



Article

On Ostrowski–Mercer’s Type Fractional Inequalities for Convex Functions and Applications

Soubhagya Kumar Sahoo ^{1,*}, Artion Kashuri ², Munirah Aljuaid ³, Soumyarani Mishra ⁴
and Manuel De La Sen ^{5,*}

¹ Department of Mathematics, C.V. Raman Global University, Bhubaneswar 752054, India

² Department of Mathematics, Faculty of Technical and Natural Sciences, University Ismail Qemali, 9400 Vlora, Albania

³ Department of Mathematics, College of Science, Northern Border University, Arar 73213, Saudi Arabia

⁴ Department of Mathematics, Institute of Technical Education and Research, Siksha ‘O’ Anusandhan University, Bhubaneswar 751030, India

⁵ Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of Basque Country, 48940 Leioa, Spain

* Correspondence: soubhagyalulu@gmail.com (S.K.S.); manuel.delasen@ehu.eus (M.D.L.S.)

Abstract: This research focuses on the Ostrowski–Mercer inequalities, which are presented as variants of Jensen’s inequality for differentiable convex functions. The main findings were effectively composed of convex functions and their properties. The results were directed by Riemann–Liouville fractional integral operators. Furthermore, using special means, q -digamma functions and modified Bessel functions, some applications of the acquired results were obtained.

Keywords: convex function; Ostrowski’s inequality; Mercer inequality; Riemann–Liouville fractional integral operators; special means; q -digamma functions; Bessel function



Citation: Sahoo, S.K.; Kashuri, A.; Aljuaid, M.; Mishra, S.; De La Sen, M. On Ostrowski–Mercer’s Type Fractional Inequalities for Convex Functions and Applications. *Fractal Fract.* **2023**, *7*, 215. <https://doi.org/10.3390/fractalfract7030215>

Academic Editor: Agnieszka B. Malinowska

Received: 27 January 2023

Revised: 17 February 2023

Accepted: 22 February 2023

Published: 25 February 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

We will begin by introducing the Ostrowski inequality, which produces an upper bound for the approximation of the integral average $\frac{1}{v-u} \int_u^v \mathcal{W}(\mathbb{k}) d\mathbb{k}$ by the value of $\mathcal{W}(\mathbb{k})$ at the point $\mathbb{k} \in [u, v]$ and has quite a lot of applications in the field of inequalities.

Let $\mathcal{W} : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° , the interior of the interval J , such that $\mathcal{W} \in L_1[u, v]$, where $u, v \in J$ with $v > u$. If $|\mathcal{W}'(\ell)| \leq M$, for all $\ell \in [u, v]$, then the following inequality holds:

$$\left| \mathcal{W}(\ell) - \frac{1}{v-u} \int_u^v \mathcal{W}(\mathbb{k}) d\mathbb{k} \right| \leq M(v-u) \left[\frac{1}{4} + \frac{(\ell - \frac{u+v}{2})^2}{(v-u)^2} \right], \quad (1)$$

New versions, generalizations and modifications of this unique inequality have been produced by many researchers in the literature (see [1–12]).

Now we will discuss a class of function that acts as one of the cornerstones of inequality theory. This class of function, called convex function, has been introduced in numerous variants and has applications in many disciplines, such as convex programming, statistics, numerical analysis, and approximation theory.

Definition 1 ([13]). *A function $\mathcal{W} : I = [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$, is called convex, if*

$$\mathcal{W}(\mathbb{k}u + (1 - \mathbb{k})v) \leq \mathbb{k}\mathcal{W}(u) + (1 - \mathbb{k})\mathcal{W}(v), \quad (2)$$

for all $u, v \in I$, $\mathbb{k} \in [0, 1]$.

If the function \mathcal{W} is concave, then $(-\mathcal{W})$ is convex.

Many aesthetic inequalities on convex functions exist in the literature, among which Jensen's inequality has a special place. This inequality is proved under fairly simple conditions, and is extensively used by researchers in fields such as information theory and inequality theory. Jensen's inequality is presented as follows:

Let $0 < \chi_1 \leq \chi_2 \leq \dots \leq \chi_n$ and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be non-negative weights such that $\sum_{\ell=1}^n \sigma_\ell = 1$. The Jensen inequality (see [14]) in the literature states that if \mathcal{W} is a convex function on the interval $[u, v]$, then

$$\mathcal{W}\left(\sum_{\ell=1}^n \sigma_\ell \chi_\ell\right) \leq \sum_{\ell=1}^n \sigma_\ell \mathcal{W}(\chi_\ell), \quad (3)$$

holds for all $\chi_\ell \in [u, v]$, $\sigma_\ell \in [0, 1]$ and $\ell = 1, 2, \dots, n$. It is a crucial inequality in information theory that aids in the extraction of bounds for useful distances (see [15–17]).

Although many researchers have focused on Jensen's inequality, the version proposed by Mercer is the most interesting and remarkable among them. Mercer [18], in 2003, introduced a new variant of Jensen's inequality given as follows:

If \mathcal{W} is a convex function on $[u, v]$, then

$$\mathcal{W}\left(u + v - \sum_{\ell=1}^n \sigma_\ell \chi_\ell\right) \leq \mathcal{W}(u) + \mathcal{W}(v) - \sum_{\ell=1}^n \sigma_\ell \mathcal{W}(\chi_\ell), \quad (4)$$

holds for all $\chi_\ell \in [u, v]$, $\sigma_\ell \in [0, 1]$ and $\ell = 1, 2, \dots, n$.

Several refinements of Jensen–Mercer inequalities were put forth by Pečarić, J. et al. [19]. Mercer's type inequalities later received many adaptations to higher dimensions by Niezgoda [20]. Recently, it has made a significant addition to inequality theory, owing to its well-known characterizations. The concept of the Jensen inequality for super quadratic functions was considered by Kian [21].

The Jensen–Mercer inequality was credited to Kian and Moslehian [22], and the following Hermite–Hadamard–Mercer inequality is as follows:

$$\begin{aligned} \mathcal{W}\left(u + v - \frac{s_1 + s_2}{2}\right) &\leq \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \mathcal{W}(u + v - k) dk \\ &\leq \frac{\mathcal{W}(u + v - s_1) + \mathcal{W}(u + v - s_2)}{2} \leq \mathcal{W}(u) + \mathcal{W}(v) - \frac{\mathcal{W}(s_1) + \mathcal{W}(s_2)}{2}, \end{aligned} \quad (5)$$

where \mathcal{W} is the convex function on $[u, v]$.

For more recent studies linked to the Jensen–Mercer inequality, one can refer to the following articles [23–26]. Although fractional analysis has a history as long as classical analysis, it has recently gained popularity among researchers. It is constantly striving to advance with its use in real-world problems, contribution to engineering sciences and opportunity for development in different dimensions. One aspect that keeps fractional analysis up to date is the definition of fractional order derivatives and integrals, as well as the contribution of each new operator to different fields. When the new operators are closely examined, various features such as singularity, locality, generalization and differences in their kernel structures become apparent. Although generalizations and inferences are the foundations of mathematical methods, the new fractional operators add new features to solutions, particularly for the time memory effect. Accordingly, various operators such as Riemann–Liouville, Grünwald Letnikov, Raina, Katugampola, Prabhakar, Hilfer, Caputo–Fabrizio and Atangana–Baleanu reveal the true potential of fractional analysis. Now we will continue by introducing the Riemann–Liouville integral operators, which have a special place among these operators.

Let $\mathcal{W} \in L_1[u, v]$. Then, Riemann–Liouville fractional integrals of order $\alpha > 0$ are defined as follows:

$$\mathbb{I}_{u^+}^\alpha \mathcal{W}(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-t)^{\alpha-1} \mathcal{W}(t) dt, \quad x > u$$

and

$$\mathbb{I}_{v^-}^\alpha \mathcal{W}(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} \mathcal{W}(t) dt, \quad x < v.$$

The Riemann–Liouville fractional integral operator is further expanded to many new integral operators, i.e., k -Riemann–Liouville fractional integral [27], ψ -Riemann–Liouville fractional integrals [28], Katugampola fractional integrals [29], k, s -Riemann–Liouville fractional integrals [30] and many such new definitions. Inspired by the Riemann–Liouville fractional integral operators, Ahmad et al. [31] introduced a new fractional integral operator involving an exponential function in its kernel and established a few generalizations of the Hermite–Hadamard type and its inequalities. It has applications in the Schrödinger Equation [32], electrical screening effect [33] and delayed nonlinear oscillator [34].

For further details, we refer to the following papers (see [35–40]).

Recently, the effect of fractional analysis has begun to be felt more in the theory of inequality. Many new inequalities and new approaches for some well-known inequalities have been introduced using fractional operators. The Hermite–Hadamard inequality has been generalized with the Riemann–Liouville integral operators, which is the most important result of this effort. This generalization is presented by Sarikaya et al. as follows (see [35]).

Theorem 1. Let $\mathcal{W} : [u, v] \rightarrow \mathbb{R}$ be a positive function with $0 \leq u < v$ and $\mathcal{W} \in L_1[u, v]$. If \mathcal{W} is a convex function on $[u, v]$, then the following inequality for fractional integral holds:

$$\mathcal{W}\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [\mathbb{I}_{u^+}^\alpha \mathcal{W}(v) + \mathbb{I}_{v^-}^\alpha \mathcal{W}(u)] \leq \frac{\mathcal{W}(u) + \mathcal{W}(v)}{2} \quad (6)$$

with $\alpha > 0$.

The Ostrowski inequality was generalized by numerous mathematicians in various ways. In particular, a number of academic studies that consider various convexities have been published in this area. Alomari et al. [1], for instance, employed the concept of s -convexity, and Icsan et al. [41] used the concept of harmonically s -convex function. The fractional variant of the Ostrowski-type inequality was first proposed by Set [42] using Riemann–Liouville fractional operators. Liu [43] developed new iterations of Ostrowski-type inequality for the MT-convex function using the equality proved in [42]. By using the Raina fractional integral operator, Agarwal et al. [44] examined a more generalised Ostrowski-type inequality. To create novel generalizations of the Ostrowski-type inequality, Sarikaya et al. [45] used local fractional integrals. For an extended form of the Ostrowski inequality, Gurbuz et al. [46] employed the Katugampola fractional operator. Atangana–Baleanu fractional operator for differentiable convex functions was used by Ahmad et al. [47] to show some innovative generalization of the Ostrowski inequality. As an advancement of this inequality, Sial et al. [48] presented Ostrowski–Mercer type inequalities for differentiable convex functions, and Ali et al. [49] used harmonically convex functions to prove new versions of Ostrowski–Mercer-type inequalities.

The major objective of this study is to create some novel Mercer–Ostrowski-type inequalities for convex functions by using Riemann–Liouville fractional integral operators with the help of a novel integral identity. Applications of the results were also presented considering numerous particular cases of the primary findings.

2. Main Results

In this section, we present Mercer–Ostrowski inequalities for the first differentiable functions on (u, v) for the Riemann–Liouville integral operators. For this, we introduce a new fractional identity that will act as an aid in establishing future findings.

Lemma 1. Suppose $\mathcal{W} : I = [u, v] \rightarrow \mathbb{R}$ be a differentiable function on (u, v) with $v > u$. If $\mathcal{W}' \in L_1[u, v]$, then for all $\ell \in [s_1, s_2]$, $s_1, s_2 \in [u, v]$ and $\alpha > 0$, the following identity holds true:

$$\begin{aligned} & \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ I_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + I_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \\ &= (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha \mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell]) d\mathbb{k} - (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha \mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell]) d\mathbb{k}. \end{aligned} \quad (7)$$

Proof. Let us start with

$$\begin{aligned} I &= (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha \mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell]) d\mathbb{k} \\ &\quad - (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha \mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell]) d\mathbb{k}. \end{aligned} \quad (8)$$

$$I = (\ell - s_1)^{\alpha+1} I_1 - (s_2 - \ell)^{\alpha+1} I_2, \quad (9)$$

where

$$\begin{aligned} I_1 &= \int_0^1 \mathbb{k}^\alpha \mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell]) d\mathbb{k} \\ &= \frac{\mathbb{k}^\alpha \mathcal{W}(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])}{\ell - s_1} \Big|_0^1 - \int_0^1 \frac{\mathbb{k}^{\alpha-1} \mathcal{W}(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])}{\ell - s_1} \cdot \alpha d\mathbb{k}. \end{aligned}$$

By substituting the variables, we obtain

$$I_1 = \frac{\mathcal{W}(\ell + u - s_1)}{\ell - s_1} - \frac{\Gamma(\alpha + 1)}{(\ell - s_1)^{\alpha+1}} I_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u), \quad (10)$$

and similarly, we get

$$\begin{aligned} I_2 &= \int_0^1 \mathbb{k}^\alpha \mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell]) d\mathbb{k} \\ &= - \frac{\mathcal{W}(\ell + v - s_2)}{s_2 - \ell} + \frac{\Gamma(\alpha + 1)}{(s_2 - \ell)^{\alpha+1}} I_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v). \end{aligned} \quad (11)$$

By placing the I_1 and I_2 with (9), we obtain (7). \square

Remark 1. Taking $s_1 = u, s_2 = v$ in Lemma 1, one has Lemma 2 in [42].

Remark 2. Choosing $s_1 = u, s_2 = v$ and $\alpha = 1$ in Lemma 1, one has Lemma 1 in [1].

Theorem 2. Suppose $\mathcal{W} : I = [u, v] \rightarrow \mathbb{R}$ be a differentiable mapping on (u, v) with $v > u$ such that $\mathcal{W}' \in L_1[u, v]$. If $|\mathcal{W}'|$ is a convex function on $[u, v]$, then under the assumptions of Lemma 1, the following inequality

$$\begin{aligned}
 & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\
 & \leq (\ell - s_1)^{\alpha+1} \left\{ \frac{1}{\alpha+1} (|\mathcal{W}'(\ell)| + |\mathcal{W}'(u)|) - \left[\frac{1}{\alpha+2} |\mathcal{W}'(s_1)| + \frac{1}{(\alpha+1)(\alpha+2)} |\mathcal{W}'(\ell)| \right] \right\} \\
 & + (s_2 - \ell)^{\alpha+1} \left\{ \frac{1}{\alpha+1} (|\mathcal{W}'(\ell)| + |\mathcal{W}'(v)|) - \left[\frac{1}{\alpha+2} |\mathcal{W}'(s_2)| + \frac{1}{(\alpha+1)(\alpha+2)} |\mathcal{W}'(\ell)| \right] \right\}, \tag{12}
 \end{aligned}$$

holds true for all $\alpha > 0$.

Proof. From Lemma 1 and the Jensen–Mercer inequality with $|\mathcal{W}'|$ is a convex function on $[u, v]$, we obtain

$$\begin{aligned}
 & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\
 & \leq (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])| d\mathbb{k} \\
 & + (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])| d\mathbb{k} \\
 & \leq (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha \{ |\mathcal{W}'(\ell)| + |\mathcal{W}'(u)| - [\mathbb{k} |\mathcal{W}'(s_1)| + (1 - \mathbb{k}) |\mathcal{W}'(\ell)|] \} d\mathbb{k} \\
 & + (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha \{ |\mathcal{W}'(\ell)| + |\mathcal{W}'(v)| - [\mathbb{k} |\mathcal{W}'(s_2)| + (1 - \mathbb{k}) |\mathcal{W}'(\ell)|] \} d\mathbb{k} \\
 & = (\ell - s_1)^{\alpha+1} \left\{ \frac{1}{\alpha+1} (|\mathcal{W}'(\ell)| + |\mathcal{W}'(u)|) - \left[\frac{1}{\alpha+2} |\mathcal{W}'(s_1)| + \frac{1}{(\alpha+1)(\alpha+2)} |\mathcal{W}'(\ell)| \right] \right\} \\
 & + (s_2 - v)^{\alpha+1} \left\{ \frac{1}{\alpha+1} (|\mathcal{W}'(\ell)| + |\mathcal{W}'(v)|) - \left[\frac{1}{\alpha+2} |\mathcal{W}'(s_2)| + \frac{1}{(\alpha+1)(\alpha+2)} |\mathcal{W}'(\ell)| \right] \right\}, \tag{13}
 \end{aligned}$$

which completes the proof. \square

Remark 3. Taking $s_1 = u, s_2 = v$ in Theorem 2, one has Theorem 7 in [42] for $s = 1$.

Corollary 1. If we set $s_1 = u, s_2 = v$ with $\alpha = 1$ in Theorem 2, then we have

$$\begin{aligned}
 & \left| \mathcal{W}(\ell) - \frac{1}{v - u} \int_u^v \mathcal{W}(\mathbb{k}) d\mathbb{k} \right| \\
 & \leq \frac{(\ell - u)^2}{3(v - u)} \left\{ \frac{1}{2} |\mathcal{W}'(u)| + |\mathcal{W}'(\ell)| \right\} + \frac{(v - \ell)^2}{3(v - u)} \left\{ \frac{1}{2} |\mathcal{W}'(v)| + |\mathcal{W}'(\ell)| \right\}.
 \end{aligned}$$

Corollary 2. Choosing $\alpha = 1$ in Theorem 2, we obtain

$$\begin{aligned}
 & \left| \left\{ \frac{\ell - s_1}{s_2 - s_1} \mathcal{W}(\ell + u - s_1) + \frac{s_2 - \ell}{s_2 - s_1} \mathcal{W}(\ell + v - s_2) \right\} - \frac{1}{s_2 - s_1} \left\{ \int_{\ell+u-\ell}^{\ell+u-s_1} \mathcal{W}(\mathbb{k}) d\mathbb{k} + \int_{\ell+v-s_2}^{\ell+v-\ell} \mathcal{W}(\mathbb{k}) d\mathbb{k} \right\} \right| \\
 & \leq \frac{(\ell - s_1)^2}{s_2 - s_1} \left\{ \frac{1}{2} (|\mathcal{W}'(\ell)| + |\mathcal{W}'(u)|) - \left[\frac{1}{3} |\mathcal{W}'(s_1)| + \frac{1}{6} |\mathcal{W}'(\ell)| \right] \right\} \\
 & + \frac{(s_2 - \ell)^2}{s_2 - s_1} \left\{ \frac{1}{2} (|\mathcal{W}'(\ell)| + |\mathcal{W}'(v)|) - \left[\frac{1}{3} |\mathcal{W}'(s_2)| + \frac{1}{6} |\mathcal{W}'(\ell)| \right] \right\}.
 \end{aligned}$$

Corollary 3. Under the assumption of $|\mathcal{W}'| \leq M$, Theorem 2 gives

$$\left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \leq \frac{M}{(s_2 - s_1)(\alpha + 1)} \left\{ (v - s_1)^{\alpha+1} + (s_2 - v)^{\alpha+1} \right\}.$$

Remark 4. Taking $s_1 = u, s_2 = v$ in Corollary 3, one has Corollary 1 in [42].

Remark 5. Choosing $s_1 = u, s_2 = v$ and $\alpha = 1$ in Corollary 3, one has Theorem 2 in [1] that yields the same result with $s = 1$.

Theorem 3. Suppose $\mathcal{W} : I = [u, v] \rightarrow \mathbb{R}$ be a differentiable mapping on (u, v) with $v > u$ such that $\mathcal{W}' \in L_1[u, v]$. If $|\mathcal{W}'|^q$ is a convex function on $[u, v]$, $q > 1$, then under the assumptions of Lemma 1, the following inequality

$$\begin{aligned} & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ & \leq (\ell - s_1)^{\alpha+1} \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left\{ (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q) - \frac{1}{2} [|\mathcal{W}'(s_1)|^q + |\mathcal{W}'(\ell)|^q] \right\}^{\frac{1}{q}} \\ & + (s_2 - \ell)^{\alpha+1} \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left\{ (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q) - \frac{1}{2} [|\mathcal{W}'(s_2)|^q + |\mathcal{W}'(\ell)|^q] \right\}^{\frac{1}{q}}, \end{aligned} \tag{14}$$

holds true for all $\alpha > 0$, where $q^{-1} + p^{-1} = 1$.

Proof. Under the hypothesis of the Hölder integral inequality and the Jensen–Mercer inequality with a convexity of $|\mathcal{W}'|^q$ for Lemma 1, we obtain

$$\begin{aligned} & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ & \leq (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])| d\mathbb{k} + (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])| d\mathbb{k} \\ & \leq (\ell - s_1)^{\alpha+1} \left(\int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^1 |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right)^{\frac{1}{q}} \\ & + (s_2 - \ell)^{\alpha+1} \left(\int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^1 |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right)^{\frac{1}{q}} \\ & \leq (\ell - s_1)^{\alpha+1} \left(\int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^1 \{ |\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q - [\mathbb{k}|\mathcal{W}'(s_1)|^q + (1 - \mathbb{k})|\mathcal{W}'(\ell)|^q] \} d\mathbb{k} \right)^{\frac{1}{q}} \\ & + (s_2 - \ell)^{\alpha+1} \left(\int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^1 \{ |\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q - [\mathbb{k}|\mathcal{W}'(s_2)|^q + (1 - \mathbb{k})|\mathcal{W}'(\ell)|^q] \} d\mathbb{k} \right)^{\frac{1}{q}} \\ & = (\ell - s_1)^{\alpha+1} \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left\{ (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q) - \frac{1}{2} [|\mathcal{W}'(s_1)|^q + |\mathcal{W}'(\ell)|^q] \right\}^{\frac{1}{q}} \\ & + (s_2 - \ell)^{\alpha+1} \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left\{ (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q) - \frac{1}{2} [|\mathcal{W}'(s_2)|^q + |\mathcal{W}'(\ell)|^q] \right\}^{\frac{1}{q}}. \end{aligned}$$

The proof is completed. \square

Remark 6. Taking $s_1 = u, s_2 = v$ in Theorem 3, one has Theorem 8 in [42] for $s = 1$.

Corollary 4. Choosing $s_1 = u, s_2 = v$ with $\alpha = 1$ in Theorem 3, then we have

$$\left| \mathcal{W}(\ell) - \frac{1}{v-u} \int_u^v \mathcal{W}(\mathbb{k}) d\mathbb{k} \right| \leq \frac{1}{2^{\frac{1}{q}}(v-u)} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[(\ell-u)^2 \left\{ |\mathcal{W}'(u)|^q + |\mathcal{W}'(\ell)|^q \right\}^{\frac{1}{q}} + (v-\ell)^2 \left\{ |\mathcal{W}'(v)|^q + |\mathcal{W}'(\ell)|^q \right\}^{\frac{1}{q}} \right].$$

Corollary 5. *Choosing $\alpha = 1$ in Theorem 3, we obtain*

$$\left| \left\{ (\ell - s_1) \mathcal{W}(\ell + u - s_1) + (s_2 - \ell) \mathcal{W}(\ell + v - s_2) \right\} - \left\{ \int_u^{\ell+u-s_1} \mathcal{W}(\mathbb{k}) d\mathbb{k} + \int_{\ell+v-s_2}^v \mathcal{W}(\mathbb{k}) d\mathbb{k} \right\} \right| \leq (\ell - s_1)^2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q \right) - \frac{1}{2} [|\mathcal{W}'(s_1)|^q + |\mathcal{W}'(\ell)|^q] \right\}^{\frac{1}{q}} + (s_2 - \ell)^2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q \right) - \frac{1}{2} [|\mathcal{W}'(s_2)|^q + |\mathcal{W}'(\ell)|^q] \right\}^{\frac{1}{q}}.$$

Corollary 6. *Theorem 3 with $|\mathcal{W}'| \leq M$, we get*

$$\left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \leq M \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left\{ (\ell - s_1)^{\alpha+1} + (s_2 - \ell)^{\alpha+1} \right\}.$$

Remark 7. *Choosing $s_1 = u, s_2 = v$ in Corollary 6, one has Corollary 2 in [42].*

Remark 8. *Taking $s_1 = u, s_2 = v$ and $\alpha = 1$ in Corollary 6, one has Theorem 3 in [1] for $s = 1$.*

Theorem 4. *Suppose $\mathcal{W} : I = [u, v] \rightarrow \mathbb{R}$ be a differentiable mapping on (u, v) with $v > u$ such that $\mathcal{W}' \in L_1[u, v]$. If $|\mathcal{W}'|^q$ is a convex function on $[u, v]$, $q \geq 1$, then under the assumptions of Lemma 1, the following inequality*

$$\left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \leq (\ell - s_1)^{\alpha+1} \left(\frac{1}{\alpha + 1} \right)^{1-\frac{1}{q}} \left\{ \frac{1}{\alpha + 1} \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q \right) - \left[\frac{1}{\alpha + 2} |\mathcal{W}'(s_1)|^q + \frac{1}{(\alpha + 1)(\alpha + 2)} |\mathcal{W}'(\ell)|^q \right] \right\}^{\frac{1}{q}} + (s_2 - \ell)^{\alpha+1} \left(\frac{1}{\alpha + 1} \right)^{1-\frac{1}{q}} \left\{ \frac{1}{\alpha + 1} \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q \right) - \left[\frac{1}{\alpha + 2} |\mathcal{W}'(s_2)|^q + \frac{1}{(\alpha + 1)(\alpha + 2)} |\mathcal{W}'(\ell)|^q \right] \right\}^{\frac{1}{q}}, \tag{15}$$

holds true for all $\alpha > 0$.

Proof. Under the assumption of the power–mean integral inequality and the Jensen–Mercer inequality with a convexity of $|\mathcal{W}'|^q$ for Lemma 1, we have

$$\begin{aligned}
 & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\
 & \leq (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])| d\mathbb{k} \\
 & + (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])| d\mathbb{k} \\
 & \leq (\ell - s_1)^{\alpha+1} \left(\int_0^1 \mathbb{k}^\alpha d\mathbb{k} \right)^{1-\frac{1}{q}} \left(\int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right)^{\frac{1}{q}} \\
 & + (s_2 - \ell)^{\alpha+1} \left(\int_0^1 \mathbb{k}^\alpha d\mathbb{k} \right)^{1-\frac{1}{q}} \left(\int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right)^{\frac{1}{q}} \\
 & \leq (\ell - s_1)^{\alpha+1} \left(\int_0^1 \mathbb{k}^\alpha d\mathbb{k} \right)^{1-\frac{1}{q}} \left(\int_0^1 \mathbb{k}^\alpha \{ |\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q - [\mathbb{k}|\mathcal{W}'(s_1)|^q + (1 - \mathbb{k})|\mathcal{W}'(\ell)|^q] \} d\mathbb{k} \right)^{\frac{1}{q}} \\
 & + (s_2 - \ell)^{\alpha+1} \left(\int_0^1 \mathbb{k}^\alpha d\mathbb{k} \right)^{1-\frac{1}{q}} \left(\int_0^1 \mathbb{k}^\alpha \{ |\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q - [\mathbb{k}|\mathcal{W}'(s_2)|^q + (1 - \mathbb{k})|\mathcal{W}'(\ell)|^q] \} d\mathbb{k} \right)^{\frac{1}{q}} \\
 & = (\ell - s_1)^{\alpha+1} \left(\frac{1}{\alpha + 1} \right)^{1-\frac{1}{q}} \left\{ \frac{1}{\alpha + 1} (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q) - \left[\frac{1}{\alpha + 2} |\mathcal{W}'(s_1)|^q + \frac{1}{(\alpha + 1)(\alpha + 2)} |\mathcal{W}'(\ell)|^q \right] \right\}^{\frac{1}{q}} \\
 & + (s_2 - \ell)^{\alpha+1} \left(\frac{1}{\alpha + 1} \right)^{1-\frac{1}{q}} \left\{ \frac{1}{\alpha + 1} (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q) - \left[\frac{1}{\alpha + 2} |\mathcal{W}'(s_2)|^q + \frac{1}{(\alpha + 1)(\alpha + 2)} |\mathcal{W}'(\ell)|^q \right] \right\}^{\frac{1}{q}},
 \end{aligned}$$

which completes the proof. \square

Remark 9. Taking $s_1 = u, s_2 = v$ in Theorem 4, one has Theorem 9 in [42] for $s = 1$.

Corollary 7. Choosing $s_1 = u, s_2 = v$ with $\alpha = 1$ in Theorem 4, then we obtain

$$\begin{aligned}
 & \left| \mathcal{W}(\ell) - \frac{1}{v - u} \int_u^v \mathcal{W}(\mathbb{k}) d\mathbb{k} \right| \\
 & \leq \frac{1}{(v - u)} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \frac{1}{3} \left[(\ell - u)^2 \left\{ \frac{1}{2} |\mathcal{W}'(u)|^q + |\mathcal{W}'(\ell)|^q \right\}^{\frac{1}{q}} + (v - \ell)^2 \left\{ \frac{1}{2} |\mathcal{W}'(v)|^q + |\mathcal{W}'(\ell)|^q \right\}^{\frac{1}{q}} \right].
 \end{aligned}$$

Corollary 8. Choosing $\alpha = 1$ in Theorem 4, we have

$$\begin{aligned}
 & \left| \left\{ (\ell - s_1) \mathcal{W}(\ell + u - s_1) + (s_2 - \ell) \mathcal{W}(\ell + v - s_2) \right\} - \left\{ \int_u^{\ell+u-s_1} \mathcal{W}(\mathbb{k}) d\mathbb{k} + \int_{\ell+v-s_2}^v \mathcal{W}(\mathbb{k}) d\mathbb{k} \right\} \right| \\
 & \leq (\ell - s_1)^2 \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \frac{1}{2} (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q) - \left[\frac{1}{3} |\mathcal{W}'(s_1)|^q + \frac{1}{6} |\mathcal{W}'(\ell)|^q \right] \right\}^{\frac{1}{q}} \\
 & + (s_2 - \ell)^2 \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \frac{1}{2} (|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q) - \left[\frac{1}{3} |\mathcal{W}'(s_2)|^q + \frac{1}{6} |\mathcal{W}'(\ell)|^q \right] \right\}^{\frac{1}{q}}.
 \end{aligned}$$

Corollary 9. Theorem 4 with $|\mathcal{W}'| \leq M$, we get

$$\begin{aligned}
 & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\
 & \leq \frac{M}{(\alpha + 1)} \left\{ (\ell - s_1)^{\alpha+1} + (s_2 - \ell)^{\alpha+1} \right\}.
 \end{aligned}$$

Remark 10. Taking $s_1 = u, s_2 = v$ in Corollary 9, one has Corollary 3 in [42].

Remark 11. Choosing $s_1 = u, s_2 = v$ and $\alpha = 1$ in Corollary 9, one has Theorem 4 in [1] for $s = 1$.

Theorem 5. Suppose $\mathcal{W} : I = [u, v] \rightarrow \mathbb{R}$ be a differentiable mapping on (u, v) with $v > u$ such that $\mathcal{W}' \in L_1[u, v]$. If $|\mathcal{W}'|^q$ is a convex function on $[u, v]$, then under the assumptions of Lemma 1, the following inequality

$$\begin{aligned} & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ & \leq (\ell - s_1)^{\alpha+1} \left\{ \frac{1}{(\alpha p + 1)p} + \frac{1}{q} \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q - \frac{1}{2} [|\mathcal{W}'(s_1)|^q + |\mathcal{W}'(\ell)|^q] \right) \right\} \\ & + (s_2 - \ell)^{\alpha+1} \left\{ \frac{1}{(\alpha p + 1)p} + \frac{1}{q} \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q - \frac{1}{2} [|\mathcal{W}'(s_2)|^q + |\mathcal{W}'(\ell)|^q] \right) \right\}, \end{aligned} \tag{16}$$

holds true for all $\alpha > 0$, where $p, q > 1$ are conjugate exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Under the assumption of Lemma 1, we have

$$\begin{aligned} & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ & \leq (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])| d\mathbb{k} + (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])| d\mathbb{k}. \end{aligned} \tag{17}$$

Using Young’s inequality, i.e.,

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q,$$

(equality holds if $x^p = y^q$)

$$\begin{aligned} & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ & \leq (\ell - s_1)^{\alpha+1} \left\{ \frac{1}{p} \int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} + \frac{1}{q} \int_0^1 |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right\} \\ & + (s_2 - \ell)^{\alpha+1} \left\{ \frac{1}{p} \int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} + \frac{1}{q} \int_0^1 |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right\}. \end{aligned} \tag{18}$$

Under the assumption of Jensen–Mercer inequality and a convexity of $|\mathcal{W}'|^q$, we obtain

$$\begin{aligned} & \left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ & \leq (\ell - s_1)^{\alpha+1} \left\{ \frac{1}{p} \int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} + \frac{1}{q} \int_0^1 \{ |\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q - [\mathbb{k} |\mathcal{W}'(s_1)|^q + (1 - \mathbb{k}) |\mathcal{W}'(\ell)|^q] \} d\mathbb{k} \right\} \\ & + (s_2 - \ell)^{\alpha+1} \left\{ \frac{1}{p} \int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} + \frac{1}{q} \int_0^1 \{ |\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q - [\mathbb{k} |\mathcal{W}'(s_2)|^q + (1 - \mathbb{k}) |\mathcal{W}'(\ell)|^q] \} d\mathbb{k} \right\} \\ & = (\ell - s_1)^{\alpha+1} \left\{ \frac{1}{(\alpha p + 1)p} + \frac{1}{q} \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(u)|^q - \frac{1}{2} [|\mathcal{W}'(s_1)|^q + |\mathcal{W}'(\ell)|^q] \right) \right\} \\ & + (s_2 - \ell)^{\alpha+1} \left\{ \frac{1}{(\alpha p + 1)p} + \frac{1}{q} \left(|\mathcal{W}'(\ell)|^q + |\mathcal{W}'(v)|^q - \frac{1}{2} [|\mathcal{W}'(s_2)|^q + |\mathcal{W}'(\ell)|^q] \right) \right\}. \end{aligned}$$

This concludes the proof. \square

Corollary 10. Theorem 5 with $|\mathcal{W}'| \leq M$, we have

$$\left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ \leq \left\{ \frac{1}{(\alpha p + 1)p} + \frac{1}{q} M^q \right\} \left[(\ell - s_1)^{\alpha+1} + (s_2 - \ell)^{\alpha+1} \right].$$

Theorem 6. Suppose $\mathcal{W} : I = [u, v] \rightarrow \mathbb{R}$ be a differentiable mapping on (u, v) with $v > u$ such that $\mathcal{W}' \in L_1[u, v]$. If $|\mathcal{W}'|^q$ is a convex function on $[u, v]$, $q > 1$, then under the assumptions of Lemma 1, the following inequality

$$\left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ \leq \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left\{ (\ell - s_1)^{\alpha+1} \left| \mathcal{W}' \left(\ell + u - \frac{s_1 + \ell}{2} \right) \right| + (s_2 - \ell)^{\alpha+1} \left| \mathcal{W}' \left(\ell + v - \frac{s_2 + \ell}{2} \right) \right| \right\}, \quad (19)$$

holds true for all $\alpha > 0$, where $q^{-1} + p^{-1} = 1$.

Proof. Under the assumption of Hölder's inequality and Lemma 1, we have

$$\left| \left\{ (\ell - s_1)^\alpha \mathcal{W}(\ell + u - s_1) + (s_2 - \ell)^\alpha \mathcal{W}(\ell + v - s_2) \right\} - \Gamma(\alpha + 1) \left\{ \mathbb{I}_{(\ell+u-s_1)^-}^\alpha \mathcal{W}(u) + \mathbb{I}_{(\ell+v-s_2)^+}^\alpha \mathcal{W}(v) \right\} \right| \\ \leq (\ell - s_1)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])| d\mathbb{k} + (s_2 - \ell)^{\alpha+1} \int_0^1 \mathbb{k}^\alpha |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])| d\mathbb{k} \\ \leq (\ell - s_1)^{\alpha+1} \left(\int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^1 |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right)^{\frac{1}{q}} \\ + (s_2 - \ell)^{\alpha+1} \left(\int_0^1 \mathbb{k}^{\alpha p} d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^1 |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \right)^{\frac{1}{q}}. \quad (20)$$

Since $|\mathcal{W}'|^q$ is a convex function, from (5), we get

$$\int_0^1 |\mathcal{W}'(\ell + u - [\mathbb{k}s_1 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \leq \left| \mathcal{W}' \left(\ell + u - \frac{s_1 + \ell}{2} \right) \right|^q \quad (21)$$

and

$$\int_0^1 |\mathcal{W}'(\ell + v - [\mathbb{k}s_2 + (1 - \mathbb{k})\ell])|^q d\mathbb{k} \leq \left| \mathcal{W}' \left(\ell + v - \frac{s_2 + \ell}{2} \right) \right|^q. \quad (22)$$

We obtain the following inequality (19) by placing inequalities (21) and (22) in (20). The proof is completed. \square

Remark 12. Choosing $s_1 = u, s_2 = v$ and $\alpha = 1$ in Theorem 6, one has Theorem 5 in [1] for $s = 1$.

3. Applications

3.1. Special Means

In the literature, the following means are well known for $0 < \Phi_1 < \Phi_2$.

The arithmetic mean:

$$A(\Phi_1, \Phi_2) = \frac{\Phi_1 + \Phi_2}{2}.$$

The logarithmic-mean:

$$L(\Phi_1, \Phi_2) = \frac{\Phi_2 - \Phi_1}{\log \Phi_2 - \log \Phi_1}.$$

The generalized logarithmic-mean:

$$L_m(\Phi_1, \Phi_2) = \left[\frac{\Phi_2^{m+1} - \Phi_1^{m+1}}{(m+1)(\Phi_2 - \Phi_1)} \right]^{\frac{1}{m}}; \quad m \in \mathbb{R} \setminus \{-1, 0\}.$$

Proposition 1. Let $u, v \in \mathbb{R}, 0 < u < v$ and $n \in \mathbb{Z}$. Then, for all $\ell \in [s_1, s_2]$, and $s_1, s_2 \in [u, v]$, we obtain

$$\begin{aligned} & \left| \left\{ (\ell - s_1)(2A(\ell, u) - s_1)^n + (s_2 - \ell)(2A(\ell, v) - s_2)^n \right\} \right. \\ & \left. - \left\{ (\ell - s_1)L_n^n(\ell + u - s_1, u) + (s_2 - \ell)L_n^n(v, \ell + v - s_2) \right\} \right| \\ & \leq n(\ell - s_1)^2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ 2A(\ell^{(n-1)q}, u^{(n-1)q}) - A(s_1^{(n-1)q}, \ell^{(n-1)q}) \right\}^{\frac{1}{q}} \\ & \quad + n(s_2 - \ell)^2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ 2A(\ell^{(n-1)q}, v^{(n-1)q}) - A(s_2^{(n-1)q}, \ell^{(n-1)q}) \right\}^{\frac{1}{q}}. \end{aligned}$$

Proof. Under the assumptions of Corollary 5 and for $\mathcal{W}(\ell) = \ell^n$, we obtain the desired result. \square

Proposition 2. Let $u, v \in \mathbb{R}, 0 < u < v$ and $n \in \mathbb{Z}$. Then, for all $\ell \in [s_1, s_2]$, and $s_1, s_2 \in [u, v]$, we have

$$\begin{aligned} & \left| \left\{ (\ell - s_1)(2A(\ell, u) - s_1)^n + (s_2 - \ell)(2A(\ell, v) - s_2)^n \right\} \right. \\ & \left. - \left\{ (\ell - s_1)L_n^n(\ell + u - s_1, u) + (s_2 - \ell)L_n^n(v, \ell + v - s_2) \right\} \right| \\ & \leq n(\ell - s_1)^2 \left\{ A(\ell^{n-1}, u^{n-1}) - \frac{1}{3}A(2s_1^{n-1}, \ell^{n-1}) \right\} \\ & \quad + n(s_2 - \ell)^2 \left\{ A(\ell^{n-1}, v^{n-1}) - \frac{1}{3}A(2s_2^{n-1}, \ell^{n-1}) \right\}. \end{aligned}$$

Proof. Under the assumptions of Corollary 2 and for $\mathcal{W}(\ell) = \ell^n$, we obtain the desired result. \square

3.2. q-Digamma Function

The q-analog of the digamma function φ (see [50,51]), i.e., q-digamma function φ_q , for $0 < q < 1$, is given as follows:

$$\begin{aligned} \varphi_q &= -\ln(1 - q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+\omega}}{1 - q^{k+\omega}} \\ &= -\ln(1 - q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k\omega}}{1 - q^{k\omega}}. \end{aligned}$$

The q-digamma function φ_q can also be written as follows:

$$\begin{aligned} \varphi_q &= -\ln(q - 1) + \ln q \left[\omega - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+\omega)}}{1 - q^{-(k+\omega)}} \right] \\ &= -\ln(q - 1) + \ln q \left[\omega - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-k\omega}}{1 - q^{-k\omega}} \right], \end{aligned}$$

for $q > 1$ and $\omega > 0$.

Proposition 3. Let $0 < u < v, q > 1, 0 < q < 1$ and $q^{-1} = 1 - p^{-1}$. Then, for all $\ell \in [s_1, s_2]$ and $s_1, s_2 \in [u, v]$, we have

$$\begin{aligned} & \left| \left\{ (\ell - s_1)\varphi_q(\ell + u - s_1) + (s_2 - \ell)\varphi_q(\ell + v - s_2) \right\} - \frac{1}{s_2 - s_1} \left\{ \int_u^{\ell+u-s_1} \varphi_q(\omega)d\omega + \int_{\ell+v-s_2}^v \varphi_q(\omega)d\omega \right\} \right| \\ & \leq (\ell - s_1)^2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(|\varphi'_q(\ell)|^q + |\varphi'_q(u)|^q \right) - \frac{1}{2} [|\varphi'_q(s_1)|^q + |\varphi'_q(\ell)|^q] \right\}^{\frac{1}{q}} \\ & + (s_2 - \ell)^2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(|\varphi'_q(\ell)|^q + |\varphi'_q(v)|^q \right) - \frac{1}{2} [|\varphi'_q(s_2)|^q + |\varphi'_q(\ell)|^q] \right\}^{\frac{1}{q}}. \end{aligned}$$

Proof. The assertion can be obtained immediately by using Corollary 5 with the $\mathcal{W} : \omega \rightarrow \varphi_q(\omega)$ is a completely monotone function on $(0, \infty)$ for all $\omega > 0$ and consequently, $\mathcal{W}'(\omega) := \varphi'_q(\omega)$ is convex. \square

4. Modified Bessel Function

The modified Bessel function of the first kind ω_τ , (see [51], p.77) is given as follows:

$$\omega_\tau(\omega) = \sum_{n \geq 0} \frac{\left(\frac{\omega}{2}\right)^{\tau+2n}}{n! \Gamma(\tau + n + 1)}.$$

where $\omega \in \mathbb{R}$ and $\tau > -1$.

The modified Bessel function of the second kind Y_τ (see [51], p.78) is defined as follows:

$$Y_\tau(\omega) = \frac{\pi}{2} \frac{\omega_{-\tau}(\omega) - \omega_\tau(\omega)}{\sin \tau \pi}.$$

The function $\mathcal{B}_\tau(\omega) : \mathbb{R} \rightarrow [1, \infty)$ can be defined as

$$\mathcal{B}_\tau(\omega) = 2^\tau \Gamma(\tau + 1) \omega^{-\tau} Y_\tau(\omega),$$

where Γ is the gamma function.

The following derivative formulas of $\mathcal{B}_\tau(\omega)$ are presented in [51]:

$$\mathcal{B}'_\tau(\omega) = \frac{\omega}{2(\tau + 1)} \mathcal{B}_{\tau+1}(\omega). \tag{23}$$

$$\mathcal{B}''_\tau(\omega) = \frac{\omega^2 \mathcal{B}_{\tau+2}(\omega)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(\omega)}{2(\tau + 1)}. \tag{24}$$

Proposition 4. Suppose that $\tau > -1$ and $0 < u < v$. Then, for all $\ell \in [s_1, s_2]$ and $s_1, s_2 \in [u, v]$, we have

$$\begin{aligned} & \left| \left\{ (\ell - s_1) \frac{\ell + u - s_1}{2(\tau + 1)} \mathcal{B}_{\tau+1}(\ell + u - s_1) + (s_2 - \ell) \frac{\ell + v - s_2}{2(\tau + 1)} \mathcal{B}_{\tau+1}(\ell + v - s_2) \right\} \right. \\ & \left. - \left\{ (\mathcal{B}_\tau(\ell + u - s_1) - \mathcal{B}_\tau(u)) + (\mathcal{B}_\tau(v) - \mathcal{B}_\tau(\ell + v - s_2)) \right\} \right| \\ & \leq (\ell - s_1)^2 \left\{ \frac{1}{2} \left(\frac{\ell^2 \mathcal{B}_{\tau+2}(\ell)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(\ell)}{2(\tau + 1)} + \frac{u^2 \mathcal{B}_{\tau+2}(u)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(u)}{2(\tau + 1)} \right) \right. \\ & \left. - \left[\frac{1}{3} \left(\frac{s_1^2 \mathcal{B}_{\tau+2}(s_1)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(s_1)}{2(\tau + 1)} \right) + \frac{1}{6} \left(\frac{\ell^2 \mathcal{B}_{\tau+2}(\ell)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(\ell)}{2(\tau + 1)} \right) \right] \right\} \\ & + (s_2 - \ell)^2 \left\{ \frac{1}{2} \left(\frac{\ell^2 \mathcal{B}_{\tau+2}(\ell)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(\ell)}{2(\tau + 1)} + \frac{v^2 \mathcal{B}_{\tau+2}(v)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(v)}{2(\tau + 1)} \right) \right. \\ & \left. - \left[\frac{1}{3} \left(\frac{s_2^2 \mathcal{B}_{\tau+2}(s_2)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(s_2)}{2(\tau + 1)} \right) + \frac{1}{6} \left(\frac{\ell^2 \mathcal{B}_{\tau+2}(\ell)}{4(\tau + 1)(\tau + 2)} + \frac{\mathcal{B}_{\tau+1}(\ell)}{2(\tau + 1)} \right) \right] \right\}. \end{aligned} \tag{25}$$

Proof. Applying the inequality in Corollary 2 to the function $\mathcal{W} : \omega = \mathcal{B}'_t(\omega), \omega > 0$ (Note that all assumptions are satisfied) and the identities (23) and (24). \square

5. Conclusions

Recently, it has been seen that researchers working in the field of inequality theory focus on obtaining new generalizations, introducing new inequalities with effective applications and extending existing inequalities to different spaces. This study focuses on new generalizations of Ostrowski–Mercer-type inequalities by taking several of these objectives into account. Furthermore, the study has been enriched with applications for special means, modified Bessel functions and q-digamma functions, which are a motivating aspect. Researchers can contribute to the development of the results based on this study by developing new integral identities. In the future, the results can be further generalized via interval-valued analysis and quantum calculus. Furthermore, one can use the concept of this paper to prove different versions of Mercer-type inequalities for subadditive functions.

Author Contributions: Conceptualization, S.K.S. and A.K.; methodology, S.K.S. and S.M.; software, S.K.S. and M.A.; validation, S.K.S. and M.D.L.S.; formal analysis, A.K. and M.A.; investigation, S.K.S. and S.M.; resources, S.M. and M.D.L.S.; data curation, S.K.S. and M.A.; writing—original draft preparation, S.K.S. and S.M.; writing—review and editing, S.K.S., A.K. and S.K.S.; supervision, S.K.S. and M.D.L.S.; project administration, S.K.S.; funding acquisition, M.D.L.S. All authors have read and agreed to the published version of the manuscript.

Funding: Basque Government: Grants IT1555-22 and KK-2022/00090; and MCIN/AEI 269.10.13039/501100011033 for Grant PID2021-1235430B-C21/C22.

Data Availability Statement: This study did not report any data.

Acknowledgments: The authors are grateful to the Basque Government for its support through Grants IT1555-22 and KK-2022/00090; and to MCIN/AEI 269.10.13039/501100011033 for Grant PID2021-1235430B-C21/C22.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Alomari, M.; Darus, M.; Dragomir, S.S.; Cerone, P. Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense. *Appl. Math. Lett.* **2010**, *23*, 1071–1076. [\[CrossRef\]](#)
- Alomari, M.; Darus, M. Some Ostrowski type inequalities for quasi-convex functions with applications to special means. *RGMA Res. Rep. Coll* **2010**, *13*, 6.
- Cerone, P.; Dragomir, S.S. Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions. *Demonstr. Math.* **2004**, *37*, 299–308. [\[CrossRef\]](#)
- Dragomir, S.S. On the Ostrowski's integral inequality for mappings with bounded variation and applications. *Math. Ineq. Appl.* **2001**, *4*, 59–66. [\[CrossRef\]](#)
- Set, E.; Sarikaya, M.Z.; Özdemir, M.E. Some Ostrowski's Type Inequalities for Functions whose Second Derivatives are s-Convex in the Second Sense and Applications. *arXiv* **2010**, arXiv:1006.2488.
- Pachpatte, B.G. On an inequality of Ostrowski type in three independent variables. *J. Math. Anal. Appl.* **2000**, *249*, 583–591. [\[CrossRef\]](#)
- Set, E.; Özdemir, M.E.; Sarikaya, M.Z. New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *arXiv* **2010**, arXiv:1005.0702.
- Sahoo, S.K.; Latif, M.A.; Alsalmi, O.M.; Treanță, S.; Sudsutad, W.; Kongson, J. Hermite-Hadamard, Fejér and Pachpatte-Type Integral Inequalities for Center-Radius Order Interval-Valued Preinvex Functions. *Fractal Fract.* **2022**, *6*, 506. [\[CrossRef\]](#)
- Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Baleanu, D.; Kodamasingh, B. Hermite-Hadamard type inequalities for interval-valued preinvex functions via fractional integral operators. *Internat. J. Comput. Intel. Syst.* **2022**, *15*, 8. [\[CrossRef\]](#)
- Sahoo, S.K.; Tariq, M.; Ahmad, H.; Kodamasingh, B.; Shaikh, A.A.; Botmart, T.; El-Shorbagy, M.A. Some Novel Fractional Integral Inequalities over a New Class of Generalized Convex Function. *Fractal Fract.* **2022**, *6*, 42. [\[CrossRef\]](#)
- Sahoo, S.K.; Mohammed, P.O.; Kodamasingh, B.; Tariq, M.; Hamed, Y.S. New Fractional Integral Inequalities for Convex Functions Pertaining to Caputo-Fabrizio Operator. *Fractal Fract.* **2022**, *6*, 171. [\[CrossRef\]](#)
- Özdemir, M.E.; Kavurmaci-Önalán, H.; Avci-Ardic, M. New Inequalities of Ostrowski Type for Mappings whose Derivatives are (α, m) -Convex via Fractional Integrals. *Thai J. Math.* **2018**, *16*, 723–731.

13. Chun, L.; Qi, F. Integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex. *J. Inequal. Appl.* **2013**, *2013*, 451. [[CrossRef](#)]
14. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis. Mathematics and Its Applications (East European Series)*; Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 1993; Volume 61.
15. N. Mehmood, S. I. Butt, D. Pečarić and J. Pečarić, Generalizations of cyclic refinements of Jensen's inequality by Lidstone's polynomial with applications in Information Theory. *J. Math. Inequal.* **2019**, *14*, 249–271. [[CrossRef](#)]
16. Butt, S.I.; Bakula, M.K.; Pečarić, D.; Pečarić, J. Jensen-Grüss inequality and its applications for the Zipf-Mandelbrot law. *Math. Methods Appl. Scis.* **2021**, *44*, 1664–1673. [[CrossRef](#)]
17. Khan, S.; Adil, Khan, M.; Butt, S.I.; Chu, Y.M. A new bound for the Jensen gap pertaining twice differentiable functions with applications. *Adv. Differ. Equ.* **2020**, *2020*, 333. [[CrossRef](#)]
18. Mercer, A.M. A variant of Jensens inequality. *J. Ineq. Pure Appl. Math.* **2003**, *4*, 73.
19. Matkovic, A.; Pečarić, J. A variant of Jensens inequality of Mercers type for operators with applications. *Linear Algebra Appl.* **2006**, *418*, 551–564. [[CrossRef](#)]
20. Niezgodą, M. A generalization of Mercer's result on convex functions. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 2771–2779. [[CrossRef](#)]
21. Kian, M. Operator Jensen inequality for superquadratic functions. *Linear Algebra Appl.* **2014**, *456* 82–87. [[CrossRef](#)]
22. Kian, M.; Moslehian, M.S. Refinements of the operator Jensen-Mercer inequality, *Electron. J. Linear Algebra* **2013**, *26*, 742–753.
23. Zhao, J.; Butt, S.I.; Nasir, J.; Wang, Z.; Tlili, I. Hermite-Jensen-Mercer type inequalities for Caputo fractional derivatives. *J. Func. Spaces* **2020**, *2020*, 7061549. [[CrossRef](#)]
24. Moradi, H.R.; Furuichi, S. Improvement and generalization of some Jensen-Mercer-type inequalities. *arXiv* **2019**, arXiv:1905.01768.
25. Ekinci, A.; Özdemir, M.E. Some New Integral Inequalities via Riemann Liouville Integral Operators. *Appl. Comput. Math.* **2019**, *3*, 288–295.
26. Set, E.; Akdemir, A.O.; Özdemir, M.E. Simpson type integral inequalities for convex functions via Riemann–Liouville integrals. *Filomat* **2017** *31*, 4415–4420 . [[CrossRef](#)]
27. Mubeen, S; Habibullah, G.M. k -fractional integrals and application. *Int. J. Contemp. Math. Sci.* **2012**, *7*, 89–94
28. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
29. Chen, H.; Katugampola, U.N. Hermite—Hadamard and Hermite—Hadamard—Fejér type inequalities for generalized fractional integrals. *J. Math. Anal. Appl.* **2017**, *446*, 1274–1291. [[CrossRef](#)]
30. Tomar, M.; Maden, S.; Set, E. (k, s) -Riemann–Liouville fractional integral inequalities for continuous random variables. *Arab. J. Math.* **2017**, *6*, 55–63. [[CrossRef](#)]
31. Ahmad, B.; Alsaedi, A.; Kirane, M.; Torebek, B.T. Hermite–Hadamard, Hermite–Hadamard–Fejér, Dragomir–Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals. *J. Comput. Appl. Math.* **2019**, *353*, 120–129. [[CrossRef](#)]
32. Das, T.; Ghosh, U.; Sarkar, S.; Das, S. Higher-dimensional fractional time-independent Schrödinger equation via fractional derivative with generalised pseudoharmonic potential. *Pramana* **2019**, *93*, 76. [[CrossRef](#)]
33. Al-Raei, M. Applying fractional quantum mechanics to systems with electrical screening effects. *Chaos Solitons Fractals* **2021**, *150*, 111209. [[CrossRef](#)]
34. El-Dib, Y.O. Modified multiple scale technique for the stability of the fractional delayed nonlinear oscillator. *Pramana* **2020**, *94*, 56. [[CrossRef](#)]
35. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.* **2013**, *57*, 2403–2407. [[CrossRef](#)]
36. Podlubni, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
37. Butt, S.I.; Umar, M.; Rashid, S.; Akdemir, A.O.; Chu, Y.M. New Hermite Jensen Mercer type inequalities via k -fractional integrals. *Adv. Differ. Equations* **2020**, *2020*, 635. [[CrossRef](#)]
38. Butt, S.I.; Nadeem, M.; Qaisar, S.; Akdemir, A.O.; Abdeljawad, T. Hermite Jensen Mercer type inequalities for conformable integrals and related results. *Adv. Differ. Equ.* **2020**, *2020*, 501. [[CrossRef](#)]
39. Sahoo, S. K.; Kodamasingh, B.; Kashuri, A.; Aydi, H.; Ameer, E. Ostrowski type inequalities pertaining to Atangana-Baleanu fractional operators and applications containing special functions. *J. Inequal. Appl.* **2022**, *2022*, 162. [[CrossRef](#)]
40. Botmart, T.; Sahoo, S. K.; Kodamasingh, B.; Latif, M.A.; Jarad, F.; Kashuri, A. Certain midpoint-type Fejér and Hermite-Hadamard inclusions involving fractional integrals with an exponential function in kernel. *AIMS Math.* **2023**, *8*, 5616–5638. [[CrossRef](#)]
41. İşcan, İ. Ostrowski type inequalities for harmonically s -convex functions. *Konuralp J. Math.* **2015**, *3*, 63–74
42. Set, E. New inequalities of Ostrowski type for mappings whose derivatives are s -convex in the second sense via fractional integrals. *Comput. Math. Appl.* **2012**, *63*, 1147–1154. [[CrossRef](#)]
43. Liu, W. Ostrowski type fractional integral inequalities for MT-convex functions. *Miskolc Math. Notes* **2015**, *16*, 249–256. [[CrossRef](#)]
44. Agarwal, R.P.; Luo, M.J.; Raina, R.K. On Ostrowski type inequalities. *Fasc. Math.* **2016**, *56*, 5–27. [[CrossRef](#)]

45. Sarikaya, M.; Budak, H. Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Am. Math. Soc.* **2017**, *145*, 1527–1538. [[CrossRef](#)]
46. Gürbüz, M.; Taşdan, Y.; Set, E. Ostrowski type inequalities via the Katugampola fractional integrals. *AIMS Math.* **2020**, *5*, 42–53. [[CrossRef](#)]
47. Ahmad, H.; Tariq, M.; Sahoo, S.K.; Askar, S.; Abouelregal, A.E.; Khedher, K.M. Refinements of Ostrowski Type Integral Inequalities Involving Atangana-Baleanu Fractional Integral Operator. *Symmetry* **2021**, *13*, 2059. [[CrossRef](#)]
48. Sial, I.B.; Patanarapeelert, N.; Ali, M.A.; Budak, H.; Sitthiwirattam, T. On Some New Ostrowski–Mercer-Type Inequalities for Differentiable Functions. *Axioms* **2022**, *11*, 132. [[CrossRef](#)]
49. Ali, M.A.; Asjad, M.I.; Budak, H.; Faridi, W.A. On Ostrowski–Mercer inequalities for differentiable harmonically convex functions with applications. *Math. Methods Appl. Sci.* **2021**. [[CrossRef](#)]
50. Jain, S.; Mehrez, K.; Baleanu, D.; Agarwal, P. Certain Hermite-Hadamard inequalities for logarithmically convex functions with applications. *Mathematics* **2019**, *7*, 163. [[CrossRef](#)]
51. Watson, G.N. *A Treatise on the Theory of Bessel Functions*; Cambridge University Press: Cambridge, UK, 1995.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.