The Role of Human Capital Accumulation in Long-Term Macroeconomic Trends

Ricardo M. López Busch

Doctorado en Economía:
Instrumentos del Análisis Económico

Directora: Ilaski Barañano Mentxaka

Departmento de Análisis Económico
Facultad de Economía y Empresa
Universidad del País Vasco / Euskal Herriko Unibertsitatea
Febrero de 2023

eman ta zabal zazu

Universidad del País Vasco | Euskal Herriko Unibertsitatea

The Role of Human Capital Accumulation in Long-Term Macroeconomic Trends

Ricardo M. López Busch

Doctoral Programme in Economics:
Tools of Economic Analysis

Supervisor: Ilaski Barañano Mentxaka

Department of Economic Analysis
Faculty of Business and Economics
University of the Basque Country
February 2023

eman ta zabal zazu

UPV EHU

UC UNIVERSIDAD DE CANTABRIA

Universidad de Oviedo

Agradecimientos

Cada persona debe recorrer su propio camino, pero por donde se empieza depende de su familia. Es por ello que la primera muestra de gratitud debe ser a mis padres, a mis tíos y a mis abuelos, por dejarme lo suficientemente cerca de mi destino como para llegar, y darme ese pequeño empujón extra cuando el camino parecía demasiado largo. Mi camino hasta aquí podría haber sido mucho más triste y lúgubre sin los amigos que me acompañan, por ello le doy las gracias a los que conozco desde la infancia, a los que conozco desde la carrera y a los que conocí como doctorando. Por último, pero no por ello menos importante, quiero mencionar a la persona que comparte la vida conmigo, porque con ella soy mi versión más feliz.

Obviamente, debo agradecerle a Ilaski Barañano por darme la oportunidad de trabajar con ella en esta tesis doctoral, sin su esfuerzo y paciencia no hubiera sido posible. También cabe mencionar la valiosa ayuda de María Paz Moral, quien nos ha ayudado a navegar el mar de incertidumbre que es realizar y defender una tesis doctoral. Otro agradecimiento debe ser para Peter Funk y demás miembros del Centro de Investigación Macroeconómica de la Universidad de Colonia por acogerme y enseñarme tanto. Por el camino me he encontrado a excelentes académicos como Jaime Alonso y Amaia Iza, que me han ayudado enormemente a perfilar y pulir las ideas presentes en esta tesis. Lo mismo se puede decir de los numerosos académicos que me he encontrado en los diferentes congresos, seminarios y workshops en los que presenté las ideas contenidas en esta tesis.

La realización de esta tesis doctoral ha sido posible gracias a la financiación recibida por el Departamento de Educación, Política Lingüística y Cultura del Gobierno Vasco (IT869-13), y por el proyecto de investigación PID2019-106146GB-I00 financiado por MCIN/AEI/10.13039/501100011033 y por "ERDF A way of making Europe".

Contents

Summary

The aim of this doctoral thesis is to broaden our understanding about the role of human capital accumulation in long-term macroeconomic trends. In the first two chapters, I investigate why parents reallocate time from work to their children's education despite the decline in fertility rates. The remaining two chapters of the thesis are devoted to investigate how individual's preferences can shape the incentives to accumulate human capital, thereby affecting the process of long-run economic growth. Although this is an eminently theoretical thesis, all the mechanisms discussed here are based on empirically tested hypotheses. It therefore contributes to the literature by extending the scope of analysis to issues that have not been addressed before, serving as a guide for future empirical research.

The first chapter of the thesis entitled "Child Survival and the Decline in Hours Worked" builds a model that depicts the interplay between child mortality factors, fertility, and working hours. In it, parents are aware that they partially control their children's chances of survival through the expenditure on food, medicines, etc. This assumption leads to the plausible prediction that having a child increases the supply of labour along the intensive margin because it stimulates the demand for goods. By contrast, consistent with the empirical evidence, improvements in child mortality encourage parents to raise fewer children and invest more in their education. The combination of these two mechanisms implies that demographic transition can cause trends in the use of time as those observed since the Industrial Revolution, explaining the co-movement between fertility and working hours.

Recently, it has been empirically proven that changes in fertility norms alter parents' decisions about the quantity and quality of children as theory suggests. In the second chapter of the thesis, "Fertility Norms and Parents' Allocation of Time", I develop a model to show that such norms may also affect the allocation of time. In it, the social penalty caused by fertility norms discourages parents from choosing smaller families when the marginal cost of rising a child increases, which I term as "rigidities in child quantity". As a result of such rigidities, parents react to new economic conditions by altering their labour supply along the intensive

margin (working hours) to compensate for the increased child-rearing cost, giving rise to trends in the use of time consistent with the empirical evidence.

The third chapter of this thesis investigates fertility and education choices when parents value their children's quality or human capital relative to their own and that of others, which is called *child quality externalities*. Within an endogenous growth model with quantity-quality trade-offs, I show that such externalities might cause economic growth and demographic changes. This is because they induce households to compete in child quality, pushing the educational spend upwards, and thereby leading to sustained human capital accumulation. The associated increase in child-rearing costs (educational spend) in turn encourage parents to choose smaller families because having a child becomes more expensive. Over the long-run, the existence of child quality externalities drives educational levels and fertility rates in opposite directions, causing economic growth

In the last chapter of the thesis entitled "Human Capital Formation via Investment and LBD: Low vs Modern Growth Dynamics", I study the incentives to invest time in human capital accumulation if individuals already build skills via learning by doing (LBD). In contrast to the well-known Uzawa-Lucas growth model, the existence of LBD ensures sustained human capital accumulation and economic growth regardless of technology, preferences, or initial conditions. Investment in human accumulation, however, makes the difference between growing at a high or low rate. Shocks to physical or human capital may therefore cause shifts in the long-run trend of the economy if they alter substantially the short-run incentives to save and invest.

Resumen (in Spanish)

Esta tesis doctoral trata de ampliar el conocimiento sobre el papel desempeñado por la acumulación de capital humano en las tendencias macroeconómicas a largo plazo. Los dos primeros capítulos se centran en estudiar por qué se reasigna tiempo del trabajo a la educación de los hijos mientras las tasas de fertilidad caen. Los dos capítulos restantes se dedican a investigar cómo afectan las preferencias a los incentivos a acumular capital humano y, por tanto, al proceso de crecimiento económico. Esta tesis eminentemente teórica contribuye a la literatura ampliando el análisis de mecanismos bien conocidos en la literatura empírica a cuestiones que no se han abordado antes, sirviendo de guía para futuras investigaciones empíricas. A continuación se ofrece un resumen de cada capítulo.

Capítulo 1: Child Survival and the Decline in Hours Worked

Es un hecho bien establecido que las sociedades desarrolladas han experimentado importantes cambios económicos y demográficos desde la Revolución Industrial. En particular, las tasas de fertilidad han caído secularmente durante este periodo de tiempo. Al mismo tiempo, la jornada laboral ha disminuido constantemente hasta niveles históricamente bajos. A pesar de que las decisiones sobre fertilidad y oferta de trabajo se toman conjuntamente en el contexto de la familia, sus tendencias se han estudiado por separado como si fueran fenómenos independientes. El primer capítulo de esta tesis contribuye a la literatura llenando el vacío existente entre las transiciones demográficas y las tendencias en el uso del tiempo, estableciendo una explicación unificada para ambas.

En la literatura previa se pueden encontrar tres posibles explicaciones para la disminución de las horas de trabajo. Por una parte, se ha postulado que las preferencias son tales que el crecimiento salarial induce el deseo de disfrutar de más tiempo libre porque el efecto renta es más fuerte que el efecto sustitución

(véase Boppart y Krusell (2020)). Otra hipótesis propuesta es que un sistema fiscal y de transferencias más amplio desincentiva la oferta de mano de obra en el margen intensivo (véase McDaniel (2011) y Bick and Fuchs-Schündeln (2018)). Más recientemente, se ha argumentado que la reducción del coste de encontrar un empleo puede dar lugar a un menor número de horas trabajadas por trabajador (véase Bick et al. (2022)).

Las teorías mencionadas son explicaciones plausibles de por qué se trabaja menos horas y se disfruta de más tiempo libre, pero solo ofrecen una visión parcial o incompleta de los mecanismos subyacentes. La razón es que no tienen en cuenta el hecho de que las decisiones sobre fertilidad y uso del tiempo están entrelazadas. En consecuencia, estas teorías no explican por qué las tasas de fecundidad disminuyen al mismo tiempo que lo hace la jornada laboral. Es más, tampoco explican por qué los padres cada vez dedican más tiempo a educar a sus hijos a pesar de tener menos vástagos (véase Gauthier et al. (2004), Guryan et al. (2008) Gimenez-Nadal and Sevilla (2012) y Doepke et al. (2019)).

La hipótesis que motiva este trabajo es que las tendencias a largo plazo en el uso del tiempo están parcialmente causadas por cambios en la demografía. Esta hipótesis se basa en dos mecanismos empíricamente plausibles. El primero es que las mejoras en las tasas de mortalidad infantil animan a los padres a criar menos hijos (véanse Herzer et al. (2012), Wilson (2015) y Ager et al. (2018)). El segundo mecanismo es que los padres con más hijos trabajan más horas para financiar un mayor consumo de bienes como alimentos, ropa, medicamentos, etc. La interacción entre estos dos mecanismos implica que los cambios en la demografía deberían afectar al uso del tiempo como se observa en los datos.

Para ilustrar la interacción entre la mortalidad infantil, fertilidad y uso del tiempo, desarrollo un modelo de generaciones solapadas de dos periodos en el que el tiempo puede dedicarse a trabajar, criar a los hijos y educarlos. En el modelo, los padres son conscientes de que gastar más renta por hijo mejora la tasa de supervivencia de su descendencia debido a una mejor nutrición, vestimenta, etc. Esto implica que los padres internalizan el impacto de su asignación de tiempo tanto en el capital humano de sus hijos como en sus posibilidades de supervivencia.

En esencia, el modelo estudiado es similar al planteado por Strulik (2004b), según el cual las posibilidades de supervivencia de los hijos dependen de factores controlables (internos) e incontrolables (externos). La principal diferencia es que los factores controlables se mejoran gastando más renta por hijo (consumo) en lugar de inviertiendo más tiempo por hijo (fracción de la renta). Esta diferencia no es trivial porque la salud de los niños se convierte en una función implícitamente creciente del número de horas trabajadas a través del consumo infantil, en lugar de ser una función explícitamente decreciente del número de horas trabajas.

La primera aportación a la literatura del modelo es la predicción de un gradiente positivo entre las horas de trabajo y las tasas de fecundidad. A diferencia de los modelos anteriores, este gradiente surge porque la demanda de bienes depende del número de miembros de la familia, que viene determinado por la elección de fertilidad de los padres. La consecuencia de este gradiente es que los cambios en la demografía se trasladan a la utilización del tiempo a través de la demanda de bienes. Además, dado que opera a través del consumo infantil, dicho gradiente es robusto a la endogeneización del ocio o el ahorro para la jubilación.

La segunda gran novedad del modelo es el efecto de supervivencia infantil interno o controlable. Como los padres con salarios por hora más altos gastan más ingresos por hijo, sus hijos tienen más posibilidades de sobrevivir gracias a una mejor nutrición, ropa, atención sanitaria, etc. A su vez, una mejor salud anima a los padres a tener menos hijos e invertir más en su educación. En consecuencia, los padres con más capital humano (salarios por hora más elevados) distribuyen su tiempo de forma diferente, invirtiendo más tiempo en la educación de sus hijos, tal y como señala la evidencia (véase Guryan et al. (2008), y Doepke et al. (2019)).

Capítulo 2: Fertility Norms and Parents' Allocation of Time

El segundo capítulo de esta tesis está motivado por estudios empíricos recientes que demuestran que las normas de fertilidad desempeñan un papel crucial a la hora de explicar los patrones demográficos históricos, como por ejemplo la transición hacia regímenes de fertilidad baja (véase Murphy (2015), Myong et al. (2020) y Spolaore and Wacziarg (2022)). Dado que la crianza de los hijos requiere tiempo y esfuerzo, es razonable suponer que la existencia de normas de fertilidad influye en la asignación de tiempo de los padres. El objetivo de este capítulo es investigar si las normas de fecundidad influyen en la distribución del tiempo entre el trabajo y el cuidado de los hijos.

Por normas de fecundidad se entienden las reglas sociales que penalizan a las familias por desviarse del comportamiento aceptado en materia de fertilidad. Estas normas pueden referirse directamente al tamaño de la familia o a cualquier práctica que influya en él. En cualquier caso, la penalización social implica un coste de ajuste en el número de hijos que entra en la función de utilidad de los padres. Dicho coste de ajuste puede tener una incidencia importante en la asignación del tiempo, ya que los padres no pueden ajustar su elección de fertilidad tanto como desearían cuando cambian las condiciones económicas, a lo que llamo "rigideces en el número de hijos".

Pese a que existe mucha literatura teórica que estudia conjuntamente las decisiones de fertilidad y de oferta de trabajo, hay muy pocos modelos que incluyan normas de fecundidad (véase Prettner y Strulik (2017) y Strulik (2017, 2019)). Sin embargo, estos modelos no encuentran ninguna implicación sobre la oferta de trabajo porque están construidos para describir cuándo cambian las normas de fecundidad, y cómo afectan tales cambios al trade-off entre cantidad y calidad de los hijos. Por el contrario, este trabajo aborda la cuestión de si la asignación de tiempo de los padres cambia en presencia de normas de fecundidad, además de explorar sus implicaciones con respecto a ciertas teorías del crecimiento unificado.

Para estudiar esta cuestión, se incorporan normas de fertilidad en un modelo de cantidad-calidad de hijos en el que los padres pueden dedicar tiempo a trabajar, criar a los hijos y educarlos. Siguiendo a Spolaore y Wacziarg (2022), las normas de fertilidad se modelizan como una fuente de desutilidad (penalización social) para aquellos padres que eligen una tasa de fecundidad inferior a un determinado umbral (ya sea el máximo biológico o alguna otra tasa socialmente aceptada). Esto nos permite captar simultáneamente la penalización asociada a las normas relativas al tamaño de la familia y a las prácticas (por ejemplo, el uso de profilácticos).

El principal resultado del modelo es la existencia de un gradiente positivo entre las horas de trabajo y las tasas de fecundidad. Este gradiente no establece causalidad en ninguna dirección, ya que ambas variables son elecciones endógenas. En cambio, este surge debido a la existencia de rigideces en la cantidad de hijos. Cuando el coste marginal de tener un hijo aumenta, los padres no reducen su tasa de fertilidad tanto como desearían debido a la penalización social. En consecuencia, se ven onbligados a reducir la jornada laboral para compensar la mayor inversión de tiempo en criar hijos, explicando por qué las tasas de fecundidad y las horas de trabajo disminuyen al mismo tiempo.

Capítulo 3: Child Quality Externalities, Fertility, and Economic Growth

El tercer capítulo de esta tesis se dedica a investigar las implicaciones para la educación, la fecundidad y el crecimiento económico de las externalidades de la calidad de los hijos. Esta última noción se refiere a la idea de que los padres valoran la calidad, el capital humano o la educación de sus hijos en relación con la suya propia y la de los demás. En parte, estas externalidades se derivan del hecho de que los padres son reacios a dejar que sus hijos se queden atrás en la escala socioeconómica, lo que induce a los hogares a competir por obtener mejores resultados educativos. Por consiguiente, las externalidades de la calidad infantil vinculan los incentivos para educar a los hijos con el estatus socioeconómico de los padres en comparación con los demás.

Los estudios empíricos revelan que las externalidades de la calidad infantil son un factor significativo en las decisiones educativas de los padres. Por ejemplo, en Guo y Qu (2022) se constata que competir por las plazas en mejores colegios anima a los padres a invertir más en la educación de sus hijos. En esa misma linea, Chen et al. (2022) muestran pruebas que relacionan el comportamiento de los padres con el estatus socioeconómico de los padres de los compañeros de clase de sus hijos. En consecuencia, no es de extrañar que los padres se impliquen cada vez más en la educación de sus hijos, una tendencia que se ha denominado "helicopter parenting" (véase Guryan et al. (2008), Gimenez-Nadal y Sevilla (2021), y Doepke et al. (2019)).

La principal contribución de este capítulo consiste en ampliar el análisis de las externalidades de la calidad infantil, estudiando su impacto sobre la fertilidad y el crecimiento económico. A tal efecto, modifico el marco Beckeriano estándar (véase Becker y Lewis (1973)) para incluir dicha característica. El resultado es un modelo que describe un problema de optimización estándar en el que los padres se enfrentan a un compromiso entre la cantidad y la calidad de los hijos sujetos a estas externalidades. Las decisiones de fertilidad y educación se sitúan en un contexto de equilibrio dinámico, en el que el crecimiento económico se determina endógenamente por la acumulación de capital humano e innovación.

La implicación más importante del modelo desarrollado es que las externalidades de la calidad infantil pueden ser una causa fundamental del crecimiento económico. Esto se debe a que dichas externalidades inducen a los padres a competir por la educación de los hijos, lo que provoca un aumento del gasto en educación y una acumulación sostenida de capital humano. Esto a su vez permite superar la ley de rendimientos decrecientes de la innovación, alimentando un progreso tecnológico sostenido. La producción por trabajador crece entonces impulsada por la innovación y la acumulación de capital humano, financiando el creciente gasto en educación de los padres.

Otra implicación interesante del modelo es que las externalidades de la calidad de los hijos ejercen un efecto directo negativo sobre la fertilidad. Dado que los padres con más capital humano invierten más por hijo, tener un hijo es obviamente más caro para ellos. En el trabajo, este efecto se denomina *efecto educación.* Por otra parte, los padres dotados de mayor capital humano también disfrutan de mayores niveles de renta, lo cual genera un efecto indirecto y positivo sobre la fertilidad a través de la renta. Esto implica que el signo neto depende de cual de los dos efectos domine (educación o renta), por lo que en este modelo son posibles relaciones no monotónicas entre renta (capital humano) y fertilidad, como sugieren algunos estudios (véase Hazan y Zoabi (2015), Zanin et al. (2015) y Vogl (2015)).

A escala macroeconómica, la relación dinámica entre el crecimiento económico y las tasas de fertilidad depende del horizonte temporal considerado. En consonancia con algunos estudios empíricos, la fecundidad reacciona de forma procíclica a los shocks de renta puros (véase Doepke et al. (2019)). A largo plazo, sin embargo, el efecto de la educación tiende a imponerse, arrastrando los niveles de fertilidad a la baja. De esta manera, el modelo es capaz de explicar por qué el crecimiento económico conlleva descensos de la fertilidad (véase Herzer et al. (2012)), pero sin asumir que la fertilidad se comporta como un bien inferior.

Capítulo 4: Human Capital Formation via Investment and LBD: Low vs Modern Growth Dynamics.

El último capítulo estudia los incentivos a invertir tiempo en la acumulación de capital humano si los individuos ya adquieren habilidades mediante el aprendizaje práctico (LBD, por sus siglas en inglés "Learning-by-Doing"). A diferencia del conocido modelo de crecimiento Uzawa-Lucas (véase Uzawa (1965) y Lucas (1988)), la existencia de LBD garantiza una acumulación sostenida de capital humano y crecimiento económico con independencia de la tecnología, las preferencias o las condiciones iniciales. Sin embargo, la inversión activa en capital humana marca la diferencia entre crecer a un ritmo alto o bajo.

En el modelo, la existencia de una senda de alto crecimiento económico depende de dos cuestiones. En primer lugar, los parámetros que gobiernan las preferencias deben de permitirlo, siendo el peso del consumo en las preferencias mayor que el ocio. Si esta condición se cumple, la economía tiene acceso a una senda de elevado crecimiento económico alimentada por la inversión en capital humano. No obstante, que dicha senda sea viable no implica que efectivamente se dé, puesto que los incentivos a corto plazo (condiciones iniciales) también deben de ser los correctos. Si el capital físico abunda más que el capital humano, los precios de los factores generan fuertes incentivos a consumir, trabajar y disfrutar de ocio. Si dichos incentivos son demasiado fuertes, la acumulación de capital humano se deprime tanto que la economía queda atrapada en un régimen de bajo crecimiento.

Chapter 1

Child Survival and the Decline in Hours Worked

1.1 Introduction

It is well known that all developed societies have experienced major economic and demographic changes since the Industrial Revolution. For example, fertility rates have followed a path of secular decline over this period of time. Likewise, working hours have steadily decreased to historically low levels. But although fertility and labour supply decisions are made jointly in the context of the family, their trends have been studied separately as independent phenomena until now. This paper contributes to the literature by filling the gap between demographic transitions and trends in the use of time, establishing a unified explanation for both.

One can find three possible explanations for the decline in working hours in the literature. The first one is that preferences are such that wage growth induces the desire to enjoy more leisure time because the income effect is stronger than the substitution effect (see Boppart and Krusell (2020)). Another proposed hypothesis is that more extensive tax and transfer system discourage the supply of labour along the intensive margin (see McDaniel (2011) and Bick and Fuchs-Schündeln (2018)). More recently, it has been argued that reducing the cost of finding a job may result in fewer hours worked per worker (see Bick et al. (2022)).

The mentioned theories seem plausible explanations for why people work fewer hours and enjoy more leisure time, but they offer only a partial or incomplete vision of the underlying mechanisms. The reason is that they do not take into account the fact that fertility and labour supply decisions are intertwined. As a result, these theories do not explain why fertility rates decline at the same time as working hours do. Moreover, they also fail to explain the upwards trend in the amount of time devoted to educational childcare time (see Gauthier et al. (2004), Guryan et al. (2008) Gimenez-Nadal and Sevilla (2012), and Doepke et al. (2019)).

The hypothesis motivating this paper is that the long-run trends in the use of time are partly caused by changes in the demography. This hypothesis builds upon two empirically plausible mechanisms. The first one is that improvements in child mortality rates encourage parents to raise fewer children (see Herzer et al. (2012), Wilson (2015), and Ager et al. (2018)). It is also likely that parents with more children work more hours to finance a higher consumption of goods such as food, clothes, medicines, etc. The interplay between these two mechanisms implies that changes to the demography should affect the use of time as observed in data.

In order to illustrate the interplay between child mortality, fertility, and labour supply decisions, I develop a two periods overlapping-generations (OLG) in which time can be devoted to work, raising children, and educating them. In the model, parents are aware that spending more income per child improves their offspring's survival rate due to better nutrition, clothing, and health investment. These two assumptions imply that parents internalize the impact of their time allocation on both their children's human capital and chances of survival.

Essentially, the model studied here is similar to the one propounded by Strulik (2004b), whereby children's chances of survival depend on controllable (internal) and uncontrollable (external) factors. The main difference is that controllable mortality elements such as nutrition, clothing, or health investment are improved by spending more income per child (consumption) instead of time (spending share). This difference is non-trivial because children's health becomes an implicitly positive function of the number of hours worked through child consumption, rather than parents having to choose between labour income or children's health.

The first main novelty of the model is the prediction of a positive gradient between working hours and fertility rates. Unlike previous models, this gradient It emerges because the demand for goods depends on the number of members in the family, which is determined by parents' fertility choice. The consequence of this gradient is that changes to the demography translate to the use of time. Moreover, given that it operates through child consumption, the gradient holds even if leisure time or saving rates are endogenized as control variables.

The second main novelty of the model is what we call "internal or controllable child survival effect". As parents with higher hourly wages spend more income per child, their children have better chances of survival due to better nutrition, clothing, healthcare, etc. Better health in turn encourages parents to have fewer children and invest more in their education. Accordingly, parents with more human capital (higher hourly wages) allocate their time differently, investing more time in their children's education as the evidence points out (see Guryan et al. (2008), and Doepke et al. (2019)).

Similarly to Strulik (2004b), this model also predicts the existence of an "external child survival effect". The latter notion refers to changes in the optimal household allocations caused by variations in external or uncontrollable mortality factors. The novelty in this model is that the external survival effect also alters parents' labour supply decisions for the same reason as the internal effect. However, this model predicts that the external survival effect is weaker for wealthier parents because their children already enjoy a better health.

One of the main predictions of the model is thus that long-run economic growth can induce demographic transitions accompanied by changes in the use of time consistent with the historical evidence. Moreover, given that the effects described above operate through internal and external mortality factors, the model can easily be extended to study the role of taxes and public spending. In summary, the model provides a description of the interplay between child mortality, fertility, and labour supply decisions that can explain their co-movement.

The remainder of the paper is organised as follows. The next section describes the model, while its predictions are presented in the third section. The last section makes some final considerations.

1.2 The Model

Time is discrete and periods are ordered by $t \in \mathbb{N}_+$. During each period, the economy is comprised by two overlapped generations called adults and children. Families are mono-parental, with all decisions being made by the sole parent in the household, while children remain passive. During adulthood, individuals chose to have b_t children, from which $0 < s_t < 1$ reach adulthood. For the sake of simplicity, we assume that children are homogeneous within households, so their parents treat them identically. Individuals may, nonetheless, differ in their human capital endowment h_t if they come from different families.

There are two features that differentiate this model from existing ones. The first difference is that parents distinguish between their own consumption and that of their children, denoted c_t and z_t respectively. Yet even more importantly, parents are aware that spending more income on consumption per child improves their children's chances of survival, that is, $\partial s/\partial z > 0$. This last assumption reflects the positive effect on children's health of better nutrition, clothing, healthcare, etc.

Parents have preferences defined over their own consumption c_t and the quantity-quality mix of their surviving children $s_t b_t h_{t+1}$, where h_{t+1} stands for children's quality or human capital. These preferences are represented by a standard logarithmic utility function of the form:

$$U[c, s, b, h] = \alpha \ln c_t + (1 - \alpha) \ln(s_t b_t h_{t+1}), \qquad 1 > \alpha > 0. \tag{1.1}$$

Since child consumption z_t already provides utility through higher child survival rates, the inclusion of an explicit term for it in the utility function simply adds redundant parameters.

Regarding the budget constraint, the sole source of income is labour $w_t h_t n_t$, which depends on the adult's human capital h_t, the number of hours worked n_t, and the wage rate per efficient hour paid in the labour market w_t. Income is spent entirely purchasing units of a homogeneous final good that is consumed by the adult and their children. There is no savings market, so the household's budget constraint reads as

$$w_t h_t n_t = c_t + b_t z_t. \tag{1.2}$$

Besides supplying labour, parents can devote time to childcare. According to the standard, raising a child involves a fixed time cost of $\bar{\epsilon}$ hours required for basic childcare. In addition, parents can invest ϵ_t hours per child to promote their quality. Denoting the total number of available hours by $T > 0$, the distribution of time between working and childcare must satisfy the time constraint

$$T = n_t + (\bar{\epsilon} + \epsilon_t) b_t. \tag{1.3}$$

Children's quality or human capital h_{t+1} depends on how much time their parents invest in their education ϵ_t according to the technology

$$h_{t+1} = (\theta + \epsilon_t) h_t. \tag{1.4}$$

Parent's human capital h_t enters the education technology as a multiplicative term to allow for sustained accumulation over generations. Parameter $\theta > 0$ measures the amount of human capital acquired through other channels, such as observation, genetics, social interactions or public school if there were. This education technology is used in the literature to stress that investing time in children's education is not an essential input in the process of skill formation (i.e., $h_{t+1} > 0$ even if $\epsilon_t = 0$). Nevertheless, parents may still invest time in their children's education because it fosters the process, which occurs if it holds that $\theta > \bar{\epsilon}$.[1]

[1]This parametric condition implies that the fixed time cost of fertility $\bar{\epsilon}$ is not so high that investing more time in educating children is always suboptimal.

Following Strulik (2004a,b), child survival rates are the composite function of controllable (internal) and uncontrollable (external) factors. Controllable elements such as the quality of nutrition or clothing are included in child consumption z_t. Uncontrollable factors such as the climate or the prevalence of diseases are in turn captured by the dynamic index X_t. For instance, public investment in the potable water infrastructure can be represented by a higher X_t. Formally, our assumptions about child mortality are represented by a survival function,

$$s_t = S[z_t, X_t], \tag{1.5}$$

which is twice-continuously differentiable, strictly increasing and concave in z_t and X_t. As argued by Strulik (2004a,b), improvements in the external mortality elements should reduce the effectiveness of the internal factors, leading to the assumption that

$$\frac{\partial^2 S[z_t, X_t]}{\partial z_t \partial X_t} \leq 0.$$

Under these hypotheses about the survival function, it is easy to verify that the spending elasticity of child survival

$$\Phi_t = \frac{\partial S[z_t, X_t]}{\partial z_t} \frac{z_t}{S[z_t, X_t]},$$

is a strictly decreasing function of both child consumption z_t and the dynamic index X_t. Finally, given that the risk of mortality can never equal zero, we impose the following Inada conditions:

$$\lim_{z \to \infty} \frac{\partial S[z_t, X_t]}{\partial z_t} = 0, \qquad \lim_{X \to \infty} \frac{\partial S[z_t, X_t]}{\partial X_t} = 0.$$

Definition 1. *Given $\{h_t, w_t, X_t\}$, an allocation $\Omega_t^* \equiv \{c_t^*, b_t^*, \epsilon_t^*, z_t^*\}$ is said to be optimal if it maximises utility* (1.1)*, subject to the budget constraint* (1.2)*, the time restriction* (1.3)*, the education technology* (1.4) *and the child survival function* (1.5)*.*

When imposing non-negativity conditions on the solution, an optimal household allocation must satisfy the following system of first order conditions (FOCs):

$$\frac{\partial U}{\partial c} = \lambda_t, \tag{1.6}$$

$$\frac{\partial U}{\partial b} = \frac{\partial U}{\partial c} \left(z_t + w_t h_t (\epsilon_t + \bar{\epsilon}) \right), \tag{1.7}$$

$$\left(\frac{\partial U}{\partial s}\frac{\partial s}{\partial z} - \frac{\partial U}{\partial c}b_t\right) z_t = 0, \tag{1.8}$$

$$\left(\frac{\partial U}{\partial h}\frac{\partial h}{\partial \epsilon} - \frac{\partial U}{\partial c}b_t w_t h_t\right) \epsilon_t = 0. \tag{1.9}$$

According to equation (1.6), the optimal adult consumption level c_t^* must be such that the marginal utility of consumption equals the marginal utility of income (shadow value) λ_t. This means that the cost of all the other choices is measured in terms of forgone parental consumption. For instance, condition (1.7) establishes that the marginal utility of having a child must equate its marginal cost in terms of forgone adult consumption. Similarly, if parents decide to spend income on their children, the optimal child consumption level must be such that its marginal utility equals the marginal cost involved (see equation (1.8)). Finally, condition (1.9) states that if parents decide to invest time in their children's education, its marginal utility and marginal cost must be the same.

Note that equations (1.6)-(1.8) uphold the following: parents always decide to have children $b_t^* > 0$, parent's consumption is positive $c_t^* > 0$, and children receive a positive amount of goods $z_t^* > 0$. Otherwise, no children would survive and raising them would therefore be pointless. By contrast, equation (1.9) yields an interior solution, namely $\epsilon_t^* > 0$, only when the optimal fertility choice is below a certain threshold (see the Appendix). If this threshold is interpreted as the biological maximum, the optimal solution is always interior. In any case, we will abstract from the corner solution and focus on the behaviour of the interior one with positive time investment in children's education $\epsilon_t^* > 0$.[2]

A shortcoming of working with a general functional form for the child survival function (1.5) is that we cannot explicitly solve the system of FOCs (1.6)-(1.9). What can be done, however, is express the optimal household allocations as functions of the elasticity measure Φ_t^* and the optimal fertility rate b_t^* as follows:

$$c_t^* = w_t h_t \alpha T \tag{1.10}$$

$$z_t^* = w_t h_t \left(\theta - \bar{\epsilon}\right), \tag{1.11}$$

$$b_t^* = \frac{(1-\alpha)T\Phi_t^*}{(\theta - \bar{\epsilon})}, \tag{1.12}$$

$$\epsilon_t^* = \frac{(1-\alpha)T}{b_t^*} - \theta, \tag{1.13}$$

$$n_t^* = \alpha T + \left(\theta - \bar{\epsilon}\right) b_t^*. \tag{1.14}$$

[2]The existence and properties of a corner solution are presented in the Appendix, as it may be of interest regarding potential unified growth dynamics.

Equations (1.10) and (1.11) show that every individual in the household consumes a constant fraction of the hourly wage. Recall that the elasticity measure Φ_t^* depends inversely on child consumption z_t^* (internal factors) and the dynamic index X_t (external factors). Equation (1.12) thus depicts the effect of internal and external survival factors on child quantity b_t^*. Finally, equations (1.13) and (1.14) describe how these effects impact upon the allocation of time through fertility.

1.3 Results

This model's main novelty is the existence of a positive gradient between working hours and fertility rates. The reason it emerges and the intuitions behind it will be discussed in the next subsection. The second novelty refers to what we call internal or controllable child survival effect, as opposed to the external or uncontrollable one, which will be discussed in the second subsection. The last subsection will address the external child survival effect.

1.3.1 Labour Supply and Fertility Decisions

There are two ways to explain the positive gradient between hours worked and fertility rates. The first one uses the budget constraint. Dividing both sides of equation (1.2) by the hourly wage $w_t h_t$ reveals that the labour supply is governed by the spending shares on consumption by parents and children:

$$n_t^* = \frac{c_t^*}{w_t h_t} + \frac{b_t^* z_t^*}{w_t h_t}. \tag{1.15}$$

Given that every individual in the household consumes a constant fraction of the hourly wage (see equations (1.10) and (1.11)), it readily follows that the household's total consumption depends on its size, which is determined by the fertility rate b_t^*. This means that parents with more children work more hours because they spend a larger fraction of their hourly wage on children's consumption.

An alternative intuition behind the positive gradient between fertility and labour supply decisions can be obtained by using the time restriction. First, one needs to decompose the labour supply into working hours to finance consumption by parents and children, denoted n_t^c and n_t^z respectively. The time constraint is then decomposed as follows

$$T = n_t^c + n_t^z + (\bar{\epsilon} + \epsilon_t) b_t. \tag{1.16}$$

This last equation shows that parents devote time to their children directly through childcare $(\bar{\epsilon}+\epsilon_t)b_t$, and indirectly through consumption n_t^z. As parents always work

αT hours to finance their own consumption (see equation (1.10)), it is necessarily the case that they use the remaining $(1-\alpha)T$ hours to satisfy their children's needs:

$$(1-\alpha)T = n_t^z + (\bar{\epsilon} + \epsilon_t)b_t. \tag{1.17}$$

Equation (1.17) thus indicates that the labour supply moves in step with fertility because the composition of the time devoted to satisfying children's needs changes. When parents raise more children b_t, they need to work longer hours n_t^z in order to purchase more food, clothes, etc. As the number of children diminishes, the need for goods decreases, and parents can reallocate time from work to childcare $(\bar{\epsilon} + \epsilon_t)b_t$. The latter becomes clear when considering the spending rule (1.11), whereby the amount of time worked for children's consumption is given by $n_t^z = (\theta - \bar{\epsilon})b_t$. Equation (1.17) can therefore be reformulated to obtain the trade-off between the quantity and quality of children in equation (1.13).

1.3.2 The Internal Child Survival Effect

One of this model's main novelties is the existence of an internal child survival effect. It emerges because parents are aware that their children's health depends on nutrition, clothing, and other controllable (internal) factors encompassed in child consumption z_t^*. As it is to be expected, parents with higher hourly wages spend more income per child, and thus, they can expect more children to survive $(\partial s_t^*/\partial w_t h_t > 0)$. In turn, greater chances of child survival encourage parents to educate each child more, causing changes in the optimal household allocations, which is what we call as the internal child survival effect.

Theorem 1. *Parents with higher hourly wages have fewer children, work fewer hours, and devote more time to children's education because their children have better survival chances.*

Proof: As parents with higher hourly wages spend more income per child (see equation (1.11)), it follows that $\partial\Phi_t^*/\partial w_t h_t < 0$. Applying the chain rule in equations (1.12), (1.13) and (1.14) therefore shows that greater chances of child survival lead to a lower fertility rate

$$\frac{\partial b_t^*}{\partial w_t h_t} = \frac{\partial b_t^*}{\partial \Phi_t^*}\frac{\partial \Phi_t^*}{\partial z_t^*}\frac{\partial z_t^*}{\partial w_t h_t} < 0,$$

more time devoted to children's education,

$$\frac{\partial \epsilon_t^*}{\partial w_t h_t} = \frac{\partial \epsilon_t^*}{\partial b_t^*}\frac{\partial b_t^*}{\partial w_t h_t} > 0,$$

and fewer hours worked

$$\frac{\partial n_t^*}{\partial w_t h_t} = \frac{\partial n_t^*}{\partial b_t^*} \frac{\partial b_t^*}{\partial w_t h_t} < 0. \quad \square$$

To understand the mechanism behind Theorem 1, it should be noted that equations (1.8) and (1.9) can be merged to obtain the optimality condition

$$\frac{\partial U}{\partial s} \frac{\partial s}{\partial z} \frac{z_t^*}{\theta - \bar{\epsilon}} = \frac{\partial U}{\partial h} \frac{\partial h}{\partial \epsilon}.$$

Following our assumptions about the child survival function and the education technology, this latter condition establishes that health and education investments go hand in hand. In other words, parents that spend more income per child will also invest more time in their education, that is, $\partial \epsilon_t^* / \partial z_t^* > 0$. As parents with higher hourly wages spend more income per child and invest more time in their education, the cost of raising a child increases (see the right hand-side of eq. (1.7)), leading them to have fewer children. A smaller family involves a lower demand for goods, allowing parents to reallocate time from work to childcare.

The internal child survival effect is a novelty compared to the study by Strulik (2004b). It emerges because the controllable mortality factors can be improved by spending more income per child instead of time, causing transversal patterns in the use of time driven by differences in human capital (h_t). Interestingly, these transversal patterns are consistent with the empirical evidence (see Guryan et al. (2008), Bick et al. (2018) and Doepke et al. (2019)). A further novelty of this model is thus that economic growth, either in the form of technological progress (w_t) or human capital accumulation (h_t), leads to a reallocation of time from work to educational childcare that is consistent with the evidence (see Gimenez-Nadal and Sevilla (2012), Doepke et al. (2019) and Boppart and Krusell (2020)).

1.3.3 The External Child Survival Effect

In line with Strulik (2004b), this model predicts the existence of an external child survival effect. This notion refers to changes in the optimal household allocations caused by variations in the external mortality factors encompassed by the dynamic index X_t. Like the internal effect, improvements in uncontrollable mortality factors (higher X_t) increase child survival rates, changing the optimal household allocations as stated in Theorem 2.

Theorem 2. *When uncontrollable mortality factors improve, parents decide to have fewer children, devote less time to work, and invest more time in educating their children.*

Proof: Since improvements in uncontrollable child mortality lead to higher child survival rates, that is, $\partial\Phi_t^*/\partial X_t < 0$, applying the chain rule in equations (1.12), (1.13) and (1.14) implies that parents react by having fewer children

$$\frac{\partial b_t^*}{\partial X_t} = \frac{\partial b_t^*}{\partial \Phi_t^*}\frac{\partial \Phi_t^*}{\partial X_t} < 0,$$

by increasing the amount of time devoted to children's education,

$$\frac{\partial \epsilon_t^*}{\partial X_t} = \frac{\partial \epsilon_t^*}{\partial b_t^*}\frac{\partial b_t^*}{\partial X_t} > 0,$$

and by working fewer hours

$$\frac{\partial n_t^*}{\partial X_t} = \frac{\partial n_t^*}{\partial b_t^*}\frac{\partial b_t^*}{\partial X_t} < 0. \quad \square$$

The mechanism behind the external survival effect is the same as in the case of the internal one. Improved child survivability encourages parents to educate each child more, raising the marginal cost of having a child. Consequently, parents choose to have a smaller family, which therefore decreases their demand for goods and labour supply. In this case, what is actually new is that the labour supply does not remain constant, as it does in Strulik (2004b). Moreover, the magnitude of the external effect varies across individuals with their hourly wage. In particular, parents with higher hourly wages are less sensitive to improvements in the external mortality factors because their children already have better chances of survival. This is confirmed by the second cross-partial derivative in equation (1.12),

$$\frac{\partial^2 b_t^*}{\partial X_t \partial w_t h_t} = \frac{(1-\alpha)T}{(\theta - \bar{\epsilon})}\frac{\partial^2 \Phi_t^*}{\partial X_t \partial w_t h_t} < 0.$$

From a macroeconomic perspective, this last result is important regarding the impact of health innovation and its diffusion. Through the external child survival effect (X_t), less developed countries may converge demographically with richer ones without achieving the same income level $w_t h_t$. In other words, two countries with different hourly wages can have similar fertility and child mortality rates because of the international diffusion of medical knowledge and public health policies. This may lead to demographic convergence despite persistent income differentials.

1.4 Conclusions

This paper shows that child survival might play a role in labour supply decisions if parents consider that spending more income per child improves their offspring's chances of survival. According to the model, both internal and external improvements in child survivability lead to a shift in the allocation of time from work to childcare that is consistent with the empirical evidence. For the sake of brevity, this analysis has been restricted to the effects upon the optimal household allocations (the demand side). However, these effects may also lead to interesting general equilibrium implications, with the following two highlights.

One of the model's predictions is that higher hourly wages discourage the labour supply of individuals. If the labour demand of firms also falls with the hourly wage, then the labour market could yield multiple equilibria. Two economies with the same fundamentals (preferences, technology, and initial conditions) may thus follow different equilibrium paths towards the same asymptotic steady state (local indeterminacy). In such case, despite the convergence in fertility, mortality and growth rates over the long-run, there would be permanent differences in income.

Another potentially fruitful avenue of research regards the impact of public healthcare upon the household allocations. Improvements in public healthcare programmes might prompt an external child survival effect resulting in a reallocation of time from work to childcare. In turn, labour income taxes have the exact opposite effect. It can therefore be expect that the efficiency with which it is spent determines the net effect of public healthcare programmes upon the allocation of time between work and childcare.

1.5 Appendix

This appendix discusses when does the model predict a corner solution without time investment in education, that is, $\epsilon_t^* > 0$. Such a corner solution may be of interest regarding unified growth dynamics because of its features on fertility and human capital accumulation, which may resemble the behaviour of a Malthusian economy. Equation (1.13) shows that the model's solution is interior if, and only if, the optimal fertility choice is below the threshold

$$b_t^* < \frac{(1-\alpha)T}{\theta}.$$

This threshold depends on the total number of available hours T, so it could be interpreted as the biological maximum, which would ensure that the optimal solution is always interior. Suppose that this is not the case, and the biological maximum exceeds this threshold, allowing the existence of a corner solution. Replacing the optimal fertility choice (1.12) on the left hand-side of the above condition then yields a reformulated condition in terms of child survivability

$$\Phi_t^* < \frac{\theta - \bar{\epsilon}}{\theta}.$$

This condition indicates that the interior solution exists only when child survival rates are sufficiently high (Φ_t^* sufficiently low), which in turn depends on the economy's developmental stage captured by the hourly wage $w_t h_t$ and external mortality factors X_t. In short, the corner solution would take place in environments of low economic development (Malthusian regime).

The properties of the corner solution can be derived if $\epsilon_t^* = 0$ is set in the system of FOCs. The combination of equations (1.12) and (1.11)

$$\frac{1-\Phi_t^*}{\Phi_t^*} z_t^* = w_t h_t \bar{\epsilon},$$

implies that child consumption z_t^* must be a strictly increasing and concave function of the hourly wage $w_t h_t$. Combining the first order condition of fertility, consumption and the budget constraint delivers the optimal fertility choice as a function of child consumption

$$b_t^* = \frac{(1-\alpha)T}{\bar{\epsilon} + \frac{z_t^*}{w_t h_t}}.$$

Given the strict concavity of z_t^* with respect to $w_t h_t$, it follows that the optimal fertility rate b_t^* becomes an increasing function of the hourly wage $w_t h_t$ at the corner solution. Finally, the optimal labour supply is computed using the budget constraint, whereby

$$n_t^* = \alpha T + (1 - \alpha)\Phi_t^* T.$$

Since Φ_t^* depends negatively on z_t^*, the labour supply decreases with higher hourly wages. In sum, the corner solution differs from the interior one in the way the internal child survival effect impacts the optimal fertility rate, turning the sign from negative to positive. Its effect upon the labour supply, however, remains the same. Regarding the external child survival effect, its impact becomes indeterminate because child consumption now depends on X_t.

Chapter 2

Fertility Norms and Parents' Allocation of Time

2.1 Introduction

This paper is motivated by recent empirical studies reporting that fertility norms play a crucial role in explaining historical demographic patterns, for instance, the transition towards low fertility regimes (see Murphy (2015), Myong et al. (2020) and Spolaore and Wacziarg (2022)). Given that parenting requires time and effort, it is reasonable to assume that the existence of fertility norms plays a role in parents' allocation of time. The aim here is to investigate whether fertility norms might indeed influence the way parents allocate time between work and childcare.

Fertility norms mean social rules that penalise families for deviating from socially accepted behaviours regarding fertility. These rules can be formal (laws) and informal (culture), and they can refer directly to family size (number of children) or to any practice that influences it such as the use of condoms. In either case, the social penalty caused by fertility norms involves an adjustment cost in the number of children that enters parents' utility function. Such adjustment cost may have an important bearing on the allocation of time because parents cannot adjust their fertility choice as much as they would like when economic conditions change, which is what we call "rigidities in child quantity" in this paper.

Although there is a great deal of theoretical literature that jointly studies fertility and labour supply decisions, there are very few models that include fertility norms (see Prettner and Strulik (2017) and Strulik (2017, 2019)). Moreover, these models do not find any impact on labour supply decisions because they are built to describe when fertility norms change and how such changes affect the trade-off between quantity and quality of children. By contrast, this paper addresses the issue of whether parents' allocation of time changes in the presence of fertility norms, and explores its implications within certain families of unified growth theories.

To study the role of fertility norms in the allocation of time, fertility norms are incorporated into a beckerian child quantity-quality optimisation model in which parents can devote time to work, raising children, and educating them (see Becker and Lewis (1973)). Following Spolaore and Wacziarg (2022), fertility norms are modeled in a generic way, as a source of disutility (social penalty) for those parents choosing a fertility rate below a certain threshold (either the biological maximum or some other socially accepted rate). This allows to simultaneously capture the social penalty associated with formal and informal norms regarding family size and practices (e.g., the use of prophylactics).

The model's main result is the existence of a positive gradient between working hours and fertility rates. This gradient does not imply causality in either direction, as both variables are endogenous choices. Instead, it emerges due to the existence of rigidities in child quantity. When the marginal cost of having a child in terms of time increases, parents do not reduce their fertility rate as much as they would like because of the social penalty. As a result, they need to reduce their labour supply to compensate for the increased child rearing cost, thereby explaining why fertility rates and working hours fall in tandem.

An interesting implication of the above gradient is that the impact of changes in fertility norms on parents' labour supply can change over time as children's education technology does. In particular, if fertility rates are sufficiently sensitive to changes in fertility norms, then the liberalisation of norms can cause a decline win working hours. As documented below, this prediction finds empirical support in the case of 1980s Ireland, when contraception and abortion laws were liberalised. Using data obtained from the World Bank Development Indicators and the Penn World Tables 10, it is shown that the mentioned changes in Irish fertility norms were immediately followed by a process of convergence in fertility rates and working hours towards the UK. Moreover, also consistent with the model, it is shown that the number of hours that Irish parents work per child increased and converged to Uk levels also right after the liberalisation of fertility norms.

Another topic addressed in this paper refers to the interaction between fertility norms and child rearing costs. This is motivated by the fact that several unified growth theories build upon the idea of children becoming more costly as engine of economic and demographic transitions. One may expect that, in the presence of fertility norms, such theories will predict a downwards trend in working hours and upwards in childcare time consistent with observed patterns since the Industrial Revolution (see Gimenez-Nadal and Sevilla (2012), Doepke et al. (2019) and Boppart and Krusell (2020)). Indeed, this paper reviews theories based on educational changes and increasing child rearing costs, showing that they effectively predict plausible trends in working hours and child care time.

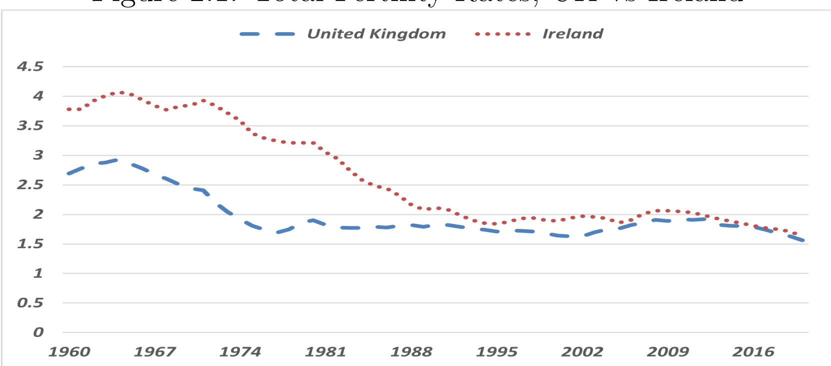
The layout of the paper is as follows. The next section discusses the case of Ireland as motivational evidence. The model is described in section three, while its predictions are discussed in the fourth section. The last section concludes.

2.2 Motivational Evidence

The aim of this section is to provide a brief data analysis to motivate the rest of the chapter. To that end, I will compare fertility rates and working hours in Ireland and the United Kingdom (UK) between 1960 and 2020. This comparison is interesting because both countries share a similar culture and have been subject to similar macroeconomic trends during this period of time. Fertility norms in Ireland, however, were much stricter than in the UK until the mid 80s.

A clear example of the differences between Ireland and the UK in fertility norms until the 80s is the use of condoms. While the production, sale and advertising of such items was widespread in the post-war UK, access to condoms and other contraceptive methods was virtually impossible in neighbouring Ireland until the mid 80s. Likewise, abortions were also illegal. Given the difference in fertility norms, it is unsurprising that Irish women had on average one more child than British women during the 60s and 70s (see Figure 2.1).

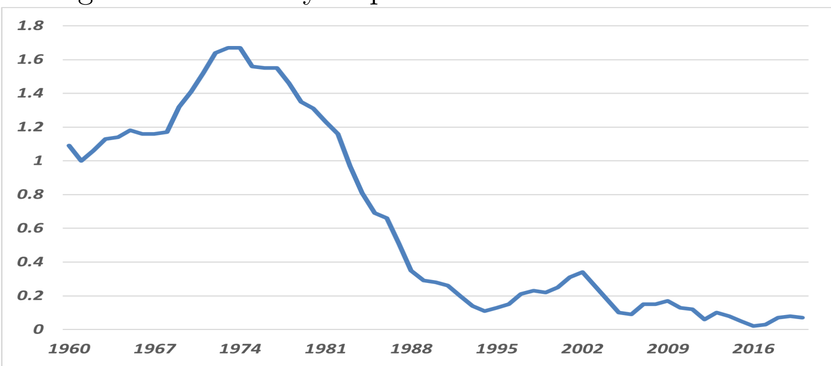
Figure 2.1: Total Fertility Rates, UK vs Ireland



Source: World Bank Development Indicators.

Although legal reforms began in the 80s, the demand for more liberal laws gained substantial strength already in the late 60s. These demands materialised in an unsuccessful legislative bill by Mary Robinson in 1971, followed by several acts of civil protest. This shift in the Irish society is reflected by the fact that the difference in fertility rates between Ireland and the United Kingdom entered a path of convergence in the mid 70s (see Figure 2.2). Thereafter, and just in one decade, the difference in fertility rates went down from around one child to almost zero. The gap has fluctuated in about 0.1 children per women since the 90s.

Figure 2.2: Fertility Gap between Ireland and the UK

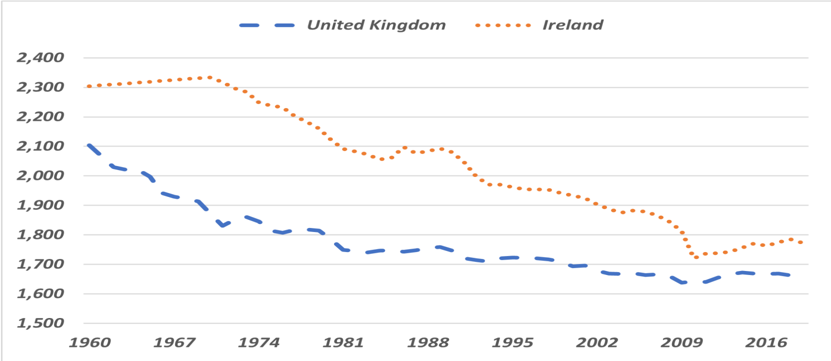


Source: World Bank Development Indicators.

The main conclusion that one can extract from Figures 2.1 and 2.2 is that changes in fertility norms had a quick and significant impact on fertility rates in Ireland, as one might expect. The question now is to asses whether those changes also altered the allocation of time. The idea is that, if fertility and labour supply decisions were not intertwined, one would expect no change in labour market dynamics. In other words, working hours in Ireland should exhibit no remarkable pattern compared to the UK during the 70s and 80s. By contrast, if having more children stimulates the supply of labour, one would expect two things: more working hours in Ireland because higher fertility rates, and convergence starting in the 70s to 80s due to convergence in fertility rates.

Figure 2.3 compares working hours in Ireland and the UK during the same period as in Figures 2.1 and 2.2. First, it is clear that working hours are higher in Ireland than in the UK over the entire period of time plotted, consistent with the hypothesis that fertility and labour supply decisions are intertwined over the long-run. Even more interesting, one can see that the difference in working hours between Ireland and the UK is not stable over time, existing a process of convergence much alike it happened in fertility rates. The difference in working hours shrinks from around 300 hours per year during the 60s and 70s to 100 hours during the last decades, again consistent with the hypothesis of this paper.

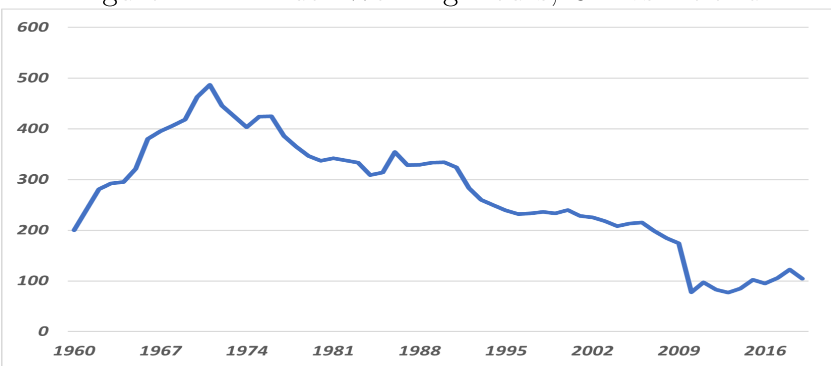
Figure 2.3: Annual Working Hours, UK vs Ireland



Source: Penn World Tables 10.

If we repeat the exercise carried out in Figure 2.2 but with working hours, it turns out that the difference between Ireland and the UK also started to narrow in the early 1970s (see Figure 2.4). This means that both fertility rates and working hours entered a path of convergence at the same time. Although these data do not imply causal evidence by themselves, it seems unlikely that such simultaneous dynamics could happen by chance, without the change in fertility norms having anything to do with the pattern in working hours.

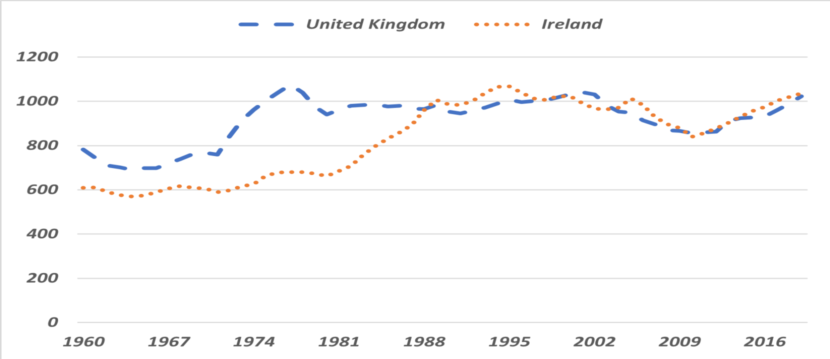
Figure 2.4: Annual Working Hours, UK vs Ireland



Source: Penn World Tables 10.

There is an extra piece of evidence before ending this section. For the next exercise, I compute the ratio between working hours and fertility rates to see how much hours work parents per child. Again, Ireland is compared against the UK during the same time window as in Figures 2.1 to 2.4. The result is displayed in Figure 2.5, where one can observe two interesting things. One is that the ratio between working hours and fertility rates is lower in Ireland than in the UK until the 80s. The second point, and most important one, is that the difference in the mentioned ratio becomes virtually zero once fertility norms change, which is one of the main predictions of the model presented here.

Figure 2.5: Annual Working Hours over Fertility Rates, UK vs Ireland



Source: Penn World Tables 10.

Of course, testing the hypothesis waged in this paper using the presented data would require an application of the synthetic control method or similar quantitative technique. However, the evidence shown above is compelling enough to study why fertility norms may affect the allocation of time in the way it seems, which is the ultimate motivation for this paper.

2.3 The Model

Time is discrete, and periods are denoted by $t \in \mathbb{N}_+$. During each period, the economy is comprised by two overlapped generations called adults and children. Families are assumed to be mono-parental, with all decisions being made by the sole parent, while children remain passive. Parents choose the number of children to be born b_t, from which only a fraction $0 < s_t < 1$ reaches adulthood. For now, we shall assume that parents take their children's survival chances as given, but this assumption will be relaxed in a latter subsection.

The model's main novelty is that it incorporates fertility norms. This means that individuals suffer a penalty (disutility) if they deviate from socially accepted behaviour regarding fertility. I study its impact on parent's allocation of time by extending a standard model in which individuals have preferences defined over consumption c_t, and the quantity-quality mix of children $b_t h_{t+1}$. These preferences are represented by the following utility function:

$$U[c, b, h] = \alpha \ln (c_t) + (1 - \alpha) \ln (b_t h_{t+1}) - \sigma_t \left(\frac{\bar{b}_t - b_t}{b_t} \right). \tag{2.1}$$

Fertility norms are captured by the last term in the utility function. Dynamic parameter σ_t measures how sensitive individuals are to such norms, which is a subjective magnitude. Parameter $\bar{b}_t$ is a reference level that can be either the biological maximum or some socially accepted birth rate. These two parameters capture social norms and are therefore taken as given by individuals, although they may vary over generations. In the presence of such norms, fertility rates expected to be less sensitive to changes in economic and demographic conditions, referred to here as rigidities in child quantity. A later subsection will show that such rigidities arise only if the stigmatisation effect alters the second order derivative of the utility function in a specific way. Otherwise, fertility norms have the implausible effect of increasing the sensitivity of fertility.

Regarding the budget constraint, the sole source of income is labour $w_t h_t n_t$, which depends on the number of hours worked n_t, human capital level h_{t+1}, and the wage rate per efficient hour paid in the labour market w_t. In turn, income is spent entirely purchasing units of an homogeneous good that is consumed by the family. There are no saving markets, so the household budget constraint is given by:

$$w_t h_t n_t = c_t. \tag{2.2}$$

Besides devoting time to work, parents spend time bringing up their children. Following the standard, raising a child requires $\bar{\epsilon}$ hours of basic childcare. In addition, parents can invest ϵ_t hours per child in educational activities. If the total number of available hours is denoted by T, then the allocation of parents' time must satisfy the following restriction:

$$T = n_t + b_t(\bar{\epsilon} + \epsilon_t). \tag{2.3}$$

As mentioned above, an offspring's quality or human capital depends on the amount of time parents invest per child ϵ_t. In addition, some authors consider that education technology may change over time or across households, leading to observed trends and patterns in fertility and education. Given the wide range of factors that might influence education technology, a dynamic index X_t is used to represent them in a generic way. Formally, education technology is assumed to be:

$$h_{t+1} = H[\epsilon_t, X_t]h_t, \tag{2.4}$$

where H is increasing and concave in ϵ_t, but decreasing and convex in X_t. Dynamic index X_t is taken as given by parents, and it measures the "erosion" effect caused by certain elements related to the economic growth process. For instance, Galor (2011) contends that faster innovation erodes the productive value of existing skills, yet complements education. Other phenomena that might exert a similar effect include task automation, certain types of biased technological change or the process of creative destruction applied to skills. In order to ensure that the erosion effect ($\partial H/\partial X < 0$) indeed encourages parents to invest more in their children's education, we assume that ϵ_t and X_t are either complementary or neutral, i.e.

$$\frac{\partial^2 H[\epsilon_t, X_t]}{\partial\epsilon\partial X} \geq 0.$$

This last assumption means that education lessens the erosion effect caused by faster technological progress or automation. Finally, ensuring the existence of a unique interior solution requires the typical Inada conditions:[1]

$$\lim_{\epsilon\to 0}\frac{\partial H[\epsilon_t, X_t]}{\partial \epsilon_t} = \infty, \quad \lim_{\epsilon\to\infty}\frac{\partial H[\epsilon_t, X_t]}{\partial \epsilon_t} = 0.$$

Definition 1. *Given* $\left\{w_t, h_t, \bar{b}_t, \sigma_t, X_t\right\}$*, an allocation* $\Omega_t^* \equiv \{c_t^*, b_t^*, \epsilon_t^*\}$ *is said to be optimal if it maximises utility* (2.1)*, subject to the budget constraint* (2.2)*, time restriction* (2.3)*, and education technology* (2.4)*.*

Along with time and budget constraints, an optimal household allocation must satisfy the following system of first order conditions (FOCs):

$$\frac{\partial U}{\partial c_t} = \lambda_t, \tag{2.5}$$

$$\frac{\partial U}{\partial b_t} = \frac{\partial U}{\partial c_t} w_t h_t(\epsilon_t + \bar{\epsilon}), \tag{2.6}$$

$$\frac{\partial U}{\partial h_{t+1}}\frac{\partial h_{t+1}}{\partial \epsilon_t} = \frac{\partial U}{\partial c_t} b_t w_t h_t. \tag{2.7}$$

According to condition (2.5), the optimal consumption level must be such that its marginal utility equals the marginal utility of income (shadow value) λ_t. This means that parents evaluate the cost of having and educating children in terms of forgone consumption. Equation (2.6) then establishes that the marginal utility of having a child must equate its marginal cost measured in forgone consumption, which is positively related to the amount of time invested per child $\epsilon_t + \bar{\epsilon}$. Finally, the marginal utility of devoting time to children's education must be the same as the marginal cost involved (see equation (2.7)), which is increasing in the number of children raised b_t.

[1] If education technology were to be taken as linear in ϵ_t, the existence of an interior solution would require a set of restrictions on the parameters. However, the intuitions would remain unchanged.

Although there is a unique optimal solution that is always interior, a closed form solution cannot be obtained without assuming a functional form for the education technology. Nevertheless, certain optimal allocations can be expressed as functions of the fertility choice b_t^* as follows:

$$c_t^* = w_t h_t \frac{\alpha T b_t^*}{b_t^* + \sigma_t \bar{b}_t}, \tag{2.8}$$

$$n_t^* = \frac{\alpha T b_t^*}{b_t^* + \sigma_t \bar{b}_t}. \tag{2.9}$$

$$\bar{\epsilon} + \epsilon_t^* = \frac{(1-\alpha) b_t^* + \sigma_t \bar{b}_t}{b_t^* + \sigma_t \bar{b}_t} \frac{T}{b_t^*}, \tag{2.10}$$

The spending rule (2.8) indicates that parents with more children devote a larger fraction of the hourly wage to consumption. In the absence of other income sources, the labour supply must coincide with the fraction of hourly wage consumed, leading to the gradient between working hours and fertility rates in equation (2.9). Expression (2.10) establishes the existence of a trade-off between the quantity and quality of children, that is, children with more siblings will receive less education from their parents. Finally, the optimal fertility choice is the root of the following expression:

$$\frac{b_t^* + \sigma_t \bar{b}_t}{T} = (1-\alpha) \frac{\partial H[\epsilon_t, X_t]}{\partial \epsilon_t} \frac{1}{H[\epsilon_t, X_t]}. \tag{2.11}$$

This last equation allows determining the effect of variations in the dynamic index X_t and child rearing costs $\bar{\epsilon}$ upon the optimal fertility choice b_t^*. The way the allocation of time varies in response to such changes may therefore be calculated through equations (2.9) and (2.10).

2.4 Results

This section is devoted to study the implications of the model presented above. The first subsection addresses the cause for the positive gradient between labour supply and fertility decisions. The second one describes the impact that educational changes have on labour supply decisions. The final subsection shows how do higher child rearing costs influence labour supply decisions.

2.4.1 The Gradient between Fertility and Labour Supply

One of this model's main contributions is the positive gradient between working hours and fertility rates stated by equation (2.9). This gradient does not establish causality in either direction, as both variables are endogenous choices. Instead, it emerges due to the existence of rigidities in child quantity. The latter means that parents do not adjust their fertility rate as much as they would do in the absence of fertility norms. To see why such rigidities matter for the allocation of time, it suffices to define the fertility choice b_t and the labour supply n_t as functions of $\epsilon_t + \bar{\epsilon}$, and differentiate the time constraint to obtain

$$\frac{\partial n_t}{\partial(\epsilon_t + \bar{\epsilon})} = -\left(\frac{\partial b_t}{\partial(\epsilon_t + \bar{\epsilon})}\frac{(\epsilon_t + \bar{\epsilon})}{b_t} + 1\right) b_t. \tag{2.12}$$

Equation (2.12) shows that the labor supply response to changes in $\epsilon_t + \bar{\epsilon}$ is governed by the elasticity of fertility b_t with respect to $\epsilon_t + \bar{\epsilon}$. In the absence of fertility norms, and of any other factor that may generate rigidities in child quantity, the labour supply remains constant because parents tend to substitute quantity and quality investment on a one-to-one basis, that is,

$$\frac{\partial b_t}{\partial\left(\epsilon_t + \bar{\epsilon}\right)}\frac{(\epsilon_t + \bar{\epsilon})}{b_t} = -1.$$

By contrast, when fertility norms exist, parents no longer substitute quantity and quality investment on a one-to-one basis, which is what I call rigidities in child quantity. The reason is that the social penalty induced by norms discourages changes in the number of children, reducing the elasticity of fertility:

$$-1 < \frac{\partial b_t}{\partial(\epsilon_t + \bar{\epsilon})}\frac{(\epsilon_t + \bar{\epsilon})}{b_t} < 0.$$

A lower elasticity of b_t with respect to $\epsilon_t + \bar{\epsilon}$ implies that the amount of time devoted to childcare $b_t(\epsilon_t + \bar{\epsilon})$ moves in step with $\epsilon_t + \bar{\epsilon}$. As a result, the effect of an increase in $\epsilon_t + \bar{\epsilon}$ upon the labour supply n_t becomes negative because parents need to comply with their time constraint. In other words, parents reallocate time from work to childcare when the marginal cost of having a child increases, explaining why fertility rates and working hours fall in tandem.

The above analysis shows that the key ingredient for obtaining a positive gradient between working hours and fertility rates involves rigidities in child quantity. This feature emerges here because the social penalty induced by norms discourages changes in the optimal fertility rate. Note, however, that such rigidities arise only if the functional form chosen to capture fertility norms fulfils certain requirements. This becomes clear by applying the implicit function theorem in equation (2.6) to obtain the elasticity of fertility:

$$\frac{\partial b_t}{\partial(\epsilon_t + \bar{\epsilon})}\frac{(\epsilon_t + \bar{\epsilon})}{b_t} = \frac{\frac{\partial U}{\partial c_t} w_t h_t}{\frac{\partial^2 U}{(\partial b_t)^2}\frac{b_t}{(\epsilon_t + \bar{\epsilon})} - \frac{\partial^2 U}{\partial b_t \partial c_t} w_t h_t b_t}.$$

This expressions shows that rigidities in child quantity arise when the denominator takes higher negative values in the presence of fertility norms. Otherwise, fertility norms would have the exact opposite impact to what one would expect, increasing the sensitivity of fertility rates. Avoiding such an implausible outcome therefore imposes certain restrictions upon the functional form that fertility norms may take. Technically, they must increase the degree of concavity of the utility function, meaning that the social penalty should gain strength as parents deviate from socially acceptable behaviour.

2.4.2 Changes in Fertility Norms

As both the empirical and theoretical literature point out, this model predicts that socio-cultural changes in the direction of more lax fertility norms (lower $\sigma_t \bar{b}_t$) lead to lower fertility rates b_t^*. The obvious reason is that parents face then milder social penalties if they choose a smaller family, discouraging them from having as many children. Given the evidence of mothers' labour supply, one would expect that more lax fertility norms should cause an increase in working hours because fertility rates go down. The gradient between fertility rates and working hours, however, does not unambiguously determine in which direction do working hours change when fertility norms do so.

Proposition 1. *An increase in $\sigma_t \bar{b}_t$ leads to a higher optimal fertility rate b_t^*.*

Proof: See the Appendix.

One can see in equation (2.9) that, if fertility rates react more than proportionally to changes in fertility norms $(\partial b_t^* / \partial \sigma_t \bar{b}_t > 1)$, then parents increase their labour supply in response $(\partial n_t^* / \partial \sigma_t \bar{b}_t > 0)$. By contrast, if fertility rates react less than proportionally $(\partial b_t^* / \partial \sigma_t \bar{b}_t < 1)$, the labour supply of parents changes in the opposite direction, $(\partial n_t^* / \partial \sigma_t \bar{b}_t < 0)$. In turn, from equation (2.11), one can easily deduce that the key feature determining the sensitivity of fertility to norms is precisely the education technology. This means that the reaction of working hours to more lax fertility norms can be positive in some historical contexts, but become negative in others, depending on how does the education technology changes over time. In any case, what gradient (2.11) does unambiguously imply is that the ratio of working hours to fertility rates n_t^* / b_t^* goes up when fertility norms become more lax, which is consistent with the above evidence.

Proposition 2. *An increase in $\sigma_t \bar{b}_t$ leads to a lower ratio n_t^*/b_t^*.*

Proof: Taking into account Proposition 1, it follows directly from dividing both sides of equation (2.9) by the optimal fertility rate b_t^*, and computing the derivative with respect to fertility norms $\sigma_t \bar{b}_t$:

$$\frac{\partial (n_t^*/b_t^*)}{\partial \sigma_t \bar{b}_t} = \alpha T \frac{(\partial b_t^*/\partial \sigma_t \bar{b}_t) + 1}{(b_t^* + \sigma_t \bar{b}_t)^2} > 0. \quad \square$$

2.4.3 Educational Changes

The first determinant of fertility considered here are the incentives to educate children, which are embedded in the dynamic index X_t. For instance, Galor (2011) argues that faster innovation (higher X_t) erodes the productive value of existing skills, increasing the marginal utility of children's education. Other factors that might cause a similar "erosion" effect are biased technological change and task automation. Such educational changes involve new incentives that alter the trade-off between quantity and quality of children as stated in Theorem 1.

Theorem 1. *A stronger erosion effect (higher X_t) encourages parents to invest more time in each child's education ϵ_t^*, have fewer children b_t^*, and devote less time to work n_t^*.*

Proof: See the Appendix.

By increasing the marginal utility of children's education (the left hand-side of equation (2.7)), the erosion effect encourages parents to invest more time in each child. Obviously, this raises the marginal cost of having a child (see the right hand-side of equation (2.6)), leading parents to have fewer children. Due to the presence of fertility norms, nevertheless, the number of children b_t^* drops less than its marginal cost $w_t h_t(\epsilon_t + \bar{\epsilon})$ increases, so the amount of time devoted to childcare $b_t^*(\epsilon_t^* + \bar{\epsilon})$ goes up. Parents thus need to reduce their labour supply to satisfy their time constraint. This means that parents react to these new incentives regarding children's education by reallocating time from work to educational childcare because of rigidities in child quantity.

From a unified growth perspective, Theorem 1 implies that those theories based on increasing incentives to educate children should predict a downwards trend in working hours and upwards in childcare time, consistent with the evidence. In the case of Galor (2011), for instance, fertility norms imply that the transition between a Malthusian and Modern regime comes along a shift in parents' allocation of time as innovation accelerates and stimulates children's education. The downwards trend in the labour supply and upwards in childcare time would stop only once the rate of innovation converges to a stable rate (X_t remains constant).

2.4.4 Child Rearing Costs

The erosion effect leads to changes in optimal household allocations because it triggers an increase in the marginal cost of having a child $\epsilon_t + \bar{\epsilon}$. Another factor with a similar effect is an increase in child rearing costs $\bar{\epsilon}$. This may happen, for instance, when parents' labour productivity grows more than children's one (see Guzmán and Weisdorf (2011)). Also, when the price of food or housing costs increase in relation to the hourly wage (e.g., Strulik and Weisdorf (2008)). Another example is that children require more food as natural and social forces select towards higher height and size (see Dalgaard et al. (2021)). Behind such increments in child rearing costs can lie other macroeconomic factors, such as increasing population density, structural transformation, urbanisation, sustained human capital accumulation, etc.

Theorem 2. *An increase in child rearing costs $\bar{\epsilon}$ leads to both lower fertility rates b_t^* and fewer hours worked n_t^*.*

Proof: See the Appendix.

As one would expect, higher child rearing costs increase the marginal cost of having a child (see the right hand-side of equation (2.6)), leading parents to have fewer children. Yet again, the number of children b_t^* drops less than its marginal cost $w_t h_t(\epsilon_t + \bar{\epsilon})$ increases, so the amount of time devoted to childcare $b_t^*(\epsilon_t^* + \bar{\epsilon})$ goes up and working hours n_t^* drop. Note that in this case, the net effect upon ϵ_t^* is ambiguous. On the one hand, higher child rearing costs lead to lower fertility rates (Theorem 2). On the other hand, they also discourage parents from investing time in each child's education (see equation (2.10)). Education technology (2.4) is thus crucial for inferring the net effect on ϵ_t^*.

2.5 Conclusions

This paper shows that fertility norms may alter the way parents allocate time between work and childcare. The reason is that fertility norms generate rigidities in the sense that parents do not adjust their fertility choice as much as they would like in response to new economic and demographic conditions. Because of the latter, both fertility rates and working hours fall when the marginal cost of raising a child increases, which is a central idea in many unified growth theories.

For the sake of expositional clarity, this paper does not address the effect of changes in fertility norms upon the labour supply because the net effect is ambiguous. Socio-cultural changes in favour of smaller families encourage parents to have fewer children and to reallocate time from work to educational childcare. Those same socio-cultural changes, however, also involve a smaller social penalty, weakening the incentives to adjust the labour supply. The net effect could thus change drastically across unified growth theories, or even within a model.

2.6 Appendix

Part 1: Existence and Uniqueness. According to equation (2.10), the investment rule in children's education ϵ_t^* can be expressed as a strictly continuous, decreasing function of the optimal fertility rate b_t^*, which I denote by $\epsilon_t^* = E[b_t^*]$. Note that there is a fertility level k such that the optimal investment becomes zero, i.e. $E[k] = 0$. This means that the solution to the optimization problem stated in Definition 1 is interior, i.e. $\infty > \epsilon_t^* > 0$, only if the optimal fertility choice lies within the interval $k > b_t^* > 0$.

If the optimal investment rule $\epsilon_t^* = E[b_t^*]$ is inserted in equation (2.11), it follows that its right hand-side behaves as a strictly continuous, increasing, and convex function of the optimal fertility rate b_t^*, taking values between 0 and ∞ for $b_t^* \in (0, \kappa]$. By contrast, the left hand-side of equation (2.11) becomes a linearly increasing function of the optimal fertility rate b_t^*, and therefore, takes finite positive values for any $b_t^* \in (0, \kappa]$. By virtue of the Intermediate Value Theorem, one can conclude that equation (2.11) has a unique root b_t^* that lies in the interval $b_t^* \in (0, \kappa)$. In other words, the left and right hand-sides of equation (2.11) intersect only once if one takes into account that $\epsilon_t^* = E[b_t^*]$, which establishes the existence and uniqueness of an interior solution.

Part 2: Changes in Fertility Norms. Given that both sides of equation (2.11) are differentiable in b_t and $\sigma_t \bar{b}_t$, the Implicit Function Theorem establishes that the derivative $\partial b_t^* / \partial \sigma_t \bar{b}_t$ exists. To infer the sign of the derivative, however, we shall resort to comparative statics. Suppose that fertility norms change from $\sigma_t \bar{b}_t = q$ to $\sigma_t \bar{b}_t = a$, with $a > q$. Under the above asserted hypotheses about education technology (2.4), it follows that the the left hand-side of equation (2.11) takes higher positive values any fertility level $b_t \in (0, \kappa)$:

$$b_t^* + a > b_t^* + q, \quad \forall b_t.$$

Now, notice in equation (2.10) that $\bar{\epsilon} + \epsilon_t^*$ takes higher positive values for any fertility level because

$$\frac{(1-\alpha)b_t^* + a}{b_t^* + a} \frac{T}{b_t^*} > \frac{(1-\alpha)b_t^* + q}{b_t^* + q} \frac{T}{b_t^*}, \quad \forall b_t > 0,$$

and thus, $\kappa_a > \kappa_q$. Consequently, the right hand-side of equation (2.11) takes lower positive values for any fertility level:

$$\frac{\partial H[\epsilon, q]}{\partial \epsilon_t} \frac{1}{H[\epsilon_t, q]} > \frac{\partial H[\epsilon_t, a]}{\partial \epsilon_t} \frac{1}{H[\epsilon, a]}, \quad \forall b_t.$$

The Intermediate Value Theorem therefore implies that the root of equation (2.11) b_t^* must take a higher value in situation a compared to q, for any $a > q$. Using the definition of derivative as a limit, one can thus conclude that $\partial b_t^* / \partial \sigma_t \bar{b}_t > 0$.

Part 3: Erosion Effect. Given that both sides of equation (2.11) are differentiable in b_t and X_t, the Implicit Function Theorem establishes that the derivative $\partial b_t^* / \partial X_t$ exists. To infer the sign, however, we shall resort to comparative statics. Suppose that the dynamic index moves from X^0 to X^1, with $X^1 > X^0$ (the erosion effect gains strength). Under the above asserted hypotheses about education technology (2.4), it follows that the right hand-side of equation (2.11) takes higher positive values for any fertility level:

$$\frac{\partial H[\epsilon, X^1]}{\partial \epsilon_t} \frac{1}{H[\epsilon_t, X^1]} > \frac{\partial H[\epsilon_t, X^0]}{\partial \epsilon_t} \frac{1}{H[\epsilon, X^0]}, \quad \forall b_t \in (0, \kappa).$$

By contrast, the left hand-side of equation (2.11) takes the same values for any fertility level $b_t \in (0, \kappa)$. The Intermediate Value Theorem therefore implies that the root of equation (2.11) b_t^* must take a lower value in situation X^1 compared to X^0, for any $X^1 > X^0$. Using the definition of derivative as a limit, one can thus conclude that $\partial b_t^* / \partial X_t < 0$. Finally, applying the chain rule to equations (2.9) and (2.10) establishes that $\partial \epsilon_t^* / \partial X_t > 0$ and $\partial n_t^* / \partial X_t < 0$.

Part 4: Child Rearing Costs. In this case, one needs to take into account that equation (2.10) defines the investment rule ϵ_t as a decreasing function of both b_t and $\bar{\epsilon}$. This implies that, if $\bar{\epsilon}^1 > \bar{\epsilon}^0$, then $\epsilon^1 < \epsilon^0$ for all b_t, and $\kappa^1 < \kappa^0$. From this point on, applying the same logic as with the erosion effect leads to the conclusion that $\partial b_t^* / \partial \bar{\epsilon} < 0$ and $\partial n_t^* / \partial \bar{\epsilon} < 0$. □

Chapter 3

Child Quality Externalities, Fertility, and Economic Growth

3.1 Introduction

This paper sets out to investigate the implications for education, fertility and economic growth of *child quality externalities*. The latter notion refers to the idea that parents value their children's quality, human capital or education relative to their own and that of others. In part, such externalities stem from the fact that parents are reluctant to let their children fall behind on the socio-economic ladder, inducing households to compete for better educational outcomes. Child quality externalities therefore link the incentives to educate children with parents' socio-economic status compared to others.

Empirical studies reveal that child quality externalities are a significant factor in parents' education decisions. For example, Guo and Qu (2022) find that competing for slots in better schools encourages parents to invest more in their children's education. Chen et al. (2022) show evidence linking parents' behaviour to the socio-economic status of the parents of their children's classmates. It is thus unsurprising that parents have become increasingly involved in their children's education, a trend that has been termed "helicopter parenting" (see Guryan et al. (2008) Gimenez-Nadal and Sevilla (2012), and Doepke et al. (2019)).

Despite the importance of this topic, the literature on child quality externalities is relatively recent and therefore scant. So far, it has focused exclusively on children's educational outcomes. However, one would expect child quality externalities to influence parents' fertility decisions because education is one of the costs associated with raising children. Through their microeconomic impact, such externalities are likely to affect economic growth and other macroeconomic dynamics. Education policies should take the existence of child quality externalities into account because both factors affect the incentives to have and educate children.

The main contribution of this paper is to extend the analysis of child quality externalities to fertility and economic growth. To that end, I modify the standard Beckerian framework (see Becker and Lewis (1973)) to include this feature. The model depicts a standard optimisation problem in which parents face a trade-off between the quantity and quality of children subject to these externalities. The resulting fertility and education choices are then placed within a dynamic equilibrium context, where economic growth is determined endogenously by both human capital accumulation and innovation.

The most important implication of the model developed is that child quality externalities may be a fundamental cause of economic growth. This is because they induce parents to compete on children's education, leading to rising spending on education and sustained human capital accumulation. The latter in turn breaks the law of decreasing returns to innovation, allowing for sustained technological progress. Output per worker then grows driven by innovation and human capital accumulation, financing the increasing education spending of parents and keeping the virtuous loop active.

Another interesting implication of the model is that child quality externalities exert a negative effect on fertility. Given that parents with more human capital invest more per child, having a child is obviously more expensive for them. I refer to this as the *education effect*. On the other hand, parents endowed with higher human capital also enjoy higher income levels, which have a positive effect on fertility. The net sign thus depends on which effect is stronger, so non-monotonic relationships are possible in this model as suggested by some studies (see Hazan and Zoabi (2015), Zanin et al. (2015) and Vogl (2015)).

At the macroeconomic scale, the dynamic relationship between economic growth and fertility rates depends on the time horizon considered. Consistent with some empirical studies, fertility reacts procyclically to pure income shocks (see Doepke et al. (2019)). In the long-run, however, the education effect exerts a negative effect on fertility that drives fertility rates down. The model is thus able to explain why economic growth brings declines in fertility (see Herzer et al. (2012)), but without assuming that fertility behaves as an inferior good.

The last question that this paper addresses is the impact of subsidies on private education spending in the presence of child quality externalities. In the absence of quality externalities, one would expect fertility to be inelastic with respect to such subsidies, because parents fully reinvest the subsidy in their children's education. Conversely, in the presence of quality externalities I find that parents exploit subsidies by increasing both the quality and quantity of children. This means that subsiding private education as a function of parents' income or socio-economic status reduces inherited inequality levels while promoting fertility. This opens the door to the design of policies that can help promote long-run economic growth and fertility in highly developed countries.

The layout of the paper is as follows. The next section presents the model and describes its main microeconomic predictions. The third section discusses the long-term implications of child quality externalities. In the forth section, the model is extended to include subsidies to private education. The last section concludes.

3.2 The Model

Time is discrete and periods are ordered by $t \in \mathbb{N}_{+}$. During each period, the economy is comprised by two overlapped generations called adults and children. Families are mono-parental, with all decisions being made by the sole parent in the household, while children remain passive. During adulthood, individuals chose to have b_t children, from which $0 < s_t < 1$ reach adulthood. For the sake of simplicity, we assume that all individuals are identical, with L_t being the number of adults and the indicator of population density in this model.

3.2.1 Demand Side

The key novelty of this model is that it includes child quality externalities. Such externalities imply that parents value their children's human capital h_{t+1} relative to their own h_t and that of others, represented here by the average stock $\bar{h}_t$. To formalize this notion, I consider an otherwise standard child quantity-quality model in which parents have preferences defined over household consumption c_t, the number of surviving children $s_t b_t$, and the quality of their children h_{t+1}. These preferences are represented by a logarithmic utility function of the form:

$$U[c, s, b, h] = \alpha \ln c_t + (1-\alpha) \ln s_t b_t + (1-\alpha)\mu \ln(h_{t+1} - k_t), \qquad 1 > \alpha > 0, \quad (3.1)$$

where index k_t measures the quality externality, and $1 > \mu > 0$ is an intergenerational discount parameter. As mentioned above, child quality externalities are a composite function of parents' human capital and the average level in the economy $\bar{h}_t$:

$$k_t = (h_t)^{\eta} \left(\bar{h}_t\right)^{(1-\eta)} \qquad 1 > \eta > 0.$$

According to this specification, quality externalities become more important as the education level of the society grows. It may therefore be almost negligible in early development regimes, but be of great significance at later stages, inducing observed changes in parenting styles (see Doepke et al. (2019)).

Parents are endowed with one unit of time, which they supply inelastically for a wage $w_t h_t$, with w_t standing for the wage rate per efficient hour worked. Labour is the sole source of income, and it can be spent on consumption c_t and the education of children $b_t d_t$, where d_t is the spending per child. Parents thus face the following budget constraint:

$$w_t h_t = c_t + b_t d_t. \tag{3.2}$$

Children's human capital h_{t+1} is determined by parents' spending according to the following education technology

$$h_{t+1} = d_t^{\phi}, \quad \frac{1}{\mu} > \phi > 0. \tag{3.3}$$

Note that the education technology indicated is allowed to exhibit some degree of increasing returns to education spending as long as it does not break the concavity of the optimisation problem, i.e. $1/\mu > \phi$. Otherwise, there would be no interior solution.

Definition 1. *Given* $\{w_t, h_t, \bar{h}_t, s_t\}$*, an allocation* $\Omega_t^* \equiv \{c_t^*, b_t^*, d_t^*\}$ *is said to be optimal if it maximises utility function* (3.1)*, subject to the budget constraint* (3.2) *and the education technology* (3.3).

An interior optimal household allocation must satisfy the following system of first order conditions (FOCs):

$$\frac{\partial U}{\partial c} = \lambda_t, \tag{3.4}$$

$$\frac{\partial U}{\partial b} = \frac{\partial U}{\partial c} d_t, \tag{3.5}$$

$$\frac{\partial U}{\partial h}\frac{\partial h}{\partial d} = \frac{\partial U}{\partial c} b_t. \tag{3.6}$$

According to equation (3.4), the optimal adult consumption level c_t^* must be such that the marginal utility of consumption equals the marginal utility of income (shadow value) λ_t. This means that the cost of all the other choices is measured in terms of forgone consumption. Condition (3.5) then establishes that the marginal utility of having a child must equal its marginal cost in terms of forgone consumption. Similarly, the marginal utility of educational spending must be the same as the marginal cost involved (see equation (3.6)).

The fact that parents are reluctant to let their children slip down the socio-economic ladder implies that the marginal utility of educating them (the left hand-side of condition (3.6)) increases with parents' human capital. In other words, parents with higher levels of human capital value their children's education more, and therefore have more incentives to invest in it. Indeed, if equations (3.5) and (3.6) are merged, it follows that the optimal education spend per child depends positively on the quality externalities,

$$d_t^* = \left(\frac{k_t}{1-\mu\phi}\right)^{\frac{1}{\phi}}. \tag{3.7}$$

This result does not depend on whether investment in education is measured in terms of goods or time, because child quality externalities always increase the marginal utility of children's education for any education level, boosting the incentives to invest in children's human capital.

Given that parents with more human capital spend more per child in education, they obviously face a higher marginal cost of fertility (see the right hand-side of equation (3.5)). Raising a child is thus more costly for them. I refer to this as the "education effect". At the same time, fertility is subject to a positive income effect because having a child involves no opportunity cost in terms of forgone income. These two insights are captured by the fertility rule:

$$b_t^* = (1-\alpha)w_t h_t \left(\frac{1-\mu\phi}{k_t}\right)^{\frac{1}{\phi}}. \tag{3.8}$$

Note that the positive income effect vanishes if raising a child has an opportunity cost in terms of forgone income (e.g., Kalemli-Ozcan (2002), Kalemli-Ozcan (2003), Strulik (2004a)). The education effect always dominate in that case, giving rise to a negative gradient between fertility and human capital. If there is no such opportunity cost, parents' human capital affects fertility through two opposing channels: the income and education effects, with the net sign being determined by parameters η and ϕ. If $\phi > \eta$, more educated parents have more children because the income effect is stronger than the education effect. By contrast, if $\phi < \eta$, the sign reverses, and the education effect dominates. If $\phi = \eta$ the effects cancel each other out.[1]

Another conclusion that can be drawn is that human capital accumulates sustainedly in the presence of child quality externalities. This is because the fear of raising children who fall behind in the socio-economic ladder encourages parents

[1] To see this, one needs merely to apply logs in equation (3.8) and compute the elasticity of b_t^* w.r.t. h_t.

to invest more and more in their children's education, ensuring increasing levels of human capital. This can be verified by plugging spending rule (3.7) into the education technology (3.3), taking into account that $h = \bar{h}$, which yields the intergenerational accumulation rate

$$g = \frac{h_{t+1} - h_t}{h_t} = \frac{\mu\phi}{1 - \mu\phi} > 0. \tag{3.9}$$

This expression indicates that child quality externalities cause sustained human capital accumulation by inducing competition between household in children's educational outcomes.

3.2.2 Firms

Following Galor (2011), we consider that the supply side of the economy admits a representative firm that produces a final good denoted by Y_t. Production is subject to constant returns to scale, perfect competition, and endogenous productivity growth in the form of human capital accumulation and technological progress. Along with units of efficient labour H_t, the firm exploits natural resources $A_t R$ to produce. Technological progress is embedded in the dynamic efficiency index A_t, while R denotes the fixed stock of natural resources (land). These features are captured by the following production function:

$$Y_t = (A_t R)^{\theta} (H_t)^{1-\theta}, \quad 1 > \theta > 0. \tag{3.10}$$

For the sake of simplicity, I consider that there are no returns to land, so the zero profit condition involves a wage rate per efficient hours worked of

$$w_t = \left(\frac{A_t R}{H_t} \right)^{\theta}. \tag{3.11}$$

An individual's income $w_t h_t$ is thus increasing in its own human capital and technology level A_t, but depends negatively on the aggregate human capital in the economy H_t. Nevertheless, taking into account that all individuals are identical, $H_t = L_t h_t$, it follows that income grows via innovation and human capital accumulation, but declines with population size (L_t):

$$y_t = w_t h_t = \left(\frac{A_t R}{L_t} \right)^{\theta} (h_t)^{1-\theta}. \tag{3.12}$$

3.2.3 Technological Progress

As in Galor (2011), I skip the microeconomic foundations of innovation, assuming that it depends on the existing stock of ideas (A_t) and the average human capital level of the population. Unlike Galor (2011), I consider that there is no population scale effect on innovation, which is governed by the following difference equation:

$$A_{t+1} = A_t + (A_t)^{\beta}(\bar{h}_t)^{1-\beta}, \quad 1 > \beta > 0. \tag{3.13}$$

According to this formulation, existing technology (ideas) A_t exhibit decreasing returns in the production of new ideas. This means that sustained human capital accumulation is required to support a constant rate of innovation, which will depend on the relation between human capital and existing technology (h_t/A_t). In other words, if education levels are too low relative to technology, innovation slows down. On the other hand, high education levels in a low technology environment result in a fast pace of innovation.

3.3 Equilibrium Dynamics

In this section, I explore the dynamic macroeconomic implications of child quality externalities. The previous section shows that the dynamics of human capital formation are affected by the existence of such externalities. The dynamics of innovation are therefore also different in the presence of quality externalities.

Theorem 1. *In the long-run, technology progresses at the same rate as human capital g^*. In the short-run, however, the rate of innovation can be higher or lower than g^* depending on whether the ratio h_t^*/A_t^* starts above or below a certain threshold.*

Proof: See the Appendix.

The main insight behind Theorem 1 is that child quality externalities are a fundamental cause of economic growth as they encourage parents to increase their spending on education from one generation to the next. This in turn leads to a sustained human capital accumulation that breaks the law of decreasing returns on innovation, thus allowing for sustained technological progress. Both factors then converge to a common rate of growth g^*. In the short-run, however, technological progress may exceed its long-run rate if the ratio h_t^*/A_t^* is above a certain steady-state level determined by parameters ϕ, μ and β (see Appendix 1).

Given that human capital and technology grow over time, the education and income effects are active at the same time and interact with each other. Moreover, since household income depends on population density, current fertility also depends on past fertility and mortality. All this can be seen merely by plugging equation (3.12) into the optimal fertility rule (3.8) to obtain the equilibrium fertility rate

$$b_t^* = \delta \left(\frac{A_t^* R}{h_t^* L_t^*} \right)^{\theta} (h_t^*)^{\frac{\phi-1}{\phi}} , \tag{3.14}$$

where δ is a combination of parameters. This equation describes how fertility rates evolve over time. In the short-run, there may be fertility booms and busts caused by shocks to technology, mortality or natural resources. In the long-run, fertility dynamics depend crucially on parameter ϕ.

Theorem 2. *Assume that child mortality converges to a constant rate $s_t = s^{ss}$. If it holds that $1/\mu > \phi > 1$, population size grows over the long-run driven by fertility levels above replacement. If $\phi = 1$, then the population stagnates in the long-run because fertility converges to the replacement level $1/s^{ss}$. Finally, if $\phi < 1$, the population collapses to zero due to below replacement fertility levels.*

Proof: See the Appendix.

The main insight derived from Theorem 2 is that the parameter measuring the returns to education spending, i.e. ϕ, plays a key role in determining long-run demographic dynamics. If the returns to spending are slightly increasing then the income effect dominates over the education effect in the long-run, permitting fertility levels above replacement and, hence, population growth. If the returns to education spending are constant or decreasing, the education effect either cancels out or offsets the income effect, leading to replacement or sub-replacement fertility levels. Recall, however, that if raising a child involves an opportunity cost in terms of forgone consumption the positive income effect on fertility may disappear. In that case, the education effect will always dominate, driving fertility levels down regardless of the returns to education.

One thing that the model does not take into account but is likely to occur during the process of economic development is that the return to education ϕ increases as argued by Galor (2011). One possible reason is that technological progress erodes the productive value of existing skills, raising the value of education. In that case, fertility exhibits a sharp decline as the returns to educational grow. Moreover, the rate of human capital accumulation would also accelerate over time, leading to faster economic growth, in other words, it would lead to unified growth dynamics.

Finally, once the dynamics of human capital, technology and fertility have been established, it is straightforward to obtain the dynamics of income per worked. If fertility converges to replacement levels, then the constant returns to scale of the production function imply that income per worker grows at the same rate as human capital in the long-run. By contrast, if fertility rates converge to below (above) replacement levels, the economy enjoys (suffers) a demographic dividend (penalty).

Theorem 3. *Suppose that child mortality remains constant $s_t = s^{ss}$. If $\phi = 1$, then output per worker grows at rate g^* in the long-run. If $1/\mu > \phi > 1$, then output per worker grows slower than g^* in the long-run. Finally, if $\phi < 1$, output per worker grows faster than g^* in the long-run.*

The proof of Theorem 3 is straightforward in the light of Theorems 1 and 2. If equation (**??**) is broken down in terms of growth rates and steady state conditions are imposed, the following expression is reached

$$g_y^{ss} = g^* - \theta(s^{ss}b^{ss} - 1),$$

where g_y^{ss} is the long-run growth rate of output per worker. Fertility levels at replacement imply that the last term cancels, i.e., $s^{ss}b^{ss} = 1$, and output per worker grows at the same rate as technology and human capital $g_y^{ss} = g^*$. By contrast, if fertility tends to sub-replacement levels, then $1 > s^{ss}b^{ss}$, and the growth rate of output per worker remains higher than the rate of innovation and human capital accumulation $g_y^{ss} > g^*$ (demographic dividend). By contrast, when fertility converges to above replacement levels, output per worker grows more slowly than human capital because the population expands (demographic penalty).

3.4 Subsidies on Private Education Spending

The aim of this section is to investigate whether the existence of child quality externalities may change the quantitative and qualitative implications of subsidies on private education. The latter are defined in this model as a fraction $1 > q_t > 0$ of parents' spend on education d_t that the public sector gives back. This policy is financed by an income tax of $1 > \tau > 0$. In order to differentiate between subsidies on fertility and education, I assume that raising children entails now a cost in terms of goods equal to z_t. The budget constraint reads then as

$$(1 - \tau)w_t h_t = c_t + b_t((1 - q_t)d_t + z_t). \tag{3.15}$$

Moreover, for the sake of simplicity, the returns to education spending are assumed to be constant, i.e. $\phi_t = 1$:

$$h_{t+1} = d_t. \tag{3.16}$$

Recall that parents' control variables remain the same, namely the household consumption level c_t, the per child education expenditure d_t, and fertility b_t. As a result, the only FOCs that change are those of fertility and child quality:

$$\frac{\partial U}{\partial b} = \frac{\partial U}{\partial c}((1-q_t)d_t + z_t), \tag{3.17}$$

$$\frac{\partial U}{\partial h}\frac{\partial h}{\partial d} = \frac{\partial U}{\partial c} b_t(1-q_t). \tag{3.18}$$

Subsidies on private education influence parents' choices because they decrease the cost of educating a child (see the right hand-side of equation (3.18)), leading parents to invest more in their children's education. This intuition is captured by the optimal educational spending rule

$$d_t^* = \frac{k_t}{1-\mu} + \frac{\mu}{1-\mu}\frac{z_t}{1-q_t}. \tag{3.19}$$

Note that the marginal effect of subsidies on private education spending, i.e. $\partial d_t^*/\partial q_t$, does not change in the presence of child quality externalities. Parents, however, spend more on education per child in the presence of such externalities, which implies a lower elasticity of d_t^* with respect to q_t. In other words, the perceptual change in education expenditure d_t^* as a response to a one percent change in subsidy q_t is smaller in the presence of child quality externalities.

Given that there is a trade-off between the quantity and quality of children, it follows that the response of fertility rate b_t^* to changes in subsidy q_t is inversely related to that of education spending d_t^*. Fertility is therefore more sensitive to subsidies in the presence of child quality externalities. This can be verified by looking at the optimal fertility choice:

$$b_t^* = \frac{(1-\alpha)(1-\mu)(1-\tau)w_t h_t}{z_t + (1-q_t)k_t}. \tag{3.20}$$

This expression indicates that, in the absence of quality externalities ($k_t = 0$), fertility is insensitive to subsidies because parents reinvest the full subsidy in their children's education. By contrast, in the presence of quality externalities ($k_t > 0$), parents react to subsidies by increasing both the quantity and quality of children. Furthermore, subsidizing the private education spend dampens the negative effect that child quality externalities have on fertility ("education effect").

The main takeaway from this section is that subsidies on private education simultaneously promote children's education and fertility in the presence of child quality externalities. Implementing subsidies as a function of parents' income or socio-economic status can therefore reduce inherited inequality and promote fertility. Moreover, since promoting human capital formation in children results in faster productivity growth, such subsidies also boost long-run economic growth. The latter becomes evident when the optimal spending rule (3.19) is plugged into the education technology (3.16), emerging the growth rate

$$g_t^* = \frac{\mu}{1-\mu} + \frac{\mu}{1-\mu}\left(\frac{z_t}{h_t}\frac{1}{1-q_t}\right).$$

The combination of child quality externalities and subsidies on private education spend thus promote long-run economic growth and encourage fertility rates simultaneously. This may therefore be a public policy of interest to highly developed countries such as Spain or Italy, which currently face economic slow down and fertility issues.

3.5 Conclusions

This paper investigates fertility and education choices when parents value their children's quality or human capital relative to their own and that of others. Within an endogenous growth model with quantity-quality trade-offs, I show that such *child quality externalities* might cause economic growth and demographic changes. The reason is that they induce households to compete in child quality, leading to rising spending on education and sustained human capital accumulation. In turn, the associated increase in child rearing costs (education) encourages parents to have fewer children.

3.6 Appendix

Proof of Theorem 1

First, I define the ratio $x_t^* = A_t^*/h_t^*$, whose law of movement is obtained as the combination of dynamic equations (3.3) and (3.13):

$$x_{t+1}^* = \frac{1}{(1+g^*)}\left(x_t^* + (x_t^*)^\beta\right). \tag{3.21}$$

Note that the right hand-side of equation (3.21) is strictly increasing, concave and differentiable. Moreover, this law of movement has a unique positive fixed point that takes the value $x^{ss} = (1/(1+g^*))^{1-\beta}$. It thus holds that $x_{t+1}^* - x_t^* > 0$, $\forall x_t^* < x^{ss}$, whereas $x_{t+1}^* - x_t^* < 0$, $\forall x_t^* > x^{ss}$, which means that x^{ss} is asymptotically global stable, i.e., $x_t^* \to x^{ss}$, $\forall x_0 > 0$. The rate of innovation therefore converges to the rate of human capital accumulation g^* for any feasible initial condition.

Proof of Theorem 2

Recall that population size is governed by the law of movement

$$L_{t+1}^* = L_t^* b_t^* s_t^*. \tag{3.22}$$

Plugging the equilibrium fertility rate (3.14) into equation (3.22), and assuming a constant child mortality rate s^{ss} gives a difference equation that governs population size in the long-run:

$$L_{t+1}^* = \delta\left(\frac{A_t^* R}{h_t^*}\right)^\theta (L_t^*)^{1-\theta} (h_t^*)^{\frac{\phi-1}{\phi}} s^{ss}. \tag{3.23}$$

Taking into account that ratio h/A remains constant in the long-run, and rewriting $L_{t+1}^*/L_t^* = a_{t+1}$, it holds that

$$a_{t+1}^* = (a_t)^{1-\theta} (1+g^*)^{\frac{\phi-1}{\phi}}. \tag{3.24}$$

Note that this transformed difference equation exhibits a unique, globally stable fixed point at

$$a^{ss} = (1+g^*)^{\frac{\phi-1}{\phi\theta}}.$$

If $1/\mu > \phi > 1$, then it holds that $a^{ss} > 1$, meaning that the population grows in the long-run and fertility converges to above replacement levels. If $\phi = 1$, it follows immediately that $a^{ss} = 1$, which implies that population size converges because long-run fertility is at replacement levels. Finally, if $\phi < 1$, then $a^{ss} < 1$, meaning that population collapses to zero driven by below replacement fertility levels.

Chapter 4

Human Capital Formation via Investment and LBD: Low vs Modern Growth Dynamics.

4.1 Introduction

One of the most extended ideas in economics is that the accumulation of human capital can sustain economic growth over time. By formalizing this idea, the Uzawa-Lucas model (Uzawa (1965) and Lucas (1988), UL henceforth) became one of the most cited and fruitful frameworks in the growth literature and related fields. But despite its appealing intuitions, long-run economic growth fails to be a general feature of the model. Here, we propose a modification that makes long-term economic growth a general feature.

Essentially, the UL model is an extension of the neoclassical framework that adds two key assumptions about human capital. The first one is that new generations inherit a proportion of the existing human capital, allowing for sustained accumulation at the aggregate scale. The second assumption is that individuals can form human capital if they invest part of their resources, measured in terms of available time, into that objective. As a result, when the economy maintains a positive level of investment, the sustained formation of human capital breaks the stagnant income story predicted by the neoclassical growth model.

The problem is that the UL model does not always predict active investment in the formation of human capital over the long-run. Caballé and Santos (1993) show that the long-run equilibrium path displays active investment only when technology and preferences combine in a certain way.[1] When the labor supply is endogenized by including leisure in the utility function, Ladrón-de Guevara et al. (1999) find that, not only the parametric conditions become even stronger, but also that the right historical conditions (initial state of the economy) are needed. So, the central message of the UL model is that long-run economic growth requires the right technology, preferences and historical conditions, pretty hard requirements.

In this paper, we argue that these predictions arise because the formation of human capital via investment is a partial description of the process of human capital accumulation. Our point is that individuals also acquire and improve skills during the production of goods and services, which is usually known as learning by doing (Arrow (1962), LBD henceforth). In fact, the coexistence of both mechanism was already mentioned in Lucas (1988), but was left unexplored, remaining so in the growth literature.[2] As a consequence, LBD displaces the investment channel as the essential source of human capital and the latter becomes a spurring mechanism.

In order to illustrate the differences between this generalized model and the pure UL model, we study the equilibrium dynamics when the labor supply is fully endogenous. To that end, we follow Ladrón-de Guevara et al. (1999) and consider a set-up in which leisure enters the preferences, where individuals distribute their available time between working, leisure activities and active formation of human capital. Moreover, we abstract from externalities and other distortions, so the equilibrium path matches the optimal solution.

Despite the potential lack of concavity and interiority, our main analytical result is to prove that the optimal solution (hence equilibrium path) entails long-run economic growth regardless of the initial condition under a minimal parametric assumption called "growth condition", which stands in sharp contrast to the pure UL model. In turn, this prediction implies that the interpretation of the model changes when LBD is acknowledged, drawing a different picture of economic growth.

Since inner paths are characterized by active investment in the formation of human capital, which result in faster dynamics, those trajectories can be identified as modern growth dynamics (modern economic growth). In contrast to the UL model, non-interior paths also exhibit long-run economic growth, but the resulting dynamics are slower due to the lack of investment, so they can be interpreted as low-growth dynamics. Thus, in this generalized model, studying the existence and

[1] Specifically, they show that the first order conditions for inner solutions yield a globally stable dynamical system, whose stable point is admissible only when the parameters of the model satisfy certain restrictions.

[2] A microfoundation is given in Killingsworth (1982). Similar notions have been also explored in the business cycles literature (see Einarsson and Marquis (1996)) and in some growth accounting studies (see Tamura et al. (2019)).

features of inner solutions is the same as studying the theoretical properties of modern economic growth.

In addition to the above result, we provide some result regarding modern economic growth. We establish that modern growth dynamics can be optimal if, and only if, they converge to a balanced growth path (BGP). Then, we show that the dynamical system derived from the first order conditions (FOCs) for inner solutions yields up to two inner BGPs, one stable and one unstable, whose existence depends on the parameterization. Also, we find that the unstable ray is a threshold in the ratio of physical to human capital above which modern economic growth ceases to be optimal, which in turn may lead to history dependent outcomes.[3]

Finally, we investigate why does the unstable ray arise. While the optimal leisure allocation is an increasing function of the physical to human capital ratio, it turns out that the optimal labor supply is a non-monotone function. This reveals that the incentives to accumulate human capital vary non-monotonically when physical capital becomes relatively more abundant than human capital due to the asymmetric effect of human capital on each use of time. The consequence is that the laws of human and physical capital accumulation may intersect more than once, leading to a multiplicity of BGPs and history dependence.

The intuition emerging from this generalized model is that modern economic growth takes place when an economy gathers the right combination of technology, preferences and historical conditions. Otherwise, the economy remains tied to low-growth dynamics. This suggests that the emergence of modern economic growth in western Europe could have been caused by a structural change (parameters) or by a major exogenous shock, which is an hypothesis that other authors have already suggested (see Voigtländer and Voth (2012)). Moreover, the model also suggests that modern economic growth may be reversible via emigration or other shocks that increase the ratio of physical to human capital.

The remainder of the paper is as follows. The next section describes the model. In the third section, we discuss the optimality of long-run economic growth. The fourth section is devoted to study the theoretical properties of modern growth dynamics. The last section concludes.

[3]Given that the UL model is nested as an extreme case, our results extend and generalizer some of the results given in Ladrón-de Guevara et al. (1999).

4.2 The model

We specify the model in continuous time running to an infinite horizon, where instants are denoted by $t \in \mathbb{R}_+$. Following the standard, we consider a closed economy that admits a representative infinitely-lived household growing at a constant exogenous rate, enabling us to define all variables in per capita terms. Moreover, we abstract from externalities and distortions, so the competitive equilibrium path coincides with the solution of a social planning problem.

In the UL class of models, human capital has the key property that it spills over generations, in other words, new generations inherit a proportion of the existing human capital. The consequence of this property is that, although human capital depreciates throughout the life cycle, the resulting law of accumulation is linear in the stock of human capital from the point of view of an infinitely-lived household. Thus, at the aggregate scale, human capital can be accumulated sustainedly, opening the door to endogenous economic growth.

In this generalized model, there are two sources of human capital. Like in Lucas (1988), if an individual invests $\epsilon(t)$ hours, it forms $A_\epsilon \epsilon(t) h(t)$ units of human capital. Additionally, we consider the existence of LBD in the production of goods, and $n(t)$ hours worked elicit $A_n n(t) h(t)$ units of human capital. As a result, the law of human capital accumulation reads as

$$\dot{h}(t) \equiv A_\epsilon \epsilon(t) h(t) + A_n n(t) h(t). \tag{4.1}$$

The remainder of the model is purely neoclassical. The economy produces an homogeneous final output denoted as $y(t)$, which is obtained by combining two essential inputs: physical capital $k(t)$, and efficient labor $n(t)h(t)$. The production function is $y(t) \equiv F[k(t), n(t)h(t)]$, where $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is a C^2, strictly increasing and concave mapping.[4] $F[\bullet]$ exhibits constant returns to scale and satisfies the following Inada conditions:

$$\lim_{k \to 0} F_k[k(t), nh] = \infty, \quad \lim_{nh \to 0} F_{nh}[k, n(t)h(t)] = \infty,$$

$$\lim_{k \to \infty} F_k[k(t), nh] = 0, \quad \lim_{nh \to \infty} F_{nh}[k, n(t)h(t)] = 0.$$

Final output can be consumed or invested, so physical capital accumulates as investment less depreciation:

$$\dot{k}(t) \equiv F[k(t), n(t)h(t)] - c(t) - \delta_k k(t), \tag{4.2}$$

[4] By essential it is meant that $F[0, nh] = F[k, 0] = 0$

where $1 > \delta_k > 0$ encompasses depreciation and population growth, while $c(t)$ denotes consumption.

Finally, the representative household derives utility from consumption and devoting time to leisure activities, which is denoted by $l(t)$, under the following preferences

$$U[c(t), l(t)] \equiv \frac{c(t)^{1-\xi} l(t)^{(1-\xi)\omega}}{1-\xi}, \quad \xi > 0, \quad \omega > 0,$$

and discounts future utility at a constant rate $1 > \rho > 0$. Moreover, the amount of time available per person is bounded and normalized to one, so the level of investment in new human capital can be rewritten as $\epsilon(t) \equiv 1 - n(t) - l(t)$.

Definition 1. Define the vector of control variables as $q(t) \equiv \{c(t), l(t), n(t)\}$ and the vector of state variables as $m(t) \equiv \{k(t), h(t)\}$. Then, given an initial date $s \in \mathbb{R}_+$ and a starting condition $m(s) \in \mathbb{R}^2_+/0$, a path $\phi \equiv \{q(t), m(t)\}_{t=s}^{\infty}$ is said to be admissible if it satisfies differential equations (4.1) and (4.2), along with constraints $n(t), l(t), c(t) > 0$, $1 \geq n(t) + l(t) > 0$, and $k(t), h(t) > 0$, $\forall t \in [s, \infty)$. The set of all admissible paths for given $\{s, m(s)\}$ is denoted as $\Phi[s, m(s)]$.

The set of admissible paths is non-empty, e.g. picking $q(t)$ so that $\dot{k}(t) = 0$ is an admissible choice for any initial condition. Each admissible path is then ranked according to the level of welfare that it attains, which is measured by the discounted stream of utility:

$$\sigma[s, k(s), h(s), \phi] \equiv \int_s^{\infty} e^{-\rho t} U[c(t), l(t)]\, dt.$$

Definition 2. An optimal plan or optimal path is an admissible trajectory $\phi^* \in \Phi$ that maximizes welfare:

$$\phi^* = ArgMax\ \sigma[s, k(s), h(s), \phi], \tag{P}$$

where the supreme welfare level is defined as the value function

$$V[s, k(s), h(s)] = Sup\ \sigma[s, k(s), h(s), \phi].$$

Before moving on, we need to impose two restrictions on the parameters. The first one regards the existence of inner solutions. Since there is LBD, investing in human capital formation can be optimal if, and only if, it generates more human capital than the LBD mechanism, i.e. $A_\epsilon > A_n$. Otherwise, the opportunity cost of active investment in the formation of human capital would be too high compared to the LBD channel.

The second restriction is usually known as "growth condition", which ensures that no admissible path generates an utility flow able to offset the inter-temporal discount. Given that the accumulation of human capital is bounded, so the supreme rate at which utility can grow is $(1-\xi)A_\epsilon$, the growth condition reads as $\rho > (1-\xi)A_\epsilon$. Since empirical estimates suggest that most economies fall in the range $\xi \geq 1$ (see Havranek et al. (2015)), we make that assumption.

Assumption 1. The parameters satisfy $\xi \geq 1$ and $A_\epsilon > A_n$. The set of all parameterizations that satisfy Assumption 1 is denoted by $\psi \in \Psi$.

Problem (P) is time-autonomous, meaning that the optimal control choice does not depend on the time index t (see chapter II in Arrow and Kurz (2013)). Therefore, we can focus on rules that choose the control vector as a function of the state vector, which we call *feed-back control* and denote by $q(m)$. In turn, the optimal feed-back allocations determine the optimal variation of the economy via laws of motion (4.1) and (4.2), i.e. $\dot{m}^*(t) = \dot{m}^*(q^*(m), m) = \dot{m}^*(m)$, where the mapping composition $\dot{m}^*(m)$ is referred to as the *policy function.*

Theorem 1. Suppose Assumption 1 holds. Then a unique optimal path ϕ^* exists, which is characterized by a unique feed-back control rule $q^*(m)$ and a unique policy function $\dot{m}^*(m)$.

Proof: Appendix 1. □

The proof of Theorem 1 requires merely standard arguments from the literature. We start by establishing the existence of an optimal plan as in Ladrón-de Guevara et al. (1999). In the second step, we prove the feed-back nature of the optimal control choice. Lastly, we exploit the first order conditions (FOCs) of the stationary Hamilton-Jacobi-Bellman (HJB) equation to prove the uniqueness part. Since the proof has little economic meaning, we relegate it to Appendix 1.

Notice that Theorem 1 has several limitations. First, it does not say whether the policy function is differentiable or continuous everywhere, which depends on the continuity of the optimal feed-back controls. Second, the existence of a unique optimal plan does not imply that any solution to the FOCs is optimal (sufficiency), which would require a concave Hamiltonian. Still, Theorem 1 is essential in what is to come as it reveals two key features about the optimal path.

The first feature is that we can reduce the dimension of the policy function to one variable. To see it, let $z(t) \equiv k(t)/h(t)$ be the univariate state of the economy, and define $x(t) \equiv c(t)/k(t)$. Then the combination of equations (4.1) and (4.2) yields the law of motion for $z(t)$

$$\frac{\dot{z}(t)}{z(t)} = F[1, n(t)/z(t)] - \delta_k - x(t) - A_\epsilon(1 - n(t) - l(t)) - A_n n(t). \tag{4.3}$$

Given that the optimal feed-back allocations are unique for a given state of the economy, we can define the relationship in terms of z as $q^*(z) = \{x^*(z), l^*(z), n^*(z)\}$, becoming the policy function one-dimensional, i.e. $\dot{z}^*(t) = \dot{z}^*(q^*(z), z) = \dot{z}^*(z)$.

Corollary 1. Define $z(t) \equiv k(t)/h(t)$ $x(t) \equiv c(t)/k(t)$. Then the policy function can be reduced to one dimension $\dot{z}^*(z)$, where the optimal feed-back controls are $q^*(z) = \{x^*(z), l^*(z), n^*(z)\}$.

Characterizing a one-dimensional policy function implies that we can resort to standard notions and techniques from the literature at the cost of losing some information. Particularly important is the fact that a unique policy function in terms of z implies that the optimal trajectory $z^*(t)$ must be monotone, which is the second key feature derived from Theorem 1.

Corollary 2. Suppose Assumption 1 holds. Then the optimal state path $\{z^*(t)\}_s^\infty$ is monotone.

Proof: Assume that the optimal state path $z^*(t)$ is not monotone. Due to the continuity of $z^*(t)$, there can be a monotonicity change if, and only if, $\exists a, b, \ \ a \neq b$ such that $z^*(a) = z^*(b)$ and $\dot{z}^*(a) \neq \dot{z}^*(b)$. However, given the uniqueness of $z^*(z)$, it follows that $z^*(z^*(a)) = z^*(z^*(b))$, reaching a contradiction. Thus, the premise is false and the optimal state path must be monotone over time. $\square$

4.3 Long-run Economic Growth

In the UL model, the formation of human capital takes the form of an intertemporal trade-off mechanism (investment). Thus, when the subjective discount rate is too high, the returns are too low, or the state of the economy sets too high opportunity costs, sustained human capital accumulation becomes a suboptimal outcome. By including the LBD mechanism, the formation of human capital is linked to the production of goods and services, making sustained human capital accumulation a general feature of the optimal solution.

However, the fact that the optimal solution exhibits human capital accumulation does not necessary imply that it also entails long-term economic growth. One reason is that the accumulation of human capital can be used to fuel an upward trend in leisure without the need to reduce consumption. Another reason is that human capital accumulation smoothes out the decline in income resulting from letting physical capital depreciate. In turn, such an over-consumption path could lead to income stagnation or decline (below the discounting rate) over the long-run.

The usual way to determine whether the incentives to economic growth dominate over the disincentives relies on the argument of a concave Hamiltonian. Under that premise, all steady states in the system of first order conditions (FOCs) are optimal and the task boils down to check if they entail sustained economic growth. Moreover, if certain interiority conditions hold, the optimal solution is continuous in the initial state of the economy and the policy function is so too. In that case, the steady states are sufficient information to provide a qualitative characterization of the policy function, hence global dynamics.

Definition 3. Steady State. A steady state is a trajectory such that $\dot{z}(t) = \dot{x}(t) = \dot{n}(t) = \dot{l}(t) = 0$, where the stationary points $\{z^{ss}, x^{ss}, n^{ss}, l^{ss}\}$ satisfy Definition 1. Moreover, if the rate of economic growth is positive, then the steady state is called balanced growth path.

In our case, the usual characterization strategy is not generally viable. As pointed out in Ladrón-de Guevara et al. (1999), the inclusion of leisure in the preferences is a potential source of non-concavity. Also, the UL class of models yields non-interior optimal solutions in many circumstances. Nevertheless, the FOCs still provide very useful information, more than it is usually exploited in the characterization of optimal growth problems. Specifically, they can be used so as to reject hypotheses about the optimal solution, and even more importantly, about the policy function, which is the characterization strategy that we apply.

Recall that an optimal steady state belongs to the policy function in the form of a root $\dot{z}^*(z^{ss}) = 0$. This implies that, when the optimal state path does not converge to a steady state, a tail of the policy function must be "divergent". Formally, the right-tail of the policy function is divergent if there is some point z^r such that $\dot{z}^*(z) > 0, \forall z \geq z^r$. Similarly, the left-tail is divergent if $\exists z^l$ such that $\dot{z}^*(z) < 0, \forall z \leq z^l$. Our characterization strategy consists then in establishing that both tails of the policy function are convergent or non-divergent.

Proving the convergent shape of the right tail is straightforward, and follows from the very same reason that blocks economic growth in the standard neoclassical model. The divergence towards infinite implies that the rate of physical capital accumulation must be permanently higher than the rate of human capital accumulation. Given the decreasing returns of the production function and depreciation of physical capital, it is impossible to sustain that situation even if the rate of human capital accumulation is zero, hence $\nexists z^r$.

More complicated is to establish that the left-tail is convergent. In this case, the premise of divergence implies that the rate of human capital accumulation is permanently higher than that of physical capital, which some admissible paths satisfy. Therefore, we need to use the system of FOCs derived from the principle of optimality to prove that those admissible paths are suboptimal.

Let the Hamiltonian to problem (P) be

$$H[t, q(t), m(t), \lambda(t)] = e^{-\rho t}U[c(t), l(t)] + \lambda_1(t)\dot{k}(t) + \lambda_2(t)\dot{h}(t) + \lambda_3(t)(1 - n(t) - l(t)),$$

where $\lambda(t) = (\lambda_1(t), \lambda_2(t))$ are the usual co-states variables and $\lambda_3(t)$ is a shadow variable included to ensure that the level of investment in human capital remains non-negative. The principle of optimality states that the optimal path must satisfy equations (4.1), (4.2),

$$\lambda_3(t)(1 - n(t) - l(t)) = 0, \tag{4.4}$$

$$e^{-\rho t}U_c[c(t), l(t)] = \lambda_1(t), \tag{4.5}$$

$$e^{-\rho t}U_l[c(t), l(t)] - \lambda_2(t)A_\epsilon h(t) = \lambda_3(t), \tag{4.6}$$

$$\lambda_1(t)F_{nh}[k(t), n(t)h(t)]h(t) - \lambda_2(t)(A_\epsilon - A_n)h(t) = \lambda_3(t), \tag{4.7}$$

$$-\frac{\dot{\lambda}_1(t)}{\lambda_1(t)} = F_k[k(t), n(t)h(t)] - \delta_k, \tag{4.8}$$

$$-\frac{\dot{\lambda}_2(t)}{\lambda_2(t)} = \frac{\lambda_1(t)}{\lambda_2(t)}F_{nh}[k(t), n(t)h(t)]n(t) + A_\epsilon(1 - n(t) - l(t)) + A_n n(t), \tag{4.9}$$

for almost every t, along with transversality conditions

$$\lim_{t\to\infty} \lambda_1(t)k(t) = 0, \tag{4.10}$$

$$\lim_{t\to\infty} \lambda_2(t)h(t) = 0. \tag{4.11}$$

Intuitively, when the state path tends towards zero, the time boundary drives the marginal productivities of physical capital and efficient labor to infinite and zero respectively. In turn, through the co-states, the incentives to save and accumulate physical capital become stronger along such a path, while the incentives to accumulate human capital decline. Thus, it comes a point in which the optimal choice is to reverse the trend, meaning that $\nexists z^l$.

Given that both tails of the policy function are convergent, there is at least one point at which the sign of the policy function changes from positive to negative. If such points are assumed to be discontinuities in the policy function, the optimal state path would enter a discontinuous loop when reaching those states of the economy, this is, it would jump between two levels again and again. On the other hand, given Theorem 1, we know that the optimal state path must be absolutely continuous and monotone. Therefore, we know that these points must be roots and the optimal trajectory must converge to a steady state for any initial condition.

We should stress out that this convergence property does not mean that all steady states in the system of FOCs are optimal. Moreover, the convergence property does not determine how much roots (steady states) has the policy function. It could be the case that the optimal path tends to the same steady state for all initial conditions (global stability), or alternatively, some initial conditions lead to a different steady state (history dependence). Though, the convergence property does imply that an unstable root in the policy function is a sufficient condition for history dependence.

Another remark about the convergence property is that we do not invoke the growth condition in any part of the proof, so the result should remain valid for $\xi < 1$. Also, since the UL model and the pure LBD model are nested as extreme cases i.e. $A_\epsilon > 0, A_n = 0$ and $A_\epsilon = 0, A_n > 0$ respectively, the convergence property holds for those pure models. Moreover, the result is easily extensible to set-ups with exogenous leisure, and should be so too when leisure enters the utility function augmented by other variables such as human capital.[5]

Finally, given that the principle of optimality guarantees that all steady states in the policy function exhibit production of goods and services, the inclusion of LBD creates a correspondence between the notions of steady state and balanced growth path. Hence, the convergence property implies that the optimal trajectory tends towards a BGP for all parameterizations and initial conditions, which is the crucial difference with respect to the UL model.

[5]The convergence of inner solution in the UL model with exogenous leisure is established in Caballé and Santos (1993), and our result just extends it to non-interior solutions. In the case that leisure is endogenous, Ladrón-de Guevara et al. (1999) establishes that convergence is a sufficient condition for optimality, while our result implies that it is a necessary condition.

Theorem 2. Suppose Assumption 1 holds. Then the optimal solution tends towards a balanced growth path for every initial condition.

Proof: Appendix 2. □

As said above, a sufficient condition for history dependence is the existence of an unstable root in the policy function. When the economy starts with a ratio of physical to human capital below an unstable root, i.e. $z(s) < z_2^{ss}$, the optimal state path leads to a stable BGP at left of the unstable BGP. But if the initial state of the economy is above, i.e. $z(s) > z_2^{ss}$, then the optimal state path leads to a different BGP at the right. Finally, if the economy starts at the unstable BGP, i.e. $z(s) = z_2^{ss}$, it will remain there until the slightest shock alters the ratio of physical to human capital.

Corollary 3. Suppose Assumption 1 holds. If $\exists z_2^{ss}$ such that $\dot{z}^*(z_2^{ss}) = 0$ with a positive slope $\partial \dot{z}^*(z_2^{ss})/\partial z > 0$, there are other two roots in the policy function such that $z_1^{ss} < z_2^{ss} < z_3^{ss}$, $z^*(t) \to z_1^{ss}$, $\forall z(s) < z_2^{ss}$, and $z^*(t) \to z_3^{ss}$, $\forall z(s) > z_2^{ss}$.

Proof: Suppose $\exists z_2^{ss}$ such that $\dot{z}^*(z_2^{ss}) = 0$, and $\partial \dot{z}^*(z_2^{ss})/\partial z > 0$. If $\nexists z_1^{ss}$ such that $z_1^{ss} < z_2^{ss}$, then Theorem 2 is violated for every $z(s) < z_2^{ss}$, hence $\exists z_1^{ss}$. The same argument applies to $\exists z_3^{ss}$, $z_2^{ss} < z_3^{ss}$. □

The most important implication of Theorem 2 is following. Since inner paths are characterized by active investment in the formation of human capital and the resulting dynamics are faster, we can identify those trajectories as modern growth dynamics. In contrast to the UL model, non-interior paths also exhibit long-run economic growth, but the resulting dynamics are slower due to the lack of investment, so they can be interpreted as low-growth dynamics. Thus, in this generalized model, studying the existence and characteristics of inner solutions is the same as identifying the theoretical properties of modern economic growth.

4.4 Modern Economic Growth

Given that all economies have access to low-growth dynamics, we will devote the reminder of the paper to study the existence and characteristics of modern economic growth. We recall that modern growth dynamics are defined as a trajectory along which the economy actively invests in the formation of human capital, i.e. $1 - n^* - l^* > 0$.

4.4.1 Existence

From Theorem 2, we know that modern growth dynamics can be optimal if, and only if, they converge to a BGP. To find such rays, we set $\lambda_3 = 0$ and impose stationary conditions in the system of FOCs. Substituting out all equations, we reach a polynomial in steady leisure, whose roots correspond to the stationary rays of the dynamical system.

It turns out that the polynomial is "quadratic", meaning that the dynamical system has either zero or two stationary points at the same time, but never only one. The stationary points, say l_1 and l_2, satisfy that $l_1 < l_2$ and $z_1 < z_2$. Moreover, making some numerical computations, we find that the stationary point with less time devoted to leisure l_1 is always saddle-path stable, whereas the stationary point with more time devoted to leisure l_2 is always unstable.

Depending on the parameterization, these stationary points satisfy Definition 1 and correspond to a BGP or not, existing three cases. In one, the combination of preferences and technology yields no inner BGP. In other case, the dynamical system exhibits one inner BGP, which is saddle-path stable. In the last case, the dynamical system gives rise to two inner BGPs, one saddle-path stable and one unstable.

Proposition 1. Suppose Assumption 1 holds. Denote the number of inner Balanced Growth Paths in system (1)-(9) for a given parameterization by $r(\psi)$. Then there are three non-empty parametric sub-spaces denoted as $\Psi_1, \Psi_2, \Psi_3 \subset \Psi$ such that: $r(\psi) = 0,\ \forall\psi \in \Psi_1$, $r(\psi) = 1,\ \forall\psi \in \Psi_2$, and $r(\psi) = 2,\ \forall\psi \in \Psi_3$. Moreover, when $\psi \in \Psi_3$, it holds that $z_1^{ss} < z_2^{ss}$, $l_1^{ss} < l_2^{ss}$, $n_1^{ss} \geq n_2^{ss}$, and $g_1^{ss} > g_2^{ss}$.[6]

Proof: Appendix 3. □

The main intuition behind Proposition 1 is that, for some economies $\psi \in \Psi_1$, modern economic growth is never optimal. This can be because the subjective discount rate ρ is too high, or because the returns to active investment A_ϵ are too low. When the welfare theorems hold and the equilibrium path matches the optimal solution, this result implies that those economies will be tied to low-growth dynamics regardless of the initial state of the economy, and exogenous interventions or shock are unable to promote modern economic growth unless they alter the underlying parameterization of the economy.

[6]This Proposition extends the result given in Ladrón-de Guevara et al. (1999) to neoclassical production functions with non-constant factor shares.

Although establishing analytically that the stable, inner BGP belongs to the policy function is quite complex, we can be pretty sure that it does. Otherwise, Problem (P) would never have an inner solution. In fact, it has been proven so when $A_n = 0$ (see Ladrón-de Guevara et al. (1999)), which is an extreme case of our generalized model. A simple continuity argument implies then that the stable, inner BGP belongs to the policy function when $\psi \in \Psi_2 \cup \Psi_3$. Regarding the equilibrium path, this result implies that some economies exhibit modern growth dynamics if the combination of technology, preferences and historical conditions is right.

About the unstable BGP, due to the potential lack of concavity, we cannot ascertain whether it belongs to the policy function as a threshold, if it indicates a discontinuity in the policy function, etc. Nevertheless, given that it arises from the same dynamical system as the stable BGP, it probably indicates when do modern growth dynamics cease to be optimal. In that case, an economy with the right combination of technology and preferences would still exhibit low-growth if the state of the economy is inadequate, which is a question that we address in the next subsection.

4.4.2 Interval of Existence

Since the optimal allocations $n^*(z)$ and $l^*(z)$ are continuous almost everywhere in z, we know that there is an arbitrarily small interval in the z axis for which modern growth dynamics are optimal, which we call interval of existence.

Definition 4. Interval of Existence. Let $z \in (\bar{z}_L, \bar{z}_U)$ be the interval os states of the economy for which it holds that $0 < n^*(z) + l^*(z) < 1$.

Out of the interval of existence, the optimal solution is non-interior. Therefore, if the interval is finite, we know that the incentives to accumulate human capital dilute when the ratio of physical to human capital becomes too unbalanced, as no active investment in the formation of human capital takes place. In turn, this may break the global stability of the inner BGP, leading to history dependent outcomes.

The finiteness of $(\bar{z}_L, \bar{z}_U)$ can be determined by playing around with the value function and the co-states. To see it, recall that the time-invariant nature of Problem (P) implies that the value function can be rewritten in stationary terms $V[t, m(t)] = e^{-\rho t} v[m(t)]$. Then, $v[m(t)]$ is the solution to the stationary Hamilton-

Jacobi-Bellman (HJB) equation:

$$v[m(t)] = U[q^*(t)] + v_k[m(t)]\dot{k}^*(t) + v_h[m(t)]\dot{h}^*(t),$$

where $\dot{k}^*(t)$ and $\dot{h}^*(t)$ are the optimal variation of each state variable (see chapter VII in Acemoglu (2009)). In turn, the partials of the value function are related to the co-states derived from the principle of optimality:

$$\lambda_1(t) = e^{-\rho t} v_k[k(t), h(t)], \quad \lambda_2(t) = e^{-\rho t} v_h[k(t), h(t)].$$

Thus, if the laws of motion (4.1) and (4.2) are combined with equations (4.8) and (4.9) so as to eliminate time in the following way:

$$\frac{\dot{\lambda}_2(t)}{\dot{h}(t)} = \frac{\partial \lambda_2(t)/\partial t}{\partial h(t)/\partial t} = \frac{\partial \lambda_2(k, h, t)}{\partial h} = e^{-\rho t} v_{hh}[k, h],$$

the sigh of each element in the Hessian matrix can be unravelled.

Setting $\lambda_3 = 0$, equations (4.5) and (4.6) equal each co-state to the marginal utility of consumption and leisure respectively, so it must hold that both co-states take positive values along an inner solution:

$$\lambda_1(t) = e^{-\rho t} U_c[\bullet] > 0, \quad \lambda_2(t) = \frac{e^{-\rho t} U_l[\bullet]}{A_\epsilon h(t)} > 0, \quad \forall t, z \in (\bar{z}_L, \bar{z}_U).$$

Taking into account this information about the co-states, equation (4.9) imposes that $\dot{\lambda}_2(t) < 0, \ \forall t, z \in (\bar{z}_L, \bar{z}_U)$. Moreover, equation (4.1) implies that $\dot{h}(t) > 0, \forall t, z$. Thus, applying the chain rule as explained above, we obtain that

$$\frac{\dot{\lambda}_2(t)}{\dot{h}(t)} = e^{-\rho t} v_{hh}[\bullet] < 0, \quad \forall t, z \in (\bar{z}_L, \bar{z}_U).$$

Notice that each element in the Hessian matrix must take finite values, if not, equations (4.5) and (4.6) would imply that some control variable takes a non-admissible value, so it must hold that

$$-\infty < v_{kh}[\bullet] < \infty, \ \forall z \in (\bar{z}_L, \bar{z}_U).$$

Since the cross-partial derivatives of the value function are given by $\dot{\lambda}_2/\dot{k}$, and the numerator takes finite positive values, it follows that physical capital must follow a monotonic path, i.e. $\dot{k}(t) \neq 0, \forall t, z \in (\bar{z}_L, \bar{z}_U)$. Moreover, given Theorem 2, it must be that physical capital accumulates along an inner optimal solution

$\dot{k}(t) > 0, \forall t, z \in (\bar{z}_L, \bar{z}_U)$. Also, from the above paragraph, we know that $\dot{\lambda}_2(t) < 0, \forall t, z \in (\bar{z}_L, \bar{z}_U)$, which combined with the previous line means that

$$\frac{\dot{\lambda}_2(t)}{\dot{k}(t)} = e^{-\rho t} v_{hk}[\bullet] < 0, \quad \forall t, z \in (\bar{z}_L, \bar{z}_U),$$

and due to symmetry, that $v_{kh}[\bullet] < 0, \forall z \in (\bar{z}_L, \bar{z}_U)$.

Now we establish the sign of the last element in the Hessian. From the above paragraph, we know that

$$e^{-\rho t} v_{kh}[\bullet] = \frac{\dot{\lambda}_1(t)}{\dot{h}(t)} < 0, \quad \forall t, z \in (\bar{z}_L, \bar{z}_U),$$

and since $\dot{h}(t) > 0, \forall t, z \in (\bar{z}_L, \bar{z}_U)$, it must hold that $\dot{\lambda}_1(t) < 0, \forall t, z \in (\bar{z}_L, \bar{z}_U)$. Moreover, from the above paragraph we also know that $\dot{k}(t) > 0, \forall t, z \in (\bar{z}_L, \bar{z}_U)$. Thus, it follows that

$$\frac{\dot{\lambda}_1(t)}{\dot{k}(t)} = e^{-\rho t} v_{kk}[\bullet] < 0, \quad \forall t, z \in (\bar{z}_L, \bar{z}_U),$$

and all elements in the Hessian matrix of the value function take negative values when the optimal trajectory is interior.

Lemma 1. All elements in the Hessian matrix of $v[k, h]$ take negative values when $z \in (\bar{z}_L, \bar{z}_U)$.

With the information given in the last couple paragraphs, we can prove that $\bar{z}_U$ is finite. On the one hand, we know that physical must accumulate along an interior solution, i.e. $\dot{k}(t) > 0$, $\forall z \in (\bar{z}_L, \bar{z}_U)$. On the other hand, since the amount of time available is bounded above, $n^*(z) < 1$, $\forall z$, we also know that the ratio y/k tends towards zero as physical capital becomes relatively more abundant than human capital, i.e. $\lim_{z\to\infty} F[1, n^*(z)/z] = 0$. This implies the existence of a finite positive threshold, say $0 < \bar{z}_U < \infty$, such that $\delta_k > F[n^*(z)/z]$, $\forall z > \bar{z}_U$. As a result, law of motion (4.2) implies that $\dot{k}(t) < 0$, $\forall z > \bar{z}_U$. The conclusion is then that physical capital must decay when it is relatively more abundant than human capital, and the solution to Problem (P) must be non-interior when the ratio of capitals is above threshold $\bar{z}_U$.

Lemma 2. Suppose Assumption 1 holds. Then the interval of existence $z \in (\bar{z}_L, \bar{z}_U)$ is finite as $\bar{z}_U < \infty$.

Lemma 2 confirms that modern economic growth is a suboptimal outcome when the ratio of physical to human capital is above a certain threshold. Paradoxically, one should expect that the incentives to accumulate human capital rise as physical capital becomes more abundant than human capital, but Lemma 2 indicates the opposite, at least for large unbalances. Regarding the equilibrium path, this result means that, even if the economy has the right combination of technology and preferences, it will display modern growth dynamics only below that threshold.

4.4.3 Optimal Feed-Back Allocations

We can learn why do the incentives to accumulate human capital dilute as physical capital becomes relatively more abundant by characterizing the optimal feed-back allocations. This can be done by exploiting the Hessian further. For expositional simplicity, we provide a characterization for a unitary elasticity of inter-temporal substitution.

Assuming $\xi = 1$ implies that the partials of the value function v_k and v_h are homogeneous of degree -1. Multiplying both sides of the FOC with respect to consumption by h, we obtain that

$$\frac{h}{c^*(k,h)} = v_k[z,1].$$

Taking into account that $v_{kk} < 0$, it follows that the ratio of consumption to human capital c/h is an increasing function of z. Consequently, consumption increases with physical capital and decreases with human capital: $\partial c^*(k,h)/\partial k > 0$ and $\partial c^*(k,h)/\partial h < 0$. Moreover, if we multiply by k instead of h, the FOC reads as

$$\frac{k}{c^*(k,h)} = v_k[1,1/z].$$

Given that $v_{hk} < 0$, it emerges that $x^*(z)$ is a decreasing function of z, and consumption increases less than proportionally with respect to physical capital, $0 < \partial c^*(k,h)/\partial k < 1$.

Moving to the optimal leisure level, the associated FOC reads as

$$\frac{1}{l^*(z)} = v_h[z,1]A_\epsilon,$$

and since $v_{hk} < 0$, the optimal leisure level must be a monotonic increasing function of z. This result implies that the incentives to devote time to leisure rise as physical capital becomes more abundant than human capital, which gives a partial answer to our opening question.

The last part regards the optimal labor supply. Computing equation (4.6) in terms of growth rates yields

$$-\frac{\dot{l}(t)}{l(t)} = \frac{\dot{\lambda}_2(t)}{\lambda_2(t)} + \frac{\dot{h}(t)}{h(t)}.$$

Substituting out equations (4.7) and (4.9) in the previous one leads to

$$\dot{l}(t) = l(t)((A_n - A_\epsilon)n(t) - \rho).$$

Eliminating time and taking into account that the optimal leisure choice is an increasing function, the last equation implies that

$$\frac{l^*(z)((A_\epsilon - A_n)n^*(z) - \rho)}{z(\dot{z}^*(z))} > 0.$$

Two things can be deduced from the above inequality. First, the optimal labor supply at an inner BGP must be $n^*(z^{ss}) = \rho/(A_\epsilon - A_n)$. Second, the slope of the optimal labor supply depends on the sign of the policy function, since $\dot{z}^*(z) < 0$ implies that $n^*(z) < \rho/(A_\epsilon - A_n)$, while $\dot{z}^*(z) > 0$ implies that $n^*(z) > \rho/(A_\epsilon - A_n)$. Thus, the slope of the optimal labor supply in the neighbourhood of a stable BGP must be negative, and positive in the vicinity of an unstable BGP.

We still have to address the monotonicity of the labor supply, whose associated FOC reads as

$$F_{nh}[z, n^*(z)] = \frac{v_h[z,1]}{v_k[z,1]}\left(A_\epsilon - A_n\right),$$

where we have exploited the homogeneity of the value function. Since the optimal labor supply is decreasing in the neighbourhood of a stable BGP, we know that the left hand-side of the previous equation is increasing in an arbitrarily small neighbourhood $(z_1^{ss} - a, z_1^{ss} + a),\ a > 0$. Consequently, the ratio v_h/v_k must be increasing within the same neighbourhood $(z_1^{ss} - a, z_1^{ss} + a)$, and since all elements in the Hessian are negative, it follows that $v_{hk}[z,1] < v_{kk}[z,1],\ \forall z \in (z_1^{ss} - a, z_1^{ss} + a)$.

Now assume that $v_{hk}[z,1] < v_{kk}[z,1]$ holds for states of the economy such that $z > z_1^{ss} + a$. Then, since both partials are bounded below by zero, i.e.

$$\lim_{z\to\infty} v_h[z,1] = \lim_{z\to\infty} v_k[z,1] = 0,$$

the premise implies that $v_h[z,1] = 0$ for some finite z. However, this is impossible along an inner solution as the FOCs would imply that some control variable takes a non-admissible value. Consequently, the premise is false and there must be some point, say $\bar{z}_M$, where the slope of ratio v_h/v_k turns to $v_{hk}[z,1] > v_{kk}[z,1]$, $\forall z > \bar{z}_M$. Given the FOC with respect to labor, this can happen if, and only if, the slope of the optimal labor supply also changes at $\bar{z}_M$, so $\partial n^*(z)/\partial z > 0$, $\forall z \in (\bar{z}_M, \bar{z}_U)$.

Summarizing the last couple paragraphs, the optimal labor supply in modern growing economies takes the shape of a skewed concave parabola "$\cup$", where the bottom of the parabola is reached at some point $\bar{z}_M$. This means that, when physical capital becomes relatively more abundant than human capital, the state of the economy encourages the supply of labor and devoting time to leisure activities simultaneously, at least for large unbalances. Consequently, the rate of human capital accumulation $\gamma_h^*(z)$ is also a sort of skewed concave parabola "$\cap$", which explains why the laws of human and physical capital accumulation intersect more than once, resulting in a multiplicity of BGP and history dependence. This completes the answer to our opening question.

4.5 Conclusions

In this paper, we show that the predictions of the Uzawa-Lucas (UL) model change notably when an additional source of human capital in the form of LBD in the production of goods is included. This is because including LBD links the formation of human capital to economic activity, making long-run economic growth a general feature of the model. In turn, this provides a more realistic picture in which all economies grow, but only those that gather the right combination of technology, preferences and historical conditions enjoy modern economic growth.

To provide a clear analysis, we have considered the simplest case without externalities of any kind. However, since the LBD mechanism alters the incentives to supply labor and save, including LBD represents a fruitful avenue for future research in several directions. For instance, the magnitude of some well known externalities affects the stability properties of the equilibrium path in this class of models (see Bella et al. (2017)). Thus, the LBD mechanism could create, increase or reduce the plausibility of chaotic or cyclical dynamics. It could also affect the local or global determinacy of the equilibrium path in other set-ups.

Lastly, another difference with respect to the UL model is that including LBD ensures development dynamics when the economy starts at a subsistence production capacity. Thus, it would be very interesting to analyze the equilibrium dynamics when the initial production capacity is close to subsistence levels as in Galor (2011) and other unified growth models. A priori, the model could display two different types of development paths depending on the initial state of the economy: physical or human capital driven take-off.

4.6 Appendices

Appendix 1: Proof of Theorem 1.

For the sake of clarity, we divide the proof into a set of Propositions. The first one regards the existence of an optimal solution, for which we replicate Ladrón-de Guevara et al. (1999).

Proposition A. Existence. $\exists\phi^*$, $\forall m(s)$, and the value function corresponds to the global maximum

$$V[s,k(s),h(s)] = Max\ \sigma[s,m(s),\phi].$$

Proof: Note that the set of admissible solutions is time-invariant, i.e. $\Phi[j,m(j)] = \Phi[g,m(g)]$, $\forall m(j)=m(g)$, so we can simplify notation to $\Phi[m]$. In order to prove that the set of admissible paths is non-empty, consider the example of a steady trajectory $\bar{c}>0$, $n+l=1$, $1>n=1-l>0$, $\forall t$ such that $\dot{k}(s)=0$, which satisfies Definition 1 for any initial condition. Notice that this trajectory attains a finite welfare level

$$\frac{U[\bar{c},l]}{\rho} > -\infty.$$

Now rewrite the problem in a standard reduced from:

$$w(m,\dot{m}) = Max_q U[q(t),m(t)]$$

$$s.t;\ \dot{k}(t) = F[\bullet] - c(t) - \delta_k k(t),$$

$$\dot{h}(t) = A_\epsilon(1-n(t)-l(t))h(t) + A_n n(t)h(t).$$

Then, $m^*(t)$ is an optimal solution to problem (P) if it solves (P'),

$$W(m,\dot{m}) = Max\int_0^\infty e^{-\rho t}w(m,\dot{m})\ dt.\quad (P')$$

As shown above, there is an admissible path that reaches a finite welfare level, i.e. $-\infty < w(m,\dot{m})$, while Assumption 1 ensures that none admissible paths reaches infinite welfare levels: $w(m,\dot{m}) < \infty$, $\forall\{s,m(s)\}$. Since our asserted hypothesis about U and F imply that $w(m,\dot{m})$ is strictly concave in the controls $\dot{m}$, the standard theory establishes the existence of an absolute continuous optimal state path $\{m^*(t)\}_{t=0}^\infty$ (see Fleming and Rishel (2012)), and the supreme welfare value can be reached

$$V[s,k(s),h(s)] = Max_\phi\sigma[s,m(s),\phi].$$

□

The next result is again standard. It states that the optimal control vector is time invariant, i.e. it does not depend on the date or period "t". From this follows that the optimal control can be chosen as a function of the state variable, establishing its feed-back nature.

Proposition B. Dimension reduction. An optimal path is characterized by an optimal feed-back rule $q^*(t) \equiv q^*(m)$, and a *policy function* $\dot{m}^*(t) \equiv \dot{m}^*(m)$.

Proof: Recall that the admissible set is time autonomous and denoted as $\Phi[m]$. Consider two paths ϕ^a and ϕ^b, which are admissible for a given initial condition $\bar{m}$. Let $D[s,\bar{m}] \equiv \sigma[s,\bar{m},\phi^a] - \sigma[s,\bar{m},\phi^b]$ be the difference in attained welfare. Due to the properties of the integral, we have that

$$D[s,\bar{m}] \equiv e^{-\rho s} \int_s^\infty e^{-\rho(t-s)} \left(U[q^a(t)] - U[q^b(t)]\right)\ dt.$$

Since the integral in the right hand-side of the previous identity is independent of s, we can differentiate with respect to the starting date and obtain the following differential equation

$$\frac{\partial D[s,\bar{m}]}{\partial s} = -\rho D[s,\bar{m}],$$

whose solution is

$$D[s,\bar{m}] = e^{-\rho s}\beta.$$

Suppose that path ϕ^a is optimal for starting date j, i.e. $D[j,\bar{m}] \geq 0,\ \forall \phi \in \Phi[\bar{m}]$, which implies that $\beta \geq 0,\ \forall \phi \in \Phi[\bar{m}]$. Now assume that path ϕ^a is no longer optimal path for starting date g, and $\exists \phi^b$ such that $D[g,\bar{m}] < 0$, which implies that $\beta < 0$. Since both lines contradict each other, it must be the case that, if ϕ^a is optimal for some initial date, then it is optimal for any initial date as long as $m(s) = \bar{m}$.

We shall extended the above stated property to any point in time. Denote the optimal control for initial condition $\bar{m}$ as $\bar{q}$. Now suppose that $m(s) \neq \bar{m}$ but there is some $j > s$ such hat $m(j) = \bar{m}$. Under that premise, and exploiting the properties of the value function, we obtain that

$$V[s,m(s)] = \int_s^j e^{-\rho t} U[q^*(t)]\ dt + \int_j^\infty e^{-\rho t} U[q^*(t)]\ dt.$$

Given that

$$\int_j^\infty e^{-\rho t} U[q^*(t)] = V[j,m(j)],$$

and $m(j) = \bar{m}$, it follows that $q^*(j) = \bar{q}$, $\forall j \geq s$, establishing the existence of an optimal feed-back rule $q^*(t) = q^*(m(t))$. Moreover, plugging the optimal feed-back into the laws of motion (1) and (2), the mapping composition establishes the existence of a policy function $\dot{m}^*(t) = \dot{m}^*(q^*(t), m(t)) = \dot{m}^*(q^*(m), m) = \dot{m}^*(m)$. □

Once we have established the existence of a policy function, it is turn to determine the number of policy functions. An implication of Proposition B is that the value function can be rewritten in stationary terms as $V[s, m(s)] \equiv e^{-\rho s} v[m(s)]$, where $v[m]$ is the so-called stationary value function (see chapter VII in Acemoglu (2009)). Then the stationary value function is the solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$v[m] = Max_q \left\{ U[q^*] + v_k[m]\dot{k}^* + v_h[m]\dot{h}^* \right\}. \tag{HJB}$$

Given that the value function is defined as a supremum, it must be unique by definition, hence its partials. Therefore, there are as many feed-back rules as solutions to the system of FOCs of the HJB equation.

Proposition C. Uniqueness. There is a unique optimal feed-back $q^*(m)$ and unique policy function $\dot{m}^*(m)$, which define a unique optimal social plan ϕ^*.

Proof: Taking the FOC of the HJB equation with respect to the labor supply yields

$$n^*(m) = f_{nh}^{-1} \left[\frac{v_h[k, h]}{v_k[k, h]} \right] \frac{k}{h},$$

where f^{-1}_{nh} is the inverse of $F_{nh}[1, nh/k]$. This implies that the optimal labor supply for a given state of the economy is unique.

The FOCs of the HJB equation with respect to consumption and leisure leads to a system of equations in leisure and consumption

$$U_c[c, l] = v_k[m], \quad U_l[c, l] = v_h[m] A_\epsilon h(t),$$

from which we obtain the optimal consumption and leisure levels

$$c^*(m) = \left(\frac{l^*(m)^{\omega(1-\xi)}}{v_k[k, h]} \right)^{\frac{1}{\xi}},$$

$$l^*(m) = \left(\omega^{(1+\omega)-2} \frac{v_k[k, h]^{(1-\xi)}}{v_h[k, h]^{(1-\xi)-1}} \right)^{\frac{1}{(1-\xi)(1+\omega)-1}}.$$

Notice that there is a unique optimal consumption and leisure levels for a given state of the economy. Taking into account this information, it follows that equations (1) and (2) determine a unique optimal variation, and there is only one policy function. □

Appendix 2: Proof of Theorem 2.

For expositional clarity, we divide the proof in a set of claims.

Claim I. The right tail of the policy function is non-divergent. Assume that $\exists z^r > 0$ such that $\dot{z}^*(z) > 0, \ \forall z \geq z^r$. The premise implies that $F[n^*(z)/z] > \delta_k + x^*(z) + \gamma_h^*(z), \ \forall z \geq z^r$. However, since $n^*(z) < 1, \ \forall z$, we have that $\lim_{z\to\infty} F[n^*(z)/z] = 0$, so it $\exists z^a > 0$ such that $\delta_k > F[n^*(z)/z], \ \forall z > z^a$, reaching a contradiction with respect to the previous line. Thus, $\nexists z^r > 0$ and the right tail of the policy function cannot be divergent.

Claim II. The left tail of the policy function is non-divergent. Assume that $\exists z^l > 0$ such that $\dot{z}^*(z) < 0, \ \forall z \leq z^l$. In that case, the premise implies that $F[n^*(z)/z] < \delta_k + x^*(z) + \gamma_h^*(z), \ \forall z \leq z^l$. Three cases can arise: a) $\lim_{z\to 0} n^*(z)/z = \infty$, b) $\lim_{z\to 0} n^*(z)/z = n/z$ and c) $\lim_{z\to 0} n^*(z)/z = 0$.

- Assume case a). Then it follows that $F_k \to \infty$, and combining equations (4.5) and (4.9), since $0 \leq l \leq 1$, it emerges that $\gamma_c \to \infty$, which is non-admissible, ruling out this case.

- Assume case b). In the case of inner allocations, i.e. $\lambda_3 = 0$, system (4.5)-(4.9) yields a steady state situation, reaching a contradiction. In the case of non-interior allocations, the premise implies that $l \to 1$, which due to equation (4.1) leads to $h(t) \to constant$, and the premise can hold if, and only if, $k(t) \to 0$. Now, equating (4.6) to (4.7) and taking into account transversality condition (4.11), we obtain that

  $$\lim_{t\to\infty} e^{-\rho t} U_l[\bullet] = \lim_{t\to\infty} \lambda_1(t) F_{nh}[\bullet] h(t).$$

  Diving both sides by λ_1 and substituting out equation (4.5) yields

  $$\lim_{t\to\infty} \frac{U_l[\bullet]}{U_c[\bullet]} = \lim_{t\to\infty} F_{nh}[\bullet] h(t).$$

  Since the premise implies that $F_{nh}[\bullet]h(t) \to constant$, it follows that $c \to constant$. On the other hand, the last line also implies that $c/k \to \infty$, and due to equation (4.2), the premise would imply that $\gamma_k(t) \to -\infty$, which is non-admissible. Thus, the premise is false.

- Assume case c). For non-interior allocation, the same argument as in point b) applies. In the case of inner allocations, i.e. $\lambda_3 = 0$, we have that $F_k[z/n, 1] \to 0$, $F_{nh}[1, n/z] \to \infty$. Due to equation (4.8), the premise implies that $\lambda_1 \to \infty$. Moreover, since $F_{nh}[1, n/z] \to \infty$, (4.7) implies that $\lambda_2 \to \infty$. However, this contradicts equation (4.9), which imposes that $\lambda_2 \to 0$.

Given that all three cases are ruled out, we conclude that $\nexists z^l$, and the left-tail of the policy function cannot be divergent.

Claim III. Convergence. From Claims I and II, it follows that $\exists z^a$ such that $\dot{z}^*(z) > 0$, $\forall z \in (z^a - b, z^a)$ and $\dot{z}^*(z) > 0$, $\forall z \in (z^a, z^a + b)$, where $b > 0$ is arbitrarily large. Then, since Theorem 1 implies that the optimal state must be continuous and monotone, it must be the case that $z^a = z^{ss}$, so $\dot{z}^*(z^{ss}) = 0$ and

$$\frac{\partial \dot{z}^*(z^{ss})}{\partial z} < 0.$$

Claim IV. Economic Growth. Notice that $g^{ss} \leq 0$ if, and only if, $l^{ss} \geq 1$. In that case, k^{ss} and h^{ss} remain constant, and since x^{ss} remains constant, so does c^{ss} too. However, from equations (4.5) and (4.9) it follows that $\gamma_c^{ss} < 0$, reaching a contradiction. Therefore, it must be the case that $l^{ss} < 1$ and all roots of the policy function are BGPs. □

Appendix 3: Proof of Proposition 1

Assume that $\lambda_3 = 0, \forall t$. First, we shall show that all steady variables are determined by steady leisure. Combining equations (4.1), (4.6), (4.7) and (4.9) under steady state conditions, it emerges that the steady labor supply is a monotonically decreasing function of steady leisure

$$n^{ss} = \frac{\rho - (1-\xi)A_\epsilon}{\xi(A_\epsilon - A_n)} + \frac{A_\epsilon}{A_\epsilon - A_n}\frac{1-\xi}{\xi} l^{ss}.$$

Moreover, plugging the stationary labor supply in (4.1), we find that g^{ss} is a continuous, linear decreasing function of steady leisure.

Due to the constant returns to scale of F, we know that F_k is homogeneous of degree zero, so we can rewrite $F_k[z/n, 1] = f_k[z/n]$. Moreover, since F is C^2 and strictly concave, we also know that $f_k[z/n]$ is strictly decreasing and continuous, which implies the existence of its inverse f_k^{-1}. Then, combining equations (4.5) and (4.8) under steady state conditions and exploiting the properties of F, we obtain the steady level of z:

$$z^{ss} = n^{ss} f_k^{-1}[\xi g^{ss} + \delta_k + \rho].$$

Due to the Inada conditions, it follows that f_k^{-1} is convex decreasing. Thus z^{ss} is a strictly increasing function of steady leisure.

We obtain x^{ss} from equation (4.2) under steady state conditions

$$x^{ss} = F\left[1, \frac{1}{f_k^{-1}[\xi g^{ss} + \delta_k + \rho]}\right] - g^{ss} - \delta_k.$$

From Euler's Theorem on homogeneous functions it follows that $F[\bullet] = F_k[\bullet]k(t) + F_{nh}[\bullet]n(t)h(t)$, which can be rewritten as

$$F[1, n(t)/z(t)] - F_k[1, n(t)/z(t)] = F_{nh}[z(t)/n(t), 1]\frac{n(t)}{z(t)}.$$

The combination of equations (4.6) and (4.7) yields

$$F_{nh}[z^{ss}/n^{ss}, 1]\frac{n^{ss}}{z^{ss}} = \omega \frac{A_\epsilon - A_n}{A_\epsilon}\frac{x^{ss}n^{ss}}{l^{ss}}.$$

By equating the two last expressions, we obtain that

$$F[1, n^{ss}/z^{ss}] - F_k[1, n^{ss}/z^{ss}] = \omega \frac{A_\epsilon - A_n}{A_\epsilon}\frac{x^{ss}n^{ss}}{l^{ss}}.$$

Now recall that $F_k[z^{ss}, n^{ss}] = \xi g^{ss} + \delta_k + \rho$ and $x^{ss} = F[1, n^{ss}/z^{ss}] - g^{ss} - \delta_k$, which implies that the previous equation can be rearranged as

$$F\left[1, \frac{1}{f_k^{-1}[\xi g^{ss} + \delta_k + \rho]}\right]\left(n^{ss}\omega\frac{A_\epsilon - A_n}{A_\epsilon} - l^{ss}\right) = \omega\frac{A_\epsilon - A_n}{A_\epsilon}n^{ss}(g^{ss}+\delta_k) - l^{ss}(g^{ss}+\delta_k+\rho).$$

Since all steady variables are functions of steady leisure, the roots of that equation are the steady states of the unconstrained system. Notice that the right hand-side is a quadratic polynomial on l^{ss}, whereas the left hand-side is strictly decreasing. This implies that the previous equation has either two or zero real roots.

Let Ψ_1 denote the set of parameterizations for which there is no inner BGP, while Ψ_2 and Ψ_3 are the set of parameterizations for which there is one or two inner BGP respectively. We prove that the three sets are non-empty with an example.

Assume that $F[\bullet] = A_y (k(t))^{\alpha} (n(t)h(t))^{1-\alpha}$, $0 < \alpha < 1$, $0 < A_y$ and $\xi = 1$. Define the following quantities:

$$\omega_i = \frac{1-\omega}{\omega}$$

$$\kappa = \frac{(\omega_i (A_\epsilon + \delta_k) + \rho(\omega_i - 1))^2}{4\rho^2(1-\omega_i)\omega_i},$$

$$\bar{\omega}_i = \frac{(2\theta - 1)\rho}{(2\theta - 1)(A_\epsilon - \delta_k) - \rho},$$

$$\eta = -\frac{(A_\epsilon(A_n - A_\epsilon)\omega_i + \rho(A_\epsilon(\omega_i - 2) - \omega_i + 1))((A_\epsilon - A_n)\delta_k + A_\epsilon)}{(\omega_i - 1)\rho^2(A_n - A_\epsilon)^2}.$$

The direct inspection of

$$\rho + \xi g^{ss} = \frac{A_\epsilon - A_n}{A_\epsilon}\frac{n^{ss}}{z^{ss}}$$

shows that it is a quadratic equation on l^{ss} with the following cases:

1. if $\omega_i \in (0, \bar{\omega}_i)$ and $0 < \alpha < 1$ is such that $\frac{\alpha}{1-\alpha} < \eta$, the economy has a unique interior steady state.
2. if $\omega_i \in [\bar{\omega}_i, 1)$ and $0 < \alpha < 1$ is such that
   (a) $\frac{\alpha}{1-\alpha} \leq \eta$ or $\frac{\alpha}{1-\alpha} = \kappa$, the economy has a unique interior steady state.
   (b) $\frac{\alpha}{1-\alpha} \in (\eta, \kappa)$, the economy has two interior steady states.
3. In other cases, the economy has no interior steady states.

This result nests those of Ladrón-de Guevara et al. (1999) Proposition 4.1. when LBD is neglected (i.e. when the specific case $A_n = 0$ is considered). □

Bibliography

Acemoglu, D. (2009). *Introduction to Modern Growth Theory*. Princeton University Press.

Ager, P., Worm Hansen, C., and Sandholt Jensen, P. (2018). Fertility and Early-Life Mortality: Evidence from Smallpox Vaccination in Sweden. *Journal of the European Economic Association*, 16(2):487–521.

Arrow, K. J. (1962). The economic implications of learning by doing. *The Review of Economic Studies*, 29(3):155–173.

Arrow, K. J. and Kurz, M. (2013). *Public investment, the rate of return, and optimal fiscal policy*. Routledge.

Becker, G. S. and Lewis, H. G. (1973). On the interaction between the quantity and quality of children. *Journal of Political Economy*, 81(2, Part 2):S279–S288.

Bella, G., Mattana, P., and Venturi, B. (2017). Shilnikov chaos in the lucas model of endogenous growth. *Journal of Economic Theory*, 172:451 – 477.

Bick, A. and Fuchs-Schündeln, N. (2018). Taxation and Labour Supply of Married Couples across Countries: A Macroeconomic Analysis. *The Review of Economic Studies*, 85(3):1543–1576.

Bick, A., Fuchs-Schündeln, N., and Lagakos, D. (2018). How do hours worked vary with income? cross-country evidence and implications. *American Economic Review*, 108(1):170–99.

Bick, A., Fuchs-Schündeln, N., Lagakos, D., and Tsujiyama, H. (2022). Structural change in labor supply and cross-country differences in hours worked. *Journal of Monetary Economics*, 130:68–85.

Boppart, T. and Krusell, P. (2020). Labor supply in the past, present, and future: A balanced-growth perspective. *Journal of Political Economy*, 128(1):118–157.

Caballé, J. and Santos, M. S. (1993). On endogenous growth with physical and human capital. *Journal of Political Economy*, 101(6):1042–1067.

Chen, L., Chung, B. W., and Wang, G. (2022). Exposure to socially influential peer parents: Evidence from cadre parents in china.

Dalgaard, C.-J., Madsen, J. B., and Strulik, H. (2021). Physiological constraints and the transition to growth: implications for comparative development. *Journal of Economic Growth*, 26(3):241–289.

Doepke, M., Sorrenti, G., and Zilibotti, F. (2019). The economics of parenting. *Annual Review of Economics*, 11(1):55–84.

Einarsson, T. and Marquis, M. H. (1996). Note on human capital externalities. *Journal of Macroeconomics*, 18(2):341 – 351.

Fleming, W. H. and Rishel, R. W. (2012). *Deterministic and stochastic optimal control*, volume 1. Springer Science & Business Media.

Galor, O. (2011). *Unified growth theory*. Princeton University Press.

Gauthier, A. H., Smeeding, T. M., and Furstenberg Jr., F. F. (2004). Are parents investing less time in children? trends in selected industrialized countries. *Population and Development Review*, 30(4):647–672.

Gimenez-Nadal, J. I. and Sevilla, A. (2012). Trends in time allocation: A cross-country analysis. *European Economic Review*, 56(6):1338–1359.

Guo, J. and Qu, X. (2022). Competition in household human capital investments: Strength, motivations and consequences. *Journal of Development Economics*, 158:102937.

Guryan, J., Hurst, E., and Kearney, M. (2008). Parental education and parental time with children. *Journal of Economic Perspectives*, 22(3):23–46.

Guzmán, R. A. and Weisdorf, J. (2011). The neolithic revolution from a price-theoretic perspective. *Journal of Development Economics*, 96(2):209–219.

Havranek, T., Horvath, R., Irsova, Z., and Rusnak, M. (2015). Cross-country heterogeneity in intertemporal substitution. *Journal of International Economics*, 96(1):100 – 118.

Hazan, M. and Zoabi, H. (2015). Do highly educated women choose smaller families? *The Economic Journal*, 125(587):1191–1226.

Herzer, D., Strulik, H., and Vollmer, S. (2012). The long-run determinants of fertility: one century of demographic change 1900-1999. *Journal of Economic Growth*, 17(4):357–385.

Kalemli-Ozcan, S. (2002). Does the mortality decline promote economic growth? *Journal of Economic Growth*, 7(4):411–439.

Kalemli-Ozcan, S. (2003). A stochastic model of mortality, fertility, and human capital investment. *Journal of Development Economics*, 70(1):103–118.

Killingsworth, M. R. (1982). “Learning by Doing” and “Investment in Training”: A Synthesis of Two “Rival” Models of the Life Cycle. *The Review of Economic Studies*, 49(2):263–271.

Ladrón-de Guevara, A., Ortigueira, S., and Santos, M. S. (1999). A Two-Sector Model of Endogenous Growth with Leisure. *The Review of Economic Studies*, 66(3):609–631.

Lucas, R. E. (1988). On the mechanics of economic development. *Journal of Monetary Economics*, 22(1):3 – 42.

McDaniel, C. (2011). Forces shaping hours worked in the oecd, 1960-2004. *American Economic Journal: Macroeconomics*, 3(4):27–52.

Murphy, T. E. (2015). Old habits die hard (sometimes). *Journal of Economic Growth*, 20(2):177–222.

Myong, S., Park, J., and Yi, J. (2020). Social norms and fertility. *Journal of the European Economic Association*, 19(5):2429–2466.

Prettner, K. and Strulik, H. (2017). It’s a sin—contraceptive use, religious beliefs, and long-run economic development. *Review of Development Economics*, 21(3):543–566.

Spolaore, E. and Wacziarg, R. (2022). Fertility and Modernity. *The Economic Journal*, 132(642):796–833.

Strulik, H. (2004a). Child Mortality, Child Labour and Economic Development. *The Economic Journal*, 114(497):547–568.

Strulik, H. (2004b). Economic growth and stagnation with endogenous health and fertility. *Journal of Population Economics*, 17(3):433–453.

Strulik, H. (2017). Contraception and development: A unified growth theory. *International Economic Review*, 58(2):561–584.

Strulik, H. (2019). Desire and development. *Macroeconomic Dynamics*, 23(7):2717–2747.

Strulik, H. and Weisdorf, J. (2008). Population, food, and knowledge: a simple unified growth theory. *Journal of Economic Growth*, 13(3):195.

Tamura, R., Dwyer, J., Devereux, J., and Baier, S. (2019). Economic growth in the long run. *Journal of Development Economics*, 137:1 – 35.

Uzawa, H. (1965). Optimum technical change in an aggregative model of economic growth. *International Economic Review*, 6(1):18–31.

Vogl, T. S. (2015). Differential Fertility, Human Capital, and Development. *The Review of Economic Studies*, 83(1):365–401.

Voigtländer, N. and Voth, H.-J. (2012). The Three Horsemen of Riches: Plague, War, and Urbanization in Early Modern Europe. *The Review of Economic Studies*, 80(2):774–811.

Wilson, N. (2015). Child mortality risk and fertility: Evidence from prevention of mother-to-child transmission of hiv. *Journal of Development Economics*, 116:74–88.

Zanin, L., Radice, R., and Marra, G. (2015). Modelling the impact of women's education on fertility in malawi. *Journal of Population Economics*, 28(1):89–111.