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Research article

Fixed point approach to the Mittag-Leffler kernel-related fractional differential equations

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Abstract: The goal of this paper is to present a new class of contraction mappings, so-called η_θ^ℓ-contractions. Also, in the context of partially ordered metric spaces, some coupled fixed-point results for η_θ^ℓ-contraction mappings are introduced. Furthermore, to support our results, two examples are provided. Finally, the theoretical results are applied to obtain the existence of solutions to coupled fractional differential equations with a Mittag-Leffler kernel.

Keywords: fractional differential equation; Atangana-Baleanu fractional operator; fixed point methodology; Riemann-Liouville fractional integral
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1. Introduction and basic facts

Fractional differential equations are thought to be the most effective models for a variety of pertinent events. This makes it possible to investigate the existence, uniqueness, controllability, stability, and other properties of analytical solutions. For example, applying conservation laws to the fractional Black-Scholes equation in Lie symmetry analysis, finding existence solutions for some conformable differential equations, and finding existence solutions for some classical and fractional differential equations on the basis of discrete symmetry analysis, for more details, see [1–5].

Atangana and Baleanu unified and extended the definition of Caputo-Fabrizio [5] by introducing exciting derivatives without singular kernel. Also, the same authors presented the derivative containing Mittag-Leffler function as a nonlocal and nonsingular kernel. Many researchers showed their interest in this definition because it opens many and sober directions and carries Riemann-Liouville and Caputo derivatives [6–13].

A variety of problems in economic theory, control theory, global analysis, fractional analysis, and nonlinear analysis have been treated by fixed point (FP) theory. The FP method contributes greatly to the fractional differential/integral equations, through which it is possible to study the existence and uniqueness of the solution to such equations [14–17]. Also, this topic has been densely studied and several significant results have been recorded in [18–21].

The concepts of mixed monotone property (MMP) and a coupled fixed point (CFP) for a contractive mapping $\Xi : \chi \times \chi \to \chi$, where χ is a partially ordered metric space (POMS) have been initiated by Bhaskar and Lakshmikantham [22]. To support these ideas, they presented some CFP theorems and determined the existence and uniqueness of the solution to a periodic boundary value problem [23–25]. Many authors worked in this direction and obtained some nice results concerned with CFPs in various spaces [26–28].

Definition 1.1. *[22]* Consider a set $\chi \neq \emptyset$. A pair $(a, b) \in \chi \times \chi$ is called a CFP of the mapping $\Xi : \chi \times \chi \to \chi$ if $a = \Xi(a, b)$ and $b = \Xi(b, a)$.

Definition 1.2. *[22]* Assume that $(\chi, \leq)$ is a partially ordered set and $\Xi : \chi\times\chi \to \chi$ is a given mapping. We say that Ξ has a MMP if for any $a, b \in \chi$,

$$a_1, a_2 \in \chi,\ a_1 \leq a_2 \Rightarrow \Xi(a_1, b) \leq \Xi(a_2, b),$$

and

$$b_1, b_2 \in \chi,\ b_1 \leq b_2 \Rightarrow \Xi(a, b_1) \geq \Xi(a, b_2).$$

Theorem 1.1. *[22] Let $(\chi, \leq, d)$ be a complete POMS and $\Xi : \chi \times \chi \to \chi$ be a continuous mapping having the MMP on χ. Assume that there is a $\tau \in [0, 1)$ so that*

$$d\left(\Xi(a, b), \Xi(k, l)\right) \leq \frac{\tau}{2}\left(d(a, k) + d(b, l)\right),$$

for all $a \geq k$ and $b \leq l$. If there are $a_0, b_0 \in \chi$ so that $a_0 \leq \Xi\left(a_0, b_0\right)$ and $b_0 \geq \Xi\left(b_0, a_0\right)$, then Ξ has a CFP, that is, there exist $a_0, b_0 \in \chi$ such that $a = \Xi(a, b)$ and $b = \Xi(b, a)$.

The same authors proved that Theorem 1.1 is still valid if we replace the hypothesis of continuity with the following: Assume χ has the property below:

(†) if a non-decreasing sequence $\{a_m\} \to a$, then $a_m \leq a$ for all m;
(‡) if a non-increasing sequence $\{b_m\} \to b$, then $b \leq b_m$ for all m.

The following auxiliary results are taken from [29,30], which are used efficiently in the next section.

Let Θ represent a family of non-decreasing functions $\theta : [0, \infty) \to [0, \infty)$ so that $\sum_{m=1}^{\infty} \theta^m(\tau) < \infty$ for all $\tau > 0$, where θ^n is the n-th iterate of θ justifying:

(i) $\theta(\tau) = 0 \Leftrightarrow \tau = 0$;
(ii) for all $\tau > 0$, $\theta(\tau) < \tau$;
(iii) for all $\tau > 0$, $\lim_{s\to\tau^+} \theta(s) < \tau$.

Lemma 1.1. *[30] If $\theta : [0, \infty) \to [0, \infty)$ is right continuous and non-decreasing, then $\lim_{m\to\infty} \theta^m(\tau) = 0$ for all $\tau \geq 0$ iff $\theta(\tau) < \tau$ for all $\tau > 0$.*

Let $\widetilde{L}$ be the set of all functions $\widetilde{\ell} : [0, \infty) \to [0, 1)$ which verify the condition:

$$\lim_{m\to\infty} \widetilde{\ell}(\tau_m) = 1 \text{ implies } \lim_{m\to\infty} \tau_m = 0.$$

Recently, Samet et al. [29] reported exciting FP results by presenting the concept of α-θ-contractive mappings.

Definition 1.3. *[29]* Let χ be a non empty-set, $\Xi : \chi \to \chi$ be a map and $\alpha : \chi \times \chi \to \mathbb{R}$ be a given function. Then, Ξ is called α-admissible if

$$\alpha(a, b) \geq 1 \Rightarrow \alpha(\Xi a, \Xi b)) \geq 1, \ \forall a, b \in \chi.$$

Definition 1.4. *[29]* Let (χ, d) be a metric space. $\Xi : \chi \to \chi$ is called an α-θ-contractive mapping, if there exist two functions $\alpha : \chi \times \chi \to [0, +\infty)$ and $\theta \in \Theta$ such that

$$\alpha(a, b)\, d(\Xi(a, b)) \leq \theta(d(a, b)),$$

for all $a, b \in \chi$.

Theorem 1.2. *[29] Let (χ, d) be a metric space, $\Xi : \chi \to \chi$ be an α-ψ-contractive mapping justifying the hypotheses below:*

(i) Ξ is α-admissible;
(ii) there is $a_0 \in \chi$ so that $\alpha(a_0, \Xi a_0) \geq 1$;
(iii) Ξ is continuous.

Then Ξ has a FP.

Moreover, the authors in [29] showed that Theorem 1.2 is also true if we use the following condition instead of the continuity of the mapping Ξ.

- If $\{a_m\}$ is a sequence of χ so that $\alpha(a_m, a_{m+1}) \geq 1$ for all m and $\lim_{m\to+\infty} a_m = a \in \chi$, then for all m, $\alpha(a_m, a) \geq 1$.

The idea of an α-admissible mapping has spread widely, and the FPs obtained under this idea are not small, for example, see [31–34].

Furthermore, one of the interesting directions for obtaining FPs is to introduce the idea of Geraghty contractions [30]. The author [30] generalized the Banach contraction principle and obtained some pivotal results in a complete metric space. It is worth noting that a good number of researchers have focused their attention on this idea, for example, see [35–37]. In respect of completeness, we state Geraghty's theorem.

Theorem 1.3. *[30] Let $\Xi : \chi \to \chi$ be an operator on a complete metric space (χ, d). Then Ξ has a unique FP if Ξ satisfies the following inequality:*

$$d(\Xi a, \Xi b) \leq \widetilde{\ell}\,(d(a,b))\, d(a,b), \text{ for any } a, b \in \chi,$$

where $\widetilde{\ell} \in \widetilde{L}$.

We need the following results in the last part.

Definition 1.5. *[5]* Let $\sigma \in H^1(s,t)$, $s < t$, and $\nu \in [0,1)$. The Atangana–Baleanu fractional derivative in the Caputo sense of σ of order ν is described by

$$\left({}_{s}^{ABC}D^{\nu}\sigma\right)(\zeta) = \frac{Q(\nu)}{1-\nu}\int_{s}^{\zeta} \sigma'(\vartheta) M_{\nu}\left(-\nu\frac{(\zeta-\vartheta)^{\nu}}{1-\nu}\right) d\vartheta,$$

where M_{ν} is the Mittag-Leffler function given by $M_{\nu}(r) = \sum\limits_{m=0}^{\infty} \frac{r^m}{\Gamma(m\nu+1)}$ and $Q(\nu)$ is a normalizing positive function fulfilling $Q(0) = Q(1) = 1$ (see [4]). The related fractional integral is described as

$$\left({}_{s}^{AB}I^{\nu}\sigma\right)(\zeta) = \frac{1-\nu}{Q(\nu)}\sigma(\zeta) + \frac{\nu}{Q(\nu)}\left({}_{s}I^{\nu}\sigma\right)(\zeta), \tag{1.1}$$

where ${}_{s}I^{\nu}$ is the left Riemann-Liouville fractional integral defined by

$$\left({}_{s}I^{\nu}\sigma\right)(\zeta) = \frac{1}{\Gamma(\nu)}\int_{s}^{\zeta} (\zeta-\vartheta)^{\nu-1}\sigma(\vartheta) d\vartheta. \tag{1.2}$$

Lemma 1.2. *[38] For $\nu \in (0,1)$, we have*

$$\left({}_{s}^{AB}I^{\nu ABC}D^{\nu}\sigma\right)(\zeta) = \sigma(\zeta) - \sigma(s).$$

The outline for this paper is as follows: In Section 1, we presented some known consequences about α-admissible mappings and some useful definitions and theorems that will be used in the sequel. In Section 2, we introduce an η_{θ}^{ℓ}-contraction type mapping and obtain some related CFP results in the context of POMSs. Also, we support our theoretical results with some examples. In Section 5, an application to find the existence of a solution for the Atangana-Baleanu coupled fractional differential equation (CFDE) in the Caputo sense is presented.

2. Main results

Let L be the set of all functions $\ell : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\lim_{m\to\infty} \ell(\tau_n) = 1 \text{ implies } \lim_{m\to\infty} \tau_n = 1.$$

We begin this part with the following definitions:

Definition 2.1. Suppose that $\Xi : \chi \times \chi \to \chi$ and $\eta : \chi^2 \times \chi^2 \to [0, \infty)$ are two mappings. The mapping Ξ is called η-admissible if

$$\eta\left((a, b), (k, l)\right) \geq 1 \Rightarrow \eta\left((\Xi(a, b), \Xi(b, a)), (\Xi(k, l), \Xi(l, k))\right) \geq 1, \ \forall a, b, k, l \in \chi.$$

Definition 2.2. Let (χ, ϖ) be a POMS and $\Xi : \chi \times \chi \to \chi$ be a given mapping. Ξ is termed as an η^{ℓ}_{θ}-coupled contraction mapping if there are two functions $\eta : \chi^2 \times \chi^2 \to [0, \infty)$ and $\theta \in \Theta$ so that

$$\eta\left((a, b), (k, l)\right) \varpi\left(\Xi(a, b), \Xi(k, l)\right) \leq \ell\left(\theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right)\right)\theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right), \tag{2.1}$$

for all $a, b, k, l \in \chi$ with $a \geq k$ and $b \leq l$, where $\ell \in L$.

Remark 2.1. Notice that since $\ell : [0, \infty) \to [0, 1)$, we have

$$\begin{aligned} & \eta\left((a, b), (k, l)\right) \varpi\left(\Xi(a, b), \Xi(k, l)\right) \\ \leq\ & \ell\left(\theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right)\right) \times \theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right) \\ <\ & \theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right), \text{ for any } a, b, k, l \in \chi \text{ with } a \neq b \neq k \neq l. \end{aligned}$$

Theorem 2.1. *Let $(\chi, \leq, \varpi)$ be a complete POMS and Ξ be an η^{ℓ}_{θ}-coupled contraction which has the mixed monotone property so that*

(i) Ξ is η-admissible;
(ii) there are $a_0, b_0 \in \chi$ so that

$$\eta\left((a_0, b_0), (\Xi(a_0, b_0), \Xi(b_0, a_0))\right) \geq 1 \text{ and } \eta\left((b_0, a_0), (\Xi(b_0, a_0), \Xi(a_0, b_0))\right) \geq 1;$$

(iii) Ξ is continuous.

If there are $a_0, b_0 \in \chi$ so that $a_0 \leq \Xi(a_0, b_0)$ and $b_0 \geq \Xi(b_0, a_0)$, then Ξ has a CFP.

Proof. Let $a_0, b_0 \in \chi$ be such that $\eta\left((a_0, b_0), (\Xi(a_0, b_0), \Xi(b_0, a_0))\right) \geq 1$, $\eta\left((b_0, a_0), (\Xi(b_0, a_0), \Xi(a_0, b_0))\right) \geq 1$, $a_0 \leq \Xi(a_0, b_0) = a_1$ (say) and $b_0 \geq \Xi(b_0, a_0) = b_1$ (say). Consider $a_2, b_2 \in \chi$ so that $\Xi(a_1, b_1) = a_2$ and $\Xi(b_1, a_1) = b_2$. Similar to this approach, we extract two sequences $\{a_m\}$ and $\{b_m\}$ in χ so that

$$a_{m+1} = \Xi(a_m, b_m) \text{ and } b_{m+1} = \Xi(b_m, a_m), \text{ for all } m \geq 0.$$

Now, we shall show that

$$a_m \leq a_{m+1} \text{ and } b_m \geq b_{m+1}, \text{ for all } m \geq 0. \tag{2.2}$$

By a mathematical induction, we have

(1) At $m=0$, because $a_0 \leq \Xi(a_0,b_0)$ and $b_0 \geq \Xi(b_0,a_0)$ and since $\Xi(a_0,b_0)=a_1$ and $\Xi(b_0,a_0)=b_1$, we obtain $a_0 \leq a_1$ and $b_0 \geq b_1$, thus (2.2) holds for $m=0$.
(2) Suppose that (2.2) holds for some fixed $m \geq 0$.
(3) Attempting to prove the validity of (2.2) for any m, by assumption (2) and the mixed monotone property of Ξ, we get

$$a_{m+2}=\Xi(a_{m+1},b_{m+1}) \geq \Xi(a_m,b_{m+1}) \geq \Xi(a_m,b_m)=a_{m+1},$$

and

$$b_{m+2}=\Xi(b_{m+1},a_{m+1}) \leq \Xi(b_m,a_{m+1}) \leq \Xi(b_m,a_m)=b_{m+1}.$$

This implies that

$$a_{m+2} \geq a_{m+1} \text{ and } b_{m+2} \leq b_{m+1}.$$

Thus, we conclude that (2.2) is valid for all $n \geq 0$.

Next, if for some $m \geq 0$, $(a_{m+1},b_{m+1})=(a_m,b_m)$, then $a_m=\Xi(a_m,b_m)$ and $b_m=\Xi(b_m,a_m)$, i.e., Ξ has a CFP. So, let $(a_{m+1},b_{m+1}) \neq (a_m,b_m)$ for all $m \geq 0$. As Ξ is η-admissible, we get

$$\eta((a_0,b_0),(a_1,b_1))=\eta((a_0,b_0),(\Xi(a_0,b_0),\Xi(b_0,a_0))) \geq 1,$$

implies

$$\eta((\Xi(a_0,b_0),\Xi(b_0,a_0)),(\Xi(a_1,b_1),\Xi(b_1,a_1)))=\eta((a_1,b_1),(a_2,b_2)) \geq 1.$$

Thus, by induction, one can write

$$\eta((a_m,b_m),(a_{m+1},b_{m+1})) \geq 1 \text{ and } \eta((b_m,a_m),(b_{m+1},a_{m+1})) \geq 1 \text{ for all } m \geq 0. \tag{2.3}$$

Using (2.1) and (2.3) and the definition of ℓ, we have

$$\begin{aligned}
\varpi(a_m,a_{m+1}) &= \varpi(\Xi(a_{m-1},b_{m-1}),\Xi(a_m,b_m))\\
&\leq \eta((a_{m-1},b_{m-1}),(a_m,b_m))\,\varpi(\Xi(a_{m-1},b_{m-1}),\Xi(a_m,b_m))\\
&\leq \ell\left(\theta\left(\frac{\varpi(a_{m-1},a_m)+\varpi(b_{m-1},b_m)}{2}\right)\right)\theta\left(\frac{\varpi(a_{m-1},a_m)+\varpi(b_{m-1},b_m)}{2}\right)\\
&\leq \theta\left(\frac{\varpi(a_{m-1},a_m)+\varpi(b_{m-1},b_m)}{2}\right).
\end{aligned} \tag{2.4}$$

Analogously, we get

$$\begin{aligned}
\varpi(b_m,b_{m+1}) &= \varpi(\Xi(b_{m-1},a_{m-1}),\Xi(b_m,a_m))\\
&\leq \eta((b_{m-1},a_{m-1}),(b_m,a_m))\,\varpi(\Xi(b_{m-1},a_{m-1}),\Xi(b_m,a_m))\\
&\leq \theta\left(\frac{\varpi(b_{m-1},b_m)+\varpi(a_{m-1},a_m)}{2}\right).
\end{aligned} \tag{2.5}$$

Adding (2.4) and (2.5) we have

$$\frac{\varpi(a_m,a_{m+1})+\varpi(b_m,b_{m+1})}{2} \leq \theta\left(\frac{\varpi(a_{m-1},a_m)+\varpi(b_{m-1},b_m)}{2}\right).$$

Continuing in the same way, we get

$$\frac{\varpi\left(a_m, a_{m+1}\right) + \varpi\left(b_m, b_{m+1}\right)}{2} \leq \theta^m \left(\frac{\varpi(a_0, a_1) + \varpi(b_0, b_1)}{2}\right), \text{ for all } m \in \mathbb{N}.$$

For $\epsilon > 0$, there exists $m(\epsilon) \in \mathbb{N}$ so that

$$\sum_{m \geq m(\epsilon)} \theta^m \left(\frac{\varpi(a_0, a_1) + \varpi(b_0, b_1)}{2}\right) < \frac{\epsilon}{2},$$

for some $\theta \in \Theta$. Let $m, j \in \mathbb{N}$ be so that $j > m > m(\epsilon)$. Then based on the triangle inequality, we obtain

$$\begin{aligned}
\frac{\varpi\left(a_m, a_j\right) + \varpi\left(b_m, b_j\right)}{2} &\leq \sum_{i=m}^{j-1} \frac{\varpi(a_i, a_{i+1}) + \varpi(b_i, b_{i+1})}{2} \\
&\leq \sum_{i=m}^{j-1} \theta^i \left(\frac{\varpi(a_0, a_1) + \varpi(b_0, b_1)}{2}\right) \\
&\leq \sum_{m \geq m(\epsilon)} \theta^m \left(\frac{\varpi(a_0, a_1) + \varpi(b_0, b_1)}{2}\right) < \frac{\epsilon}{2},
\end{aligned}$$

this leads to $\varpi\left(a_m, a_j\right) + \varpi\left(b_m, b_j\right) < \epsilon$. Because

$$\varpi\left(a_m, a_j\right) \leq \varpi\left(a_m, a_j\right) + \varpi\left(b_m, b_j\right) < \epsilon,$$

and

$$\varpi\left(b_m, b_j\right) \leq \varpi\left(a_m, a_j\right) + \varpi\left(b_m, b_j\right) < \epsilon,$$

hence $\{a_m\}$ and $\{b_m\}$ are Cauchy sequences in χ. The completeness of χ implies that the sequences $\{a_m\}$ and $\{b_m\}$ are convergent in χ, that is, there are $a, b \in \chi$ so that

$$\lim_{m \to \infty} a_m = a \text{ and } \lim_{m \to \infty} b_m = b.$$

Since Ξ is continuous, $a_{m+1} = \Xi\left(a_m, b_m\right)$ and $b_{m+1} = \Xi\left(b_m, a_m\right)$, we obtain after taking the limit as $m \to \infty$ that

$$a = \lim_{m \to \infty} a_m = \lim_{m \to \infty} \Xi\left(a_{m-1}, b_{m-1}\right) = \Xi(a, b),$$

and

$$b = \lim_{m \to \infty} b_m = \lim_{m \to \infty} \Xi\left(b_{m-1}, a_{m-1}\right) = \Xi(b, a).$$

Therefore, Ξ has a CFP and this ends the proof.

In the above theorem, when omitting the continuity assumption on Ξ, we derive the following theorem.

Theorem 2.2. *Let $(\chi, \leq, \varpi)$ be a complete POMS and Ξ be an η_θ^ℓ-coupled contraction and having the mixed monotone property so that*

(a) Ξ is η-admissible;

(b) there are $a_0, b_0 \in \chi$ *so that*

$$\eta\left((a_0, b_0), (\Xi\left(a_0, b_0\right), \Xi\left(b_0, a_0\right))\right) \geq 1 \text{ and } \eta\left((b_0, a_0), (\Xi\left(b_0, a_0\right), \Xi\left(a_0, b_0\right))\right) \geq 1;$$

(c) if $\{a_m\}$ *and* $\{b_m\}$ *are sequences in* χ *such that*

$$\eta\left((a_m, b_m), (a_{m+1}, b_{m+1})\right) \geq 1, \ \eta\left((b_m, a_m), (b_{m+1}, a_{m+1})\right) \geq 1$$

for all $m \geq 0$, $\lim_{m\to\infty} a_m = a \in \chi$ *and* $\lim_{m\to\infty} b_m = b \in \chi$, *then*

$$\eta\left((a_m, b_m), (a, b)\right) \geq 1 \text{ and } \eta\left((b_m, a_m), (b, a)\right) \geq 1.$$

If $a_0, b_0 \in \chi$ *are that* $a_0 \leq \Xi\left(a_0, b_0\right)$ *and* $b_0 \geq \Xi\left(b_0, a_0\right)$, *then* Ξ *has a CFP.*

Proof. With the same approach as for the proof of Theorem 2.1, the sequences $\{a_m\}$ and $\{b_m\}$ are Cauchy sequences in χ. The completeness of χ implies that there are $a, b \in \chi$ so that

$$\lim_{m\to\infty} a_m = a \text{ and } \lim_{m\to\infty} b_m = b.$$

According to the assumption (c) and (2.3), one can write

$$\eta\left((a_m, b_m), (a, b)\right) \geq 1 \text{ and } \eta\left((b_m, a_m), (b, a)\right) \geq 1, \text{ for all } m \in \mathbb{N}. \tag{2.6}$$

It follows by (2.3), the definition of ℓ and the property of $\theta(\tau) < \tau$ for all $\tau > 0$, that

$$\begin{aligned}
\varpi\left(\Xi(a, b), a\right) &\leq \varpi\left(\Xi(a, b), \Xi(a_m, b_m)\right) + \varpi\left(\Xi(a_m, b_m), a\right) \\
&\leq \eta\left((a_m, b_m), (a, b)\right) \varpi\left(\Xi(a_m, b_m), \Xi(a, b)\right) + \varpi\left(a_{m+1}, a\right) \\
&\leq \ell\left(\theta\left(\frac{\varpi(a_m, a) + \varpi(b_m, b)}{2}\right)\right)\theta\left(\frac{\varpi(a_m, a) + \varpi(b_m, b)}{2}\right) + \varpi\left(a_{m+1}, a\right) \\
&\leq \theta\left(\frac{\varpi(a_m, a) + \varpi(b_m, b)}{2}\right) + \varpi\left(a_{m+1}, a\right) \\
&< \frac{\varpi(a_m, a) + \varpi(b_m, b)}{2} + \varpi\left(a_{m+1}, a\right).
\end{aligned} \tag{2.7}$$

Similarly, we find that

$$\begin{aligned}
\varpi\left(\Xi(b, a), b\right) &\leq \varpi\left(\Xi(b, a), \Xi(b_m, a_m)\right) + \varpi\left(\Xi(b_m, a_m), b\right) \\
&\leq \eta\left((b_m, a_m), (b, a)\right) \varpi\left(\Xi(b_m, a_m), \Xi(b, a)\right) + \varpi\left(b_{m+1}, b\right) \\
&\leq \ell\left(\theta\left(\frac{\varpi(b_m, b) + \varpi(a_m, a)}{2}\right)\right)\theta\left(\frac{\varpi(b_m, b) + \varpi(a_m, a)}{2}\right) + \varpi\left(b_{m+1}, b\right) \\
&\leq \theta\left(\frac{\varpi(b_m, b) + \varpi(a_m, a)}{2}\right) + \varpi\left(b_{m+1}, b\right) \\
&< \frac{\varpi(b_m, b) + \varpi(a_m, a)}{2} + \varpi\left(b_{m+1}, b\right).
\end{aligned} \tag{2.8}$$

As $m \to \infty$ in (2.7) and (2.8), we have

$$\varpi\left(\Xi(a, b), a\right) = 0 \text{ and } \varpi\left(\Xi(b, a), b\right) = 0.$$

Hence, $a = \Xi(a, b)$ and $b = \Xi(b, a)$. Thus, Ξ has a CFP and this completes the proof.

In order to show the uniqueness of a CFP, we give the theorem below. If $(\chi, \leq)$ is a partially ordered set, we define a partial order relation $\leq$ on the product $\chi \times \chi$ as follows:

$$(a, b) \leq (k, l) \Leftrightarrow a \leq k \text{ and } b \geq l, \text{ for all } (a, b), (k, l) \in \chi \times \chi.$$

Theorem 2.3. *In addition to the assertions of Theorem 2.1, assume that for each $(a, b), (y, z)$ in $\chi \times \chi$, there is $(k, l) \in \chi \times \chi$ so that*

$$\eta\left((a, b), (k, l)\right) \geq 1 \textit{ and } \eta\left((y, z), (k, l)\right) \geq 1.$$

Suppose also (k, l) is comparable to (a, b) and (y, z). Then Ξ has a unique CFP.

Proof. Theorem 2.1 asserts that the set of CFPs is non-empty. Let (a, b) and (y, z) be CFPs of the mapping Ξ, that is, $a = \Xi(a, b)$, $b = \Xi(b, a)$ and $y = \Xi(y, z)$, $z = \Xi(z, y)$. By hypothesis, there is $(k, l) \in \chi \times \chi$ so that (k, l) is comparable to (a, b) and (y, z). Let $(a, b) \leq (k, l)$, $k = k_0$ and $l = l_0$. Choose $k_1, l_1 \in \chi \times \chi$ so that $k_1 = \Xi(k_1, l_1)$, $l_1 = \Xi(l_1, k_1)$. Thus, we can construct two sequences $\{k_m\}$ and $\{l_m\}$ as

$$k_{m+1} = \Xi(k_m, l_m) \text{ and } l_{m+1} = \Xi(l_m, k_m).$$

Since (k, l) is comparable to (a, b), in an easy way we can prove that $a \leq k_1$ and $b \geq l_1$. Hence, for $m \geq 1$, we have $a \leq k_m$ and $b \geq l_m$. Because for every $(a, b), (y, z) \in \chi \times \chi$, there is $(k, l) \in \chi \times \chi$ so that

$$\eta\left((a, b), (k, l)\right) \geq 1 \text{ and } \eta\left((y, z), (k, l)\right) \geq 1. \tag{2.9}$$

Because Ξ is η-admissible, then by (2.9), we get

$$\eta\left((a, b), (k, l)\right) \geq 1 \text{ implies } \eta\left((\Xi(a, b), \Xi(b, a)), (\Xi(k, l), \Xi(l, k))\right) \geq 1.$$

Since $k = k_0$ and $l = l_0$, we obtain

$$\eta\left((a, b), (k, l)\right) \geq 1 \text{ implies } \eta\left((\Xi(a, b), \Xi(b, a)), (\Xi(k_0, l_0), \Xi(l_0, k_0))\right) \geq 1.$$

Hence,

$$\eta\left((a, b), (k, l)\right) \geq 1 \text{ implies } \eta\left((a, b), (k_1, l_1)\right) \geq 1.$$

So, by induction, we conclude that

$$\eta\left((a, b), (k_m, l_m)\right) \geq 1, \tag{2.10}$$

for all $m \in \mathbb{N}$. Analogously, one can obtain that $\eta\left((b, a), (l_m, k_m)\right) \geq 1$. Therefore, the obtained results hold if $(a, b) \leq (k, l)$. Based on (2.9) and (2.10), we can write

$$\begin{aligned}
\varpi\left(a, k_{m+1}\right) &= \varpi\left(\Xi(a, b), \Xi(k_m, l_m)\right) \\
&\leq \eta\left((a, b), (k_m, l_m)\right) \varpi\left(\Xi(a, b), \Xi(k_m, l_m)\right) \\
&\leq \ell\left(\theta\left(\frac{\varpi(a, k_m) + \varpi(b, l_m)}{2}\right)\right) \theta\left(\frac{\varpi(a, k_m) + \varpi(b, l_m)}{2}\right) \\
&\leq \theta\left(\frac{\varpi(a, k_m) + \varpi(b, l_m)}{2}\right). \qquad (2.11)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}\varpi\left(b, l_{m+1}\right) &= \varpi\left(\Xi(b, a), \Xi(l_m, k_m)\right)\\ &\leq \eta\left((b, a), (l_m, k_m)\right) \varpi\left(\Xi(b, a), \Xi(l_m, k_m)\right)\\ &\leq \ell\left(\theta\left(\frac{\varpi(b, l_m) + \varpi(a, k_m)}{2}\right)\right)\theta\left(\frac{\varpi(b, l_m) + \varpi(a, k_m)}{2}\right)\\ &\leq \theta\left(\frac{\varpi(b, l_m) + \varpi(a, k_m)}{2}\right). \end{aligned} \tag{2.12}$$

Adding (2.11) and (2.12), we have

$$\frac{\varpi\left(a, k_{m+1}\right) + \varpi\left(b, l_{m+1}\right)}{2} \leq \theta\left(\frac{\varpi(b, l_m) + \varpi(a, k_m)}{2}\right).$$

Thus,

$$\frac{\varpi\left(a, k_{m+1}\right) + \varpi\left(b, l_{m+1}\right)}{2} \leq \theta^m\left(\frac{\varpi(b, l_1) + \varpi(a, k_1)}{2}\right), \tag{2.13}$$

for each $n \geq 1$. As $m \to \infty$ in (2.13) and by Lemma 1.1, we have

$$\lim_{m\to\infty}\left(\varpi\left(a, k_{m+1}\right) + \varpi\left(b, l_{m+1}\right)\right) = 0,$$

which yields that

$$\lim_{m\to\infty} \varpi\left(a, k_{m+1}\right) = \lim_{m\to\infty} \varpi\left(b, l_{m+1}\right) = 0. \tag{2.14}$$

Similarly, one obtains

$$\lim_{m\to\infty} \varpi\left(y, k_{m+1}\right) = \lim_{m\to\infty} \varpi\left(z, l_{m+1}\right) = 0. \tag{2.15}$$

It follows from (2.14) and (2.15), we find that $a = y$ and $b = z$. This proves that the CFP is unique.

Examples below support the theoretical results.

Example 2.1. (Linear case) Let $\varpi : \chi \times \chi \to \mathbb{R}$ be a usual metric on $\chi = [0, 1]$. Define the mappings $\Xi : \chi \times \chi \to \chi$ and $\eta : \chi^2 \times \chi^2 \to [0, \infty)$ by $\Xi(a, b) = \frac{(a-b)}{32}$ and

$$\eta\left((a, b), (k, l)\right) = \begin{cases} \frac{3}{2}, & \text{if } a \geq b,\ k \geq l, \\ 0 & \text{otherwise,} \end{cases}$$

for all $a, b, k, l \in \chi$, respectively. Consider

$$\begin{aligned} &\ell\left(\theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right)\right)\theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right) - \eta\left((a, b), (k, l)\right) \varpi\left(\Xi(a, b), \Xi(k, l)\right)\\ =\ & \ell\left(\theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right)\right)\theta\left(\frac{\varpi(a, k) + \varpi(b, l)}{2}\right) - \frac{3}{2}\varpi\left(\Xi(a, b), \Xi(k, l)\right)\\ =\ & \left(\frac{1}{\frac{|a-k|}{4} + \frac{|b-l|}{4}}\right)\left(\frac{|a-k|}{4} + \frac{|b-l|}{4}\right) - \frac{3}{2}\left|\Xi(a, b) - \Xi(k, l)\right|\\ =\ & 1 - \frac{3}{2}\left|\frac{1}{32}(a-b) - \frac{1}{32}(k-l)\right| \end{aligned}$$

$$= \quad 1 - \frac{3}{64}\left|(a-b)-(k-l)\right| \geq 0,$$

which implies that

$$\eta\left((a,b),(k,l)\right)\varpi\left(\Xi(a,b),\Xi(k,l)\right) \leq \ell\left(\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right)\right)\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right).$$

Therefore, (2.1) is fulfilled with $\ell(\tau) = \frac{1}{\tau}$ and $\theta(\tau) = \frac{\tau}{2}$, for all $\tau > 0$. Also, all the hypotheses of Theorem 2.1 are satisfied and $(0,0)$ is the unique CFP of Ξ.

Example 2.2. *(Nonlinear case)* Let $\varpi : \chi \times \chi \to \mathbb{R}$ be the usual metric on $\chi = [0,1]$. Define the mappings $\Xi : \chi \times \chi \to \chi$ and $\eta : \chi^2 \times \chi^2 \to [0,\infty)$ by $\Xi(a,b) = \frac{1}{32}\left(\ln(1+a) - \ln(1+b)\right)$ and

$$\eta\left((a,b),(k,l)\right) = \begin{cases} \frac{4}{3}, & \text{if } a \geq b,\ k \geq l, \\ 0 & \text{otherwise,} \end{cases}$$

for all $a,b,k,l \in \chi$, respectively. Then, we have

$$\begin{aligned}
& \ell\left(\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right)\right)\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right) - \eta\left((a,b),(k,l)\right)\varpi\left(\Xi(a,b),\Xi(k,l)\right) \\
= \; & \ell\left(\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right)\right)\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right) - \frac{4}{3}\varpi\left(\Xi(a,b),\Xi(k,l)\right) \\
= \; & \left(\frac{1}{\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right)}\right)\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right) - \frac{4}{3}\left|\Xi(a,b)-\Xi(k,l)\right| \\
= \; & 1 - \frac{4}{3\times 32}\left|\left(\ln(1+a)-\ln(1+b)\right) - \left(\ln(1+k)-\ln(1+l)\right)\right| \\
= \; & 1 - \frac{1}{24}\left|\left(\ln\left(\frac{1+a}{1+k}\right) + \ln\left(\frac{1+l}{1+b}\right)\right)\right| \geq 1 - \frac{1}{24}\left(\ln\left(1+|a-k|\right) + \ln\left(1+|l-b|\right)\right) \geq 0.
\end{aligned}$$

Note that we used the property $\ln\left(\frac{1+a}{1+k}\right) \leq \ln(1+(a-k))$. Hence,

$$\ell\left(\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right)\right)\theta\left(\frac{\varpi(a,k)+\varpi(b,l)}{2}\right) \geq \eta\left((a,b),(k,l)\right)\varpi\left(\Xi(a,b),\Xi(k,l)\right).$$

Therefore, (2.1) holds with $\ell(\tau) = \frac{1}{\tau}$ and $\theta(\tau) = \frac{\tau}{2}$, for all $\tau > 0$. Furthermore, all the hypotheses of Theorem 2.1 are fulfilled and $(0,0)$ is the unique CFP of Ξ.

3. CFDEs with Mittag-Leffler kernel

In this section, we apply Theorem 2.2 to discuss the existence solution for the following Atangana–Baleanu fractional differential equation in the Caputo sense:

$$\begin{cases} \left({}_{0}^{ABC}D^{\nu}\sigma\right)(\zeta) = \varphi(\zeta,\sigma(\zeta),\rho(\zeta)), & \zeta \in I = [0,1], \\ \left({}_{0}^{ABC}D^{\nu}\rho\right)(\zeta) = \varphi(\zeta,\rho(\zeta),\sigma(\zeta)), & 0 \leq \nu \leq 1, \\ \sigma(0) = \sigma_0 \text{ and } \rho(0) = \rho_0, & \end{cases} \tag{3.1}$$

where D^ν is the Atangana-Baleanu derivative in the Caputo sense of order ν and $\varphi : I \times \chi \times \chi \to \chi$ is a continuous function with $\varphi(0, \sigma(0), \rho(0)) = 0$.

Let $\varpi : \chi \times \chi \to [0, \infty)$ be a function defined by

$$\varpi(\sigma, \rho) = \|\sigma - \rho\|_\infty = \sup_{\zeta \in I} |\sigma(\zeta) - \rho(\zeta)|,$$

where $\chi = C(I, \mathbb{R})$ represents the set of continuous functions. Define a partial order $\leq$ on χ by

$$(a, b) \leq (k, l) \Leftrightarrow a \leq k \text{ and } b \geq l, \text{ for all } a, b, k, l \in \chi.$$

It is clear that $(\chi, \leq, \varpi)$ is a complete POMS.

Now, to discuss the existence solution to the problem (3.1), we describe our hypotheses in the following theorem:

Theorem 3.1. *Assume that:*

(h_1) *there is a continuous function $\varphi : I \times \chi \times \chi \to \chi$ so that*

$$\begin{aligned} & |\varphi(\aleph, \sigma(\aleph), \rho(\aleph)) - \varphi(\aleph, \sigma^*(\aleph), \rho^*(\aleph))| \\ \leq \; & \frac{Q(\nu)\Gamma(\nu)}{(1-\nu)\Gamma(\nu)+1} \ell \left(\theta \left(\frac{|\sigma(\aleph) - \sigma^*(\aleph)| + |\rho(\aleph) - \rho^*(\aleph)|}{2} \right) \right) \times \\ & \theta \left(\frac{|\sigma(\aleph) - \sigma^*(\aleph)| + |\rho(\aleph) - \rho^*(\aleph)|}{2} \right), \end{aligned}$$

for $\aleph \in I$, $\ell \in L$, $\theta \in \Theta$ and $\sigma, \rho, \sigma^*, \rho^* \in \chi$. Moreover, there exists $\Im : C^2(I) \times C^2(I) \to C(I)$ such that $\Im((\sigma(\aleph), \rho(\aleph)), (\sigma^*(\aleph), \rho^*(\aleph))) \geq 0$ and $\Im((\rho(\aleph), \sigma(\aleph)), (\rho^*(\aleph), \sigma^*(\aleph))) \geq 0$, for each $\sigma, \rho, \sigma^*, \rho^* \in C(I)$ and $\aleph \in I$;

(h_2) *there exist $\sigma_1, \rho_1 \in C(I)$ with $\Im((\sigma_1(\aleph), \rho_1(\aleph)), (\Xi(\sigma_1(\aleph), \rho_1(\aleph)), \Xi(\rho_1(\aleph), \sigma_1(\aleph)))) \geq 0$ and $\Im((\rho_1(\aleph), \sigma_1(\aleph)), (\Xi(\rho_1(\aleph), \sigma_1(\aleph)), \Xi(\sigma_1(\aleph), \rho_1(\aleph)))) \geq 0$, for $\aleph \in I$, where $\Xi : C(I) \times C(I) \to C(I)$ is defined by*

$$\Xi(\rho, \sigma)(\aleph) = \sigma_0 + {}_0^{AB} I^\nu \varphi(\aleph, \sigma(\aleph), \rho(\aleph));$$

(h_3) *for $\sigma, \rho, \sigma^*, \rho^* \in C(I)$ and $\aleph \in I$, $\Im((\sigma(\aleph), \rho(\aleph)), (\sigma^*(\aleph), \rho^*(\aleph))) \geq 0$ and $\Im((\rho(\aleph), \sigma(\aleph)), (\rho^*(\aleph), \sigma^*(\aleph))) \geq 0$ implies*

$$\Im((\Xi(\sigma(\aleph), \rho(\aleph)), \Xi(\rho(\aleph), \sigma(\aleph))), (\Xi(\sigma^*(\aleph), \rho^*(\aleph)), \Xi(\rho^*(\aleph), \sigma^*(\aleph)))) \geq 0$$

and

$$\Im((\Xi(\rho(\aleph), \sigma(\aleph)), \Xi(\sigma(\aleph), \rho(\aleph))), (\Xi(\rho^*(\aleph), \sigma^*(\aleph)), \Xi(\sigma^*(\aleph), \rho^*(\aleph)))) \geq 0;$$

(h_4) *if $\{\sigma_m\}, \{\rho_m\} \subseteq C(I)$, $\lim_{m\to\infty} \sigma_m = \sigma$, $\lim_{m\to\infty} \rho_m = \rho$ in $C(I)$, $\Im((\sigma_m, \rho_m), (\sigma_{m+1}, \rho_{m+1})) \geq 0$ and $\Im((\rho_m, \sigma_m), (\rho_{m+1}, \sigma_{m+1})) \geq 0$, then $\Im((\sigma_m, \rho_m), (\sigma, \rho)) \geq 0$ and $\Im((\rho_m, \sigma_m), (\rho, \sigma)) \geq 0$, for all $m \in \mathbb{N}$.*

Then there is at least one solution for the problem (3.1).

Proof. Effecting the Atangana-Baleanu integral to both sides of (3.1) and applying Lemma 1.2, we have

$$\sigma(\aleph) = \sigma_0 +_0^{AB} I^\nu \varphi\left(\aleph, \sigma(\aleph), \rho(\aleph)\right),$$

and

$$\rho(\aleph) = \rho_0 +_0^{AB} I^\nu \varphi\left(\aleph, \rho(\aleph), \sigma(\aleph)\right).$$

Now, we shall prove that the mapping $\Xi : C(I)\times C(I) \to C(I)$ has a CFP. From (1.1) and (1.2) and (h_1), we get

$$\begin{aligned}
& |\Xi(\rho,\sigma)(\aleph) - \Xi(\rho^*,\sigma^*)(\aleph)| \\
= & \left|{}_0^{AB}I^\nu \left[\varphi\left(\aleph,\sigma(\aleph),\rho(\aleph)\right) - \varphi\left(\aleph,\sigma^*(\aleph),\rho^*(\aleph)\right)\right]\right| \\
= & \left|\frac{1-\nu}{Q(\nu)}\left[\varphi\left(\aleph,\sigma(\aleph),\rho(\aleph)\right) - \varphi\left(\aleph,\sigma^*(\aleph),\rho^*(\aleph)\right)\right]\right. \\
& \left. +\frac{\nu}{Q(\nu)}\, {}_0I^\nu \left[\varphi\left(\aleph,\sigma(\aleph),\rho(\aleph)\right) - \varphi\left(\aleph,\sigma^*(\aleph),\rho^*(\aleph)\right)\right]\right| \\
\leq & \frac{1-\nu}{Q(\nu)}\left|\varphi\left(\aleph,\sigma(\aleph),\rho(\aleph)\right) - \varphi\left(\aleph,\sigma^*(\aleph),\rho^*(\aleph)\right)\right| \\
& +\frac{\nu}{Q(\nu)}\, {}_0I^\nu \left|\varphi\left(\aleph,\sigma(\aleph),\rho(\aleph)\right) - \varphi\left(\aleph,\sigma^*(\aleph),\rho^*(\aleph)\right)\right| \\
\leq & \frac{1-\nu}{Q(\nu)}\times\frac{Q(\nu)\Gamma(\nu)}{(1-\nu)\Gamma(\nu)+1}\ell\left(\theta\left(\frac{|\sigma(\aleph)-\sigma^*(\aleph)|+|\rho(\aleph)-\rho^*(\aleph)|}{2}\right)\right)\times \\
& \theta\left(\frac{|\sigma(\aleph)-\sigma^*(\aleph)|+|\rho(\aleph)-\rho^*(\aleph)|}{2}\right) \\
& +\frac{\nu}{Q(\nu)}\times\frac{Q(\nu)\Gamma(\nu)}{(1-\nu)\Gamma(\nu)+1}\, {}_0I^\nu(1)\ell\left(\theta\left(\frac{|\sigma(\aleph)-\sigma^*(\aleph)|+|\rho(\aleph)-\rho^*(\aleph)|}{2}\right)\right)\times \\
& \theta\left(\frac{|\sigma(\aleph)-\sigma^*(\aleph)|+|\rho(\aleph)-\rho^*(\aleph)|}{2}\right) \\
= & \left\{\frac{Q(\nu)\Gamma(\nu)}{(1-\nu)\Gamma(\nu)+1}\ell\left(\theta\left(\frac{|\sigma(\aleph)-\sigma^*(\aleph)|+|\rho(\aleph)-\rho^*(\aleph)|}{2}\right)\right)\times\right. \\
& \left.\theta\left(\frac{|\sigma(\aleph)-\sigma^*(\aleph)|+|\rho(\aleph)-\rho^*(\aleph)|}{2}\right)\right\}\left(\frac{1-\nu}{Q(\nu)}+\frac{\nu}{Q(\nu)\nu\Gamma(\nu)}\right) \\
\leq & \left\{\frac{Q(\nu)\Gamma(\nu)}{(1-\nu)\Gamma(\nu)+1}\ell\left(\theta\left(\frac{\sup\limits_{\aleph\in I}|\sigma(\aleph)-\sigma^*(\aleph)|+\sup\limits_{\aleph\in I}|\rho(\aleph)-\rho^*(\aleph)|}{2}\right)\right)\times\right. \\
& \left.\theta\left(\frac{\sup\limits_{\aleph\in I}|\sigma(\aleph)-\sigma^*(\aleph)|+\sup\limits_{\aleph\in I}|\rho(\aleph)-\rho^*(\aleph)|}{2}\right)\right\}\left(\frac{1-\nu}{Q(\nu)}+\frac{\nu}{Q(\nu)\nu\Gamma(\nu)}\right) \\
= & \left(\frac{Q(\nu)\Gamma(\nu)}{(1-\nu)\Gamma(\nu)+1}\ell\left(\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right)\right)\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right)\right)\times \\
& \left(\frac{1-\nu}{Q(\nu)}+\frac{1}{Q(\nu)\Gamma(\nu)}\right)
\end{aligned}$$

$$= \ell\left(\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right)\right)\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right).$$

Hence, for $\sigma,\rho\in C(I)$, $\aleph\in I$, with $\Im\left((\sigma(\aleph),\rho(\aleph)),(\sigma^*(\aleph),\rho^*(\aleph))\right)\geq 0$ and $\Im\left((\rho(\aleph),\sigma(\aleph)),(\rho^*(\aleph),\sigma^*(\aleph))\right)\geq 0$, we get

$$\varpi(\Xi(\rho,\sigma)(\aleph),\Xi(\rho^*,\sigma^*))\leq \ell\left(\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right)\right)\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right).$$

Define $\eta : C^2(I)\times C^2(I)\to[0,\infty)$ by

$$\eta\left((\sigma(\aleph),\rho(\aleph)),(\sigma^*(\aleph),\rho^*(\aleph))\right)=\begin{cases}1, & \text{if } \Im\left((\sigma(\aleph),\rho(\aleph)),(\sigma^*(\aleph),\rho^*(\aleph))\right)\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

So

$$\begin{aligned}&\eta\left((\sigma(\aleph),\rho(\aleph)),(\sigma^*(\aleph),\rho^*(\aleph))\right)\varpi(\Xi(\rho,\sigma)(\aleph),\Xi(\rho^*,\sigma^*))\\ \leq\ & \ell\left(\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right)\right)\theta\left(\frac{\varpi(\sigma,\sigma^*)+\varpi(\rho,\rho^*)}{2}\right).\end{aligned}$$

Then, Ξ is an η_θ^ℓ-coupled contraction mapping. Now, for each $\rho,\sigma,\rho^*,\sigma^*\in C(I)$ and $\aleph\in I$, we have

$$\eta\left((\sigma(\aleph),\rho(\aleph)),(\sigma^*(\aleph),\rho^*(\aleph))\right)\geq 1,$$

due to definition of $\Im$ and η. So, hypothesis (h_3) gives

$$\begin{cases}\eta\left((\Xi(\sigma(\aleph),\rho(\aleph)),\Xi(\rho(\aleph),\sigma(\aleph))),(\Xi(\sigma^*(\aleph),\rho^*(\aleph)),\Xi(\rho^*(\aleph),\sigma^*(\aleph)))\right)\geq 1,\\ \eta\left((\Xi(\rho(\aleph),\sigma(\aleph)),\Xi(\sigma(\aleph),\rho(\aleph))),(\Xi(\rho^*(\aleph),\sigma^*(\aleph)),\Xi(\sigma^*(\aleph),\rho^*(\aleph)))\right)\geq 1,\end{cases}$$

for $\rho,\sigma,\rho^*,\sigma^*\in C(I)$. Therefore, Ξ is η-admissible. From (h_2), there are $\sigma_0,\rho_0\in C(I)$ with $\eta\left((\sigma_0(\aleph),\rho_0(\aleph)),\Xi(\sigma_0(\aleph),\rho_0(\aleph))\right)\geq 1$ and $\eta\left((\rho_0(\aleph),\sigma_0(\aleph)),\Xi(\rho_0(\aleph),\sigma_0(\aleph))\right)\geq 1$. Using (h_4) and Theorem 2.2, we conclude that there is $(\widehat{\sigma},\widehat{\rho})\in C(I)$ with $\widehat{\sigma}=\Xi(\widehat{\sigma},\widehat{\rho})$ and $\widehat{\rho}=\Xi(\widehat{\rho},\widehat{\sigma})$, that is, Ξ has a CFP, which is a solution of the system (3.1).

4. Conclusions

Many physical phenomena can be described by nonlinear differential equations (both ODEs and PDEs), so the study of numerical and analytical methods used in solving nonlinear differential equations are an interesting topic for analyzing scientific engineering problems. From this perspective, some coupled fixed point results for the class of η_θ^ℓ-contractions in POMSs are obtained. These results are reinforced by their applications in a study of the existence of a solution for a CFDE with the Mittag-Leffler kernel. In the future, our findings may be applied to differential equations of an arbitrary fractional order, linear and nonlinear fractional integro-differential systems, Hadamard fractional derivatives, Caputo-Fabrizio's kernel, and so on.

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Conflict of interest

The authors declare that they have no competing interests.

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