symmetry

MDPI

*Article*

# Double Sawi Transform: Theory and Applications to Boundary Values Problems

**Shahzeb Khan [1], Aman Ullah [1], Manuel De la Sen [2] and Shabir Ahmad [1,*]**

[1] Department of Mathematics, University of Malakand, Chakdara 18800, Pakistan; shahzebkhan251998@gmail.com (S.K.); amanswt@gmail.com (A.U.)
[2] Department of Electricity and Electronics, Institute of Research and Development of Processes, Faculty of Science and Technology, Campus of Leioa (Bizkaia), University of the Basque Country, 48940 Leioa, Spain; manuel.delasen@ehu.eus
* Correspondence: shabirahmad2232@gmail.com

**Abstract:** Symmetry can play an important role in the study of boundary value problems, which are a type of problem in mathematics that involves finding the solutions to differential equations subject to given boundary conditions. Integral transforms play a crucial role in solving ordinary differential equations (ODEs), partial differential equations (PDEs), and integral equations. This article focuses on extending a single-valued Sawi transform to a double-valued ST, which we call the double Sawi (DS) transform. We derive some fundamental features and theorems for the proposed transform. Finally, we study the applications of the proposed transform by solving some boundary value problems such as the Fourier heat equation and the D'Alembert wave equation.

**Keywords:** integral transforms; Sawi transform; boundary values problems

check for updates

**Citation:** Khan, S.; Ullah, A.; De la Sen, M.; Ahmad, S. Double Sawi Transform: Theory and Applications to Boundary Values Problems. *Symmetry* **2023**, *15*, 921. https://doi.org/10.3390/sym15040921

Academic Editor: Wei-Shih Du

Received: 22 March 2023
Revised: 3 April 2023
Accepted: 12 April 2023
Published: 15 April 2023



## 1. Introduction

Symmetry is an important concept in mathematics and physics, and it plays a crucial role in many areas of science and engineering. Integral transforms, such as the Fourier transform, Laplace transform, and the Sawi transform, are powerful tools for analyzing mathematical functions and systems. There is a close relationship between symmetry and integral transforms, and this relationship has important implications for the analysis of physical and mathematical systems. Symmetry is often associated with the idea of invariance under certain transformations. For example, a function is said to be symmetric if it is invariant under certain types of transformations, such as rotations or reflections. Integral transforms can be used to analyze the symmetry properties of a function by transforming it into a different domain, where the symmetry properties may be more apparent.

Integral transform methods are an effective approach to address ODEs and partial differential equations (PDEs) owing to their ability to simplify intricate PDEs and reduce them to more manageable equations [1–3]. These methods entail applying a mathematical transformation to a PDE, which then converts it to an algebraic or differential equation, making it easier to solve. Fourier transforms [4], Laplace transforms, and the method of characteristics are among the different types of transform methods available. A key advantage of transform methods is their applicability to solve PDEs that are either tough or impossible to resolve using other methods. For instance, transform methods may be more practical for PDEs that have nonlinear terms or variable coefficients, which can be extremely challenging to solve using conventional techniques [5–11]. Recently, the authors introduced another integral transform called the Sawi transform [12]. Some properties and applications of the Sawi transform have been studied in the literature [13–15].

Transform methods are not only limited to solving PDEs in one dimension but can also be applied to PDEs in multiple dimensions, which is particularly significant in fields such as fluid mechanics and electromagnetics, where several problems involve PDEs in three or more dimensions [16]. Some PDEs that have two independent variables may not be easily solved using single transform methods. Specifically, if a PDE involves both temporal and spatial derivatives, a single transform may not be adequate to eliminate both variables and simplify the equation to a solvable form. Therefore, double transform methods were developed to solve such complex PDEs that arise in mathematical physics, including wave propagation, heat conduction, and diffusion problems.

Double transform methods, also known as double Laplace transforms or double Fourier transforms, are employed to resolve specific types of partial differential equations (PDEs) having two independent variables [16]. These methods involve the application of two successive transforms to the PDE, which simplifies it to an algebraic equation that can be solved. By utilizing two successive transforms, double transform methods can remove both independent variables from the equation, leading to a more straightforward algebraic equation that can be solved using standard techniques. In conclusion, transform methods are an effective and versatile approach to addressing PDEs since they can simplify complex equations, making them more solvable.

The main purpose of this research article is to explore the DS transform and its various features, test problems, and its use in solving PDEs. In addition to providing examples, we establish several theorems that address the advanced features of the DS transform. Furthermore, we discuss the convolution of $\mathscr{Q}(x,y)$ and $\mathscr{V}(x,y)$ and its related theorems, including the convolution theorem, which is accompanied by a detailed proof. The applications of the proposed transform are to demonstrate the utility of the DS transform method for resolving some boundary value problems.

The rest of the paper is organized as follows: Section 2 provides a detailed definition of the double Sawi transform and its mathematical properties. Section 3 is dedicated to the applications of the double Sawi transform to solving specific types of boundary value problems. We summarize the results of our work in the Section 4. Finally, in Appendix A, we provide the double Sawi transform for several functions.

## 2. Definition and Properties of the Double Sawi Transform

**Definition 1.** *A function $\mathscr{Q}(t,x)$ is called of exponential order $(\mathcal{K} > 0, \Upsilon > 0)$ on interval $0 \leq t < \infty$ and $0 \leq x < \infty$ iff $\exists$ a constant $\mathcal{C}$, such that for $t > \mathcal{B}$ and for $x > \mathcal{X}$ $\mid \mathscr{Q}(t,x) \mid \leq \mathcal{C}\exp(\mathcal{K}t + \Upsilon x)$, when $t \to \infty, x \to \infty$; then, we can write $f(t,x) = O(\exp(\mathcal{K}t + \Upsilon x))$, where $O$ denotes a exponential order.*

**Definition 2.** *A function $\mathscr{Q}(t,x)$ is continuous and of exponential order (EO); then, the DS transform of $\mathscr{Q}(t,x)$ is given as $\mathcal{B}(\alpha_1,\ell_1)$, where $\ell_1 > 0$ and is expressed by:*

$$\mathcal{B}(\alpha_1,\ell_1) = \mathcal{S}_2\{\mathscr{Q}(t,x); t \to \alpha_1, x \to \ell_1\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1} - \frac{x}{\ell_1})dtdx.$$

*The single Sawi transform $\mathcal{B}(\alpha_1) = \mathcal{S}\{\mathscr{Q}(t); t \to \alpha_1\}$ of $\mathscr{Q}(t)$ and is defined by*

$$\mathcal{B}(\alpha_1) = \mathcal{S}\{\mathscr{Q}(t); t \to \alpha_1\} = \frac{1}{\alpha_1^2}\int_0^\infty \exp(-\frac{t}{\alpha_1})\mathscr{Q}(t)dt, \alpha_1 > 0.$$

**Theorem 1** (Existence Condition of the Double Sawi Transform)**.** *If a function $\mathscr{Q}(t,x)$ is continuous in every finite interval $(0,\mathcal{B}),(0,\mathcal{X})$ and is of EO $(\mathcal{K} > 0, \Upsilon > 0)$, then the DS transform of $\mathscr{Q}(t,x)$ exists $\forall$ $\alpha_1$ and $\ell_1$, for $Re(\alpha_1) > \mathcal{K}$ and $Re(\ell_1) > \Upsilon$, respectively.*

**Proof.** Consider

$$\begin{aligned} |\mathcal{B}(\alpha_1,\ell_1)| &= \left|\frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \exp(-\frac{t}{\alpha_1} - \frac{x}{\ell_1})\mathscr{Q}(t,x)dtdx\right| \\ &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \exp(-\frac{t}{\alpha_1})\exp(-\frac{x}{\ell_1})|\mathscr{Q}(t,x)|dtdx. \end{aligned}$$

Since $|\mathcal{Q}(t,x)| \leq \mathcal{C}\exp(\mathcal{K}t + \Upsilon x)$, then

$$\begin{aligned}
|\mathcal{B}(\alpha_1,\ell_1)| &\leq \frac{\mathcal{C}}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \exp(-\frac{t}{\alpha_1})\exp(-\frac{x}{\ell_1})\exp(\mathcal{K}t+\Upsilon x)dtdx \\
&= \frac{\mathcal{C}}{\alpha_1^2\ell_1^2}\int_0^\infty \exp -t(\frac{1}{\alpha_1}-\mathcal{K})dt\int_0^\infty \exp -x(\frac{1}{\ell_1}-\Upsilon)dx \\
&= \frac{\mathcal{C}}{\alpha_1^2\ell_1^2}\left[\frac{\exp -t(\frac{1-\mathcal{K}\alpha_1}{\alpha_1})}{-t(\frac{1-\mathcal{K}\alpha_1}{\alpha_1})} * \frac{\exp -x(\frac{1-\Upsilon\ell_1}{\ell_1})}{-x(\frac{1-\Upsilon\ell_1}{\ell_1})}\right]_0^\infty \\
&= \frac{\mathcal{C}}{\alpha_1^2\ell_1^2}\left(\frac{\alpha_1\ell_1}{(1-\mathcal{K}\alpha_1)(1-\Upsilon\ell_1)}\right).
\end{aligned}$$

We reach:

$$|\mathcal{B}(\alpha_1,\ell_1)| \leq \frac{\mathcal{C}}{\alpha_1\ell_1(1-\mathcal{K}\alpha_1)(1-\Upsilon\ell_1)}, \qquad \text{for } \mathrm{Re}(\alpha_1) > \mathcal{K} \text{ and } \mathrm{Re(j)} > \Upsilon.$$

If $\alpha_1 \to \infty$ and $\ell_1 \to \infty$, then

$$\lim_{\alpha_1\to\infty,\ell_1\to\infty}|\mathcal{B}(\alpha_1,\ell_1)| = 0.$$

Thus, we obtain

$$\lim_{\alpha_1\to\infty,\ell_1\to\infty}\mathcal{B}(\alpha_1,\ell_1) = 0.$$

□

**Theorem 2** (Linearity Property). *Let $\mathcal{Q}_1(t,x)$ and $\mathcal{Q}_2(t,x)$ be two functions and $\mathcal{B}_1(\alpha_1,\ell_1) = \mathcal{S}_2\{\mathcal{Q}_1(t,x)\}$ and $\mathcal{B}_2(\alpha_1,\ell_1) = \mathcal{S}_2\{\mathcal{Q}_2(t,x)\}$ be the DS transform of $\mathcal{Q}_1(t,x)$ and $\mathcal{Q}_2(t,x)$ respectively; then, the DS transform of $\mathcal{Q}_1(t,x)+\mathcal{Q}_2(t,x)$, written as $\mathcal{S}_2\{\mathcal{Q}_1(t,x)+\mathcal{Q}_2(t,x)\}$, is given by*

$$\mathcal{S}_2\{\mathcal{Q}_1(t,x)+\mathcal{Q}_2(t,x)\} = \mathcal{B}_1(\alpha_1,\ell_1)+\mathcal{B}_2(\alpha_1,\ell_1).$$

**Proof.** Since $\mathcal{B}_1(\alpha_1,\ell_1) = \mathcal{S}_2\{\mathcal{Q}_1(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}_1(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx$ and

$$\mathcal{B}_2(\alpha_1,\ell_1) = \mathcal{S}_2\{\mathcal{Q}_2(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}_2(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx,$$

then

$$\mathcal{S}_2\{\mathcal{Q}_1(t,x)+\mathcal{Q}_2(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty(\mathcal{Q}_1(t,x)+\mathcal{Q}_2(t,x))\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx$$

$$= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}_1(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx + \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}_2(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx$$

$$= \mathcal{B}_1(\alpha_1,\ell_1)+\mathcal{B}_2(\alpha_1,\ell_1).$$

□

**Theorem 3** (Scaling Property). *If $\mathcal{B}(\alpha_1,\ell_1)$ is the double DS transform of $\mathcal{Q}(t,x)$, then $kl\mathcal{B}(k\alpha_1,l\ell_1)$ is the DS transform of $\mathcal{Q}(kt,lx)$.*

**Proof.** Since $\mathcal{S}_2\{\mathcal{Q}(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx$, then

$$\mathcal{S}_2\{\mathcal{Q}(kt,lx)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(kt,lx)\exp(-\frac{kt}{\alpha_1}-\frac{lx}{\ell_1})dtdx.$$

Let $kt = p$ and $lx = q$ followed by $kdt = dp$ and $ldx = dq$; therefore,

$$\begin{aligned}\mathcal{S}_2\{\mathcal{Q}(kt,lx)\} &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(p,q)\exp(-\frac{p}{\alpha_1}-\frac{q}{\ell_1})\frac{dp}{k}\frac{dq}{l}\\ &= \frac{lk}{k^2l^2\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(p,q)\exp(-\frac{p}{\alpha_1}-\frac{q}{\ell_1})dpdq\\ &= lk\mathcal{B}(k\alpha_1,l\ell_1).\end{aligned}$$

Thus,

$$\mathcal{S}_2\{\mathcal{Q}(kt,lx)\} = lk\mathcal{B}(k\alpha_1,l\ell_1).$$

□

**Theorem 4** (Shifting Property)**.** *If $\mathcal{B}(\alpha_1,\ell_1)$ is the double DS transform of $\mathcal{Q}(t,x)$, then $\frac{1}{(1-k\alpha_1-l\ell_1)^2}\mathcal{B}(\frac{\alpha_1}{1-k\alpha_1},\frac{\ell_1}{1-l\ell_1})$ is the DS transform of* $\exp(kt+lx)\mathcal{Q}(t,x)$.

**Proof.** By definition $\mathcal{S}_2\{\mathcal{Q}(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx$; then,

$$\begin{aligned}\mathcal{S}_2\{\exp(kt,lx)\mathcal{Q}(t,x)\} &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(kt,lx)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx\\ &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp\left(-t(\frac{1}{\alpha_1}-k)-x(\frac{1}{\ell_1}-l)\right)dtdx\\ &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp\left(-t(\frac{1-k\alpha_1}{\alpha_1})-x(\frac{1-l\ell_1}{\ell_1})\right)dtdx\\ &= \frac{1}{(1-k\alpha_1)^2(1-l\ell_1)^2}\left[\frac{(1-k\alpha_1)^2(1-l\ell_1)^2}{\alpha_1^2\ell_1^2}\right.\\ &\quad\left.\times\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp\left(-t(\frac{1-k\alpha_1}{\alpha_1})-x(\frac{1-l\ell_1}{\ell_1})\right)dtdx\right]\\ &= \frac{1}{(1-k\alpha_1)^2(1-l\ell_1)^2}\mathcal{B}(\frac{\alpha_1}{1-k\alpha_1},\frac{\ell_1}{1-l\ell_1}).\end{aligned}$$

Hence,

$$\mathcal{S}_2\{\exp(kt,lx)\mathcal{Q}(t,x)\} = \frac{1}{(1-k\alpha_1)^2(1-l\ell_1)^2}\mathcal{B}(\frac{\alpha_1}{1-k\alpha_1},\frac{\ell_1}{1-l\ell_1}).$$

□

### *2.1. Relationship between the Double Laplace and the Double Sawi Transform*

The relationship between the double Laplace and the DS transform has been presented in the literature [17]. However, we recall it here for the ease and interest of the readers. By definition of the DS transform

$$\mathcal{B}(\alpha_1,\ell_1) = \mathcal{S}_2\{\mathcal{Q}(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx,$$

and we have $\mathcal{M}(\alpha_1, \ell_1) = \mathcal{L}_2\{\mathscr{Q}(t,x)\} = \int_0^\infty \int_0^\infty \mathscr{Q}(t,x)\exp(-\alpha_1 t - \ell_1 x)dtdx$, which is the double Laplace transform of $\mathscr{Q}(t,x)$; then, we can say

$$\mathcal{M}(\alpha_1, \ell_1) = \alpha_1^2 \ell_1^2 \mathcal{B}(\frac{1}{\alpha_1}, \frac{1}{\ell_1}).$$

Indeed,

$$\begin{aligned} \mathcal{L}_2\{\mathscr{Q}(t,x)\} &= \int_0^\infty \int_0^\infty \mathscr{Q}(t,x)\exp(-\alpha_1 t - \ell_1 x)dtdx \\ \mathcal{M}(\alpha_1, \ell_1) &= \frac{1}{\alpha_1^2 \ell_1^2}\left[\alpha_1^2 \ell_1^2 \int_0^\infty \int_0^\infty \mathscr{Q}(t,x)\exp(-\alpha_1 t - \ell_1 x)dtdx\right] \\ &= \alpha_1^2 \ell_1^2 \mathcal{B}(\alpha_1, \ell_1). \end{aligned}$$

### 2.2. Convolution Property of the DS Transform

The convolution of $\mathscr{Q}(t,x)$ and $\mathscr{V}(t,x)$ is expressed as $\mathscr{Q} * *\mathscr{V}(t,x)$ and is defined as

$$\mathscr{Q} * *\mathscr{V}(t,x) = \int_0^\infty \int_0^\infty \mathscr{Q}(t-\xi, x-\eta)\mathscr{V}(\xi,\eta)d\xi d\eta.$$

Some features of the convolution of the two functions are listed below:

- $\mathscr{Q} * *\mathscr{V}(t,x) = \mathscr{V} * *\mathscr{Q}(t,x)$ (Commutative);
- $[\mathscr{Q} * *\{\mathscr{V} * *\zeta\}](t,x) = [\{\mathscr{Q} * *\mathscr{V}\} * *\zeta](t,x)$ (Associative);
- $[\mathscr{Q} * *\{\mathscr{Q}\mathscr{V} + \Upsilon\zeta\}](t,x) = \mathscr{Q}\{\mathscr{Q} * *\mathscr{V}\}(t,x) + \Upsilon\{\mathscr{Q} * *\zeta\}(t,x)$ (Distributive);
- $\mathscr{Q} * *\circ(t,x) = \mathscr{Q}(t,x) = \circ * *\mathscr{Q}(t,x)$ (Identity), where $\circ(t,x)$ denotes the dirac delta function.

As the convolution of the DS transform satisfies the properties of a commutative semi group, it forms a commutative semi group.

**Theorem 5.** *If $\mathcal{S}_2\{\mathscr{Q}(t,x)\} = \mathcal{B}(\alpha_1, \ell_1)$, then $\mathcal{S}_2\{\mathscr{Q}(t-\xi, x-\eta)\mathcal{H}(t-\xi, x-\eta)\} = \exp(-\frac{\xi}{\alpha_1} - \frac{\eta}{\ell_1})\mathcal{B}(\alpha_1, \ell_1)$, where $\mathcal{H}(t,x)$ is the Heaviside unit step function and is expressed as $\mathcal{H}(t-a, x-b) = 1$, when $t > a$, $x > b$, and $\mathcal{H}(t-a, x-b) = 0$, when $t < a, x < b$.*

**Proof.** As $\mathcal{S}_2\{\mathscr{Q}(t,x)\} = \frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1} - \frac{x}{\ell_1})dtdx$, then

$$\begin{aligned} \mathcal{S}_2\{\mathscr{Q}(t-\xi, x-\eta)\mathcal{H}(t-\xi, x-\eta)\} &= \frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \mathscr{Q}(t-\xi, x-\eta) \\ &\quad \times \mathcal{H}(t-\xi, x-\eta)\exp(-\frac{t}{\alpha_1} - \frac{x}{\ell_1})dtdx \\ &= \frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \mathscr{Q}(t-\xi, x-\eta) \\ &\quad \exp(-\frac{t}{\alpha_1} - \frac{x}{\ell_1})dtdx, \text{ for } t > \xi, x > \eta. \end{aligned}$$

Let $t - \xi = \theta$ and $x - \eta = \vartheta$ followed by $dt = d\theta$ and $dx = d\vartheta$; so,

$$\begin{aligned} \mathcal{S}_2\{\mathscr{Q}(t-\xi, x-\eta)\mathcal{H}(t-\xi, x-\eta)\} &= \\ &\frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \exp\left(-\frac{1}{\alpha_1}(\theta+\xi) - \frac{1}{\ell_1}(\vartheta+\eta)\right)\mathscr{Q}(\theta,\vartheta)d\theta d\vartheta \\ &= \exp(-\frac{1}{\alpha_1}\xi - \frac{1}{\ell_1}\eta)\left[\frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \exp(-\frac{1}{\alpha_1}\theta - \frac{1}{\ell_1})\mathscr{Q}(\theta,\vartheta)d\theta d\vartheta\right] \\ &= \exp(-\frac{1}{\alpha_1}\xi - \frac{1}{\ell_1}\eta)\mathcal{B}(\alpha_1, \ell_1). \end{aligned}$$

We reach:

$$\mathcal{S}_2\{\mathscr{Q}(t-\xi,x-\eta)\mathcal{H}(t-\xi,x-\eta)\} = \exp(-\frac{1}{\alpha_1}\xi - \frac{1}{\ell_1}\eta)\mathcal{B}(\alpha_1,\ell_1).$$

□

**Theorem 6.** *Let $\mathscr{Q}(t,x)$ denote a periodic function whose period is $\mathcal{E},\Upsilon$(that is, $\mathscr{Q}(t+\mathcal{E},x+\Upsilon) = \mathscr{Q}(t,x)\ \forall$ t and x), and if $\mathcal{S}_2\{\mathscr{Q}(t,x)\}$ exists, then*

$$\mathcal{S}_2\{\mathscr{Q}(t,x)\} = 1-\exp(-\frac{\mathcal{E}}{\alpha_1}-\frac{\Upsilon}{\ell_1})^{-1}\left[\int_0^{\mathscr{Q}}\int_0^{\Upsilon}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx\right].$$

**Proof.** By definition

$$\begin{aligned}\mathcal{S}_2\{\mathscr{Q}(t,x);t\to\alpha_1,x\to\ell_1\} &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^{\infty}\int_0^{\infty}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx\\ &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^{\mathscr{Q}}\int_0^{\Upsilon}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx\\ &\quad+\frac{1}{\alpha_1^2\ell_1^2}\int_{\mathscr{Q}}^{\infty}\int_{\Upsilon}^{\infty}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx.\end{aligned}$$

Substituting $t=u+\mathcal{E}$ and $x=v+\Upsilon$ by $dt=du$ and $dx=dv$ in the second integral

$$\begin{aligned}\mathcal{S}_2\{\mathscr{Q}(t,x);t\to\alpha_1,x\to\ell_1\} &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^{\mathscr{Q}}\int_0^{\Upsilon}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx\\ &\quad+\frac{1}{\alpha_1^2\ell_1^2}\int_0^{\infty}\int_0^{\infty}\mathscr{Q}(u+\mathcal{E},v+\Upsilon)\exp\left(-\frac{1}{\alpha_1}(u+\mathcal{E})-\frac{1}{\ell_1}(v+\Upsilon)\right)dudv\\ &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^{\mathscr{Q}}\int_0^{\Upsilon}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx\\ &\quad+\frac{\exp(-\frac{\mathcal{E}}{\alpha_1}-\frac{\Upsilon}{\ell_1})}{\alpha_1^2\ell_1^2}\int_0^{\infty}\int_0^{\infty}\mathscr{Q}(u,v)\exp(-\frac{u}{\alpha_1}-\frac{v}{\ell_1})dudv\\ &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^{\mathscr{Q}}\int_0^{\Upsilon}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx+\exp(-\frac{\mathcal{E}}{\alpha_1}-\frac{\Upsilon}{\ell_1})\mathcal{B}(\alpha_1,\ell_1).\end{aligned}$$

Hence,

$$\mathcal{B}(\alpha_1,\ell_1) = 1-\exp(-\frac{\mathcal{E}}{\alpha_1}-\frac{\Upsilon}{\ell_1})^{-1}\left[\int_0^{\mathscr{Q}}\int_0^{\Upsilon}\mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx\right].$$

□

**Theorem 7** (Convolution Theorem). *If $\mathcal{S}_2\{\mathscr{Q}(t,x)\} = \mathcal{B}_1(\alpha_1,\ell_1)$ and $\mathcal{S}_2\{\mathscr{V}(t,x)\} = \mathcal{B}_2(\alpha_1,\ell_1)$, then*

$$\mathcal{S}_2\{\mathscr{Q}**\mathscr{V}(t,x)\} = \mathcal{S}_2\{\mathscr{Q}(t,x)\}\mathcal{S}_2\{\mathscr{V}(t,x)\} = \mathcal{B}_1(\alpha_1,\ell_1)\mathcal{B}_2(\alpha_1,\ell_1),$$

*or equivalently,*

$$\mathcal{S}_2^{-1}\{\mathcal{B}_1(\alpha_1,\ell_1)\mathcal{B}_2(\alpha_1,\ell_1)\} = \mathscr{Q}**\mathscr{V}(t,x).$$

**Proof.** Since $\mathcal{S}_2\{\mathscr{Q}**\mathscr{V}(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^{\infty}\int_0^{\infty}\mathscr{Q}**\mathscr{V}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx$, where

$$\mathscr{Q}**\mathscr{V}(t,x) = \int_0^{\infty}\int_0^{\infty}\mathscr{Q}(t-\xi,x-\eta)\mathscr{V}(\xi,\eta)d\xi d\eta,$$

then,

$$\mathcal{S}_2\{\mathcal{Q} * * \mathcal{V}(t,x)\} = \left[\frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})\int_0^\infty\int_0^\infty \mathcal{Q}(t-\xi,x-\eta)\mathcal{V}(\xi,\eta)d\xi d\eta\right]dtdx.$$

Using the Heaviside Unit Step function $\mathcal{H}(t-\xi,x-\eta)$, where $t>\xi, x>\eta$, we have

$$\begin{aligned}
\mathcal{S}_2\{\mathcal{Q} * * \mathcal{V}(t,x)\} &= \left[\frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})\right. \\
&\quad \left.\times\int_0^\infty\int_0^\infty \mathcal{Q}(t-\xi,x-\eta)\mathcal{H}(t-\xi,x-\eta)\mathcal{V}(\xi,\eta)d\xi d\eta dtdx\right] \\
&= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{V}(\xi,\eta)d\xi d\eta \\
&\quad \times\left[\int_0^\infty\int_0^\infty \exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})\mathcal{Q}(t-\xi,x-\eta)\mathcal{H}(t-\xi,x-\eta)dtdx\right] \\
&= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{V}(\xi,\eta)\exp(-\frac{1}{\alpha_1}\xi-\frac{1}{\ell_1}\eta)\mathcal{B}_1(\alpha_1,\ell_1)d\xi d\eta \\
&= \mathcal{B}_1(\alpha_1,\ell_1)\frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{V}(\xi,\eta)\exp(-\frac{1}{\alpha_1}\xi-\frac{1}{\ell_1}\eta)d\xi d\eta.
\end{aligned}$$

Thus,

$$\mathcal{S}_2\{\mathcal{Q} * * \mathcal{V}(t,x)\} = \mathcal{B}_1(\alpha_1,\ell_1)\mathcal{B}_2(\alpha_1,\ell_1).$$

□

### 2.3. Double Sawi Transform for Some Functions

Here, we discuss the double Sawi transform for the exponential function and partial derivatives. For other functions, we provide the double Sawi transform in Appendix A, Table A1.

(1) $\mathcal{S}_2\{\exp(-\mathcal{K}t-\Upsilon x)\mathcal{Q}(t,x)\} = \mathcal{B}(\mathcal{Q}+\alpha_1,\Upsilon+\ell_1)$.

**Proof.** As $\mathcal{S}_2\{\mathcal{Q}(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx$, then

$$\begin{aligned}
\mathcal{S}_2\{\exp(-\mathcal{K}t-\Upsilon x)\mathcal{Q}(t,x)\} &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \exp(-\mathcal{K}t-\Upsilon x)\mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx \\
&= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \exp-(\mathcal{Q}+\frac{1}{\alpha_1})t-(\Upsilon+\frac{1}{\ell_1})x\mathcal{Q}(t,x)dtdx \\
&= \frac{1}{(\mathcal{Q}+\alpha_1)^2(\Upsilon+\ell_1)^2}\int_0^\infty\int_0^\infty \exp-(\mathcal{Q}+\frac{1}{\alpha_1})t-(\Upsilon+\frac{1}{\ell_1})x\mathcal{Q}(t,x)dtdx.
\end{aligned}$$

So, we have

$$\mathcal{S}_2\{\exp(-\mathcal{K}t-\Upsilon x)\mathcal{Q}(t,x)\} = \mathcal{B}(\mathcal{Q}+\alpha_1,\Upsilon+\ell_1).$$

□

(2) $\mathcal{S}_2\{\frac{\partial\mathcal{Q}}{\partial t}\} = \frac{1}{\alpha_1}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\alpha_1^2}\mathcal{B}_1(\ell_1)\,,\mathcal{B}_1(\ell_1)=\mathcal{S}\{\mathcal{Q}(0,x)\}$.

**Proof.** Since

$$\mathcal{S}_2\{\mathcal{Q}(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx,$$

then

$$\begin{aligned}
\mathcal{S}_2\{\frac{\partial \mathscr{Q}}{\partial t}\} &= \frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \frac{\partial \mathscr{Q}}{\partial t}\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx \\
&= \frac{1}{\ell_1^2}\int_0^\infty \exp(-\frac{x}{\ell_1})dx * \frac{1}{\alpha_1^2}\int_0^\infty \frac{\partial \mathscr{Q}}{\partial t}\exp(-\frac{t}{\alpha_1})dt \\
&= \frac{1}{\ell_1^2}\int_0^\infty \exp(-\frac{x}{\ell_1})dx * \frac{1}{\alpha_1^2}[-\mathscr{Q}(0,x)+\frac{1}{\alpha_1}\int_0^\infty \exp(-\frac{t}{\alpha_1})\mathscr{Q}(t,x)dt] \\
&= \frac{1}{\alpha_1}\frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty \int_0^\infty \mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx - \frac{1}{\alpha_1^2}\frac{1}{\ell_1^2}\int_0^\infty \exp(-\frac{x}{\ell_1})\mathscr{Q}(0,x)dx.
\end{aligned}$$

Thus,

$$\mathcal{S}_2\{\frac{\partial \mathscr{Q}}{\partial t}\} = \frac{1}{\alpha_1}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\alpha_1^2}\mathcal{B}_1(\ell_1).$$

□

(3) $\mathcal{S}_2\{\frac{\partial \mathscr{Q}}{\partial x}\} = \frac{1}{\ell_1}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\ell_1^2}\mathcal{B}_2(\ell_1), \mathcal{B}_2(\alpha_1) = \mathcal{S}\{\mathscr{Q}(t,0)\}$.

**Proof.** As

$$\mathcal{S}_2\{\mathscr{Q}(t,x)\} = \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty \int_0^\infty \mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx,$$

then

$$\begin{aligned}
\mathcal{S}_2\{\frac{\partial \mathscr{Q}}{\partial_x}\} &= \frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \frac{\partial \mathscr{Q}}{\partial x}\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx \\
&= \frac{1}{\alpha_1^2}\int_0^\infty \exp(-\frac{t}{\alpha_1})dt * \frac{1}{\ell_1^2}\int_0^\infty \frac{\partial \mathscr{Q}}{\partial x}\exp(-\frac{x}{\ell_1})dx \\
&= \frac{1}{\alpha_1^2}\int_0^\infty \exp(-\frac{t}{\alpha_1})dt * \frac{1}{\ell_1^2}\left[-\mathscr{Q}(t,0)+\frac{1}{\ell_1}\int_0^\infty \exp(-\frac{t}{\alpha_1})\mathscr{Q}(t,x)dx\right] \\
&= \frac{1}{\ell_1}\frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty \int_0^\infty \mathscr{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx - \frac{1}{\ell_1^2}\frac{1}{\alpha_1^2}\int_0^\infty \exp(-\frac{t}{\alpha_1})\mathscr{Q}(t,0)dt.
\end{aligned}$$

Therefore,

$$\mathcal{S}_2\{\frac{\partial \mathscr{Q}}{\partial x}\} = \frac{1}{\ell_1}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\ell_1^2}\mathcal{B}_2(\alpha_1).$$

□

(4) $\mathcal{S}_2\{\frac{\partial^2 \mathscr{Q}}{\partial x^2}\} = \frac{1}{\ell_1^2}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\ell_1^2}\mathcal{B}_3(\alpha_1) - \frac{1}{\ell_1^3}\mathcal{B}_2(\alpha_1); \mathcal{B}_3(\alpha_1) = \mathcal{S}\{\mathscr{Q}_x(t,0)\}, \mathcal{B}_2(\alpha_1) = \mathcal{S}\{\mathscr{Q}(t,0)\}$.

**Proof.**

$$
\begin{aligned}
\mathcal{S}_2\{\frac{\partial^2\mathcal{Q}}{\partial x^2}\} &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \frac{\partial^2\mathcal{Q}}{\partial x^2}\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx \\
&= \frac{1}{\alpha_1^2}\int_0^\infty \exp(-\frac{t}{\alpha_1})dt[\frac{1}{\ell_{1^2}}\int_0^\infty \frac{\partial^2\mathcal{Q}}{\partial x^2}\exp(-\frac{x}{\ell_1})dx] \\
&= \frac{1}{\alpha_1^2}\int_0^\infty \exp(-\frac{t}{\alpha_1})dt[\frac{1}{\ell_1^2}(\exp(-\frac{x}{\ell_1})\frac{\partial\mathcal{Q}}{\partial x}\mid_0^\infty+\frac{1}{\ell_1}\int_0^\infty \frac{\partial\mathcal{Q}}{\partial x}\exp(-\frac{x}{\ell_1})dx)] \\
&= \frac{1}{\alpha_1^2}\int_0^\infty \exp(-\frac{t}{\alpha_1})dt[\frac{1}{\ell_1^2}(-\frac{\partial\mathcal{Q}(t,0)}{\partial x}+\frac{1}{\ell_1}\{-\mathcal{Q}(t,0)+\frac{1}{\ell_1}\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{x}{\ell_1})dx\})] \\
&= \frac{1}{\alpha_1^2\ell_1^4}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx-\frac{1}{\ell_1^2}\frac{1}{\alpha_1^2}\int_0^\infty \frac{\partial\mathcal{Q}(t,0)}{\partial x}\exp(-\frac{t}{\alpha_1})dt \\
&- \frac{1}{\ell_1^3}\frac{1}{\alpha_1^2}\int_0^\infty \mathcal{Q}(t,0)\exp(-\frac{t}{\alpha_1})dt.
\end{aligned}
$$

Thus,

$$
\mathcal{S}_2\{\frac{\partial^2\mathcal{Q}}{\partial x^2}\} = \frac{1}{\ell_1^2}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\ell_1^2}\mathcal{B}_3(\alpha_1)-\frac{1}{\ell_1^3}\mathcal{B}_2(\alpha_1).
$$

□

(5) $\mathcal{S}_2\{\frac{\partial^2\mathcal{Q}}{\partial t^2}\} = \frac{1}{\alpha_1^2}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\alpha_1^2}\mathcal{B}_4(\ell_1)-\frac{1}{\ell_1^3}\mathcal{B}_1(\ell_1); \mathcal{B}_4(\ell_1)=\mathcal{S}\{\mathcal{Q}_x(0,x)\}, \mathcal{B}_1(\ell_1)=\mathcal{S}\{\mathcal{Q}(0,x)\}$.

**Proof.** Consider

$$
\begin{aligned}
\mathcal{S}_2\{\frac{\partial^2\mathcal{Q}}{\partial t^2}\} &= \frac{1}{\alpha_1^2\ell_1^2}\int_0^\infty\int_0^\infty \frac{\partial^2\mathcal{Q}}{\partial t^2}\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx \\
&= \frac{1}{\ell_1^2}\int_0^\infty \exp(-\frac{x}{\ell_1})dx\left[\frac{1}{\alpha_1^2}\int_0^\infty \frac{\partial^2\mathcal{Q}}{\partial t^2}\exp(-\frac{t}{\alpha_1})dt\right] \\
&= \frac{1}{\ell_1^2}\int_0^\infty \exp(-\frac{x}{\ell_1})dx\left[\frac{1}{\alpha_1^2}(-\frac{\partial\mathcal{Q}(0,x)}{\partial t}+\frac{1}{\alpha_1}\{-\mathcal{Q}(0,x)+\frac{1}{\alpha_1}\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1})dt\})\right] \\
&= \frac{1}{\alpha_1^2\ell_1^4}\int_0^\infty\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx-\frac{1}{\alpha_1^2}\frac{1}{\ell_1^2}\int_0^\infty \frac{\partial\mathcal{Q}(0,x)}{\partial t}\exp(-\frac{x}{\ell_1})dx \\
&-\frac{1}{\alpha_1^3}\frac{1}{\ell_1^2}\int_0^\infty \mathcal{Q}(0,x)\exp(-\frac{x}{\ell_1})dx.
\end{aligned}
$$

So, we have

$$
\Rightarrow \mathcal{S}_2\{\frac{\partial^2\mathcal{Q}}{\partial t^2}\} = \frac{1}{\alpha_1^2}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\alpha_1^2}\mathcal{B}_4(\ell_1)-\frac{1}{\ell_1^3}\mathcal{B}_1(\ell_1).
$$

□

(6) $\mathcal{S}_2\{\frac{\partial\mathcal{Q}}{\partial x\partial t}\} = \frac{1}{\alpha_1\ell_1}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\alpha_1^2\ell_1}\mathcal{B}_1(\ell_1)-\frac{1}{\alpha_1\ell_1^2}\mathcal{B}_2(\alpha_1)+\frac{1}{\ell_1^2}\mathcal{B}(0,0)$.

**Proof.** Consider

$$\begin{aligned}
\mathcal{S}_2\{\frac{\partial \mathcal{Q}}{\partial x \partial t}\} &= \frac{1}{\alpha_1^2 \ell_1^2}\int_0^\infty \int_0^\infty \frac{\partial \mathcal{Q}}{\partial x \partial t}\exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})dtdx \\
&= \frac{1}{\alpha_1^2 \ell_1^2}\left[\int_0^\infty \exp(-\frac{t}{\alpha_1})dt\{\int_0^\infty \exp(-\frac{x}{\ell_1})\frac{\partial}{\partial x}(\frac{\partial \mathcal{Q}}{\partial t})dx\}\right] \\
&= \frac{1}{\alpha_1^2 \ell_1^2}\left[\int_0^\infty \exp(-\frac{t}{\alpha_1})dt\{\exp(-\frac{x}{\ell_1})\frac{\partial \mathcal{Q}}{\partial t}\mid_0^\infty + \frac{1}{\ell_1}\int_0^\infty \exp(-\frac{x}{\ell_1})\frac{\partial \mathcal{Q}}{\partial t}dx\}\right] \\
&= \frac{1}{\alpha_1^2 \ell_1^2}[\int_0^\infty \exp(-\frac{t}{\alpha_1})\frac{\partial \mathcal{Q}(t,0)}{\partial t}dt + \frac{1}{\ell_1}\int_0^\infty \int_0^\infty \exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})\frac{\partial \mathcal{Q}}{\partial t}dtdx \\
&= \frac{1}{\alpha_1^2 \ell_1^2}\left[-(\frac{1}{\alpha_1}\mathcal{B}_2(\alpha_1)-\mathcal{B}(0,0)) + \frac{1}{\ell_1}\{\int_0^\infty \exp(-\frac{x}{\ell_1})dx(\exp(-\frac{t}{\alpha_1})\mathcal{Q}(t,x)\mid_0^\infty\right. \\
&\left. +\frac{1}{\alpha_1}\int_0^\infty \mathcal{Q}(t,x)\exp(-\frac{t}{\alpha_1})dt)\}\right] \\
&= \frac{1}{\alpha_1^2 \ell_1^2}[-(\frac{1}{\alpha_1}\mathcal{B}_2(\alpha_1)-\mathcal{B}(0,0)) - \frac{1}{\ell_1}\int_0^\infty \exp(-\frac{x}{\ell_1})\mathcal{Q}(0,x)dx \\
&+\frac{1}{\alpha_1 \ell_1}\int_0^\infty \int_0^\infty \exp(-\frac{t}{\alpha_1}-\frac{x}{\ell_1})\mathcal{Q}(t,x)dtdx \\
&= \frac{1}{\alpha_1 \ell_1}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\ell_1^2}(\frac{1}{\alpha_1}\mathcal{B}_2(\alpha_1)-\mathcal{B}(0,0)) - \frac{1}{\alpha_1^2 \ell_1}\mathcal{B}_1(\ell_1).
\end{aligned}$$

Hence,

$$\mathcal{S}_2\{\frac{\partial \mathcal{Q}}{\partial x \partial t}\} = \frac{1}{\alpha_1 \ell_1}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\alpha_1^2 \ell_1}\mathcal{B}_1(\ell_1) - \frac{1}{\alpha_1 \ell_1^2}\mathcal{B}_2(\alpha_1) + \frac{1}{\ell_1^2}\mathcal{B}(0,0).$$

□

## 3. Applications to Some Boundary Value Problems

In this section, the applications of the double Sawi transform are discussed. We solve some boundary values problems [16] by using the DS transform to show the applicability of the proposed transform.

**Example 1.** *Consider a BVP as:*

$$\mathcal{Q}\mathcal{U}_t + \Upsilon \mathcal{U}_x = 0, \tag{1}$$

$$\mathcal{U}(t,0) = \mathcal{Q}(t), t > 0; \mathcal{U}(0,x) = 0, x > 0.$$

*Taking the DS transform of (1), we have*

$$\begin{aligned}
\mathcal{Q}\left[\frac{1}{\alpha_1}[\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\alpha_1^2}\mathcal{B}_1(\ell_1)] + \Upsilon[\frac{1}{\ell_1}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\ell_1^2}\mathcal{B}_2(\alpha_1)]\right] &= 0 \\
\left(\frac{\mathcal{Q}}{\alpha_1}+\frac{\Upsilon}{\ell_1}\right)\mathcal{B}(\alpha_1,\ell_1) &= \frac{\Upsilon}{\ell_1^2}\mathcal{B}_2(\alpha_1) \\
&= \frac{\Upsilon \alpha_1}{\ell_1}\frac{\mathcal{Q}_s(\alpha_1)}{(\mathcal{Q}\ell_1 + \Upsilon \alpha_1)} \\
\mathcal{B}(\alpha_1,\ell_1) &= \frac{\mathcal{Q}_s(\alpha_1)}{\ell_1(1+\frac{\mathcal{Q}}{\Upsilon \alpha_1}\ell_1)}. \qquad (2)
\end{aligned}$$

*Taking the double inverse Sawi transform of (2) with respect to $\ell_1$ gives us*

$$\mathcal{B}(\alpha_1, x) = \mathcal{Q}_s(\alpha_1)\exp(-\frac{\mathcal{Q}}{\Upsilon\alpha_1}x). \tag{3}$$

*Now, using the Convolution theorem and taking the double inverse Sawi transform of (3) with respect to $\alpha_1$, we obtain*

$$\begin{aligned}
\mathcal{U}(t,x) &= \mathcal{Q}(t) * \mathcal{V}(t - \frac{\mathcal{Q}x}{\Upsilon\alpha_1}) \\
\mathcal{U}(t,x) &= \int_0^x \mathcal{Q}(t-\gamma)\mathcal{V}(\gamma - \frac{\mathcal{Q}x}{\Upsilon\alpha_1})d\gamma.
\end{aligned}$$

*Thus,*

$$\mathcal{U}(t,x) = \mathcal{Q}(t - \frac{\mathcal{Q}x}{\Upsilon\alpha_1}).$$

**Example 2.** *Consider a Fourier heat equation as:*

$$\mathcal{U}_x = \mathcal{Q}\mathcal{U}_{tt}; t > 0, x > 0, \tag{4}$$

$$\mathcal{U}(t,0) = 0, \mathcal{U}(0,x) = 2\mathcal{B}_o, \mathcal{U}_t(0,x) = 0, \mathcal{U}(t,x) \to 0 \text{ as } t \to \infty.$$

*Taking the DS transform of (4), we obtain*

$$\begin{aligned}
\frac{1}{\ell_1}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\ell_1^2}\mathcal{B}_2(\alpha_1) &= \mathcal{Q}\left[\frac{1}{\alpha_1^2}\mathcal{B}(\alpha_1,\ell_1) - \frac{1}{\alpha_1^2}\mathcal{B}_4(\ell_1) - \frac{1}{\alpha_1^3}\mathcal{B}_1(\ell_1)\right] \\
\left(\frac{1}{\ell_1} - \frac{\mathcal{Q}}{\alpha_1^2}\right)\mathcal{B}(\alpha_1,\ell_1) &= -\mathcal{Q}\frac{2\mathcal{B}_o}{\alpha_1^3} \\
\mathcal{B}(\alpha_1,\ell_1) &= \frac{2\mathcal{B}_o}{\alpha_1^3}\frac{\ell_1\alpha_1^2\mathcal{Q}}{(\mathcal{Q}\ell_1 - \alpha_1^2)} \\
\mathcal{B}(\alpha_1,\ell_1) &= 2\mathcal{B}_o\frac{2}{\alpha_1(1 - \frac{\alpha_1^2}{\mathcal{Q}\ell_1})}.
\end{aligned} \tag{5}$$

*Taking the inverse DS transform of (5),*

$$\mathcal{U}(t,x) = 2\mathcal{B}_o\mathcal{S}^{-1}\{\cosh(\frac{1}{\sqrt{\mathcal{Q}\ell_1}}t)\}$$

$$\mathcal{U}(t,x) = \mathcal{B}_o\mathcal{S}^{-1}\{\exp(\frac{1}{\sqrt{\mathcal{Q}\ell_1}}t) + \exp(-\frac{1}{\sqrt{\mathcal{Q}\ell_1}}t)\}.$$

*The first term vanishes as $\mathcal{U}(t,x) \to 0, t \to \infty$,*

$$\mathcal{U}(t,x) = \mathcal{B}_o\mathcal{S}^{-1}\{\exp(-\frac{1}{\sqrt{\mathcal{Q}\ell_1}}t)\}$$

$$\mathcal{U}(t,x) = \mathcal{B}_o erfc(\frac{1}{2\sqrt{\mathcal{Q}\ell_1}}t).$$

**Example 3.** *Consider a D'Alembert's wave equation in a quarter plane as:*

$$\mathcal{C}^2\mathcal{U}_{tt} = \mathcal{U}_{xx}; t \geq 0, x > 0, \tag{6}$$

$$\mathcal{U}(t,0) = \mathcal{Q}(t), \mathcal{U}_x(t,0) = \mathcal{V}(t), \mathcal{U}(0,x) = 0, \mathcal{U}_t(0,x) = 0.$$

*Taking the DS transform of (6), we obtain*

$$
\begin{aligned}
\mathcal{C}^2\left[\frac{1}{\alpha_1^2}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\alpha_1^2}\mathcal{B}_4(\ell_1)-\frac{1}{\alpha_1^3}\mathcal{B}_1(\ell_1)\right] &= \frac{1}{\ell_1^2}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\ell_1^2}\mathcal{B}_3(\alpha_1)-\frac{1}{\ell_1^3}\mathcal{B}_2(\alpha_1)\\
\mathcal{C}^2\frac{1}{\alpha_1^2}\mathcal{B}(\alpha_1,\ell_1) &= \frac{1}{\ell_1^2}\mathcal{B}(\alpha_1,\ell_1)-\frac{1}{\ell_1^2}\mathscr{V}_s(\alpha_1)-\frac{1}{\ell_1^3}\mathscr{Q}_s(\alpha_1)\\
\mathcal{B}(\alpha_1,\ell_1) &= \frac{\alpha_1^2}{\ell_1(\alpha_1^2-\mathcal{C}^2\ell_1^2)}(\ell_1\mathscr{V}_s(\alpha_1)+\mathscr{Q}_s(\alpha_1))\\
\mathcal{B}(\alpha_1,\ell_1) &= \frac{1}{(1-(\frac{\mathcal{C}}{\alpha_1})^2\ell_1^2)}\mathscr{V}_s(\alpha_1)+\frac{1}{\ell_1(1-(\frac{\mathcal{C}}{\alpha_1})^2\ell_1^2)}\mathscr{Q}_s(\alpha_1).
\end{aligned}
\tag{7}
$$

*Taking the inverse DS transform of (7) with respect to $\ell_1$, we have*

$$
\begin{aligned}
\mathcal{U}(t,x) &= \frac{1}{2\pi\iota\mathcal{C}}\int_{\mathcal{C}-\iota\infty}^{\mathcal{C}+\iota\infty}\exp(\frac{t}{\alpha_1})\alpha_1\sinh(\frac{\alpha_1}{\mathcal{C}}x)\mathscr{V}_s(\alpha_1)d\alpha_1\\
&\quad+\frac{1}{2\pi\iota\mathcal{C}}\int_{\mathcal{C}-\iota\infty}^{\mathcal{C}+\iota\infty}\exp(\frac{t}{\alpha_1})\cosh(\frac{\alpha_1}{\mathcal{C}}x)\mathscr{Q}_s(\alpha_1)d\alpha_1\\
&= \frac{1}{2\mathcal{C}}\left[\mathcal{S}^{-1}\{\alpha_1\mathscr{V}_s(\alpha_1)(\exp(\frac{\mathcal{C}}{\alpha_1}x)-\exp(-\frac{\mathcal{C}}{\alpha_1}x))\}\right]\\
&\quad+\frac{1}{2}\mathcal{S}^{-1}\{\mathscr{Q}_s(\alpha_1)\exp(\frac{\mathcal{C}}{\alpha_1}x)+\mathscr{Q}_s(\alpha_1)\exp(-\frac{\mathcal{C}}{\alpha_1}x)\}\\
&= \frac{1}{2\mathcal{C}}[\int_0^{t+\mathcal{C}x}\mathscr{V}(\theta)d\theta-\int_0^{t-\mathcal{C}x}\mathscr{V}(\theta)d\theta]+\frac{1}{2}[\mathscr{Q}(t+\mathcal{C}x)+\mathscr{Q}(t-\mathcal{C}x)].
\end{aligned}
$$

*Thus,*

$$
\mathcal{U}(t,x)=\frac{1}{2}[\mathscr{Q}(t+\mathcal{C}x)+\mathscr{Q}(t-\mathcal{C}x)]+\frac{1}{2\mathcal{C}}\int_{t-\mathcal{C}x}^{t+\mathcal{C}x}\mathscr{V}(\theta)d\theta.
$$

## 4. Conclusions and Future Work

Partial differential equations (PDEs) have received tremendous attention from researchers due to their widespread applications in modeling phenomena that depend on more than one variable. However, it is well known that solving PDEs along with boundary conditions using single-valued integral transforms can be quite challenging. In recognition of this difficulty, researchers have sought to develop multi-valued integral transforms as an alternative solution approach. By extending the range of values that can be transformed, these multi-valued transforms can provide more efficient and effective solutions to PDEs and other complex mathematical problems. In this article, the focus is on extending a single-valued Sawi transform to a double-valued DS transform, which is referred to as the DS transform. The proposed transform has been thoroughly explored, with some fundamental features and theorems derived. Furthermore, the article delves into the DS transform of some functions and its relation with the double Laplace transform, providing readers with a deeper understanding of the topic. To better illustrate the practical applications of the proposed transform, the article presents some boundary value problems, including the Fourier heat equation and the D'Alembert wave equation, that have been solved using this method. For those interested in further exploration, the Appendix includes the DS transform of several basic functions, providing additional examples and insights.

Currently, fractional calculus has the focus of researchers due to its vast applications in mathematical physics [18,19], bifurcation [20–22], and many more [23–26]. In the future, we will try to solve fractional order boundary values using the double Sawi transform.

**Author Contributions:** Methodology, S.A.; Formal analysis, A.U.; Writing-original draft, S.K. and S.A.; Visualization, M.D.l.S.; Supervision, A.U.; Funding acquisition, M.D.l.S. All authors have read and agreed to the published version of the manuscript.

**Funding:** The APC is supported by the Basque Government, Grant IT1555-22.

**Data Availability Statement:** Not Applicable.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Appendix A

**Table A1.** The Sawi transforms of some functions.

| S. NO | $\mathcal{Q}(t,x)$ | Double Sawi Transform | $\mathcal{Q}(t)$ | Double Sawi Transform |
|---|---|---|---|---|
| 1 | 1 | $\frac{1}{\alpha_1\ell_1}$ | $t$ | $\frac{1}{\ell_1}$ |
| 2 | $tx$ | 1 | $x$ | $\frac{1}{\alpha_1}$ |
| 3 | $(tx)^n$ | $(n!)^2(\alpha_1\ell_1)^{n-1}$ | $t^2$ | $\frac{2\alpha_1}{\ell_1}$ |
| 4 | $t^m x^n$ | $m!n!(\alpha_1)^{m+1}(\ell_1)^{n+1}$ | $t^n$ | $\frac{n!(\alpha_1)^{n-1}}{\ell_1}$ |
| 5 | $x^{\mathcal{Q}}t^{\Upsilon}; \mathcal{Q} > -1, \Upsilon > -1$ | $\Gamma(\mathcal{Q}+1)(\alpha_1)^{\mathcal{Q}-1}\Gamma(\Upsilon+1)(\ell_1)^{\Upsilon-1}$ | $x^n$ | $\frac{n!(\ell_1)^{n-1}}{\alpha_1}$ |
| 6 | $\exp(\mathcal{K}t+\Upsilon x)$ | $\frac{1}{\alpha_1(1-\mathcal{Q}\alpha_1)}\frac{1}{\ell_1(1-\Upsilon\ell_1)}$ | $\exp(\mathcal{K}t)$ | $\frac{1}{\alpha_1\ell_1(1-\mathcal{Q}\alpha_1)}$ |
| 7 | $\exp\{\iota(\mathcal{K}t+\Upsilon x)\}$ | $\frac{(1-\mathcal{Q}\Upsilon\alpha_1\ell_1)+\iota(\mathcal{Q}\alpha_1+\Upsilon\ell_1)}{\alpha_1\ell_1(1+\alpha_1^2\alpha_1^2)(1+\Upsilon^2\ell_1^2)}$ | $\sin(\mathcal{K}t)$ | $\frac{\mathcal{Q}}{\ell_1}[1-\frac{\alpha_1^2\alpha_1^2}{(1+\alpha_1^2\alpha_1^2)}]$ |
| 8 | $\cos(\mathcal{K}t+\Upsilon x)$ | $\frac{(1-\mathcal{Q}\Upsilon\alpha_1\ell_1)}{\alpha_1\ell_1(1+\alpha_1^2\alpha_1^2)(1+\Upsilon^2\ell_1^2)}$ | $\sin(\Upsilon x)$ | $\frac{\Upsilon}{\alpha_1}[1-\frac{\Upsilon^2\ell_1^2}{(1+\Upsilon^2\ell_1^2)}]$ |
| 9 | $\sin(\mathcal{K}t+\Upsilon x)$ | $\frac{(\mathcal{Q}\alpha_1+\Upsilon\ell_1)}{\alpha_1\ell_1(1+\alpha_1^2\alpha_1^2)(1+\Upsilon^2\ell_1^2}$ | $\cos(\mathcal{K}t)$ | $\frac{1}{\ell_1}[\frac{1}{\alpha_1}-\frac{\alpha_1^2\alpha_1^2}{\alpha_1(1+\alpha_1^2\alpha_1^2)}]$ |
| 10 | $\cosh(\mathcal{K}t+\Upsilon x)$ | $\frac{1}{2}[\frac{1}{\alpha_1\ell_1(1-\mathcal{Q}\alpha_1)(1-\Upsilon\ell_1)}+\frac{1}{\alpha_1\ell_1(1+\mathcal{Q}\alpha_1)(1+\Upsilon\ell_1)}]$ | $\cos(\Upsilon x)$ | $\frac{1}{\alpha_1}[\frac{1}{\ell_1}-\frac{\Upsilon^2\ell_1^2}{\ell_1(1+\Upsilon^2\ell_1^2)}]$ |
| 11 | $\sinh(\mathcal{K}t+\Upsilon x)$ | $\frac{1}{2}[\frac{1}{\alpha_1\ell_1(1-\mathcal{Q}\alpha_1)(1-\Upsilon\ell_1)}-\frac{1}{\alpha_1\ell_1(1+\mathcal{Q}\alpha_1)(1+\Upsilon\ell_1)}]$ | $\sinh(\mathcal{K}t)$ | $\frac{1}{\ell_1}[\mathcal{Q}+\frac{\alpha_1^2\alpha_1^2}{\alpha_1(1-\alpha_1^2\alpha_1^2)}]$ |
| 12 | $\exp-(\mathcal{K}t+\Upsilon x)\mathcal{Q}(t,x)$ | $\mathcal{B}(\mathcal{Q}+\alpha_1,\Upsilon+\ell_1)$ | $\sinh(\Upsilon x)$ | $\frac{1}{\alpha_1}[\Upsilon+\frac{\Upsilon^2\ell_1^2}{\ell_1(1-\Upsilon^2\ell_1^2)}]$ |

## References

1. Debnath, L.; Bhatta, D. *Integral Transforms and Their Applications*, 3rd ed.; CRC Press, Chapman & Hall: Boca Raton, FL, USA, 2015.
2. Schiff, J.L. *The Laplace Transforms*; Springer: New York, NY, USA, 1999.
3. Aggarwal, S.; Gupta, A.R.; Singh, D.P.; Asthana, N.; Kumar, N. Application of Laplace transform for solving population growth and decay problems. *Int. J. Latest Technol. Eng. Manag. Appl. Sci.* **2018**, *7*, 141–145.
4. Sneddon, I.N. *Fourier Transforms*; Mc Graw Hill: New York, NY, USA, 1951.
5. Aggarwal, S.; Pandey, M.; Asthana, N.; Singh, D.P.; Kumar, A. Application of Mahgoub transform for solving population growth and decay problems. *J. Comput. Math. Sci.* **2018**, *9*, 1490–1496. [CrossRef]
6. Aggarwal, S.; Asthana, N.; Singh, D.P. Solution of population growth and decay problems by using Aboodh transform method. *Int. J. Res. Advent. Technol.* **2018**, *6*, 2706–2710.
7. Aggarwal, S.; Singh, D.P.; Asthana, N.; Gupta, A.R. Application of Elzaki transform for solving population growth and decay problems. *J. Emerg. Technol. Innov. Res.* **2018**, *5*, 281–284.
8. Gupta, A.R. Solution of Abel's integral equation using Mahgoub transform method. *J. Emerg. Technol. Innov. Res.* **2019**, *6*, 252–260.
9. Aggarwal, S.; Chauhan, R. A comparative study of Mohand and Aboodh transforms. *Int. J. Res. Advent. Technol.* **2019**, *7*, 520–529. [CrossRef]
10. Aggarwal, S.; Sharma, S.D. A comparative study of Mohand and Sumudu transforms. *J. Emerg. Technol. Innov. Res.* **2019**, *6*, 145–153.
11. Aggarwal, S. A comparative study of Mohand and Mahgoub transforms. *J. Adv. Res. Appl. Math. Statist.* **2019**, *4*, 1–7.
12. Abdelrahim Mahgoub, M.M. The new integral transform Sawi transform. *Adv. Theor. Appl. Math.* **2019**, *14*, 81–87.

13. Singh, G.P.; Aggarwal, S. Sawi transform for population growth and decay problems. *Int. J. Latest Technol. Eng. Manag. Appl. Sci.* **2019**, *8*, 157–162.
14. Aggarwal, S.; Gupta, A.R. Dualities between some useful integral transforms and Sawi transform. *Int. J. Recent. Technol. Eng.* **2019**, *8*, 5978–5982. [CrossRef]
15. Higazy, M.; Aggarwal, S. Sawi transformation for system of ordinary differential equations with application. *Ain Shams Eng. J.* **2021**, *12*, 3173–3182. [CrossRef]
16. Debnath, L. The double Laplace transforms and their properties with appplications to functional, integral and partial differential equations. *Int. J. Appl. Comput. Math.* **2015**, *2*, 1–19.
17. Patil, D.P. Dualities Between Double Integral Transforms. *Int. Adv. Res. J. Sci. Eng. Technol.* **2020**, *7*, 57–64.
18. Khan, A.; Ali. A; Ahmad, S; Saifullah, S.; Nonlaopon, K.; Akgül, A. Nonlinear Schrödinger equation under non-singular fractional operators: A computational study. *Result Phys.* **2022**, *43*, 106062. [CrossRef]
19. Ali, S.; Ullah, A.; Ahmad, S.; Nonlaopon, K.; Akgül, A. Analysis of Kink Behaviour of KdV-mKdV Equation under Caputo Fractional Operator with Non-Singular Kernel. *Symmetry* **2022**, *14*, 2316. [CrossRef]
20. Jiang, X.; Li, J.; Li; B., Yin, W.; Sun, L.; Chen, X. Bifurcation, chaos, and circuit realisation of a new four-dimensional memristor system. *Int. J. Nonlinear Sci. Numer. Simul.* **2022**. [CrossRef]
21. Li, B.; Eskandari, Z.; Avazzadeh, Z. Strong resonance bifurcations for a discrete-time prey-predator model. *J. Appl. Math. Comput.* **2023**. [CrossRef]
22. Li, B.; Eskandari, Z.; Avazzadeh, Z. Dynamical Behaviors of an SIR Epidemic Model with Discrete Time. *Fractal Fract.* **2022**, *6*, 659. [CrossRef]
23. Xu, C.; Zhang, W.; Aouiti, C.; Liu, Z.; Yao, L. Bifurcation insight for a fractional-order stage-structured predator- prey system incorporating mixed time delays. *Math. Methods Appl. Sci.* **2023**. [CrossRef]
24. Xu, C.; Mu, D.; Liu, Z.; Pang, Y.; Liao, M.; Aouiti, C. New insight into bifurcation of fractional-order 4D neural networks incorporating two different time delays. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *118*, 107043. [CrossRef]
25. Xu, C.; Liu, Z.; Li, P.; Yan, J.; Yao, L. Bifurcation mechanism for fractional-order three-triangle multi-delayed neural networks. *Neural Process. Lett*. **2022**. [CrossRef]
26. Xu, C.; Liao, M.; Li, P.; Guo, Y.; Liu, Z. Bifurcation properties for fractional order delayed BAM neural networks. *Cogn. Comput.* **2021**, *13*, 322–356. [CrossRef]