identified with the same elements in both quotients. Thus, it is a homeomorphism.

As $\mathbb{CP}^{n-1} = {}^{\mathbb{S}^{2n-1}}\!/\!\sim\, \subseteq {}^{D}\!/\!\sim$, from this description of $\mathbb{CP}^n$ as the quotient ${}^{D}\!/\!\sim$ it follows that

$$\mathbb{CP}^n = D \cup_q \mathbb{CP}^{n-1},$$

where $q : \mathbb{S}^{2n-1} \to {}^{\mathbb{S}^{2n-1}}\!/\!\sim\, = \mathbb{CP}^{n-1}$ is the quotient map. Observe that D is a closed cell of dimension $2n$. Therefore, by induction we get that $\mathbb{CP}^n$ is a CW complex with a $2k$-cell for each $k = 0, 1, \ldots, n$.

Continuing this inductive construction, the union $\mathbb{CP}^\infty = \cup_{n\geq 0}\mathbb{CP}^n$ has a CW complex structure with a cell in each even dimension.

Chapter 4

Cellular Homology

In this final chapter we will study the homology groups of CW complexes. We will see that the homology groups of a CW complex are closely related to the CW decomposition of the space.

4.1 The cellular chain complex

We aim to apply the results in Chapter 2 to get an alternative chain complex for CW complexes whose homology groups are equivalent to those of the usual singular chain complex.

Lemma 4.1.1. *Let X be a CW complex. Then, for every $n \geq 1$, (X_n, X_{n-1}) is a good pair.*

Proof. By Theorem 3.2.4 we know that X_{n-1} is closed in X_n. Let $\{e_i^n\}_{i\in I_n}$ be the family of n-cells of X. Choose a point $z_i \in e_i^n$ in each cell and let $Z = \{z_i\}_{i\in I_n}$. The intersection of Z with the closure of the cells of dimension smaller than n is empty, and the intersection with each $\mathrm{cl}(e_i^n)$ is $\{z_i\}$ which is closed in $\mathrm{cl}(e_i^n)$. Thus, Z is closed in X_n.

Set $U = X - Z \subset X_n$ such that $X_{n-1} \subseteq U$ and it is open in X_n. Since each $e_i^n - \{z_i\}$ deforms to $\mathrm{cl}(e_i^n) - e_i^n \subseteq X_{n-1}$, we conclude that U deformation retracts strongly to X_{n-1}. □

The next lemma enables us to build the cellular chain complex.

Lemma 4.1.2. *Let X be a CW complex. Then,*

(i) *$H_m(X_n, X_{n-1})$ is trivial for $m \neq n$ and is free abelian for $m = n$, with a basis in one-to-one correspondence with the n-cells of X.*

(ii) *$H_m(X_n)$ is trivial for $m > n$. In particular, if X is finite dimensional then $H_m(X) = \{0\}$ for $m > \dim X$.*

(iii) *The map $H_m(X_n) \to H_m(X)$ induced by the inclusion $X_n \hookrightarrow X$ is an isomorphism for $m < n$ and surjective for $m = n$.*

Proof. For $n \geq 0$, let $\{e_i^n\}_{i \in I_n}$ be the collection of n-cells of X. To prove (i), we know by Lemma 4.1.1 that (X_n, X_{n-1}) is a good pair and in Example 3.4.7 we showed that ${X_n}/{X_{n-1}}$ is homeomorphic to $\bigvee_{i \in I_n} \mathbb{S}^n$. Thus, by Corollaries 2.2.4 and 2.2.6 we get that for every $m \geq 0$,

$$\begin{aligned} H_m(X_n, X_{n-1}) &\cong \widetilde{H}_m({X_n}/{X_{n-1}}) \cong \widetilde{H}_m(\bigvee_{i \in I_n} \mathbb{S}^n) \\ &\cong \bigoplus_{i \in I_n} \widetilde{H}_m(\mathbb{S}^n) = \begin{cases} \oplus_{i \in I_n} \mathbb{Z} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \end{aligned}$$

Thus (i) holds. Now, let $\imath_k : X_k \hookrightarrow X_{k+1}$ be the inclusion map. Consider the exact sequence of the pair (X_n, X_{n-1}),

$$\ldots \to H_{m+1}(X_n, X_{n-1}) \to H_m(X_{n-1}) \overset{(\imath_{n-1})_*}{\longrightarrow} H_m(X_n) \to H_m(X_n, X_{n-1}) \to \ldots$$

If $m \neq n$, we know that $H_m(X_n, X_{n-1}) = 0$ so the exact sequence tells us that $(\imath_{n-1})_*$ is surjective. If $m \neq n-1$, $H_{n+1}(X_n, X_{n-1}) = 0$ and the exact sequence tells us that $(\imath_{n-1})_*$ is injective. Fixing m, consider the following sequence of induced homomorphisms:

$$\begin{aligned} H_m(X_0) \overset{(\imath_0)_*}{\longrightarrow} H_m(X_1) \overset{(\imath_1)_*}{\longrightarrow} \ldots \overset{(\imath_{m-2})_*}{\longrightarrow} H_m(X_{m-1}) \overset{(\imath_{m-1})_*}{\longrightarrow} H_m(X_m) \\ \overset{(\imath_m)_*}{\longrightarrow} H_m(X_{m+1}) \overset{(\imath_{m+1})_*}{\longrightarrow} \ldots \end{aligned} \tag{4.1}$$

As we showed in the previous paragraph, $(\imath_k)_*$ is an isomorphism for every $k \neq m-1, m$. We also know that $(\imath_{m-1})_*$ is injective and $(\imath_m)_*$ is surjective. Therefore, as X_0 is a discrete set of points, if $m > n$,

$$0 \cong H_m(X_0) \cong H_m(X_1) \cong \ldots \cong H_m(X_n).$$

Thus (ii) holds.

If X is finite dimensional, (iii) holds from (4.1). The proof of (iii) when X is infinite dimensional requires more work. Observe that for any $\sigma \in \Omega_m(X)$, $\sigma(\Delta^m)$ is compact so by Theorem 3.2.9 it is contained in the union of finitely many cells. Thus, for every singular chain $\sum_{\sigma \in \Omega_m(X)} \lambda_\sigma \sigma \in C_m(X)$ there exists some $k \geq 0$ such that $\sum_{\sigma \in \Omega_m(X)} \lambda_\sigma \sigma \in \Omega_m(X_k)$.

Write, for short, $\partial_m^k = \partial_{m|_{C_m(X_k)}}$. Let $\jmath_n : X_n \hookrightarrow X$ be the inclusion map. We first show that $(\jmath_n)_*$ is surjective if $m \leq n$. As said before, for any m-cycle $c \in \operatorname{Ker} \partial_m$ there exists some $k \geq 0$ such that $c \in C_m(X_k)$. Since $\operatorname{Ker} \partial_m^k = \operatorname{Ker} \partial_m \cap C_m(X_k)$, we have that $c \in \operatorname{Ker} \partial_m^k$. There are two options:

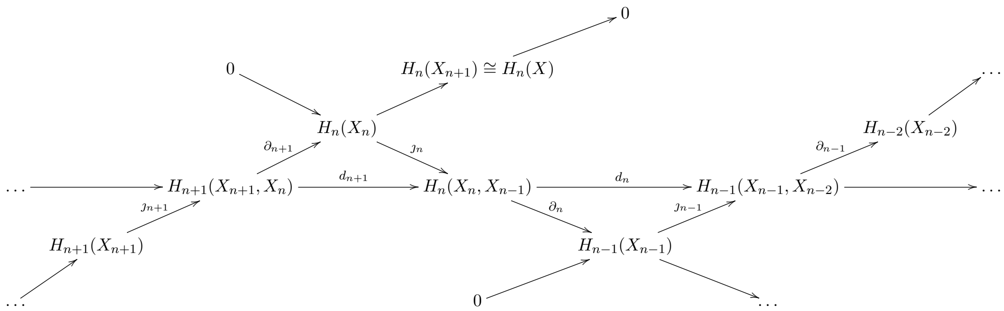
- If $k \leq n$, we are done because $\operatorname{Ker} \partial_m^k \subset \operatorname{Ker} \partial_m^n$ and the homology class of c is in $H_m(X_n)$.
- If $k > n$, then X_k is a finite dimensional CW complex and $X_n \subset X_k$, so by the finite dimensional case of (iii) there is a surjection $H_m(X_n) \to H_m(X_k)$ induced by the inclusion $X_n \hookrightarrow X_k$. Thus, there is a cycle $c' \in \operatorname{Ker} \partial_m^n$ homologous to c, and so, $(\jmath_n)_*(c' + \operatorname{Im} \partial_{m+1}^n) = c + \operatorname{Im} \partial_{m+1}$.

Finally, we show that $(\jmath_n)_*$ is injective if $m < n$. If we have that

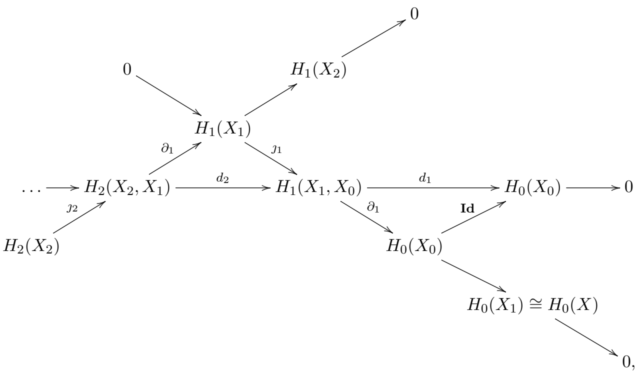
$$(\jmath_n)_*\Big(\sum_{\sigma \in \Omega(X_n)} \lambda_\sigma \sigma + \operatorname{Im} \partial_{m+1}^n \Big) = \sum_{\sigma \in \Omega(X_n)} \lambda_\sigma (\sigma \circ \jmath_n) + \operatorname{Im} \partial_{m+1} = 0,$$

this means that $\sum_{\sigma \in \Omega(X_n)} \lambda_\sigma (\sigma \circ \jmath_n) \in \operatorname{Im} \partial_{m+1}$ and there is some $k \geq n$ such that $\sum_{\sigma \in \Omega(X_n)} \lambda_\sigma (\sigma \circ \jmath_n) + \operatorname{Im} \partial_{m+1}^k \in H_m(X_k)$. From the finite dimensional case of (iii), if $n > m$ there is an isomorphism $H_m(X_n) \to H_m(X_k)$ induced by inclusion $X_n \hookrightarrow X_k$. Thus, if the homology class of $\sum_{\sigma \in \Omega(X_n)} \lambda_\sigma (\sigma \circ \jmath_n)$ is zero in $H_m(X_k)$, the homology class of $\sum_{\sigma \in \Omega(X_n)} \lambda_\sigma \sigma$ is zero in $H_m(X_n)$. □

Let X be a CW complex. Using Lemma 4.1.2, the long exact sequences for the pairs (X_{n+1}, X_n), (X_n, X_{n-1}) and (X_{n-1}, X_{n-2}) fit into the following diagram:



(4.2)



where d_n is defined to be $d_n = \jmath_{n-1} \circ \partial_n$ for every $n \geq 2$, $d_1 = \partial_1$ and $d_0 = 0$. The composition $d_n \circ d_{n+1}$ contains two succesive maps in one of the exact sequences and thus, it is zero. Therefore, the horizontal row in diagram (4.2) is a chain complex, called **cellular chain complex**. The homology groups of this chain complex are called **cellular homology groups** of X. We temporarily denote them by $H_n^{CW}(X)$. We will later show that, in fact, these groups are isomorphic to the singular homology groups.

Remark 4.1.1. We proved in Lemma 4.1.2 that $H_n(X_n, X_{n-1})$ is free with basis in one-to-one correspondence with the n-cells of X, so one can think of the elements of $H_n(X_n, X_{n-1})$ as formal linear combinations of the n-cells of X.

Theorem 4.1.3. *Let X be a CW complex. For all $n \geq 0$, the following isomorphism holds:*

$$H_n(X) \cong H_n^{CW}(X).$$

Proof. If $n = 0$, from the exact sequence of the good pair (X_1, X_0) in diagram (4.2) we get that

$$H_0^{CW}(X) = \frac{H_0(X_0)}{\operatorname{Im}\partial_1} \cong H_0(X_1) \cong H_0(X).$$

Let $n \geq 1$. We know by Lemma 4.1.2 that $H_n(X_{n+1}) \cong H_n(X)$. Moreover, from the exact sequence of the pair (X_{n+1}, X_n) in diagram (4.2), we get that

$$H_n(X) \cong \frac{H_n(X_n)}{\operatorname{Im}\partial_{n+1}}.$$

We can also observe in diagram (4.2) that $\jmath_n$ is injective. Thus, $\jmath_n(\operatorname{Im}\partial_{n+1}) = \operatorname{Im}(\jmath_n \circ \partial_{n+1}) = \operatorname{Im} d_{n+1}$. Moreover, by the first isomorphism theorem, we get $H_n(X_n) \cong \operatorname{Im}\jmath_n = \operatorname{Ker}\partial_n$.

Thus, if $n = 1$ we have that $\jmath_1$ induces the following isomorphism:

$$H_1(X) \cong \frac{H_1(X_1)}{\operatorname{Im}\partial_2} \longrightarrow \frac{\operatorname{Ker}\partial_1}{\operatorname{Im} d_2} = H_1^{CW}(X).$$

Finally, if $n \geq 2$, the map $\jmath_{n-1}$ is injective in the same way so $\operatorname{Ker} d_n = \operatorname{Ker}(\jmath_{n-1} \circ \partial_n) = \operatorname{Ker}\partial_n$. Thus, $\jmath_n$ induces an isomorphism

$$H_n(X) \cong \frac{H_n(X_n)}{\operatorname{Im}\partial_{n+1}} \longrightarrow \frac{\operatorname{Ker}\partial_n}{\operatorname{Im} d_{n+1}} = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}} = H_n^{CW}(X).$$

□

Remark 4.1.2. We list some direct consequences of the previous result:

(i) Most of the time we will no more use the notation $H_n^{CW}(X)$ to distinguish cellular homology groups from singular homology groups as we have proved that they are isomorphic.

(ii) The homology groups do not depend on the CW complex structure of X.

(iii) If X is a CW complex with no n-cells, then $H_n(X) = 0$.

(iv) In general, if X is a CW complex with k n-cells $H_n(X_n, X_{n-1})$ is free abelian on k generators so the subgroup $\operatorname{Ker} d_n$ must be generated by at most k elements, hence also the quotient $\frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$. Thus, $H_n(X)$ is generated by at most k elements.

4.2 Homology groups of some CW complexes

We finish by computing the homology groups of two CW complexes given in Section 3.4.2 directly from the cellular chain complex.

Example 4.2.1. The homology groups of the sphere $\mathbb{S}^n$ with $n \geq 2$ follow immediately from the cellular chain complex. We consider $\mathbb{S}^n$, $n \geq 2$, with the CW decomposition given in Example 3.4.5. It has a 0-cell and a n-cell. The cellular chain complex is the following:

$$\dots \xrightarrow{d_{n+2}} 0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} 0 \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0.$$

We conclude that if $n \geq 2$,

$$H_m(\mathbb{S}^n) = \frac{\operatorname{Ker} d_m}{\operatorname{Im} d_{m+1}} \cong \begin{cases} \mathbb{Z} & \text{if } m = n, 0, \\ 0 & \text{otherwise.} \end{cases}$$

The homology groups of the sphere $\mathbb{S}^1$ also follow from the cellular chain complex. Its cellular chain complex is the following:

$$\dots \xrightarrow{d_3} 0 \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0.$$

We know that $H_m(\mathbb{S}^1) = 0$ for every $m \geq 2$ as $\mathbb{S}^1$ only has a 0-cell and a 1-cell. We also know that $H_0(\mathbb{S}^1) \cong \mathbb{Z}$ as $\mathbb{S}^1$ is path-connected, so $d_1 = 0$ and $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Observe that the result agrees with what we obtained in Example 2.2.1.

Example 4.2.2. The cellular chain complex can be used to compute the homology groups of the complex projective space $\mathbb{CP}^n$. In Example 3.4.9 we saw that $\mathbb{CP}^n$ has a CW complex structure with a $2k$-cell for each $k = 0, \dots, n$.. Thus, the cellular chain complex is the following:

$$0 \xrightarrow{d_{2n+1}} \mathbb{Z} \xrightarrow{d_{2n}} 0 \xrightarrow{d_{2n-1}} \mathbb{Z} \xrightarrow{d_{2n-3}} \dots \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0.$$

Therefore,

$$H_m(\mathbb{CP}^n) = \frac{\operatorname{Ker} d_m}{\operatorname{Im} d_{m+1}} \cong \begin{cases} \mathbb{Z} & \text{if } m = 0, 2, \ldots 2n, \\ 0 & \text{otherwise.} \end{cases}$$

In the same way we can obtain the homology groups of $\mathbb{CP}^\infty$.

$$H_m(\mathbb{CP}^\infty) = \frac{\operatorname{Ker} d_m}{\operatorname{Im} d_{m+1}} \cong \begin{cases} \mathbb{Z} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

4.3 Final remarks

It is clear from Section 4.2 that the cellular chain complex is a great advantage to compute homology groups of CW complexes, even if the CW complex is infinite-dimensional. The homology groups of many more CW complexes can be computed using cellular homology, but a more explicit formula for the boundary maps d_n is needed. It is possible to get an explicit formula for d_n that depends only on the n-cells of a CW complex and their characteristic maps, called **cellular boundary formula**. However, due to the extent of this work it has not been possible to include it here, but the reader can learn more in [1, Page 140].

In conclusion, although the definition of the cellular chain complex requires many preliminary results, cellular homology is undoubtedly an efficient tool for computing the homology groups of CW complexes.

Appendix A

Solved Exercises

Exercise 1. Let $X = \{p\}$ be a point. Show that the homology groups of X are the following:

$$H_n(X) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Solution. First observe that for any $n \geq 0$, a map $\sigma : \Delta^n \longrightarrow \{p\}$ must be the constant map sending all Δ^n to p. Constant maps are continuous, so $\Omega_n(X)$ contains just the constant map. If we call σ_n to this map, $\Omega_n(X) = \{\sigma_n\}$. Therefore,

$$C_n(X) = \{\ \lambda\sigma_n \mid \lambda \in \mathbb{Z}\ \} \underset{\lambda\sigma_n \mapsto \lambda}{\cong} \mathbb{Z}.$$

If $n \geq 1$, take any $(\lambda_0, \dots, \lambda_{n-1}) \in \Delta^{n-1}$. Then,

$$\begin{aligned}
[\sigma_n]_i(\lambda_0, \dots, \lambda_{n-1}) &= \sigma_n(\varphi_{i,n}(\lambda_0, \dots, \lambda_{n-1})) \\
&= \sigma_n(\lambda_0, \dots, \lambda_{i-1}, \overset{i}{0}, \lambda_i, \dots, \lambda_{n-1}) = p
\end{aligned}$$

for all $i = 0, \dots, n$. This means that $[\sigma_n]_i = \sigma_{n-1}$, for all $i = 0, \dots n$. Knowing this, for any $\lambda \in \mathbb{Z}$, we get that

$$\begin{aligned}
\partial_n(\lambda\sigma_n) = \lambda\partial_n(\sigma_n) = \lambda\sum_{i=0}^{n}(-1)^i[\sigma_n]_i &= \lambda\sum_{i=0}^{n}(-1)^i\sigma_{n-1} \\
&= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \lambda\sigma_{n-1}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus, if n is odd, $\operatorname{Ker}\partial_n = \{0\}$ and, as $n+1$ is even, $\operatorname{Im}\partial_{n+1} = \{0\}$. If n is even, in the same way, $\operatorname{Ker}\partial_n = C_n(X)$ and $\operatorname{Im}\partial_{n+1} = C_n(X)$.

Thus, n-th homology group if $n \geq 1$ is $H_n(X) = \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}} = 0$.

If $n = 0$, as $\partial_0 = 0$, $\operatorname{Ker}\partial_0 = C_0(X)$ and $\operatorname{Im}\partial_0 = \{0\}$. Then, the homology group is

$$H_0(X) = \frac{C_0(X)}{\{0\}} \cong C_0(X) \cong \mathbb{Z}.$$

□

Exercise 2. Use the exact sequence for the pair $(\mathbb{D}^n, \mathbb{S}^n)$ and the fact that ${}^{\mathbb{D}^n}\!/_{\mathbb{S}^n} \cong \mathbb{S}^{n-1}$ to show that

$$\widetilde{H}_m(\mathbb{S}^n) \cong \widetilde{H}_{m-1}(\mathbb{S}^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Solution. Let $(X, A) = (\mathbb{D}^n, \mathbb{S}^n)$. Observe that ${}^{\mathbb{D}^n}\!/_{\mathbb{S}^n} \cong \mathbb{S}^{n-1}$. $\mathbb{D}^n$ is contractible so $\widetilde{H}_m(\mathbb{D}^n) \cong \{0\}$ for any m. Thus, applying Theorem 2.2.5 to the good pair $(\mathbb{D}^n, \mathbb{S}^n)$ we get the following exact sequence:

$$\{0\} \to \widetilde{H}_m(\mathbb{S}^n) \longrightarrow \widetilde{H}_{m-1}(\mathbb{S}^{n-1}) \to \{0\} \tag{A.1}$$

Therefore, $\widetilde{H}_m(\mathbb{S}^n) \cong \widetilde{H}_{m-1}(\mathbb{S}^{n-1})$ for any $m \geq 1$. We compute the groups by induction on n.

For $n = 0$, $\mathbb{S}^0$ consists of two disconnected points so $\widetilde{H}_0(\mathbb{S}^0) \cong \mathbb{Z}$ and $\widetilde{H}_m(\mathbb{S}^0) \cong \{0\}$ for any $m \geq 1$.

If $n \geq 1$, $\mathbb{S}^n$ has one connected component so $\widetilde{H}_0(\mathbb{S}^n) = \frac{H_0(\mathbb{S}^n)}{\mathbb{Z}} = \frac{\mathbb{Z}}{\mathbb{Z}} \cong 0$ and by induction hypothesis and (A.1)

$$\widetilde{H}_m(\mathbb{S}^n) \cong \widetilde{H}_{m-1}(\mathbb{S}^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

The non-reduced homology groups are

$$H_m(\mathbb{S}^0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } m = 0, \\ 0 & \text{if } m \geq 1 \end{cases}$$

and for $n \geq 1$,

$$H_m(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

□

Exercise 3. Let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a homeomorphism. Show that $f(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}$ and $f(\mathbb{B}^n) = f(\mathbb{B}^n)$.

Solution. If we show that $f(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}$. Then,

$$f(\mathbb{B}^n) = f(\mathbb{D}^n - \mathbb{S}^{n-1}) = f(\mathbb{D}^n) - f(\mathbb{S}^{n-1}) = \mathbb{D}^n - \mathbb{S}^{n-1} = \mathbb{B}^n.$$

Suppose that there is some $q \in f(\mathbb{S}^{n-1})$ such that $q \notin \mathbb{S}^{n-1}$, that is, $q \in \mathbb{B}^n$. Let $q = f(p)$, $p \in \mathbb{S}^{n-1}$. Then, the restriction

$$f_{|_{\mathbb{D}^n - \{p\}}} : \mathbb{D}^n - \{p\} \longrightarrow \mathbb{D}^n - \{q\}$$

is a homeomorphism and induces the isomorphism

$$H_m(\mathbb{D}^n - \{p\}) \to H_m(\mathbb{D}^n - \{q\})$$

for any $m \geq 0$. Since $p \in \mathbb{S}^{n-1}$, $\mathbb{D}^n - \{p\}$ is homotopy equivalent to a point. Since $q \in \mathbb{B}^n$, $\mathbb{D}^n - \{q\}$ is homotopy equivalent to $\mathbb{S}^{n-1}$. Thus,

$$0 \cong H_m(\mathbb{D}^n - \{p\}) \cong H_m(\mathbb{D}^n - \{q\}) \cong \mathbb{Z},$$

which is imposible. Therefore, $f(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}$.

□

Exercise 4 (Theorem of invariance of dimension)**.** Let $n, m \geq 1$. Show that if two nonempty subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homeomorphic, then $m = n$.

Solution. Let $p \in U$. For any $k \geq 0$, by the Excision Theorem,

$$H_k\big(\mathbb{R}^n, \mathbb{R}^n - \{p\}\big) \cong H_k\big(U, (\mathbb{R}^n - \{p\}) \cap U\big) = H_k\big(U, U - \{p\}\big).$$

From the long exact sequence (2.6) for the pair $(\mathbb{R}^n, \mathbb{R}^n - \{p\})$, we get the following exact sequence for any $k \geq 1$:

$$0 \to H_k(\mathbb{R}^n, \mathbb{R}^n - \{p\}) \to \widetilde{H}_k(\mathbb{R}^n - \{p\}) \to 0.$$

Thus, for any $k \geq 1$, $H_k(\mathbb{R}^n, \mathbb{R}^n - \{p\}) \cong \widetilde{H}_k(\mathbb{R}^n - \{p\})$. Since $\mathbb{R}^n - \{p\}$ is homotopy equivalent to $\mathbb{S}^{n-1}$, we get the following result:

$$H_k\big(U, U - \{p\}\big) \cong H_k\big(\mathbb{R}^n, \mathbb{R}^n - \{p\}\big) \cong \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise .} \end{cases} \tag{A.2}$$

In the same way one can prove that for any $q \in V$, $k \geq 1$,

$$H_k\big(V, V - \{q\}\big) \cong \begin{cases} \mathbb{Z} & \text{if } k = m, \\ 0 & \text{otherwise .} \end{cases} \tag{A.3}$$

A homeomorphism $h : U \to V$ induces isomorphisms

$$H_k\big(U, U - \{p\}\big) \to H_k\big(V, V - \{h(p)\}\big),$$

for any $k \geq 0$. Thus, from (A.2) and (A.3) we must have $n = m$.

□

Exercise 5. Let $n \geq 1$.

(i) Show that $\mathbb{S}^{n-1}$ is not a retract of $\mathbb{D}^n$.

(ii) (Brouwer's fixed point theorem). Show that any continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

Solution. Let $\imath : \mathbb{S}^{n-1} \hookrightarrow \mathbb{D}^n$ be the inclusion map. If $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ is a retraction, then $r \circ \imath = \mathbf{Id}_{\mathbb{S}^{n-1}}$. Then, by Proposition 1.3.1, the composition

$$\widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \xrightarrow{\imath_*} \widetilde{H}_{n-1}(\mathbb{D}^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(\mathbb{S}^{n-1})$$

is the identity map on $\widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$. But this is not possible since $\imath_* = r_* = 0$ because $\widetilde{H}_{n-1}(\mathbb{D}^n) \cong 0$. Thus, (i) holds.

Knowing this, we show Brouwer's fixed point theorem. By contradiction, suppose that $f : \mathbb{D}^n \to \mathbb{D}^n$ does not have a fixed point. That is, $f(x) \neq x$ for any $x \in \mathbb{D}^n$. Then, we can construct a unique ray from $f(x)$ to x and follow the ray until it intersects the boundary $\mathbb{S}^{n-1}$ (see Figure A.1). Calling this intersection point $F(x)$, we define a function $F : \mathbb{D}^n \to \mathbb{S}^{n-1}$ by $x \mapsto F(x)$. This function is continuous and observe that if $x \in \mathbb{S}^{n-1}$, the intersection point $F(x)$ is x itself, so $F(x) = x$. Therefore, we have a retraction of $\mathbb{D}^n$ to $\mathbb{S}^{n-1}$, which is impossible by (i).

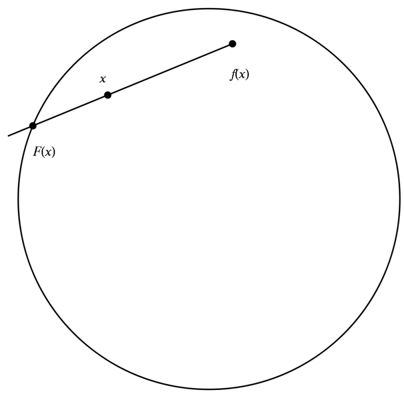


Figure A.1: An illustration of the retraction F for $n = 2$. Picture taken from **here**.

□

Exercise 6. Let $D \subseteq \mathbb{R}^n$ be a compact convex subset with nonempty interior. Show that given any point $p \in \operatorname{int} D$, there exists a homeomorphism $F : \mathbb{D}^n \longrightarrow D$ that sends 0 to p, $\mathbb{B}^n$ to $\operatorname{int} D$ and $\mathbb{S}^{n-1}$ to $\operatorname{fr} D$. In particular, D is a closed n-cell and its interior is an open n-cell.

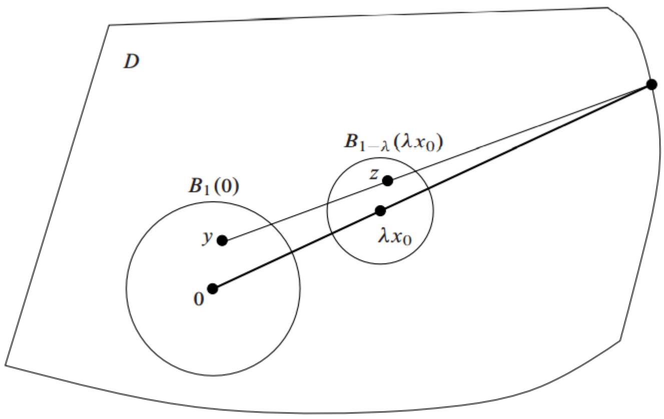


Figure A.2: Each closed ray starting at the origin intersects the boundary in a point. Picture taken from [2, Page 128].

Solution. Let $p \in \operatorname{int} D$. We can replace D with its image under the translation $x \mapsto x - p$ which is a homeomorphism of $\mathbb{R}^n$ with itself, so we can assume that $p = 0 \in \operatorname{int} D$. Then, there is some $\epsilon > 0$ such that the open ball $B_\epsilon(0)$ is contained in D. Using the dilatation $x \mapsto \frac{x}{\epsilon}$, we may assume that $\mathbb{B}^n = B_1(0) \subseteq D$.

The main claim of this proof is that each closed ray starting at the origin intersects $\operatorname{fr} D$ in exactly one point (see Figure A.2).

Let R be such a closed ray. As D is compact, $D \cap R$ is compact. Thus, there is a point $x_0 \in D \cap R$ at which the distance to the origin takes its maximum. This point lies clearly in $\operatorname{fr} D$.

To show it is unique, we show that the line segment from 0 to x_0 consists entirely of interior points of D, except for x_0 itself. As $\operatorname{fr} D = D - \operatorname{int} D$, this proves that x_0 is unique. Any point on this segment other than x_0 can be written as λx_0 for some $0 \leq \lambda < 1$. Take any $z \in B_{1-\lambda}(\lambda x_0)$ and let $y = \frac{z - \lambda x_0}{1-\lambda}$. Notice that

$$|y| = \frac{|z - \lambda x_0|}{|1-\lambda|} < \frac{1-\lambda}{1-\lambda} = 1$$

Thus, $y \in \mathbb{B}^n \subseteq D$. Since y and x_0 are both in D and $z = \lambda x_0 + (1-\lambda)y$, by convexity, $z \in D$. Thus the open ball $B_{1-\lambda}(\lambda x_0)$ is contained in D, which means that λx_0 is an interior point.

We now define a map

$$\begin{array}{rccc} f: & \operatorname{fr} D & \longrightarrow & \mathbb{S}^{n-1} \\ & x & \mapsto & f(x) = \frac{x}{|x|} \end{array}$$

It maps the point x to the point where the line segment from the origin to x intersects the unit sphere $\mathbb{S}^{n-1}$. Since we have proved that each such segment intersects $\operatorname{fr} D$ in exactly one point, the map is a bijection. Moreover, f is the restriction of a continuous map so it is continuous. Therefore, as f is a bijection between a compact and a Hausdorff space, it is a homeomorphism. Finally, we define

$$\begin{array}{rccc} F: & \mathbb{D}^n & \longrightarrow & D \\ & x & \mapsto & F(x) = \begin{cases} |x| f^{-1}(\frac{x}{|x|}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

The map F is continuous if $x \neq 0$ because f^{-1} is. Observe that as $D \subseteq \mathbb{R}^n$ is compact, it is bounded so $f^{-1} : \mathbb{S}^{n-1} \longrightarrow \operatorname{fr} D$ is a bounded function. This implies that $F(x) \to 0$ as $x \to 0$, so F is continuous at the origin.

Geometrically, F maps each radial line segment connecting 0 with a point $y \in \mathbb{S}^{n-1}$ linearly onto the radial segment from 0 to the point $f^{-1}(y) \in \operatorname{fr} D$. Thus, by convexity, it is clear that F takes values in D.

Since points on disctinct rays are mapped to distinct rays and each radial segment is mapped linearly, F is injective. It is also surjective because each point in D is on some ray from 0.

Again, since F is a continuous bijection between a compact and a Hausdorff space, we conclude that it is a homeomorphism. □

Appendix B

Preliminaries in Algebra

B.1 Free modules

In all this work R is a unitary commutative ring.

Definition B.1.1. Let A be a set and let R be a ring. We define the following set:

$$R^{(A)} = \Big\{ \sum_{a\in A} \lambda_a a \;\Big|\; \lambda_a \in R,\ \lambda_a = 0_R \text{ for all } a \in A \text{ except for finitely many of them} \Big\}.$$

Proposition B.1.1. *Let A be a set and let R be a ring. The set $R^{(A)}$ is an R-module with the following operations: for any $\sum_{a\in A} \lambda_a a$, $\sum_{a\in A} \mu_a a \in R^{(A)}$ and $r \in R$,*

(i) $\sum_{a\in A} \lambda_a a \ + \ \sum_{a\in A} \mu_a a \ = \sum_{a\in A} (\lambda_a + \mu_a) a,.$

(ii) $r \cdot \sum_{a\in A} \lambda_a a \ = \ \sum_{a\in A} (r\lambda_a) a,.$

Remark B.1.1. Notice that the elements $\sum_{a\in A} \lambda_a a$ are just **formal sums**. Moreover, one can embed any $B \subseteq A$ into $R^{(A)}$ via the following embedding:

$$\begin{array}{rccl} \iota\colon & B & \longrightarrow & R^{(A)} \\ & x & \mapsto & \sum_{a\in A} \lambda_a a, \quad \lambda_a = \begin{cases} 1_R, & if\ a = x \\ 0_R, & if\ a \neq x, \end{cases} \end{array}$$

and we can identify B with $\iota(B)$. Abusing the notation we will say that $B \subset R^{(A)}$. In the same way, if $B \subseteq A$, we can embed $R^{(B)} \subseteq R^{(A)}$ via the following map:

$$\begin{array}{ccl} R^{(B)} & \longrightarrow & R^{(A)} \\ \sum_{b\in B} \lambda_b b & \mapsto & \sum_{a\in A} \lambda_a a \quad \text{with } \lambda_a = \begin{cases} \lambda_b, & if\ a = b \\ 0_R, & if\ a \neq x. \end{cases} \end{array}$$

Proposition B.1.2. *Let A be a set and R be a ring. Then, A is a R-basis of $R^{(A)}$.*

Proof. First notice that $A \subset R^{(A)}$ and each element of $R^{(A)}$ is a finite R-linear combination of elements in A, so A is a generating set of $R^{(A)}$. We now show that A is linearly independent. Recall that

$$0_{R^{(A)}} = \sum_{a\in A} \lambda_a a \;\Leftrightarrow\; \lambda_a = 0,\ \forall a \in A.$$

Let $I \subset A$ be a finite subset of A. Let $r_a \in R$ such that $\sum_{a\in I} r_a a = 0_{R^{(A)}}$. Then,

$$0_{R^{(A)}} = \sum_{a\in I} r_a a = \sum_{a\in A} \lambda_a a \text{ where } \lambda_a = \begin{cases} r_a, & \text{if } a \in I \\ 0_R, & \text{if } a \notin I \end{cases}$$

and this happens if and only if $\lambda_a = 0,\ \forall a \in A$. Therefore, $r_a = 0$ for any $a \in I$. Therefore, A is linearly independent in $R^{(A)}$. □

Corollary B.1.3. *Let A be a set and R be a ring. $R^{(A)}$ is a free R-module generated by A.*

Being $R^{(A)}$ a free R-module is equivalent to saying that all maps from the basis A to a module can be extended to the whole $R^{(A)}$, as the next Theorem shows.

Theorem B.1.4. *Let A be a set, R be a ring and M an R-module. Let $f : A \longrightarrow M$ be a map. Then, there is a unique R-module homomorphism $\tilde{f} : R^{(A)} \longrightarrow M$ such that $\tilde{f}_{|A} = f$ and that is given by*

$$\begin{array}{ccc} \tilde{f} : R^{(A)} & \longrightarrow & M \\ \sum_{a\in A} \lambda_a a & \mapsto & \sum_{a\in A} \lambda_a f(a) \end{array}$$

Proof. In each $\sum_{a\in A} \lambda_a a \in R^{(A)}$ only a finite number of λ_a are non-zero. Therefore, the image $\sum_{a\in A} \lambda_a f(a)$ is a finite R-linear combination of elements from M, which is on M. So the map is well defined.

Let us prove now that the map is a R-module homomorphism. For any $\sum_{a\in A} \lambda_a a, \sum_{a\in A} \mu_a a \in R^{(A)}$ and $r \in R$ we have:

$$\begin{aligned} \tilde{f}(\sum_{a\in A} \lambda_a a + \sum_{a\in A} \mu_a a) &= \tilde{f}(\sum_{a\in A} (\lambda_a + \mu_a) a) = \sum_{a\in A} (\lambda_a + \mu_a) f(a) \\ &= \sum_{a\in A} \lambda_a f(a) + \sum_{a\in A} \mu_a f(a) = \tilde{f}(\sum_{a\in A} \lambda_a a) + \tilde{f}(\sum_{a\in A} \mu_a a), \end{aligned}$$

$$\tilde{f}(r \sum_{a\in A} \lambda_a a) = \tilde{f}(\sum_{a\in A} (r\lambda_a) a) = \sum_{a\in A} (r\lambda_a) f(a) = r \sum_{a\in A} \lambda_a f(a) = r\tilde{f}(\sum_{a\in A} \lambda_a a).$$

Thus, $\tilde{f}$ is a well defined R-module homomorphism.

Finally, we prove that $\tilde{f}$ is unique. Suppose that there exist $\tilde{f}$, $\tilde{g}$: $R^{(A)} \longrightarrow M$ such that $\tilde{f}_{|A}$, $\tilde{g}_{|A} = f$. Then, $\tilde{f}(a) = f(a) = \tilde{g}(a)$, $\forall a \in A$, and then,

$$\tilde{f}(\sum_{a\in A} \lambda_a a) = \sum_{a\in A} \lambda_a f(a) = \tilde{g}(\sum_{a\in A} \lambda_a a).$$

That is, $\tilde{f} = \tilde{g}$.

□

Corollary B.1.5. *Let A, B be sets, R a ring and $f : A \longrightarrow B$ a map. Then, the following map:*

$$\begin{array}{cccc} \tilde{f}: & R^{(A)} & \longrightarrow & R^{(B)} \\ & \sum_{a\in A} \lambda_a a & \longmapsto & \sum_{a\in A} \lambda_a f(a) \end{array}$$

is an R-module homomorphism. Moreover:

(i) *If f is injective, then $\tilde{f}$ is injective.*

(ii) *If f is surjective, then $\tilde{f}$ is surjective.*

(iii) *If f is bijective, then $\tilde{f}$ is an R-isomorphism.*

Proof. Notice that:

$$\tilde{f}(\sum_{a\in A} \lambda_a a) = \sum_{a\in A} \lambda_a f(a) = \sum_{b\in B} \; (\sum_{a\in f^{-1}(b)} \lambda_a) \;\; b \; \in R^{(B)},$$

so the map is well defined. Abusing the notation we can say that $B \subset R^{(B)}$. By Theorem B.1.4 the map $\tilde{f}$ is a well defined R-module homomorphism.

Let f be injective. Then, if we have $\tilde{f}(\sum_{a\in A} \lambda_a a) = \sum_{b\in B} \mu_b b$ and $\tilde{f}(\sum_{a\in A} \lambda'_a a) = \sum_{b\in B} \mu'_b b$ such that

$$\mu_b = \begin{cases} \lambda_a & \text{if } b = f(a) \\ 0 & \text{otherwise} \end{cases}$$

$$\mu'_b = \begin{cases} \lambda'_a & \text{if } b = f(a) \\ 0 & \text{otherwise} \end{cases}$$

and if $\tilde{f}(\sum_{a\in A} \lambda_a a) = \tilde{f}(\sum_{a\in A} \lambda'_a a)$, then,

$$\tilde{f}(\sum_{a\in A} \lambda_a a) = \tilde{f}(\sum_{a\in A} \lambda'_a a) \iff \mu_b = \mu'_b, \; \forall b \in B \iff \lambda_a = \lambda'_a, \; \forall a \in A.$$

Thus, $\sum_{a\in A} \lambda_a a = \sum_{a\in A} \lambda'_a a$. That is, $\tilde{f}$ is injective.

Now suppose that f is surjective. Take any $\sum_{b\in B}\mu_b b \in R^{(B)}$. As f is surjective, for any $b \in B$ there is some $a \in A$ such that $f(a) = b$. Thus,

$$\tilde{f}(\sum_{a\in A}\mu_{f(a)}a) = \sum_{b\in B}\mu_b b$$

and $\tilde{f}$ is surjective. (iii) folows from (i) and (ii). □

B.1.1 Direct sums of modules

Definition B.1.2. Let R be a ring and $\{M_i\}_{i\in I}$ be a family of R-modules. Their direct sum is defined as follows:

$$\bigoplus_{i\in I} M_i = \left\{ \sum_{i\in I} a_i \,\middle|\, a_i \in M_i, a_i = 0_{M_i} \text{ for all } i \in I \text{ except for finitely many of them} \right\},$$

Remark B.1.2. The direct sum $\bigoplus_{i\in I} M_i$ has obviously a R-module structure defining:

(i) $\sum_{i\in I} a_i + \sum_{i\in I} b_i = \sum_{i\in I}(a_i + b_i)$.

(ii) $r \cdot \sum_{i\in I} a_i = \sum_{i\in I}(ra_i)$.

In this module, $\sum_{i\in I} a_i = 0$ if and only if $a_i = 0_{M_i}$ for any $i \in I$.

Theorem B.1.6. *Let A be a set. If we write $A = \bigsqcup_{i\in I} A_i$ as a disjoint union of subsets of A, then,*

$$R^{(A)} = \bigoplus_{i\in I} R^{(A_i)}.$$

Proof. For any $i \in I$, $R^{(A_i)} \subset R^{(A)}$ so the sum is contained in $R^{(A)}$ and as the union $A = \bigsqcup_{i\in I} A_i$ is disjoint it is clear that it is a direct sum because $R^{(A_i)} \cap R^{(A_j)} = \{0\}$ if $i \neq j$. Therefore,

$$\bigoplus_{i\in I} R^{(A_i)} \subseteq R^{(A)}.$$

On the other hand, for any $\sum_{a\in A}\lambda_a a \in R^{(A)}$, as $A = \bigsqcup_{i\in I} A_i$ we can write:

$$\sum_{a\in A}\lambda_a a = \sum_{i\in I}\sum_{a\in A_i}\lambda_a a \quad \text{and} \quad R^{(A)} \subseteq \bigoplus_{i\in I} R^{(A_i)}.$$

□

Definition B.1.3. Let $\{f_i : M_i \to N_i\}_{i\in I}$ be a family of R-homomorphisms. The direct sum of this family of R-homomorphisms is the map

$$\begin{array}{rccc} \oplus_{i\in I} f_i : & \bigoplus_{i\in I} M_i & \longrightarrow & \bigoplus_{i\in I} N_i \\ & \sum_{i\in I} a_i & \longmapsto & \sum_{i\in I} f_i(a_i). \end{array}$$

It is easy to check that it is a R-homomorphism.

Proposition B.1.7. *Let $\{f_i : M_i \to N_i\}_{i \in I}$ be a family of R-homomorphisms. Then,*

(i) $\operatorname{Ker}\big(\oplus_{i\in I} f_i\big) = \oplus_{i\in I} \operatorname{Ker} f_i$

(ii) $\operatorname{Im}\big(\oplus_{i\in I} f_i\big) = \oplus_{i\in I} \operatorname{Im} f_i$

Proof. The first statement holds because $0 = \oplus_{i\in I} f_i\big(\sum_{i\in I} a_i\big) = \sum_{i\in I} f_i(a_i)$ if and only if $f_i(a_i) = 0$ for any $i \in I$. The second statement holds because

$$\begin{aligned}\operatorname{Im}\big(\oplus_{i\in I} f_i\big) &= \Big\{ \sum_{i\in I} f_i(a_i) \mid a_i \in M_i, a_i = 0\ \forall i \in I \text{ except for finitely many of them } \Big\} \\ &= \bigoplus_{i\in I} \{ f_i(a_i) \mid a_i \in M_i \} = \bigoplus_{i\in I} \operatorname{Im} f_i.\end{aligned}$$

□

B.2 Chain complexes

In this section we define chain complexes and homology groups and study their basic properties.

Definition B.2.1. Let R be a ring and for any integer $n \geq 0$ let A_n be a R-module. Let α_0 be the zero map and let $\alpha_n : A_n \to A_{n-1}$ be a sequence of homomorphisms such that $\alpha_n \circ \alpha_{n+1} = 0$, for any $n \geq 0$. The chain

$$\dots \xrightarrow{\alpha_{n+2}} A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_2} A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\alpha_0} \{0\}$$

is called a **chain complex** and is denoted by the pair (A_*, α_*).

Remark B.2.1. Notice that for any $n \in \mathbb{N} \cup \{0\}$, $\alpha_n \circ \alpha_{n+1} = 0$ is equivalent to saying that $\operatorname{Im} \alpha_{n+1} \subseteq \operatorname{Ker} \alpha_n$.

Definition B.2.2. For a chain complex (A_*, α_*) we define the n^{th} **homology group** to be the quotient

$$H_n(A_*) = \frac{\operatorname{Ker} \alpha_n}{\operatorname{Im} \alpha_{n+1}}.$$

Elements of $\operatorname{Ker} \alpha_n$ are called **cycles** and elements of $\operatorname{Im} \alpha_{n+1}$ are called **boundaries**. Elements of $H_n(A_*)$ are called **homology classes**. Two cycles with the same homology class are said to be **homologous**. This means that their difference is a boundary.

The following lemma enables us to define maps between chain complexes.

Lemma B.2.1. *Let R be a ring and A, A', B, B', C, C' be R-modules. Let $f_A : A \longrightarrow A'$, $f_B : B \longrightarrow B'$, $f_C : C \longrightarrow C'$ be R-module homomorphisms and let $\varphi : A \longrightarrow B$, $\gamma : B \longrightarrow C$, $\varphi' : A' \longrightarrow B'$, $\gamma' : B' \longrightarrow C'$ be R-module homomorphisms such that $\gamma \circ \varphi = \gamma' \circ \varphi' = 0$, $\varphi' \circ f_A = f_B \circ \varphi$ and $\gamma' \circ f_B = f_C \circ \gamma$. Then, f_B induces the following homorphism:*

$$\begin{array}{rccc} f_* : & \frac{\operatorname{Ker}\gamma}{\operatorname{Im}\varphi} & \longrightarrow & \frac{\operatorname{Ker}\gamma'}{\operatorname{Im}\varphi'} \\ & x + \operatorname{Im}\varphi & \longmapsto & f_B(x) + \operatorname{Im}\varphi'. \end{array}$$

Moreover, if f_A, f_B and f_C are isomorphisms, f_* is an isomorphism too.

Proof. We know that the following diagrams commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & \xrightarrow{\gamma} & C \\ \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} \\ A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\gamma'} & C'. \end{array}$$

Notice from $\gamma \circ \varphi = 0$ and $\gamma' \circ \varphi' = 0$, we have $\operatorname{Im}\varphi \subseteq \operatorname{Ker}\gamma$ and $\operatorname{Im}\varphi' \subseteq \operatorname{Ker}\gamma'$, so the quotients are well defined. We will first prove that the map f_* is well defined. On the one hand, we show that $f_B(\operatorname{Ker}\gamma) \subseteq \operatorname{Ker}\gamma'$. Take $b \in \operatorname{Ker}\gamma$. Then,

$$\gamma'(f_B(b)) = f_C(\gamma(b)) = f_C(0) = 0,$$

as every module homomorphism maps zero to zero. So, $f_B(\operatorname{Ker}\gamma) \subseteq \operatorname{Ker}\gamma'$. On the other hand, notice that

$$f_B(\operatorname{Im}\varphi) = \{\ f_B(\varphi(a)) \mid a \in A\ \} = \{\ \varphi'(f_A(a)) \mid a \in A\ \} \subseteq \operatorname{Im}\varphi'$$

because $f_A(a) \in A'$ and $\operatorname{Im}\varphi' = \{\varphi'(a') | a' \in A'\}$. Thus, if $x + \operatorname{Im}\varphi = y + \operatorname{Im}\varphi$ then $x - y \in \operatorname{Im}\varphi$ and as $f_B(\operatorname{Im}\varphi) \subseteq \operatorname{Im}\varphi'$, $f_B(x-y) = f_B(x) - f_B(y) \in \operatorname{Im}\varphi'$. Therefore, $f_B(x) + \operatorname{Im}\varphi' = f_B(y) + \operatorname{Im}\varphi'$.

Moreover, as f_B is a R-module homomorphism it is clear that f_* too. We conclude that f_* is a well defined homomorphism.

Finally, suppose that f_A, f_B and f_C are R-module isomorphisms. We will show that $f_B(\operatorname{Ker}\gamma) = \operatorname{Ker}\gamma'$ and $f_B(\operatorname{Im}\varphi) = \operatorname{Im}\varphi'$.

For any $b' \in \operatorname{Ker}\gamma' \subset B'$, as f_B is surjective, there is some $b \in B$ such that $f_B(b) = b'$. Then,

$$0 = \gamma'(b') = \gamma'(f_B(b)) = f_C(\gamma(b)),$$

and as f_B is injective, $f_C(\gamma(b)) = 0$ if and only if $\gamma(b) = 0$, which means that $b \in \operatorname{Ker}\gamma$. Therefore,

$$b' = f_B(b) \in f_B(\operatorname{Ker}\gamma) = \{\ f_B(b) \in B' \mid \gamma(b) = 0\gamma\ \}$$

and $\operatorname{Ker}\gamma' \subseteq f_B(\operatorname{Ker}\gamma)$, getting the first equality.

To show the other equality, for any $b' \in \operatorname{Im}\varphi'$, there is some $a' \in A'$ such that $b' = \varphi'(a')$ and for that a', as f_A is surjective, there is some $a \in A$ such that $f_A(a) = a'$. Thus, we have that for any $b' \in \operatorname{Im}\varphi'$ there is some $a \in A$ such that:

$$\varphi'(f_A(a)) = b' \iff f_B(\varphi(a)) = b',$$

which means that $b' \in f_B(\operatorname{Im}\varphi)$. Therefore, $\operatorname{Im}\varphi' \subseteq f_B(\operatorname{Im}\varphi)$, getting the equality.

Knowing this, it follows that f_* is an isomorphism.

Injectivity follows because

$$f_B(x) + \operatorname{Im}\varphi' = f_B(y) + \operatorname{Im}\varphi' \iff f_B(x) - f_B(y) = f_B(x-y) \in \operatorname{Im}\varphi',$$

and we know that $\operatorname{Im}\varphi' \subseteq f_B(\operatorname{Im}\varphi)$, so $x-y \in \operatorname{Im}\varphi$ and $x+\operatorname{Im}\varphi = y+\operatorname{Im}\varphi$.

To show surjectivity, take any $x' + \operatorname{Im}\varphi' \in \frac{\operatorname{Ker}\gamma'}{\operatorname{Im}\varphi'}$. As $\operatorname{Ker}\gamma' \subseteq f_B(\operatorname{Ker}\gamma)$, there is some $x \in \operatorname{Ker}\gamma$ such that $f_B(x) = x'$ and $f_*(x + \operatorname{Im}\varphi) = f_B(x) + \operatorname{Im}\varphi' = x' + \operatorname{Im}\varphi'$.

□

Definition B.2.3. Consider two chain complexes (A_*, α_*), (B_*, β_*). For every integer $n \geq 0$, let $F_n : A_n \longrightarrow B_n$ be a homomorphism. We say that this collection of homomorphisms defines a **chain map** from (A_*, α_*) to (B_*, β_*) if $F_n \circ \alpha_{n+1} = \beta_{n+1} \circ F_{n+1}$ for every $n \geq 0$. That is, if the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\alpha_{n+2}} & A_{n+1} & \xrightarrow{\alpha_{n+1}} & A_n & \xrightarrow{\alpha_n} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & \dots \\
 & & \downarrow{\scriptstyle F_{n+1}} & & \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & \\
\dots & \xrightarrow{\beta_{n+2}} & B_{n+1} & \xrightarrow{\beta_{n+1}} & B_n & \xrightarrow{\beta_n} & B_{n-1} & \xrightarrow{\beta_{n-1}} & \dots
\end{array}$$

We will denote the chain map as $F : A_* \longrightarrow B_*$.

By Lemma B.2.1 a chain map induces homomorphisms on homology groups

$$\begin{array}{rccc} F_*: & H_n(A_*) & \longrightarrow & H_n(B_*) \\ & c + \operatorname{Im}\alpha_{n+1} & \mapsto & F(c) + \operatorname{Im}\beta_{n+1}, \end{array}$$

for any $n \geq 0$.

Remark B.2.2. Whenever there is no real need to specify each subscript we will not write them in order to simplify the notation. For example, we would write the commutativity condition as $F \circ \alpha = \beta \circ F$ with simplified notation. Similarly, it is also common to simplify the chain complex condition $\alpha_n \circ \alpha_{n+1} = 0$ as $\alpha^2 = 0$.

We next define the notion of homotopy for chain complexes.

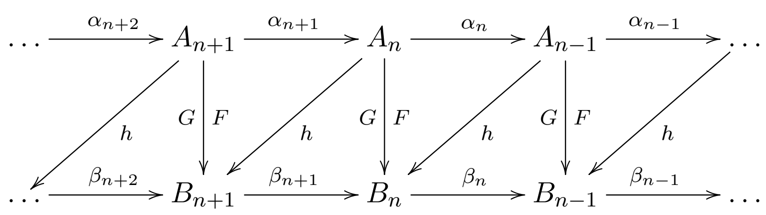
Definition B.2.4. Let (A_*, α_*), (B_*, β_*) be chain complexes and $F,\ G : A_* \longrightarrow B_*$ be chain maps. A collection of homomorphisms $h : A_n \longrightarrow B_{n+1}$ is called a **chain homotopy from** F **to** G if the following identity is satisfied in each group A_n:

$$h \circ \alpha + \beta \circ h = G - F.$$

If such map exists, F and G are said to be **chain homotopic**.

Proposition B.2.2. *Chain homotopic chain maps induce the same homomorphism on homology groups. That is, if $F,\ G : A_* \longrightarrow B_*$ are chain homotopic chain maps, then $F_* = G_* : H_n(A_*) \longrightarrow H_n(B_*)$ for every $n \geq 0$.*

Proof. We have the following diagram



As F, G are chain homotopic, for any $c \in \operatorname{Ker}\alpha_n$,

$$G(c) - F(c) = h(\alpha(c)) + \beta(h(c)) = h(0) + \beta(h(c)) = \beta(h(c))$$

So, $G(c) - F(c) \in \operatorname{Im}\beta_{n+1}$ and $G_*(c + \operatorname{Im}\alpha_{n+1}) = G(c) + \operatorname{Im}\beta_{n+1} = F(c) + \operatorname{Im}\beta_{n+1} = F_*(c + \operatorname{Im}\alpha_{n+1})$. □

B.2.1 Exact sequences

In this subsection we give the basic definitions and properties of exact sequences.

Definition B.2.5. A chain complex (A_*, α_*) is said to be an **exact sequence** if $\operatorname{Ker}\alpha_n = \operatorname{Im}\alpha_{n+1}$ for any $n \geq 0$.

Definition B.2.6. A 5-term exact sequence of the form

$$\{0\} \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow \{0\}$$

is called a **short exact sequence**.

Remark B.2.3. Consider the following exact sequence:

$$\dots \xrightarrow{\alpha_{n+2}} A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_1} A_0 \xrightarrow{\alpha_0} 0.$$

If there is an isomorphism $\varphi : A_n \longrightarrow B$ we can "substitute" the module A_n in the chain by B. Then, it is easy to check that the following sequence

$$\dots \xrightarrow{\alpha_{n+2}} A_{n+1} \xrightarrow{\varphi \circ \alpha_{n+1}} B \xrightarrow{\alpha_n \circ \varphi^{-1}} A_{n-1} \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_1} A_0 \xrightarrow{\alpha_0} \{0\}$$

is also exact.

The following lemma will give us the tool to build exact sequences on homology groups induced by short exact sequences of chain complexes.

Lemma B.2.3 (Zig-zag lemma)**.** *Let* (A_*, α_*), (B_*, β_*), (C_*, γ_*) *be chain complexes and let* $F : A_* \longrightarrow B_*$, $G : B_* \longrightarrow C_*$ *be chain maps such that for each* $n \geq 0$, *there is a short exact sequence*

$$0 \longrightarrow A_n \xrightarrow{F} B_n \xrightarrow{G} C_n \longrightarrow 0$$

Then, for each $n \geq 1$, *there is a map* $\partial_* : H_n(C_*) \longrightarrow H_{n-1}(A_*)$, *called* ***the connecting homomorphism***, *such that the following sequence is exact:*

$$\dots \xrightarrow{\partial_*} H_n(A_*) \xrightarrow{F_*} H_n(B_*) \xrightarrow{G_*} H_n(C_*) \xrightarrow{\partial_*} H_{n-1}(A_*) \xrightarrow{F_*} \dots$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{\alpha_{n+2}} & A_{n+1} & \xrightarrow{\alpha_{n+1}} & A_n & \xrightarrow{\alpha_n} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & A_{n-2} & \xrightarrow{\alpha_{n-2}} & \dots \\
 & & \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F & & \\
\dots & \xrightarrow{\beta_{n+2}} & B_{n+1} & \xrightarrow{\beta_{n+1}} & B_n & \xrightarrow{\beta_n} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_{n-2} & \xrightarrow{\beta_{n-2}} & \dots \\
 & & \downarrow G & & \downarrow G & & \downarrow G & & \downarrow G & & \\
\dots & \xrightarrow{\gamma_{n+2}} & C_{n+1} & \xrightarrow{\gamma_{n+1}} & C_n & \xrightarrow{\gamma_n} & C_{n-1} & \xrightarrow{\gamma_{n-1}} & C_{n-2} & \xrightarrow{\gamma_{n-2}} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

The hypothesis is that the diagram commutes and that the vertical columns are exact.

To define the homomorphism ∂_*, let $c + \operatorname{Im}\gamma_{n+1} \in H_n(C_*)$ be arbitrary. This means that $c \in C_n$ and $\gamma_n(c) = 0$. As the columns form short exact sequences, the map G is surjective and F is injective. Surjectivity of $G : B_n \longrightarrow C_n$ means that there exists $b \in B_n$ such that $G(b) = c$, and by the commutativity of the diagram $G(\beta_n(b)) = \gamma_n(G(b)) = \gamma_n(c) = 0$. Thus, $\beta_n(b) \in \operatorname{Ker} G$. By exactness at B_{n-1}, there exists an element $a \in A_{n-1}$ such that $F(a) = \beta_n(b)$ and again by commutativity of the diagram $F(\alpha_{n-1}(a)) = \beta_{n-1}(F(a)) = \beta_{n-1}(\beta_n(b)) = 0$. Since $F : A_{n-2} \longrightarrow B_{n-2}$ is injective, $\alpha_{n-1}(a) = 0$. Therefore $a + \operatorname{Im}\alpha_n \in H_{n-1}(A_*)$.

We wish to set $\partial_*(c + \operatorname{Im}\gamma_{n+1}) = a + \operatorname{Im}\alpha_{n-1}$, but to do so we need to check that the homology class of a does not depend on any of the choices we made along the way.

Suppose that $c' + \operatorname{Im}\gamma_{n+1} = c + \operatorname{Im}\gamma_{n+1} \in H_n(C_*)$, then there exists $\tilde{c} \in C_{n+1}$ such that $c - c' = \gamma_{n+1}(\tilde{c})$. Let $b' \in B_n$ be such that $G(b') = c'$, and let $a' \in A_{n-1}$ be such that $F(a') = \beta_n(b')$. As G is surjective, there is some $\tilde{b} \in B_{n+1}$ such that $G(\tilde{b}) = \tilde{c}$. Then, $G(\beta_{n+1}(\tilde{b})) = \gamma_{n+1}(G(\tilde{b})) = \gamma(\tilde{c}) = c - c'$ so $G(b - b') = G(b) - G(b') = c - c' = G(\beta_{n+1}(\tilde{b}))$ which is equivalent to $G(b - b') - G(\beta_{n+1}(\tilde{b})) = G(b - b' - \beta(\tilde{b})) = 0$, that is, $b - b' - \beta(\tilde{b}) \in \operatorname{Ker} G$. By exactness, there exists $\tilde{a} \in A_n$ such that $F(\tilde{a}) = b - b' - \beta_{n+1}(\tilde{b})$, and $F(\alpha_n(\tilde{a})) = \beta_n(F(\tilde{a})) = \beta_n(b - b' - \beta(\tilde{b})) = \beta_n(b) - \beta_n(b') = F(a) - F(a') = F(a - a')$. Since F is injective, this means that $\alpha_n(\tilde{a}) = a - a'$ so $a + \operatorname{Im}\alpha_n = a' + \operatorname{Im}\alpha_n$ and the map is well defined.

In summary, we have defined a map $\partial_* : H_n(C_*) \longrightarrow H_{n-1}(A_*)$ defined as $\partial_*(c + \operatorname{Im}\gamma_{n+1}) = a + \operatorname{Im}\alpha_n$, using that there is some $b \in B_n$ such that $G(b) = c$ and $F(a) = \beta_n(b)$.

We now prove that the map is a homomorphism. If $\partial_*(c + \operatorname{Im}\gamma_{n+1}) = a + \operatorname{Im}\alpha_n$ and $\partial_*(c' + \operatorname{Im}\gamma_{n+1}) = a' + \operatorname{Im}\alpha_n$, there exist $b, b' \in B_n$ such that $G(b) = c$, $G(b') = c'$ and $F(a) = \beta_n(b)$, $F(a') = \beta_n(b')$. It follows that $G(b + b') = G(b) + G(b') = c + c'$ and $F(a + a') = F(a) + F(a') = \beta_n(b) + \beta_n(b') = \beta_n(b + b')$. Hence,

$$\partial_*\big((c + \operatorname{Im}\gamma_{n+1}) \;+\; (c' + \operatorname{Im}\gamma_{n+1})\big) = \partial_*\big((c + c') + \operatorname{Im}\gamma_{n+1}\big)$$

$$= (a + a') + \operatorname{Im}\alpha_n = (a + \operatorname{Im}\alpha_n) \;+\; (a' + \operatorname{Im}\alpha_n)$$

as we proved that the map is well defined making these choices.

It is left to prove the exactness of the following sequence:

$$\dots \xrightarrow{\partial_*} H_n(A_*) \xrightarrow{F_*} H_n(B_*) \xrightarrow{G_*} H_n(C_*) \xrightarrow{\partial_*} H_{n-1}(A_*) \xrightarrow{F_*} \dots$$

We will start by looking at $H_n(A_*)$. Suppose that $\partial_*(c + \operatorname{Im}\gamma_{n+2}) = a + \operatorname{Im}\alpha_{+1}$. Then, looking at the definition of ∂_* there is some $b \in B_{n+1}$ such that $F(a) = \beta(b)$, so $F_*(\partial_*(c + \operatorname{Im}\gamma_{n+1})) = F_*(a + \operatorname{Im}\alpha_n) = F(a) + \operatorname{Im}\beta_n = \beta_{n+1}(b) + \operatorname{Im}\beta_{n+1} = 0$. Thus, $\operatorname{Im}\partial_* \subseteq \operatorname{Ker} F_*$. Conversely, if $F_*(a + \operatorname{Im}\alpha_{n+1}) = F(a) + \operatorname{Im}\beta_{n+1} = 0$, there is some $b \in B_{n+1}$ such that $F(a) = \beta_{n+1}(b)$ and then $\gamma_{n+1}(G(b)) = G(\beta_{n+1}(b)) = G(F(a)) = 0$. This means that $G(b) + \gamma_{n+2} \in H_{n+1}(C_*)$ and by the definition of ∂_* we find that $\partial_*(G(b) + \operatorname{Im}\gamma_{n+1}) = a + \operatorname{Im}\alpha_{n+1}$. Thus, $\operatorname{Ker} F_* \subseteq \operatorname{Im}\partial_*$.

Next we prove exactness at $H_n(B_*)$. From $G \circ F = 0$ it follows that $G_* \circ F_* = 0$ and thus, $\operatorname{Im} F_* \subseteq \operatorname{Ker} G_*$. If $G_*(b + \operatorname{Im}\beta_{n+1}) = G(b) + \operatorname{Im}\gamma_{n+1} = 0$ for some $b + \operatorname{Im}\beta_{n+1} \in H_n(B_*)$, there exists $c \in C_{n+1}$ such that $\gamma_{n+1}(c) = G(b)$. By surjectivity of G, there is some $b' \in B_{n+1}$ such that $G(b') = c$, and then $G(\beta_{n+1}(b')) = \gamma_{n+1}(G(b')) = \gamma_{n+1}(c) = G(b)$. This is equivalent to $G(b) - G(\beta_{n+1}(b')) = G(b - \beta_{n+1}(b')) = 0$, so $b - \beta_{n+1}(b') \in \operatorname{Ker} G = \operatorname{Im} F$. Hence, there exists $a \in A_n$ with $F(a) = b - \beta_{n+1}(b')$. Moreover, $F(\alpha_n(a)) = \beta_n(F(a)) = \beta_n(b - \beta_{n+1}(b')) = \beta_n(b) = 0$ as $b \in \operatorname{Ker}\beta$, so by injectivity of F, $\alpha_n(a) = 0$. This means that $a \in \operatorname{Ker}\alpha_n$ and $a + \operatorname{Im}\alpha_{n+1} \in H_n(A_*)$. We get that, $F_*(a + \operatorname{Im}\alpha_{n+1}) = F(a) + \operatorname{Im}\beta_{n+1} = (b - \beta_{n+1}(b')) + \operatorname{Im}\beta_{n+1} = b + \operatorname{Im}\beta_{n+1}$ and thus, $\operatorname{Ker} G_* \subseteq \operatorname{Im} F_*$.

Finally, we prove exactness at $H_n(C_*)$. Let $c + \operatorname{Im}\gamma_{n+1} \in \operatorname{Im} G_*$. This means that $c + \operatorname{Im}\gamma_{n+1} = G_*(b + \operatorname{Im}\beta_{n+1}) = G(b) + \operatorname{Im}\gamma_{n+1}$ for some $b \in B_n$ with $\beta_n(b) = 0$, so $c = G(b) + \gamma_{n+1}(c')$ for some $c' \in C_{n+1}$. As $c + \operatorname{Im}\gamma_{n+1} = (c - \gamma_{n+1}(c')) + \operatorname{Im}\gamma_{n+1} = G(b) + \operatorname{Im}\gamma_{n+1}$ we can assume that $G(b) = c$. Then, by definition, $\partial_*(c + \operatorname{Im}\gamma_{n+1}) = a + \operatorname{Im}\alpha_n$, where $a \in A_{n-1}$ is chosen so that $F(a) = \beta_n(b)$. Since F is injective and $\beta_n(b) = 0$, we have that $a = 0$, and therefore $\partial_*(c + \operatorname{Im}\gamma_{n+1}) = 0$. That is, $c + \operatorname{Im}\gamma_{n+1} \in \operatorname{Ker}\partial_*$. Conversely, if $\partial_*(c + \operatorname{Im}\gamma_{n+1}) = 0$, it means that there is some $b \in B_n$ such that $G(b) = c$ and $a \in A_{n-1}$, $a \in \operatorname{Im}\alpha_n$ such that $F(a) = \beta_n(b)$. Writing $a = \alpha(a')$ for some $a' \in A_n$, we find that $\beta_n(F(a')) = F(\alpha_n(a')) = F(a) = \beta_n(b)$, which is equivalent to saying that $\beta_n(b) - \beta_n(F(a')) = \beta_n(b - F(a')) = 0$ so $b - F(a') \in \operatorname{Ker}\beta_n$ and $G_*\big((b - F(a')) + \operatorname{Im}\beta_{n+1} \big) = \big(G(b) - G(F(a')) \big) + \operatorname{Im}\gamma_{n+1} = G(b) + \operatorname{Im}\gamma_{n+1} = c + \operatorname{Im}\gamma_{n+1}$. Therefore, $\operatorname{Ker}\partial_* \subseteq \operatorname{Im} G_*$, which concludes the proof.

□

Appendix C

Preliminaries in Topology

Here we collect a number of topological constructions and properties to be used in the work.

C.1 Disjoint union topology

Definition C.1.1. Let $\{X_i\}_{i\in I}$ be a family of sets. The set

$$\bigsqcup_{i\in I} X_i = \{\ (x,i) \mid x \in X_i\ \}$$

is called the **disjoint union** of the family. The elements of this set are ordered pairs (x,i). Here i serves as an auxiliary index that indicates which X_i the element x comes from.

Definition C.1.2. Let $\{X_i\}_{i\in I}$ be a family of sets. The map

$$\begin{array}{cccc} \imath_i : & X_i & \longrightarrow & \bigsqcup_{i\in I} X_i \\ & x & \longmapsto & (x,i) \end{array}$$

is called the **canonical injection** of X_i.

Remark C.1.1. It is clear that $\imath_i$ is injective. The image set

$$\imath_i(X_i) = \{(x,i)|x \in X_i\}$$

is a "copy" of X_i in the disjoint union and can be identified with it. Abusing the notation we may write $X_i \subset \bigsqcup_{i\in I} X_i$ and $x_i \in \bigsqcup_{i\in I} X_i$ for $x_i \in X_i$.

Observe that for $i \neq j$ the sets $\imath_i(X_i)$ and $\imath_j(X_j)$ are disjoint even if the sets X_i and X_j are not.

Now we define a topology over the disjoint union of topological spaces. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Over $\bigsqcup_{i\in I} X_i$ we can take the finest topology for which all canonical injections are continuous. Explicitly, the family of open sets is

$$\tau = \{\ U \subseteq \sqcup_{i \in I} X_i \mid \imath_i^{-1}(U) \subseteq X_i \text{ is open for all } i \in I\ \}.$$

Then, $(\bigsqcup_{i \in I} X_i, \tau)$ is a toplogical space.

Remark C.1.2. Observe that two components $\imath_i(X_i)$ and $\imath_j(X_j)$ are always disconnected for $i \neq j$.

Proposition C.1.1. *Each canonical injection $\imath_i : X_i \to \bigsqcup_{i \in I} X_i$ is a topological embedding and an open and closed map.*

Proof. First we prove that $\imath_i$ is open. For any open $U \subseteq X_i$, $\imath_i(U) = U \times \{i\}$ is open because

$$\imath_j^{-1}(U \times \{i\}) = \begin{cases} U & \text{if } j = i, \\ \emptyset & \text{if } j \neq i, \end{cases}$$

is open. Now we show that the canonical injections are closed. Let $C = X_i - U \subseteq X_i$ be a closed set (U open). As $\imath_i$ is injective and $\imath_i(X_i) \cap \imath_j(X_j) = \emptyset$, for any $i \neq j$,

$$\imath(C) = \imath_i(X_i) - \imath_i(U) = \bigsqcup_{i \in I} X_i - ((\cup_{j \neq i} \imath_j(X_j)) \cup \imath_i(U)).$$

The set $(\cup_{j \neq i} \imath_j(X_j)) \cup \imath_i(U)$ is a union of open sets as we have proved that the canonical injections are open maps. Therefore, $\imath(C)$ is closed.

Finally we observe that if we restrict the codomain, $\imath_i : X_i \longrightarrow \imath_i(X_i)$ is a homeomorphism. Indeed, we know it is bijective by definition of the map and continuous by definition of the topology. Moreover, we have proved it is an open map to the whole domain so it is an open map. Therefore it is a homeomorphism.

□

Proposition C.1.1 tells us that there is a homeomorphic copy of each X_i in $\bigsqcup_{i \in I} X_i$. Therefore we may write $X_i \hookrightarrow \bigsqcup_{i \in I} X_i$ or, abusing the notation, we could also write $X_i \subseteq \bigsqcup_{i \in I} X_i$.

Proposition C.1.2. *A set $C \subseteq \bigsqcup_{i \in I} X_i$ is closed if and only if $\imath_i^{-1}(C)$ is closed for any $i \in I$.*

Proof. Suppose that $C = \bigsqcup_{i \in I} X_i - U$ is closed (U open). Then, for any $i \in I$,

$$\imath_i^{-1}(C) = \imath_i^{-1}(\sqcup_{i \in I} X_i) - \imath_i^{-1}(U) = X_i - \imath_i^{-1}(U)$$

is closed as $\imath_i^{-1}(U)$ is open.

Suppose now that for any $i \in I$, $\imath_i^{-1}(C)$ is closed in X_i. That is, $\imath_i^{-1}(C) = X_i - U_i$ for some open set $U_i \subseteq X_i$. Then,

$$\begin{aligned} C &= \bigsqcup_{i\in I} X_i \cap C = \bigcup_{i\in I}(C \cap \imath_i(X_i)) = \bigcup_{i\in I}\{(x,i) | \imath_i(x) \in C\} \\ &= \bigcup_{i\in I}(\imath_i^{-1}(C) \times \{i\}) = \bigcup_{i\in I}(X_i - U_i) \times \{i\} = \bigcup_{i\in I}(X_i \times \{i\} - U_i \times \{i\}) \\ &= \bigcup_{i\in I}(\imath_i(X_i) - \imath_i(U_i)) = \bigcup_{i\in I}\imath_i(X_i) - \bigcup_{i\in I}\imath_i(U_i) = \bigsqcup_{i\in I} X_i - \bigcup_{i\in I}\imath_i(U_i), \end{aligned}$$

where we have used that $\imath_i(X_i) \cap \imath_j(X_j) = \emptyset$ if $i \neq j$. As we proved in Proposition C.1.1 the maps $\imath_i$ are open and the union of open sets is open, thus, $\bigcup_{i\in I} \imath_i(U_i)$ is open. Therefore C is closed. □

Finally we give the characterization of continuity of maps from the disjoint union space.

Theorem C.1.3 (Characteristic property of disjoint union spaces)**.** *Let $\{X_i\}_{i\in I}$ be a family of topological spaces and Y be any topological space. A map $f : \bigsqcup_{i\in I} X_i \longrightarrow Y$ is continuous if and only if $f \circ \imath_i$ is continuous for any $i \in I$.*

Proof. It is clear that if f is continuous then each $f \circ \imath_i$ is continuous as it is the composition of two continuous maps. Suppose that $X_i \xrightarrow{\imath_i} \bigsqcup_{i\in I} X_i \xrightarrow{f} Y$ is continuous for any $i \in I$. Then, if $U \subseteq Y$ is open, we know that $(f \circ \imath_i)^{-1}(U) = \imath_i^{-1}(f^{-1}(U))$ is open for every $i \in I$. Thus, by definition of the open sets in the disjoint union, $f^{-1}(U)$ is open in $\bigsqcup_{i\in I} X_i$. Therefore f is continuous. □

Remark C.1.3. As we explained before, there is no problem in considering each X_i as a subspace of $\bigsqcup_{i\in I} X_i$. Thus, we can rewrite the definition of the topology, Proposition B.1.2 and Theorem B.1.3 as follows:

- A set $U \subseteq \bigsqcup_{i\in I} X_i$ is open if and only if $X_i \cap U$ is open in X_i for any $i \in I$.
- A set $C \subseteq \bigsqcup_{i\in I} X_i$ is closed if and only if $X_i \cap C$ is closed in X_i for any $i \in I$.
- A map $f : \bigsqcup_{i\in I} X_i \longrightarrow Y$ is continuous if and only if $f_{|X_i}$ is continuous in X_i for any $i \in I$.

C.2 Quotient by a set

In this section we study some special kind of quotient topological spaces. We want to define the notion of collapsing some subspace $A \subseteq X$ to a point. These quotient spaces are used in Chapter 2.

Definition C.2.1. Let X be a topological space and $A \subseteq X$. We define the quotient of X by the subset A as the quotient space under the following relation:

$$x \sim y \iff x = y \text{ or } x, y \in A,$$

and denote the quotient space as ${}^{X}\!/_{A}$.

Remark C.2.1. As a set, ${}^{X}\!/_{A}$ is formed by equivalence classes $[x]$ for every $x \in X$. Observe that if $x \in X - A$, $[x] = \{x\}$ and if $x \in A$, $[x] = A$. That is,

$${}^{X}\!/_{A} = \{\ \{x\} \mid x \in X - A\} \cup \{A\}.$$

Consider the quotient map:

$$\begin{array}{rccc} q: & X & \longrightarrow & {}^{X}\!/_{A} \\ & x & \mapsto & [x]. \end{array}$$

By definition of quotient spaces, $U \subseteq {}^{X}\!/_{A}$ is open if and only if $q^{-1}(U)$ is open in X. We also know that $C \subseteq {}^{X}\!/_{A}$ is closed if and only if $q^{-1}(C)$ is closed in X.

Let $U \subseteq {}^{X}\!/_{A}$. As q is surjective, there is some $V \subseteq X$ such that $q(V) = U$. There are two options:

- If $A \in U$, or equivalently, $A \cap V \neq \emptyset$, $q^{-1}(U) = V \cup A$ so

$$U \text{ is open/closed in } {}^{X}\!/_{A} \iff V \cup A \text{ is open/closed in } X$$

- If $A \notin U$, or equivalently, $A \cap V = \emptyset$, $q^{-1}(U) = V$ so

$$U \text{ is open/closed in } {}^{X}\!/_{A} \iff V \text{ is open/closed in } X$$

It is reasonable to think that we can identify $X - A$ with ${}^{X}\!/_{A} - {}^{A}\!/_{A}$. Although we do have a bijection in order to get a homeomorphism we must be careful with the topology. We want to know when $q_{|X-A}$ is a topological embedding.

Example C.2.1. Consider $X = \mathbb{R}$ and $A = \mathbb{Q}$. Which are the open sets in ${}^{\mathbb{R}}\!/_{\mathbb{Q}}$? Let $q : \mathbb{R} \to {}^{\mathbb{R}}\!/_{\mathbb{Q}}$ be the quotient map. Let $U = q(V) \subseteq {}^{\mathbb{R}}\!/_{\mathbb{Q}}$.

If $V \cap \mathbb{Q} = \emptyset$, we have seen that U is open if and only if V is open. By density of $\mathbb{Q}$ if V is open and nonempty it must intersect $\mathbb{Q}$, so V must be the empty set in this case. Thus, $U = \emptyset$ and all nonempty open sets in the quotient contain $\mathbb{Q}$.

Therefore, since any two nonempty open sets have nonempty intersection, the quotient is not Hausdorff. Since the space $\mathbb{R} - \mathbb{Q}$ is Hausdorff (subspace of a Hausdorff space), $\mathbb{R}/\mathbb{Q} - \mathbb{Q}/\mathbb{Q}$ can not be homeomorphic to $\mathbb{R} - \mathbb{Q}$.

The next result gives the conditions under which $q_{|X-A}$ is a topological embedding.

Proposition C.2.1. *Let X be a topological space and $A \subseteq X$. Let $q : X \longrightarrow X/A$ be the quotient map. If A is either open or closed, then $q_{|X-A}$ is a topological embedding.*

Proof. It is clear that $q_{|X-A}$ is a continuous bijection. We will prove that $q_{|X-A}$ is an open map when A is closed and in the same way one can prove that $q_{|X-A}$ is a closed map when A is open. Therefore, in either case we get that it is a homeomorphism.

Let A be closed. As $X - A$ is open, any subset $U \subseteq X - A$ that is open in the subspace $X - A$ is also open in X. Since $A \cap U = \emptyset$, we have that $q^{-1}(q(U)) = U$. Therefore, as U is open in X, $q(U)$ is open in $X/A - A/A$. □

C.3 Wedge sum

We defined the disjoint union of topological spaces in section B.1. Now we want to join topological spaces gluing a point from each one. The idea is to identify a point from each topological space.

Definition C.3.1. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and let $x_i \in X_i$ for each $i \in I$. The following quotient space is called the **wedge sum** of the family $\{X_i\}_{i \in I}$:

$$\bigvee_{i \in I} X_i = \sqcup_{i \in I} X_i / \sqcup_{i \in I} \{x_i\}.$$

C.4 Adjunction spaces

Adjunction spaces come from the idea of attaching a topological space to another.

Definition C.4.1. Let X and Y be topological spaces, $A \subseteq Y$ be a closed subset and $f : A \longrightarrow X$ be a continuous map. We define the following relation on $X \sqcup Y$: for any $x, y \in X \sqcup Y$, $x \sim y$ if and only if

- If $x, y \in X$, $x = y$.
- If $x, y \in Y$, there is some $z \in X$ such that $x, y \in f^{-1}(z)$.
- If $x \in X$ and $y \in Y$, $f(y) = x$.

It is an equivalence relation. The quotient space $X \sqcup Y/\sim$ is called the **adjunction space**, and it is denoted by $X \cup_f Y$. We say that Y *has been attached to* X *by* f.

Remark C.4.1. Roughly speaking, we identify all points in the sets $\{x\} \cup f^{-1}(x)$ for each $x \in f(A)$. The union $X \sqcup Y$ is disjoint so we have the following options for any $x \in X \sqcup Y$:

- If $x \in X$, $[x] = \{x\} \cup f^{-1}(x)$.
- If $x \in A \subseteq Y$, $[x] = \{f(x)\} \cup \{y \in A | f(y) = f(x)\} = \{f(x)\} \cup f^{-1}(f(x))$.
- If $x \in Y - A$, $[x] = \{x\}$.

Therefore if $q : X \sqcup Y \longrightarrow X \cup_f Y$ is the quotient map it is clear that $X \cup_f Y$ is the disjoint union of $q(X)$ and $q(Y - A)$.

Proposition C.4.1. *Let $X \cup_f Y$ be an adjunction space and let $q : X \sqcup Y \longrightarrow X \cup_f Y$ be the associated quotient map. Then,*

(i) *$q_{|X}$ is a topological embedding whose image set $q(X)$ is a closed subspace of $X \cup_f Y$.*

(ii) *$q_{|Y-A}$ is a topological embedding whose image set $q(Y - A)$ is an open subspace of $X \cup_f Y$.*

Proof. We begin showing (i). Observe that the equivalence relation does not identify any points in X with each other so $q_{|X} : X \longrightarrow q(X)$ is a bijection. Moreover the restriction of a continuous map is continuous so it is continuous too. We will show that the map is closed to conclude that it is a topological embedding.

Let $C \subseteq X$ be a closed subspace. To show that $q(C)$ is closed we need to show that $q^{-1}(q(C))$ is closed in $X \sqcup Y$, which is equivalent to showing that its intersections with X and Y are closed in X and Y respectively. From Remark C.4.1, $q^{-1}(q(C)) = C \cup f^{-1}(C)$. Thus,

- $q^{-1}(q(C)) \cap X = C$ is closed in X by assumption.
- $q^{-1}(q(C)) \cap Y = f^{-1}(C)$ is closed in A by continuity of f and also closed in Y because A is closed in Y.

So q_X is a closed map. It follows, in particular, that $q(X)$ is closed in $X \cup_f Y$.

To prove (ii), it is clear by Remark C.4.1 that $q_{|Y-A}$ is a bijection and it is continuous as it is the restriction of a continuous map. We now show it is an open map to conclude that it is an embedding. Let $U \subseteq Y - A$ be an open set. Observe that $q^{-1}(q(U)) = U$. Thus,

- $q^{-1}(q(U)) \cap X = \emptyset$ which is open in X.
- $q^{-1}(q(U)) \cap Y = U$ which is open in Y by assumption.

Therefore $q(U)$ is open in $X \cup_f Y$ and $q_{|Y-A}$ is an open map. It follows, in particular, that $q(Y - A)$ is open in $X \cup_f Y$. □

C.5 Local finiteness

Definition C.5.1. Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be **locally finite** if each point of X has an open neighborhood that intersects at most finitely many of the sets in $\mathcal{A}$.

Here are some elementary properties of local finiteness.

Proposition C.5.1. *Let X be a topological space and $\mathcal{A}$ be a collection of subsets of X. Consider the collection*

$$\mathrm{cl}(\mathcal{A}) = \{ \ \mathrm{cl}(A) \mid A \in \mathcal{A} \ \}$$

Then, $\mathcal{A}$ is locally finite if and only if $\mathrm{cl}(\mathcal{A})$ *is locally finite.*

Proof. If $\mathrm{cl}(\mathcal{A})$ is locally finite, since $A \subseteq \mathrm{cl}(A)$ for any $A \in \mathcal{A}$, it follows immediately that $\mathcal{A}$ is locally finite.

Conversely, suppose that $\mathcal{A}$ is locally finite. Given $x \in X$, let V be an open neighbourhood of x that intersects only finitely many sets $\{A_1, \dots, A_n\}$ in $\mathcal{A}$. If V contains a point y of $\mathrm{cl}(A)$ for some $A \in \mathcal{A}$, then every open neighbourhood of y contains a point of A. The neighbourhood V is also an open neighbourhood of y and it contains a point of A, so A must be one of the sets $A_1, \dots, A_n$. Thus, the same neighbourhood V intersects $\mathrm{cl}(A)$ for finitely many $\mathrm{cl}(A) \in \mathrm{cl}(\mathcal{A})$.

□

Proposition C.5.2. *Let X be a topological space and $\mathcal{A}$ a locally finite collection of subsets of X. Then,*

$$\mathrm{cl}(\bigcup_{A\in\mathcal{A}} A) = \bigcup_{A\in\mathcal{A}} \mathrm{cl}(A).$$

Proof. It is true in general that the right-hand side is contained in the left hand side so we only need to prove the reverse containment.

We will prove the contrapositive: assuming $x \in X$ is not an element of $\bigcup_{A\in\mathcal{A}} \mathrm{cl}(A)$, we show it is not an element of $\mathrm{cl}(\bigcup_{A\in\mathcal{A}} A)$ either. By Proposition C.5.1, x has a neighbourhood U that intersects only finitely many sets in $\mathrm{cl}(\mathcal{A})$, say $\mathrm{cl}(A_1), \dots, \mathrm{cl}(A_n)$. Then, $U - (\bigcup_{i=1}^{n} \mathrm{cl}(A_i))$ is an open neighbourhood of x that intersects none of the sets in $\mathcal{A}$. Therefore, $x \notin \mathrm{cl}(\bigcup_{A\in\mathcal{A}} A)$. □

C.6 Coherent topologies

Definition C.6.1. Let X be a topological space and $\mathcal{B}$ be a family of subspaces of X whose union is X. We say that the topology of X is **coherent with** $\mathcal{B}$ if a set $U \subseteq X$ is open in X if and only if $U \cap B$ is open in B for every $B \in \mathcal{B}$.

Remark C.6.1. An equivalent definition would be that X is coherent with $\mathcal{B}$ if a set $C \subseteq X$ is closed if and only if $C \cap B$ is closed in B for any $B \in \mathcal{B}$.

To show this let $C = X - U \subseteq X$. Just notice that for any $B \in \mathcal{B}$ the set $C \cap B = (X - U) \cap B = (X \cap B) - U \cap B = B - U \cap B$ is closed in B if and only if $U \cap B$ is open in B. So it is clear that they are equivalent.

In either case, the "only if" implication always holds by definition of the subspace topology on B so it is the "if" part that is significant.

Example C.6.1. If $\{X_i\}_{i\in I}$ is an indexed family of topological spaces, the disjoint union topology on $\bigsqcup_{i\in I} X_i$ is coherent with the family $\{X_i\}_{i\in I}$, thought of as subspaces of the disjoint union.

The next proposition expresses some basic properties of coherent topologies.

Proposition C.6.1. *Let X be a topological space whose topology is coherent with a family $\mathcal{B}$ of subspaces. Then,*

(i) *If Y is another topological space, a map $f : X \longrightarrow Y$ is continuous if and only if $f_{|_B}$ is continuous for every $B \in \mathcal{B}$.*

(ii) *The map $\bigsqcup_{B\in\mathcal{B}} B \longrightarrow X$ induced by inclusion of each set $B \hookrightarrow X$ is an identification map.*

Proof. To prove (i), notice that if $f : X \longrightarrow Y$ is continuous it is clear that $f_{|_B}$ is continuous for every $B \in \mathcal{B}$ as it is the restriction of a continuous map. Suppose that we know that $f_{|_B}$ is continuous for every $B \in \mathcal{B}$. Let $U \subseteq Y$ open. Since $X = \bigcup_{B\in\mathcal{B}} B$, we conclude that

$$\begin{aligned} f^{-1}(U) &= f^{-1}(U) \bigcap X = f^{-1}(U) \bigcap \left(\cup_{B\in\mathcal{B}} B\right) \\ &= \bigcup_{B\in\mathcal{B}} f^{-1}(U) \ \cap \ B = \bigcup_{B\in\mathcal{B}} f_{|_B}^{-1}(U) \end{aligned}$$

is open because it is a union of open sets in X.

To prove (ii), observe that since $X = \bigcup_{B\in\mathcal{B}} B$, the map $\mathbf{I} : \bigsqcup_{B\in\mathcal{B}} B \longrightarrow X$ is surjective. Finally notice that by definition of a coherent topology, $U \subseteq X$ is open if and only if $U \cap B$ is open in B for any $B \in \mathcal{B}$, which is equivalent to saying that $\mathbf{I}^{-1}(U)$ is open in the disjoint union $\bigsqcup_{B\in\mathcal{B}} B$. Therefore $\mathbf{I}$ is an identification map. □

Remark C.6.2. From Proposition C.6.1 it is clear that the topology of X is coherent with a family of subspaces $\mathcal{B}$ if and only if it is the finest topology on X for which all the inclusion maps $B \hookrightarrow X$ are continuous.

C.7 Separability of inductively built CW complexes

In the proof of Theorem 3.3.3 it is left to prove that the inductively built space $X = \bigcup_{n\geq 0} X_n$ is Hausdorff. To show this we need some lemmas. These three lemmas appear as exercises in [2].

Lemma C.7.1. *Let X be a topological space. If for every $x \in X$ there is a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(0) = \{x\}$, then X is Hausdorff.*

Proof. Let $x, y \in X$ be distinct. There is some function $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(0) = \{x\}$. Since $x \neq y$, $f(y) \neq 0$ and as $\mathbb{R}$ with the usual topology is Hausdorff, there are some open sets U, V with empty intersection such that $0 \in U, f(y) \notin U$ and $f(y) \in V, 0 \notin V$. Then, the preimages $f^{-1}(U)$ and $f^{-1}(V)$ are open and disjoint such that $x \in f^{-1}(U), y \notin f^{-1}(U)$ and $y \in f^{-1}(V), x \notin f^{-1}(V)$. □

Lemma C.7.2. *Let D be a closed n-cell with $n \geq 1$. Given any point $p \in \operatorname{int} D$, there is a continuous function $F : D \to [0, 1]$ such that $F^{-1}(1) = \operatorname{fr} D$ and $F^{-1}(0) = \{0\}$.*

Proof. As D is an n-cell, there is a homeomorphism $D \xrightarrow{f} \mathbb{D}^n$ sending $f(\operatorname{int} D) = \mathbb{B}^n$ and $f(\operatorname{fr} D) = \mathbb{S}^{n-1}$. If $p \in \operatorname{int} D$, $f(p) \in \mathbb{D}^n$ and by Proposition 3.1.1 there is a homeomorphism $g : \mathbb{D}^n \to \mathbb{D}^n$ that sends $f(p)$ to 0,

$g(\operatorname{int}\mathbb{D}^n) = g(\mathbb{B}^n) = \mathbb{B}^n$ and $g(\operatorname{fr}\mathbb{D}^n) = g(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}$. We finally define $F : D \to [0,1]$ as $F(x) = ||g(f(x))||$. It is a composition of continuous functions so it is continuous. Moreover,

- $||g(f(x))|| = 1$ if and only if $g(f(x)) \in \mathbb{S}^{n-1}$, and since $f(\operatorname{fr} D) = \mathbb{S}^{n-1} = g^{-1}(\mathbb{S}^{n-1})$, this happens if and only if $x \in \operatorname{fr} D$.
- $||g(f(x))|| = 0$ if and only if $g(f(x)) = 0$, and as $f(p) = g^{-1}(0)$ this happens if and only if $x = p$.

□

Lemma C.7.3. *Let D be a closed n-cell with $n \geq 1$. Any continuous function $f : \operatorname{fr} D \to [0,1]$ extends to a continuous function $F : D \to [0,1]$ that is strictly positive in* $\operatorname{int} D$.

Proof. There is a homeomorphism $g : \mathbb{D}^n \longrightarrow D$ that sends $g(\mathbb{B}^n) = \operatorname{int} D$ and $g(\mathbb{S}^{n-1}) = \operatorname{fr} D$. We define $h : \mathbb{D}^n \to [0,1]$ as

$$h(x) = \begin{cases} ||x|| f(g(\frac{x}{||x||})) + \frac{1-||x||}{2}, & \text{if } x \neq 0, \\ \frac{1}{2}, & \text{if } x = 0, \end{cases}$$

and finally $F = h \circ g^{-1} : D \to [0,1]$. Observe that h is continuous since $h(x) \to 1/2$ when $x \to 0$, so F is clearly continuous. Moreover,

- If $x \in \operatorname{fr} D$, $F(x) = h(g^{-1}(x)) = f(x)$ since $g^{-1}(x) \in \mathbb{S}^{n-1}$.
- Observe that $h(x) > 0$ for any $x \in \mathbb{B}^n$, so $F(x) > 0$ for any $x \in \operatorname{int} D$.

□

Knowing these three lemmas, we prove the result.

Recall that in Theorem 3.3.3 we have a sequence of topological spaces $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ satisfying the following conditions:

(i) X_0 is a nonempty discrete space.

(ii) For each $n \geq 1$, X_n is obtained from X_{n-1} by attaching a (possibly empty) collection of n-cells.

Until now we have proved that $X = \bigcup_{n\geq 0} X_n$,

(a) has a unique topology coherent with $\{X_n\}$: A subset $C \subseteq X$ is closed if and only if each $C \cap X_n$ is closed,

(b) has a cell decomposition where the n-cells are defined to be the components of $X_n - X_{n-1}$, and for each n-cell e_j^n the characteristic map is defined to be

$$D_j^n \hookrightarrow X_{n-1} \bigsqcup \left(\sqcup_{i \in I_n} D_i^n \right) \xrightarrow{q_n} X_n \hookrightarrow X$$

where the first and last maps are inclusions and $q_n : X_{n-1} \bigsqcup \left(\sqcup_{i \in I_n} D_i^n \right) \to X_n$ is the quotient map of the adjunction space.

By Lemma C.7.1 it is sufficient to show that for each $p \in X$ there is a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = \{p\}$. Let $p \in X$ be arbitrary, and let $e_{i_0}^m$ be the unique cell of dimension m containing p. Let $\Phi_{i_0}^m : D_{i_0}^m \to X$ be the characteristic map of $e_{i_0}^m$. We will define the map inductively. We start by defining a map $f_m : X_m \to [0,1]$ as follows: if $m = 0$, just let $f_m(p) = 0$ and $f_m(x) = 1$ for $x \neq p$. If $m \geq 1$, let $\tilde{p} = (\Phi_{i_0}^m)^{-1}(p) \in \operatorname{int} D_{i_0}^m$. By Lemma C.7.2 there is a continuous function $F : D_{i_0}^m \to [0,1]$ that is equal to 1 in $\operatorname{fr} D_{i_0}^m$ and is equal to 0 exactly at $\tilde{p}$. Define a function

$$\tilde{f}_m : X_{m-1} \bigsqcup \left(\sqcup_{i \in I} D_i^m \right) \longrightarrow [0,1]$$

by letting $\tilde{f}_m = F$ on $D_{i_0}^m$ and $\tilde{f}_m = 1$ everywhere else. Then, since $\tilde{f}$ is continuous in each component of the union, by Theorem C.1.3 it is continuous. Passing to the quotient, there is a unique map $f_m : X_m \to [0,1]$ such that $\tilde{f}_m = f_m \circ q_m$ and $f_m^{-1}(0) = \{p\}$.

Now suppose by induction that for $n > m$ we have defined a continuous map $f_{n-1} : X_{n-1} \to [0,1]$ such that $(f_{n-1})^{-1}(0) = \{p\}$.

We want to define a map $\tilde{f}_n : X_{m-1} \bigsqcup \left(\sqcup_{i \in I} D_i^m \right) \to [0,1]$. Lemma C.7.3 shows that for each closed n-cell D_i^n the function $f_{n-1} \circ \Phi^n_{i|\operatorname{fr} D_i^n} : \operatorname{fr} D_i^n \to [0,1]$ can be extended to a continuous function $F_i^n : D_i^n \to [0,1]$ that has no zeros in $\operatorname{int} D_i^n$. If we define $\tilde{f}_n$ by $\tilde{f}_n = f_{n-1}$ on X_{n-1} and $\tilde{f}_n = F_i^n$ on D_i^n, the map is continuous and passes to the quotient to give us $f_n : X_n \to [0,1]$ such that $\tilde{f}_n = f_n \circ q_n$ whose zero set is $\{p\}$.

Finally we just define $f : X \to [0,1]$ by letting $f(x) = f_n(x)$ if $x \in X_n$. We have constructed those f_n inductively in a way that the map is well defined, it is continuous since the restriction to each X_n is continuous and $f^{-1}(0) = \{p\}$.

Therefore X is Hausdorff.

Appendix D

The prism operator

The following Lemma was left without proof in Chapter 1. We used it to prove the homotopy invariance of singular homology groups.

Lemma D.0.1. *Let X be a topological space and $I = [0, 1]$. The chain maps induced by*

$$\begin{array}{rccc} \iota_0 : & X & \longrightarrow & X \times I \\ & x & \mapsto & (x, 0) \end{array} \quad \textit{and} \quad \begin{array}{rccc} \iota_1 : & X & \longrightarrow & X \times I \\ & x & \mapsto & (x, 1) \end{array}$$

are chain-homotopic.

Proof. We are in the following situation:

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \dots \\ & & \downarrow{\scriptstyle (\iota_i)_\#} & & \downarrow{\scriptstyle (\iota_i)_\#} & & \downarrow{\scriptstyle (\iota_i)_\#} & & \\ \dots & \xrightarrow{\partial} & C_{n+1}(X \times I) & \xrightarrow{\partial} & C_n(X \times I) & \xrightarrow{\partial} & C_{n-1}(X \times I) & \xrightarrow{\partial} & \dots \end{array}$$

Our goal is to define a chain homotopy between $(\iota_0)_\#$ and $(\iota_1)_\#$. For each $n \geq 0$, we would like to define a homomorphism

$$h : C_n(X) \longrightarrow C_{n+1}(X \times I)$$

that satisfies

$$\partial \circ h + h \circ \partial = (\iota_1)_\# - (\iota_0)_\# \tag{D.1}$$

For the standard n-simplex $\Delta^n = [e_0, e_1, \dots, e_n] \subset \mathbb{R}^{n+1}$ we define

$$E_i = (e_i, 0), \quad E'_i = (e_i, 1) \in \mathbb{R}^{n+2}$$

They are the vertices of the following (n+1)-simplices:

$$\Delta^n \times \{0\} = [E_0, \dots, E_n] \subset \mathbb{R}^{n+2}$$

$$\Delta^n \times \{1\} = [E'_0, \dots, E'_n] \subset \mathbb{R}^{n+2}$$

Let $\Gamma_{i,n} = \varphi_{[E_0, \dots, E_i, E'_i, \dots, E'_n]}$ for any $n \geq 0$ and $i = 0, \dots, n$. That is,

$$\begin{array}{rccc} \Gamma_{i,n}: & \Delta_{n+1} & \longrightarrow & [E_0,\ldots,E_i,E'_i,\ldots,E'_n] \subset \Delta_n \times I \\ & (\lambda_0,\ldots,\lambda_{n+1}) & \mapsto & \sum_{k=0}^{i} \lambda_k E_k \;+\; \sum_{k=i+1}^{n+1} \lambda_k E'_{k-1} \end{array}$$

For any $n \geq 0$ we define the map $h : C_n(X) \longrightarrow C_{n+1}(X \times I)$ by

$$h(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma \times \mathbf{Id}) \circ \Gamma_{i,n},$$

for each $\sigma \in \Omega_n(X)$ in the basis, and we extend to $C_n(X)$ linearly. Notice that

$$\Delta_{n+1} \overset{\Gamma_{i,n}}{\longrightarrow} \Delta^n \times I \overset{\sigma\times\mathbf{Id}}{\longrightarrow} X \times I$$

so $(\sigma \times \mathbf{Id}) \circ \Gamma_{i,n} \in \Omega_{n+1}(X \times I)$ and h is a well defined homomorphism. This map is called the **prism operator**. Before continuing, we observe how the face maps and $\Gamma_{i,n}$ combine. We first show that

$$\Gamma_{i,n} \circ \varphi_{i,n+1} = \Gamma_{i-1,n} \circ \varphi_{i,n+1}, \tag{D.2}$$

for any $i = 1, \ldots, n$. Indeed,

$$\begin{aligned} \Gamma_{i,n}(\varphi_{i,n+1}(\lambda_0,\ldots,\lambda_n)) &= \Gamma_{i,n}(\lambda_0,\ldots,\lambda_{i-1},\overset{i}{0},\lambda_{i+1},\ldots,\lambda_n) \\ &= \sum_{j=0}^{i-1} \lambda_j E_j + \sum_{j=i+1}^{n} \lambda_j E'_j \\ &= \Gamma_{i-1,n}(\lambda_0,\ldots,\lambda_{i-1},\overset{i}{0},\lambda_j,\ldots,\lambda_n) \\ &= \Gamma_{i-1,n}(\varphi_{i,n+1}(\lambda_0,\ldots,\lambda_n)). \end{aligned}$$

Moreover,

$$(\varphi_{j,n} \times \mathbf{Id}) \circ \Gamma_{i,n-1} = \begin{cases} \Gamma_{i+1,n} \circ \varphi_{j,n+1} & \text{if } i \geq j, \\ \Gamma_{i,n} \circ \varphi_{j+1,n+1} & \text{if } i < j. \end{cases} \tag{D.3}$$

To show this, if $i \geq j$,

$$\begin{aligned}
(\varphi_{j,n} \times \mathbf{Id})(\Gamma_{i,n-1}(\lambda_0, \dots, \lambda_n)) &= (\varphi_{j,n} \times \mathbf{Id})(\sum_{k=0}^{i} \lambda_k E_k + \sum_{k=i+1}^{n} \lambda_k E'_{k-1}) \\
&= \sum_{k=0}^{i} \lambda_k (\varphi_{j,n} \times \mathbf{Id})(e_k, 0) \\
&\quad\quad + \sum_{k=i+1}^{n} \lambda_k (\varphi_{j,n} \times \mathbf{Id})(e_{k-1}, 1) \\
&= \sum_{k=0}^{j-1} \lambda_k (e_k, 0) + \sum_{k=j}^{i} \lambda_k (e_{k+1}, 0) + \sum_{k=i+1}^{n} \lambda_k (e_k, 1) \\
&= \sum_{k=0}^{j-1} \lambda_k E_k + \sum_{k=j}^{i} \lambda_k E_{k+1} + \sum_{k=i+1}^{n} \lambda_k E'_k \\
&= \Gamma_{i+1,n}(\lambda_0, \dots, \lambda_{j-1}, \overset{j}{0}, \lambda_j, \dots, \lambda_n) \\
&= \Gamma_{i+1,n}(\varphi_{j,n+1}(\lambda_0, \dots, \lambda_n)),
\end{aligned}$$

and if $i < j$,

$$\begin{aligned}
(\varphi_{j,n} \times \mathbf{Id})(\Gamma_{i,n-1}(\lambda_0, \dots, \lambda_n)) &= (\varphi_{j,n} \times \mathbf{Id})(\sum_{k=0}^{i} \lambda_k E_k + \sum_{k=i+1}^{n} \lambda_k E'_{k-1}) \\
&= \sum_{k=0}^{i} \lambda_k (\varphi_{j,n} \times \mathbf{Id})(e_k, 0) \\
&\quad\quad + \sum_{k=i+1}^{n} \lambda_k (\varphi_{j,n} \times \mathbf{Id})(e_{k-1}, 1) \\
&= \sum_{k=0}^{i} \lambda_k (e_k, 0) + \sum_{k=i+1}^{j-1} \lambda_k (e_{k-1}, 0) + \sum_{k=j}^{n} \lambda_k (e_k, 1) \\
&= \sum_{k=0}^{i} \lambda_k E_k + \sum_{k=i+1}^{j-1} \lambda_k E_{k-1} + \sum_{k=j}^{n} \lambda_k E'_k \\
&= \sum_{k=0}^{i} \lambda_k E_k + \sum_{k=i+1}^{j-1} \lambda_k E_{k-1} + \sum_{k=j+1}^{n+1} \lambda_{k-1} E'_{k-1} \\
&= \Gamma_{i+1,n}(\lambda_0, \dots, \lambda_{j-1}, \overset{j}{0}, \lambda_j, \dots, \lambda_n) \\
&= \Gamma_{i,n}(\varphi_{j+1,n+1}(\lambda_0, \dots, \lambda_n)).
\end{aligned}$$

Now we check if h satisfies (D.1). By linearity, it is enough to check if the equality holds for the elements of the basis. On the one hand, by (D.3), for any $\sigma \in \Omega_n(X)$,

$$\begin{aligned}
h(\partial(\sigma)) &= h(\sum_{j=0}^{n}(-1)^j[\sigma]_j) = \sum_{j=0}^{n}(-1)^j h([\sigma]_j) \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n-1}(-1)^{i+j}((\sigma \circ \varphi_{j,n}) \times \mathbf{Id}) \circ \Gamma_{i,n-1} \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n-1}(-1)^{i+j}((\sigma \times \mathbf{Id}) \circ (\varphi_{j,n} \times \mathbf{Id})) \circ \Gamma_{i,n-1} \\
&= \sum_{0\leq j\leq i\leq n-1}(-1)^{i+j}(\sigma \times \mathbf{Id}) \circ \Gamma_{i+1,n} \circ \varphi_{j,n+1} \\
&\quad + \sum_{0\leq i<j\leq n}(-1)^{i+j}(\sigma \times \mathbf{Id}) \circ \Gamma_{i,n} \circ \varphi_{j+1,n+1}.
\end{aligned}$$

On the other hand,

$$\partial(h(\sigma)) = \partial(\sum_{i=0}^{n}(-1)^i(\sigma \times \mathbf{Id}) \circ \Gamma_{i,n}) = \sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+j}[(\sigma \times \mathbf{Id}) \circ \Gamma_{i,n}]_j$$

separating terms where $i < j$, $i = j-1$,$i = j$ and $i > j$ this becomes

$$\begin{aligned}
\partial(h(\sigma)) = &\sum_{0\leq i<j-1<j\leq n+1}(-1)^{i+j}(\sigma \times \mathbf{Id}) \circ \Gamma_{i,n} \circ \varphi_{j,n+1} \\
&- \sum_{1\leq j\leq n+1}(\sigma \times \mathbf{Id}) \circ \Gamma_{j-1,n} \circ \varphi_{j,n+1} \\
&+ \sum_{0\leq j\leq n}(\sigma \times \mathbf{Id}) \circ \Gamma_{j,n} \circ \varphi_{j,n+1} \\
&+ \sum_{0\leq j<i\leq n}(-1)^{i+j}(\sigma \times \mathbf{Id}) \circ \Gamma_{i,n} \circ \varphi_{j,n+1},
\end{aligned}$$

rearranging indices $j = j' + 1$ in the first sum and $i = i' + 1$ in the last,

$$\begin{aligned}
\partial(h(\sigma)) = &\sum_{0\leq i<j'<j'+1\leq n}(-1)^{i+j'+1}(\sigma \times \mathbf{Id}) \circ \Gamma_{i,n} \circ \varphi_{j'+1,n+1} \\
&- \sum_{1\leq j\leq n+1}(\sigma \times \mathbf{Id}) \circ \Gamma_{j-1,n} \circ \varphi_{j,n+1} \\
&+ \sum_{0\leq j\leq n}(\sigma \times \mathbf{Id}) \circ \Gamma_{j,n} \circ \varphi_{j,n+1} \\
&+ \sum_{0\leq j<i'\leq n-1}(-1)^{i'+j+1}(\sigma \times \mathbf{Id}) \circ \Gamma_{i'+1,n} \circ \varphi_{j,n+1}.
\end{aligned}$$

If we add both computations, we observe that the first and last sum cancel out and the terms in the middle sums by (D.2) cancel out too except those where $j = 0$ and $j = n + 1$. That is,

$$h(\partial(\sigma)) + \partial(h(\sigma)) = -(\sigma \times \mathbf{Id}) \circ \Gamma_{n,n} \circ \varphi_{n+1,n+1} + (\sigma \times \mathbf{Id}) \circ \Gamma_{0,n} \circ \varphi_{0,n+1}$$

and finally notice that

$$\begin{aligned}(\sigma \times \mathbf{Id})(\Gamma_{n,n}(\varphi_{n+1,n+1}(\lambda_0, \ldots, \lambda_n))) &= (\sigma \times \mathbf{Id})(\Gamma_{n,n}(\lambda_0, \ldots, \lambda_n, 0)) \\ &= (\sigma \times \mathbf{Id})(\sum_{j=0}^{n} \lambda_k E_k) \\ &= (\sigma \times \mathbf{Id})((\lambda_0, \ldots, \lambda_n), 0) \\ &= (\sigma(\lambda_0, \ldots, \lambda_n), 0) \\ &= \iota_0(\sigma(\lambda_1, \ldots, \lambda_n)),\end{aligned}$$

$$\begin{aligned}(\sigma \times \mathbf{Id})(\Gamma_{0,n}(\varphi_{0,n+1}(\lambda_0, \ldots, \lambda_n))) &= (\sigma \times \mathbf{Id})(\Gamma_{0,n}(\lambda_0, \ldots, \lambda_n, 1)) \\ &= (\sigma \times \mathbf{Id})(\sum_{j=0}^{n} \lambda_k E'_k) \\ &= (\sigma \times \mathbf{Id})((\lambda_0, \ldots, \lambda_n), 1) \\ &= (\sigma(\lambda_0, \ldots, \lambda_n), 1) \\ &= \iota_1(\sigma(\lambda_1, \ldots, \lambda_n)).\end{aligned}$$

Thus, $(\iota_0)_\#(\sigma) = \iota_0 \circ \sigma = (\sigma \times \mathbf{Id}) \circ \Gamma_{n,n} \circ \varphi_{n+1,n+1}$ and $(\iota_1)_\#(\sigma) = \iota_1 \circ \sigma = (\sigma \times \mathbf{Id}) \circ \Gamma_{0,n} \circ \varphi_{0,n+1}$ which concludes the proof. □

Appendix E

Excision theorem

E.1 The barycentric subdivision

Definition E.1.1. Let X be a topological space and let $\mathcal{U}$ be a collection of subspaces of X whose interiors cover X. A singular n-chain $c \in C_n(X)$ is said to be $\boldsymbol{\mathcal{U}}$**-small** if every singular simplex that appears in c has an image lying entirely in one of the subsets in $\mathcal{U}$.

Let $C_n^{\mathcal{U}}(X)$ denote the subgroup of $C_n(X)$ consisting of $\mathcal{U}$-small chains. They form a chain complex $(C_*^{\mathcal{U}}(X), \partial_*)$. Let $H_n^{\mathcal{U}}(X)$ denote the n-th homology group of $(C_*^{\mathcal{U}}(X), \partial_*)$ for every $n \geq 0$. The goal of this section is to prove the following result.

Proposition E.1.1. *Let X be a topological space and let $\mathcal{U}$ be a collection of subspaces of X whose interior cover X. Then, the inclusion map $C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ induces a homology isomorphism $H_n^{\mathcal{U}}(X) \to H_n(X)$ for any $n \geq 0$.*

The idea of the proof is to show that if $\sigma : \Delta^n \longrightarrow X$ is any singular n-simplex, there is a homologous n-chain obtained by "subdividing" σ into n-simplices with smaller images. If we divide sufficiently finely, we can ensure that each of the resulting simplices will be $\mathcal{U}$-small. The tricky part is to do this in a way that allows us to keep track of the boundary operators.

Definition E.1.2. For any n-simplex $[p_0, \dots, p_n] \subseteq \mathbb{R}^m$, we define the **barycenter** of $[p_0, \dots, p_n]$ to be the point

$$b_{[p_0,\dots,p_n]} = \sum_{i=0}^{n} \frac{1}{n+1} p_i \in \operatorname{int}([p_0, \dots, p_n]).$$

Let $[p_0, \dots, p_n] \subseteq \mathbb{R}^m$ be an n-simplex. The canonical homeomorphism $\varphi_{[p_0,\dots,p_n]} : \Delta^n \to [p_0, \dots, p_n]$ is a singular n-simplex in $\Omega_n(\mathbb{R}^m)$. We now give such singular simplices a name.

Definition E.1.3. A singular n-simplex $\alpha \in \Omega_n(\mathbb{R}^m)$ is called **affine n-simplex** if $\alpha = \varphi_{[p_0,\dots,p_n]}$ for some n-simplex $[p_0, \dots, p_n] \subseteq \mathbb{R}^m$.

A chain in $C_n(\mathbb{R}^m)$ formed exclusively by affine n-simplices is called an **affine n-chain**.

Remark E.1.1. Observe that the boundary of an affine n-simplex is an affine $(n-1)$-chain. Indeed, if $\alpha = \varphi_{[p_0,\dots,p_n]}$,

$$\partial(\alpha) = \sum_{i=0}^{n} (-1)^i \varphi_{[p_0,\dots,\hat{p_i},\dots,p_n]}.$$

Definition E.1.4. Let $q \in \mathbb{R}^m$. For any affine n-simplex $\alpha = \varphi_{[p_0,\dots,p_n]} \in \Omega_n(\mathbb{R}^m)$ we define an affine $(n+1)$-simplex $q * \alpha$ called the **cone** on α from q by

$$q * \alpha = \varphi_{[q,p_0,\dots,p_n]}.$$

We can extend this operator to all affine n-chains by linearity.

Remark E.1.2. Observe that the cone $q*\alpha$ is the affine simplex that sends e_0 to q and whose 0-th face map is equal to α.

Lemma E.1.2. *Let $c \in C_n(\mathbb{R}^m)$ be an affine chain. Then, for any $q \in \mathbb{R}^m$,*

$$\partial(q * c) + q * \partial(c) = c.$$

Proof. We will prove the identity for an affine n-simplex $\alpha = \varphi_{[p_0,\dots,p_n]}$. The result for affine n-chains follows by linearity. Indeed,

$$\begin{aligned}\partial(q * \alpha) = \partial(\varphi_{[q,p_0,\dots,p_n]}) &= \sum_{i=0}^{n+1} (-1)^i [\varphi_{[q,p_0,\dots,p_n]}]_i \\ &= \varphi_{[p_0,\dots,p_n]} + \sum_{i=0}^{n} (-1)^{i+1} \varphi_{[q,p_0,\dots,\hat{p_i},\dots,p_n]} = \alpha + q * \partial(\alpha).\end{aligned}$$

□

Definition E.1.5. We define the **singular subdivision operator** S on affine n-chains inductively. For $n = 0$, set $S = \mathbf{Id}$. For $n \geq 1$, assume that S has been defined for chains of dimension less than n and for any affine n-simplex $\alpha = \varphi_{[p_0,\dots,p_n]}$ we set

$$S(\alpha) = \alpha(b_{\Delta^n}) * S(\partial(\alpha)),$$

and extend linearly to affine n-chains.

Definition E.1.6. Let $A \subseteq \mathbb{R}^m$ be a compact subset of the euclidean space. The **diameter** of A is

$$\operatorname{diam} A = \sup \{ ||x - y|| \mid x, y \in A \}$$

Lemma E.1.3. *Let $\alpha = \varphi_{[p_0,\ldots,p_n]}$ be an affine n-simplex. Let β be any of the affine n-simplices that appear in the affine n-chain $S(\alpha)$. Then,*

(i) $\beta = \varphi_{[b_n,\ldots,b_0]}$, *where each b_i is the barycenter of an i-dimensional face* of $[p_0, \ldots, p_n]$.*

(ii) $\operatorname{diam}([b_n, \ldots, b_0]) \leq \frac{n}{n+1} \operatorname{diam}([p_0, \ldots, p_n])$.

Proof. We start by proving (i) by induction. If $n = 0$, $\alpha = \varphi_{[p_0]}$ and the claim holds because $S(\alpha) = \alpha$ and $b_{[p_0]} = p_0$.

If $n \geq 1$, let $\alpha = \varphi_{[p_0,\ldots,p_n]}$. Let $b_n = \alpha(b_{\Delta^n})$. Notice that b_n is the barycenter of $[p_0, \ldots, p_n]$. Then, $\partial(\alpha)$ is an $(n-1)$-chain, so by induction hypothesis each affine singular $(n-1)$-simplex in $S(\partial(\alpha))$ is of the form $\varphi_{[b_{n-1},\ldots,b_0]}$, where each b_i is the barycenter of an i-dimensional face of $[p_0, \ldots, p_n]$. Thus, each affine n-simplex in $S(\alpha)$ is of the form $b_n * \varphi_{[b_{n-1},\ldots,b_0]} = \varphi_{[b_n,b_{n-1},\ldots,b_0]}$ as desired.

To prove (ii), notice that since a simplex is the convex hull of its vertices, the diameter of $[b_n, \ldots, b_0]$ is equal to the maximum of the distances between its vertices. Thus, it suffices to show that

$$||b_i - b_j|| \leq \frac{n}{n+1} \operatorname{diam}([p_0, \ldots, p_n])$$

whenever b_i and b_j are barycenters of faces of $[p_0, \ldots, p_n]$. We will prove it by induction. For $n = 0$, there is nothing to prove. Assume the claim is true for simplices of dimension less than n. For $i, j < n$, both vertices b_i, b_j lie in some m-dimensional face $[q_1, \ldots, q_m] \subseteq [p_0, \ldots p_n]$ with $m < n$. By induction, we get

$$||b_i - b_j|| \leq \frac{m}{m+1} \operatorname{diam}([q_0, \ldots, q_m]) \leq \frac{n}{n+1} \operatorname{diam}([p_0, \ldots, p_n]).$$

It remains only to bound the distance between b_n and the other vertices. But since b_n is the barycenter of $[p_0, \ldots, p_n]$ itself and every other vertex b_j lies in some j-dimensional face of $[p_0, \ldots, p_n]$ with $j < n$, the distance from b_n to b_j is bounded by the maximum of the distance from b_n to any of the

*We call "i-dimensional face" to the simplices obtained after removing $n - i$ vertices from $[p_0, \ldots, p_n]$. A n-dimensional face would be the whole simplex, and a 0-dimensional face the singleton $[p_i]$.

vertices p_j of $[p_0, \dots, p_n]$. Then, for any vertex p_j,

$$||b_n - p_j|| = ||\sum_{i=0}^{n} \frac{1}{n+1} p_i - p_j|| = ||\sum_{i=0}^{n} \frac{1}{n+1} p_i - \sum_{i=0}^{n} \frac{1}{n+1} p_j||$$
$$\leq \sum_{i=0}^{n} \frac{1}{n+1} ||p_i - p_j|| \leq \frac{n}{n+1} \operatorname{diam}([p_0, \dots, p_n]).$$

The maximum of such distances bounds $||b_n - b_j||$, so this completes the proof. □

Now we need to extend the singular subdivision operator to arbitrary (not necessarily affine) singular chains. Let X be a topological space and $\sigma \in \Omega_n(X)$. Notice that $\sigma = \sigma_{\#}(\imath_n)$, where $\imath_n = \mathbf{Id}_{\Delta^n} = \varphi_{[e_0,\dots,e_n]}$ is an affine n-simplex and $\sigma_{\#} : C_n(\Delta^n) \to C_n(X)$ is the chain map obtained from the continuous map $\sigma : \Delta^n \to X$. We define

$$S(\sigma) = \sigma_{\#}\big(S(\imath_n)\big),$$

and we extend linearly to all $C_n(X)$. We may iterate S to obtain operators $S^2 = S \circ S$ and more generally $S^k = S \circ S^{k-1}$.

Definition E.1.7. The **mesh** of an affine n-chain $c = \sum_{i \in I} \lambda_i \alpha_i$ in $C_n(\mathbb{R}^m)$ is the maximum of the diameters of the images of the affine simplices that appear in c. That is,

$$\operatorname{mesh}(c) = \max\left\{ \operatorname{diam}(\alpha_i(\Delta^n)) \;\middle|\; i \in I \right\}.$$

Remark E.1.3. By (ii) of Lemma E.1.3, observe that choosing k large enough we can make the mesh of $S^k(c)$ arbitrarily small for an affine n-chain c, because

$$\operatorname{mesh}(S^k(c)) \leq \frac{n^k}{(n+1)^k} \operatorname{mesh}(c).$$

Lemma E.1.4. *The singular subdivision operators $S : C_n(X) \to C_n(X)$ have the following properties:*

(i) *For any continuous map $f : X \to Y$, $S \circ f_{\#} = f_{\#} \circ S$.*

(ii) $\partial \circ S = S \circ \partial$.

(iii) *Given any open cover $\mathcal{U}$ of X and any chain $c \in C_n(X)$, there exists some $m \geq 1$ such that $S^m(c) \in C_n^{\mathcal{U}}(X)$.*

Proof. (i) follows inmediately from the definition of S. For any $\sigma \in \Omega_n(X)$,

$$S(f_*(\sigma)) = S(f \circ \sigma) = (f \circ \sigma)_*(S(\imath_n)) = f_*(\sigma_*(S(\imath_n))) = f_*(S(\sigma)).$$

The result for any chain in $C_n(X)$ follows by linearity.

We prove identity (ii) by induction on n. For $n = 0$ it is inmediate because S acts as the identity on 0-chains. For $n \geq 1$, let $\sigma \in \Omega_n(X)$. By (i), Lemma E.1.2 and the induction hypothesis,

$$\begin{aligned}\partial(S(\sigma)) &= \partial\big(\ \sigma_\#(S(\imath_n))\ \big) = \partial\big(\ \sigma_\#(b_{\Delta^n} * S(\partial(\imath_n)))\ \big)\\ &= \sigma_\#\big(\ \partial(b * S(\partial(\imath_n)))\ \big) = \sigma_\#\big(\ S(\partial(\imath_n)) - b_{\Delta^n} * \partial(S(\partial(\imath_n)))\ \big)\\ &= S\big(\ \sigma_\#(\partial(\imath_n))\ \big) - \sigma_\#\big(\ b_{\Delta^n} * S(\partial^2(\imath_n))\ \big) = S\big(\ \partial(\sigma_\#(\imath_n))\ \big) - 0\\ &= S(\partial(\sigma)).\end{aligned}$$

Again, the identity for a singular n-chain follows by linearity.

To prove (iii) observe that by Remark E.1.3 by choosing m large enough we can make the mesh of $S^m(\imath_n)$ arbitrarilly small. If σ is any singular simplex in X, by the Lebesgue number lemma, there is some $\delta > 0$ such that any subset of Δ^n of diameter less than δ lies entirely in $\sigma^{-1}(U)$ for one of the sets $U \in \mathcal{U}$. In particular, if c is an affine chain in $C_n(\Delta^n)$ whose mesh is less than δ, every singular n-simplex in $\sigma_\#(c)$ is contained entirely in one of the open sets $U \in \mathcal{U}$. Then, $\sigma_\#(c) \in C_n^{\mathcal{U}}(X)$. Therefore, for any $c = \sum_{\sigma\in\Omega_n(X)} \lambda_\sigma \sigma \in C_n(X)$ if we choose δ to be the minimum of the Lebesgue numbers for all the singular simplices $\sigma \in \Omega_n(X)$ appearing in c, and choose m large enough so that $S^m(\imath_n)$ has mesh less than δ, we get that for each σ in the chain c, $S^m(\sigma) = \sigma_*(S^m(\imath_n)) \in C_n^{\mathcal{U}}(X)$ and $S^m(c) \in C_n^{\mathcal{U}}(X)$ by linearity. □

With all the machinery we have built up, we finally prove the main result of the section.

Proof of Proposition D.1.1. The crux of the prove is the construction of a chain homotopy between the singular subdivision operator S and the identity map of $C_n(X)$. That is, we aim to build a homomorphism $h : C_n(X) \to C_{n+1}(X)$ satisfying

$$\partial \circ h + h \circ \partial = \mathbf{Id} - S \tag{E.1}$$

We define h by induction on n. For $n = 0$, h is the zero homomorphism. For $n \geq 1$, $\sigma \in \Omega_n(X)$ we define

$$h(\sigma) = \sigma_\#\big(b_{\Delta^n} * (\imath_n - S(\imath_n) - h(\partial(\imath_n)))\big),$$

and extend it to the whole $C_n(X)$ linearly. Consider a continuous map $f : X \to Y$. If $n \geq 1$, for any $\sigma \in \Omega_n(X)$ we have that

$$\begin{aligned}h(f_\#(\sigma)) &= h(f \circ \sigma) = (f \circ \sigma)_\#\big(\ b_{\Delta^n} * (\imath_n - S(\imath_n) - h(\partial(\imath_n)))\ \big)\\ &= f_\#\Big(\sigma_\#\big(b_{\Delta^n} * (\imath_n - S(\imath_n) - h(\partial(\imath_n)))\big)\Big) = f_\#(h(\sigma)).\end{aligned}$$

If $n = 0$ the identity $h \circ f_\# = f \circ h$ is trivially true.

Moreover, if $n \geq 1$ and if σ is a $\mathcal{U}$-small singular n-simplex, for any $\tau \in \Omega_n(\Delta^n)$ we have that $\sigma \circ \tau$ is also $\mathcal{U}$ small. Thus, $\operatorname{Im}\sigma_* \subseteq C_n^{\mathcal{U}}(X)$ and $h(\sigma) \in C_n^{\mathcal{U}}(X)$. This means that h maps $C_n^{\mathcal{U}}(X)$ to $C_{n+1}^{\mathcal{U}}(X)$. If $n = 0$ this fact is also trivially true.

The identity (E.1) is proven by induction on n. For $n = 0$ it is inmediate because $h = \partial = 0$ and $S = \mathbf{Id}$. Suppose it holds for $(n-1)$ chains in all spaces. If $\sigma \in \Omega_n(X)$, then by Lemma E.1.2 and since $\partial(\imath_n)$ is a $(n-1)$ chain,

$$\begin{aligned}
\partial(h(\sigma)) &= \partial\Big(\ \sigma_\#\big(\ b_{\Delta^n} * (\imath_n - S(\imath_n) - h(\partial(\imath_n)))\ \big)\ \Big) \\
&= \sigma_\#\Big(\ \partial\big(\ b_{\Delta^n} * (\ \imath_n - S(\imath_n) - h(\partial(\imath_n)))\ \big)\ \Big) \\
&= \sigma_\#\big(\ \imath_n - S(\imath_n) - h(\partial(\imath_n))\ \big) \\
&\qquad - \sigma_\#\Big(\ b_{\Delta^n} * \big(\ \partial(\imath_n) - \partial(S(\imath_n)) - \partial(h(\partial(\imath_n)))\ \big)\ \Big) \\
&= \sigma_\#\big(\ \imath_n - S(\imath_n) - h(\partial(\imath_n))\ \big) \\
&\qquad - \sigma_\#\Big(\ b_{\Delta^n} * \big(\ \partial(\imath_n) - S(\partial(\imath_n)) - \partial(h(\partial(\imath_n))) - h(\partial^2(\imath_n))\ \big)\ \Big) \\
&= \sigma_\#\big(\ \imath_n - S(\imath_n) - h(\partial(\imath_n))\ \big) \\
&\qquad - \sigma_\#\Big(\ b_{\Delta^n} * \big(\ (\mathbf{Id} - S - \partial\circ h - h \circ \partial)\big(\partial(\imath_n)\big)\ \big)\ \Big) \\
&= \sigma_\#\big(\ \imath_n - S(\imath_n) - h(\partial(\imath_n))\ \big) - 0 \\
&= \sigma_\#(\imath_n) - S(\sigma_\#(\imath_n)) - h(\partial(\sigma_\#(\imath_n))) \\
&= \sigma - S(\sigma) - h(\partial(\sigma)),
\end{aligned}$$

which proves identity (E.1). Let $c \in C_n(X)$ be a cycle. (E.1) shows that

$$c - S(c) = \partial(h(c)) + h(\partial(c)) = \partial(h(c)),$$

so $S(c)$ differs from c by a boundary. If $c \in C_n^{\mathcal{U}}(X)$, the difference is the boundary of a chain in $C_{n+1}^{\mathcal{U}}(X)$. The same holds for any $S^m(c)$. If $m \geq 1$, by induction, if $c - S^{m-1}(c) = \partial(a)$ for some $a \in C_{n+1}(X)$, then

$$\begin{aligned}
c - S^m(c) = c - S^{m-1}(c - \partial(h(c))) &= c - S^{m-1}(c) - S^{m-1}(\partial(c)) \\
= \partial(h(a)) - \partial(S^{m-1}(c)) &= \partial(h(a) - S^{m+1}(c)).
\end{aligned}$$

Hence, the difference of c and $S^m(c)$ is a boundary. Moreover, $S^m(c)$ is also a cycle because S commutes with ∂.

The inclusion map $\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is clearly a chain map so it induces a homology homomorphism $\iota_* : H_n^{\mathcal{U}}(X) \to H_n(X)$. The homomorphism ι_* is surjective because, by Lemma E.1.4, for any $c \in C_n(X)$ we can choose m large enough so that $S^m(c) \in C_n^{\mathcal{U}}(X)$, and have showed above that c is homologous to $S^m(c)$. To prove injectivity, let $c + \operatorname{Im}\partial \in H_n^{\mathcal{U}}(X)$ be such that $\iota_*(c + \operatorname{Im}\partial) = 0$. This means that there is some $(n+1)$-chain $b \in C_{n+1}(X)$ such that $c = \partial(b)$. Choose m large enough so that $S^m(b) \in C_{n+1}(X)$. Then,

$$\partial(S^m(b)) = S^m(\partial(b)) = S^m(c),$$

which differs from c by a boundary of a chain in $C_{n+1}^{\mathcal{U}}(X)$ as showed before. Thus, $c + \operatorname{Im}\partial = 0$. □

E.2 The Excision Theorem

Using the machinery built in the previous chapter, we prove the Excision Theorem which was stated without proof in Chapter 2.

Theorem E.2.1 (Excision theorem). *Let X be a topological space and $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A. Then, the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \to H_n(X, A)$ for all $n \geq 0$.*

Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all $n \geq 0$.

Remark E.2.1. The translation between the two assertions of the theorem is obtained as follows:

- To get the second from the first, assume the hypotheses of the second version and set $Z = X - B$. Then, as $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, we get $Z \subseteq A \subseteq X$ and $\operatorname{cl}(Z) \subseteq \operatorname{int}(A)$ since $X - \operatorname{int}(B) = \operatorname{cl}(Z)$. Thus, as $A \cap B = A - Z$, $H_n(B, A \cap B) = H_n(X - Z, A - Z) \cong H_n(X, A)$ for all n.

- To get the first from the second, assume the hypotheses of the first version and set $B = Z - X$. Since $X - \operatorname{int}(B) = \operatorname{cl}(Z) \subseteq \operatorname{int}(A)$, it is clear that $X = \operatorname{int}(A) \cup \operatorname{int}(B)$. Therefore, as $A \cap B = A - Z$, $H_n(X - Z, A - Z) = H_n(B, A \cap B) \cong H_n(X, A)$ for all n.

Proof of the Excision Theorem. We prove the second version. Let $A, B \subseteq X$ be two subspaces whose interiors cover X. Let $\mathcal{U} = \{A, B\}$. We know that the inclusion $C_n^{\mathcal{U}} \hookrightarrow C_n(X)$ induces an isomorphism on homology by Proposition E.1.1. This inclusion sends $C_n(A) \subseteq C_n^{\mathcal{U}}$ to $C_n(A) \subseteq C_n(X)$, so it induces a map in the quotient $\iota : \frac{C_n^{\mathcal{U}}(X)}{C_n(A)} \to C_n(X, A)$. The singular subdivision operator S defined in Section E.1 and the homeomorphism h defined in the proof of Proposition E.1.1 also send chains in A to chains in A, so they induce maps $S : C_n(X, A) \to C_n(X, A)$ and $h : C_n(X, A) \to C_{n+1}(X, A)$ in the quotients. These maps also satisfy equation (E.1) because they satisfied it before passing to the quotient. Thus, in the same way as in Proposition E.1.1 one can prove that $\iota : \frac{C_n^{\mathcal{U}}(X)}{C_n(A)} \to C_n(X, A)$ induces an isomorphism on homology.

Moreover, the inclusion $C_n(B) \hookrightarrow C_n^{\mathcal{U}}(X)$ also induces a map in the quotient $\jmath : C_n(B, A \cap B) \to \frac{C_n^{\mathcal{U}}(X)}{C_n(A)}$ since $C_n(A \cap B) \subseteq C_n(A)$. $\jmath$ is clearly an isomorphism since both quotient groups are free $\mathbb{Z}$-modules with basis the singular n-simplices in B that do not lie in A. Thus $\jmath$ also induces an isomorphism on homology.

Finally observe that the map induced by the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ on homology groups is precisely the composition $\iota_* \circ \jmath_*$. Thus, the map $H_n(B, A \cap B) \to H_n(X, A)$ induced by inclusion is an isomorphism.

□

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