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Applications in Integral Equations through Common Results in C*-Algebra-Valued S_b-Metric Spaces

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Abstract: We study some common results in C*-algebra-valued S_b-metric spaces. We also present an interesting application of an existing and unique result for one type of integral equation.

Keywords: integral equation; C* algebra; S_b-metric space; common fixed point; compatible; weakly compatible

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1. Introduction

A metric space is suitable for those interested in analysis, mathematical physics, or applied sciences. Thus, various extensions of metric spaces have been studied, and several results related to the existence of fixed points were obtained (see [1–3]).

In 2014, Ma et al. introduced C*-algebra-valued metric spaces [4], and in 2015, they introduced the concept of C*-algebra-valued b-metric spaces and studied some results in this space [5]. In addition, Razavi and Masiha investigated some common principles in C*-algebra-valued b-metric spaces [6].

Recently, Sedghi et al. defined the concept of an S-metric space [7]. Additionally, Ege and Alaca introduced the concept of C*-algebra-valued S-metric spaces [8].

Inspired by the work of Souayah and Mlaiki in [9], we introduced the C*-algebra-valued S_b-metric space in [10]. In this paper, we study some common fixed-point principles in this space. We also investigate the existence and uniqueness of the result for one type of integral equation.

2. Preliminaries

This section provides a short introduction to some realities about the theory of C* algebras [11]. First, suppose that $\mathcal{A}$ is a unital C* algebra with the unit $1_{\mathcal{A}}$. Set $\mathcal{A}_h = \{t \in \mathcal{A} \ : \ t = t^*\}$. The element $t \in \mathcal{A}$ is said to be positive, and we write $t \succeq 0_{\mathcal{A}}$ if and only if $t = t^*$ and $\sigma(t) \subseteq [0, \infty)$, in which $0_{\mathcal{A}}$ in $\mathcal{A}$ is the zero element and the spectrum of t is $\sigma(t)$.

On $\mathcal{A}_h$, we can find a natural partial ordering given by $u \preceq v$ if and only if $v - u \succeq 0_{\mathcal{A}}$. We denote with $\mathcal{A}_+$ and $\mathcal{A}'$ the sets of $\{t \in \mathcal{A} \ : \ t \succeq 0_{\mathcal{A}}\}$ and $\{t \in \mathcal{A} \ : tk = kt\,, \ \forall k \in \mathcal{A}\}$, respectively.

In 2015, Ma et al. [5] introduced the notion of C*-algebra-valued b-metric spaces as follows:

Definition 1. *Let $\mathcal{X}$ be a nonempty set and $\mathcal{A}$ be a C* algebra. Suppose that $k \in \mathcal{A}'$ such that $||k|| \geq 1$. A function $\delta_b : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ is called a C*-algebra-valued b metric on $\mathcal{X}$ if for all $u, v, t \in \mathcal{A}$, the following apply:*

(1) $\delta_b(u,v) \succeq 0_{\mathcal{A}}$ *for every* u *and* v *in* $\mathcal{X}$, *and* $\delta_b(u,v) = 0$ *if and only if* $u = v$;
(2) $\delta_b(u,v) = \delta_b(v,u)$;
(3) $\delta_b(u,v) \preceq k[\delta_b(u,t) + \delta_b(t,v)]$.

Therefore, $(\mathcal{X}, \mathcal{A}, \delta_b)$ *is a C*-algebra-valued b-metric space (in short, a C*-AV-BM space) with a coefficient* k.

In 2015, Kalaivani et al. [12] presented the notion of a C*-algebra-valued S-metric space:

Definition 2. *Assume that* $\mathcal{X}$ *is a nonempty set and* $\mathcal{A}$ *is a C* algebra. A function* $\sigma : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ *is called a C*-algebra-valued S metric on* $\mathcal{X}$ *if for all* $u, v, t, a \in \mathcal{X}$, *the following apply:*
(1) $\sigma(u,v,t) \succeq 0_{\mathcal{A}}$;
(2) $\sigma(u,v,t) = 0$ *if and only if* $u = v = t$;
(3) $\sigma(u,v,t) \preceq \sigma(u,u,a) + \sigma(v,v,a) + \sigma(t,t,a)$.

Then, $(\mathcal{X}, \mathcal{A}, \sigma)$ *is a C*-algebra-valued S-metric space (in short, a C*-AV-SM space).*

In fact, in 2016, Souayah et al. [9] presented the notion of an S_b-metric space:

Definition 3. *Assume that* $\mathcal{X}$ *is a nonempty set and* $s \geq 1$ *is a given number. A function* $\gamma_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *is an* S_b *metric on* $\mathcal{X}$ *if for every* $u, v, t, a \in \mathcal{X}$, *the following apply:*
(1) $\gamma_b(u,v,t) = 0$ *if and only if* $u = v = t$;
(2) $\gamma_b(u,v,t) \preceq s[\gamma_b(u,u,a) + \gamma_b(v,v,a) + \gamma_b(t,t,a)]$.

Then, $(\mathcal{X}, \gamma_b)$ *is called an* S_b*-metric space (in short, an* S_b*M space) with a coefficient* s.

Definition 4. *An* S_b*-metric* γ_b *is called symmetric if*

$$\gamma_b(u,u,v) = \gamma_b(v,v,u), \ \forall u, v \in \mathcal{X}.$$

Razavi and Masiha [10] introduced the notion of a C*-algebra-valued S_b-metric space as follows:

Definition 5. *Assume that* $\mathcal{X}$ *is a nonempty set and* $k \in \mathcal{A}'$ *such that* $||k|| \geq 1$. *A function* $\sigma_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ *is called a C*-algebra-valued* S_b *metric on* $\mathcal{X}$ *if for every* $u, v, t, a \in \mathcal{X}$, *the following apply:*
(1) $\sigma_b(u,v,t) \succeq 0_{\mathcal{A}}$;
(2) $\sigma_b(u,v,t) = 0$ *if and only if* $u = v = t$;
(3) $\sigma_b(u,v,t) \preceq k[\sigma_b(u,u,a) + \sigma_b(v,v,a) + \sigma_b(t,t,a)]$.

Then, $(\mathcal{X}, \mathcal{A}, \sigma_b)$ *is called a C*-algebra-valued* S_b*-metric space (in short, a C*-AV-*S_b*M space) with a coefficient* k.

Definition 6. *A C*-AV-*S_b*M* σ_b *is symmetric if*

$$\sigma_b(u,u,v) = \sigma_b(v,v,u), \ \forall u, v \in \mathcal{X}.$$

Under the above definitions, we give an example in a C*-AV-S_bM space:

Example 1. *Let* $\mathcal{X} = \mathbb{R}$ *and* $\mathcal{A} = M_2(\mathbb{R})$ *be all* 2×2 *matrices with the usual operations of addition, scalar multiplication, and matrix multiplication. It is clear that*

$$||A|| = (\sum_{i,j=1}^{2} |a_{ij}|^2)^{\frac{1}{2}}$$

defines a norm on $\mathcal{A}$, where $A = (a_{ij}) \in \mathcal{A}$. $* : \mathcal{A} \to \mathcal{A}$ defines an involution on $\mathcal{A}$ and where $\mathcal{A}^* = \mathcal{A}$. Then, $\mathcal{A}$ is a C^* algebra. For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{A}$, a partial order on $\mathcal{A}$ can be given as follows:

$$A \leq B \Leftrightarrow (a_{ij} - b_{ij}) \leq 0 \ \ \forall i, j = 1, 2$$

Let $(\mathcal{X}, d)$ be a b-metric space where, $||k|| \geq 1$ and $\sigma_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to M_2(\mathbb{R})$, fulfilling

$$\sigma_b(u, v, t) = \begin{bmatrix} d(u,v) + d(v,t) + d(u,t) & 0 \\ 0 & d(u,v) + d(v,t) + d(u,t) \end{bmatrix}$$

Then, this is a C-AV-S_bM space. Now, we check condition (3) of Definition 5:*

$$\begin{aligned}
\sigma_b(u, v, t) &= \begin{bmatrix} d(u,v) + d(v,t) + d(u,t) & 0 \\ 0 & d(u,v) + d(v,t) + d(u,t) \end{bmatrix} \\
&\preceq k \begin{bmatrix} 2d(u,a) & 0 \\ 0 & 2d(u,a) \end{bmatrix} + k \begin{bmatrix} 2d(v,a) & 0 \\ 0 & 2d(v,a) \end{bmatrix} + k \begin{bmatrix} 2d(t,a) & 0 \\ 0 & 2d(t,a) \end{bmatrix} \\
&= k[2 \begin{bmatrix} d(u,a) & 0 \\ 0 & d(u,a) \end{bmatrix} + 2 \begin{bmatrix} d(v,a) & 0 \\ 0 & d(v,a) \end{bmatrix} + 2 \begin{bmatrix} d(t,a) & 0 \\ 0 & d(t,a) \end{bmatrix}] \\
&= k[\sigma_b(u,u,a) + \sigma_b(v,v,a) + \sigma_b(t,t,a)]
\end{aligned}$$

Thus, for all $u, v, t, a \in \mathcal{X}$, $(\mathcal{X}, \mathcal{A}, \sigma_b)$ is a C-AV-S_bM space.*

3. Definitions and Basic Properties

We define some concepts in a C*-AV-S_bM space and present some lemmas which will be needed in the follow-up:

Definition 7. *Let $(\mathcal{X}, \mathcal{A}, \sigma_b)$ be a C*-AV-S_bM space and $\{u_n\}$ be a sequence in $\mathcal{X}$:*

(1) If $||\sigma_b(u_n, u_n, u)|| \to 0$, where $n \to \infty$, then $\{u_n\}$ converges to u, and we present it with $\lim_{n\to\infty} u_n = u$.

(2) If for all $p \in \mathbb{N}$, $||\sigma_b(u_{n+p}, u_{n+p}, u_n)|| \to 0$, where $n \to \infty$, then $\{u_n\}$ is a Cauchy sequence in $\mathcal{X}$.

(3) If every Cauchy sequence is convergent in $\mathcal{X}$, then $(\mathcal{X}, \mathcal{A}, \sigma_b)$ is a complete C-AV-S_bM space.*

Definition 8. *Suppose that $(\mathcal{X}, \mathcal{A}, \sigma_b)$ and $(\mathcal{X}_1, \mathcal{A}_1, \sigma_{b_1})$ are C*-AV-S_bM spaces, and let $f : (\mathcal{X}, \mathcal{A}, \sigma_b) \to (\mathcal{X}_1, \mathcal{A}_1, \sigma_{b_1})$ be a function. Then, f is continuous at a point $u \in \mathcal{X}$ if, for every sequence, $\{u_n\}$ in $\mathcal{X}$, $\sigma_b(u_n, u_n, u) \to 0_{\mathcal{A}}$, $(n \to \infty)$ implies $\sigma_{b_1}(f(u_n), f(u_n), f(u)) \to 0_{\mathcal{A}}$, where $n \to \infty$. A function f is continuous at $\mathcal{X}$ if and only if it is continuous at all $u \in \mathcal{X}$.*

The next lemmas will be used tacitly in the follow-up:

Lemma 1 ([13])**.** *Suppose that $\mathcal{A}$ is a unital C^* algebra with a unit $1_{\mathcal{A}}$:*

1) If $\{u_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ and $\lim_{n\to\infty} u_n = 0_{\mathcal{A}}$, then for any $u \in \mathcal{A}$, $\lim_{n\to\infty} u^* u_n u = 0_{\mathcal{A}}$.

2) If $u, v \in \mathcal{A}_h$ and $t \in \mathcal{A}'_+$, then $u \preceq v$ yields $tu \preceq tv$, in which $\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.

3) If $u \in \mathcal{A}_+$ with $||u|| < \frac{1}{2}$, then $1_{\mathcal{A}} - u$ is invertible, and $||u(1_{\mathcal{A}} - u)^{-1}|| < 1$.

4) If $u, v \in \mathcal{A}_+$ such that $uv = vu$, then $uv \succeq 0_{\mathcal{A}}$.

Lemma 2. *Let $(\mathcal{X}, \mathcal{A}, \sigma_b)$ be a symmetric C*-AV-S_bM space and $\{u_n\}$ be a sequence in $\mathcal{X}$. If $\{u_n\}$ converges to u and v, then $u = v$.*

Proof. Let $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} u_n = v$. Under condition (3) of Definitions 5 and 6, we have

$$\begin{aligned}\sigma_b(u,u,v) &\preceq k[\sigma_b(u,u,u_n) + \sigma_b(u,u,u_n), \sigma_b(v,v,u_n)] \\ &= k[\sigma_b(u_n,u_n,u) + \sigma_b(u_n,u_n,u) + \sigma_b(u_n,u_n,v)] \\ &= 2k\sigma_b(u_n,u_n,u) + k\sigma_b(u_n,u_n,v) \\ &\to 0_{\mathcal{A}}, (n \to \infty).\end{aligned}$$

as $||\sigma_b(u,u,v)|| = 0$ if and only if $u = v$. □

Due to the following definition, we extend the concept of compatible mappings of Jungck [14] to C*-algebra-valued metric spaces:

Definition 9. *Let $(\mathcal{X}, \mathcal{A}, \sigma_b)$ be a C*-AV-S_bM space. A pair $\{\psi, \varphi\}$ is called compatible if and only if $\sigma_b(\psi\varphi u_n, \psi\varphi u_n, \varphi\psi u_n) \to 0_{\mathcal{A}}$ whenever $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \psi u_n = \lim_{n\to\infty} \varphi u_n = u$ for some $u \in \mathcal{X}$.*

Definition 10. *A point $u \in \mathcal{X}$ is a coincidence point of ψ and φ if and only if $\psi u = \varphi u$. Herein, $t = \psi u = \varphi u$ is a point of coincidence of ψ and φ. If ψ and φ commute at all of their coincidence points, then they are weakly compatible, but the converse is not true.*

If mappings T and S are compatible, then they are weakly compatible in metric spaces. Provided that the converse is not true [15], the same holds for the C*-algebra-valued S_b-metric spaces:

Theorem 1. *If mappings ψ and φ on the C*-AV-S_bM space $(\mathcal{X}, \mathcal{A}, \sigma_b)$ are compatible, then they are weakly compatible.*

Proof. Let $\psi u = \varphi u$ for some $u \in \mathcal{X}$. It suffices to present that $\psi\varphi u = \varphi\psi u$. By setting $u_n \equiv u$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \psi u_n = \lim_{n\to\infty} \varphi u_n$. Since ψ and φ are compatible, we achieve $\lim_{n\to\infty} \sigma_b(\psi\varphi u_n, \psi\varphi u_n, \varphi\psi u_n) \to 0_{\mathcal{A}}$ as $n \to \infty$; that is, $||\sigma_b(\psi\varphi u_n, \psi\varphi u_n, \varphi\psi u_n)|| \to 0$, where $n \to \infty$. Hence, $\sigma_b(\psi\varphi u_n, \psi\varphi u_n, \varphi\psi u_n) = 0_{\mathcal{A}}$, which means $\psi\varphi u = \varphi\psi u$. □

The subsequent lemma can be seen in [15]:

Lemma 3 ([15])**.** *Let ψ and φ be weakly compatible mappings of a set $\mathcal{X}$. If ψ and φ have a unique point of coincidence, then it is the unique common fixed point (FP) of ψ and φ.*

4. Main Results

Here, we present an extension of the common principles for the mappings which applies to variant contractive conditions in complete symmetric C*-valued S_b-metric spaces:

Theorem 2. *Suppose that $(\mathcal{X}, \mathcal{A}, \sigma_b)$ is a a complete symmetric C*-AV-S_bM space and $\psi, \varphi : \mathcal{X} \to \mathcal{X}$ satisfies*

$$\sigma_b(\psi u, \psi u, \varphi v) \preceq a^* \sigma_b(u,u,v) a, \tag{1}$$

for all $u, v \in \mathcal{X}$, where $a \in \mathcal{A}$ in which $||a|| < 1$. Hence, ψ and φ have a unique common FP in $\mathcal{X}$.

Proof. Suppose that $u_0 \in \mathcal{X}$ and $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $u_{2n+1} = \psi u_{2n}$, $u_{2n+2} = \varphi u_{2n+1}$. From Equation (1), we have

$$\begin{aligned}\sigma_b(u_{2n+2}, u_{2n+2}, u_{2n+1}) &= \sigma_b(\varphi u_{2n+1}, \varphi u_{2n+1}, \psi u_{2n}) \\ &\preceq a^* \sigma_b(u_{2n+1}, u_{2n+1}, u_{2n}) a \\ &\preceq (a^*)^2 \sigma_b(u_{2n}, u_{2n}, u_{2n-1})(a)^2 \\ &\vdots \\ &\preceq (a^*)^{2n+1} \sigma_b(u_1, u_1, u_0)(a)^{2n+1},\end{aligned}$$

By remembering the property where if $t, k \in \mathcal{A}_h$, then $t \preceq k$ yields $u^* t u \preceq u^* k u$, we see the following for each $n \in \mathbb{N}$:

$$\sigma_b(u_{2n+1}, u_{2n+1}, u_{2n}) \preceq (a^*)^{2n} \sigma_b(u_1, u_1, u_0)(a)^{2n}.$$

Similarly, we have

$$\sigma_b(u_{n+1}, u_{n+1}, u_n) \preceq (a^*)^n \sigma_b(u_1, u_1, u_0)(a)^n.$$

Let $\sigma_b(u_1, u_1, u_0) = B_0$ for some $B_0 \in \mathcal{A}_+$. For any $p \in \mathbb{N}$, we achieve

$$\begin{aligned}\sigma_b(u_{n+p}, u_{n+p}, u_n) &\preceq b[\sigma_b(u_{n+p}, u_{n+p}, u_{n+p-1}) + \sigma_b(u_{n+p}, u_{n+p}, u_{n+p-1}) \\ &+ \sigma_b(u_n, u_n, u_{n+p-1})] \\ &= 2b\sigma_b(u_{n+p}, u_{n+p}, u_{n+p-1}) + b\sigma_b(u_n, u_n, u_{n+p-1}) \\ &= 2b\sigma_b(u_{n+p}, u_{n+p}, u_{n+p-1}) + b\sigma_b(u_{n+p-1}, u_{n+p-1}, u_n) \\ &\preceq 2b\sigma_b(u_{n+p}, u_{n+p}, u_{n+p-1}) \\ &+ 2b^2\sigma_b(u_{n+p-1}, u_{n+p-1}, u_{n+p-2}) \\ &+ b^2\sigma_b(u_{n+p-2}, u_{n+p-2}, u_n) \\ &\vdots \\ &\preceq 2b\sigma_b(u_{n+p}, u_{n+p}, u_{n+p-1}) \\ &+ 2b^2\sigma_b(u_{n+p-1}, u_{n+p-1}, u_{n+p-2}) \\ &+ 2b^3\sigma_b(u_{n+p-2}, u_{n+p-2}, u_{n+p-3}) \\ &+ \cdots + 2b^p\sigma_b(u_{n+1}, u_{n+1}, u_n) \\ &\preceq 2b(a^*)^{n+p-1}\sigma_b(u_1, u_1, u_0)(a)^{n+p-1} \\ &+ 2b^2(a^*)^{n+p-2}\sigma_b(u_1, u_1, u_0)(a)^{n+p-2} \\ &+ 2b^3(a^*)^{n+p-3}\sigma_b(u_1, u_1, u_0)(a)^{n+p-3} \\ &+ \cdots + 2b^p(a^*)^n\sigma_b(u_1, u_1, u_0)(a)^n \\ &\preceq 2\sum_{k=1}^{p-1} b^k (a^*)^{n+p-k}\sigma_b(u_1, u_1, u_0)(a)^{n+p-k} \\ &= 2\sum_{k=1}^{p-1} b^k (a^*)^{n+p-k} B_0 (a)^{n+p-k} \\ &= 2\sum_{k=1}^{p-1} ((a^*)^{n+p-k} b^{\frac{k}{2}} B_0^{\frac{1}{2}})(B_0^{\frac{1}{2}} b^{\frac{k}{2}} a^{n+p-k})\end{aligned}$$

$$
\begin{aligned}
&\preceq 2\sum_{k=1}^{p-1}(B_0^{\frac{1}{2}}b^{\frac{k}{2}}a^{n+p-k})^*(B_0^{\frac{1}{2}}b^{\frac{k}{2}}a^{n+p-k}) \\
&\preceq 2\sum_{k=1}^{p-1}||B_0^{\frac{1}{2}}b^{\frac{k}{2}}a^{n+p-k}||^2 1_{\mathcal{A}} \\
&\leq 2||B_0^{\frac{1}{2}}||^2\sum_{k=1}^{p-1}||a||^{2(n+p-k)}||b||^k 1_{\mathcal{A}} \\
&\leq 2||B_0||\frac{||b||^p||a||^{2(n+1)}}{||b||-||a||^2}1_{\mathcal{A}} \\
&\longrightarrow 0\ (n\to\infty),
\end{aligned}
$$

in which $1_{\mathcal{A}}$ is the unit element in $\mathcal{A}$.

As $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$, and $\mathcal{X}$ is complete, there exists $u \in \mathcal{X}$ such that $\lim_{n\to\infty} u_n = u$.

By using condition (3) of Definitions 5 and 6 as well as Equation (1), we have

$$
\begin{aligned}
\sigma_b(u,u,\varphi u) &\preceq b[\sigma_b(u,u,u_{2n+1})+\sigma_b(u,u,u_{2n+1})+\sigma_b(u_{2n+1},u_{2n+1},\varphi u)] \\
&= 2b\sigma_b(u,u,u_{2n+1})+b\sigma_b(u_{2n+1},u_{2n+1},\varphi u) \\
&= 2b\sigma_b(u_{2n+1},u_{2n+1},u)+b\sigma_b(\psi u_{2n},\psi u_{2n},\varphi u) \\
&\preceq 2b\sigma_b(u_{2n+1},u_{2n+1},u)+ba^*\sigma_b(u_n,u_n,u)a \\
&\longrightarrow 0_{\mathcal{A}}\ (n\to\infty).
\end{aligned}
$$

Hence, $\varphi u = u$. Again, we note that

$$0_{\mathcal{A}} \preceq \sigma_b(\psi u,\psi u,u) = \sigma_b(\psi u,\psi u,\varphi u) \preceq a^*\sigma_b(u,u,u)a = 0_{\mathcal{A}},$$

In other words, $\sigma_b(\psi u,\psi u,u) = 0_{\mathcal{A}}$, and hence $\psi u = u$.

For the uniqueness of the common FP in $\mathcal{X}$, let there be another point $v \in \mathcal{X}$ such that $\psi v = \varphi v = v$. From Equation (1), we achieve

$$0_{\mathcal{A}} \preceq \sigma_b(u,u,v) = \sigma_b(\psi u,\psi u,\psi v) \preceq a^*\sigma_b(u,u,v)a$$

which, together with $||a|| < 1$, yields that

$$
\begin{aligned}
0 \preceq ||\sigma_b(u,u,v)|| &\preceq ||a^*\sigma_b(u,u,v)a|| \\
&\preceq ||a^*||||\sigma_b(u,u,v)||||a|| \\
&\preceq ||a||^2||\sigma_b(u,u,v)|| \\
&\preceq ||\sigma_b(u,u,v)||
\end{aligned}
$$

Thus, $||\sigma_b(u,u,v)|| = 0$ and $\sigma_b(u,u,v) = 0_{\mathcal{A}}$, which gives $u = v$. Hence, ψ and φ have a unique common FP in $\mathcal{X}$. □

With the proof of Theorem 2, the relevant results are as follows:

Corollary 1. *Assume that $(\mathcal{X},\mathcal{A},\sigma_b)$ is a complete symmetric C*-AV-S_bM space, and suppose that $\psi,\varphi : \mathcal{X} \to \mathcal{X}$ represent two mappings such that*

$$||\sigma_b(\psi u,\psi u,\varphi v) \preceq ||a||||\sigma_b(u,u,v)||,$$

for all $u, v \in \mathcal{X}$, where $a \in \mathcal{A}$ and $||a|| < 1$. Then, ψ and φ have a unique common FP in $\mathcal{X}$.

Corollary 2. *Assume that* $(\mathcal{X}, \mathcal{A}, \sigma_b)$ *is a complete symmetric C*-AV-S_bM space and the mapping* $\psi : \mathcal{X} \to \mathcal{X}$ *satisfies*

$$\sigma_b(\psi^m u, \psi^m u, \psi^n v) \preceq a^* \sigma_b(u, u, v) a,$$

for all $u, v \in \mathcal{X}$, *in which* $a \in \mathcal{A}$ *and* $||a|| < 1$, *and* m *and* n *are fixed positive integers. Thus,* ψ *has a unique FP in* $\mathcal{X}$.

Proof. Set $\psi = \psi^m$ and $\varphi = \psi^n$ in Equation (1). The result is obtained using Theorem 2. □

Remark 1. *By substituting* $\psi = \varphi$ *into Equation (1), we have*

$$\sigma_b(\psi u, \psi u, \psi v) \preceq a^* \sigma_b(u, u, v) a,$$

for all $u, v \in \mathcal{X}$, *where* $a \in \mathcal{A}$ *and* $||a|| < 1$. *Thus, we conclude the next corollary.*

Corollary 3. *Suppose that* $(\mathcal{X}, \mathcal{A}, \sigma_b)$ *is a complete symmetric C*-AV-S_bM space and the mapping* $\psi : \mathcal{X} \to \mathcal{X}$ *satisfies*

$$\sigma_b(\psi u, \psi u, \psi v) \preceq a^* \sigma_b(u, u, v) a,$$

for all $u, v \in \mathcal{X}$, *where* $a \in \mathcal{A}$ *and* $||a|| < 1$. *Then,* ψ *has a unique FP in* $\mathcal{X}$.

Theorem 3. *Suppose that* $(\mathcal{X}, \mathcal{A}, \sigma_b)$ *is a complete symmetric C*-AV-S_bM space and* ψ, $\varphi : \mathcal{X} \to \mathcal{X}$ *satisfies*

$$\sigma_b(\psi u, \psi u, \psi v) \preceq a^* \sigma_b(u, u, v) a, \tag{2}$$

for all $u, v \in \mathcal{X}$, *where* $a \in \mathcal{A}$ *and* $||a|| < 1$. *If* $R(\psi)$, *contained in* $R(\varphi)$ *and* $R(\varphi)$, *is complete in* $\mathcal{X}$, *then* ψ *and* φ *have a unique point of coincidence in* $\mathcal{X}$. *Additionally, if* ψ *and* φ *are weakly compatible, then* ψ *and* φ *have a unique common FP in* $\mathcal{X}$.

Proof. Suppose that $u_0 \in \mathcal{X}$ is arbitrary. Choose $u_1 \in \mathcal{X}$ such that $\varphi u_1 = \psi u_0$. This is correct because $R(\psi) \subseteq R(\varphi)$. Let $u_2 \in \mathcal{X}$ such that $\varphi u_2 = \psi u_1$. In the same way, we obtain a sequence $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ satisfying $\varphi u_n = \psi u_{n-1}$. Therefore, with Equation (2), we have

$$\begin{aligned}
\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) &= \sigma_b(\psi u_n, \psi u_n, \psi u_{n-1}) \\
&\preceq a^* \sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1}) a \\
&\vdots \\
&\preceq (a^*)^n \sigma_b(\varphi u_1, \varphi u_1, \varphi u_0)(a)^n,
\end{aligned}$$

which shows that $\{\varphi u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $R(\varphi)$. Since $R(\varphi)$ is complete in $\mathcal{X}$, there exists $q \in \mathcal{X}$ such that $\lim_{n\to\infty} \varphi u_n = \varphi q$, and thus

$$\begin{aligned}
\sigma_b(\varphi u_n, \varphi u_n, \psi q) &= \sigma_b(\psi u_{n-1}, \psi u_{n-1}, \psi q) \\
&\preceq a^* \sigma_b(\varphi u_{n-1}, \varphi u_{n-1}, \varphi q) a,
\end{aligned}$$

From $\lim_{n\to\infty} \varphi u_n = \varphi q$ and Lemma 1, we obtain $a^* \sigma_b(\varphi u_{n-1}, \varphi u_{n-1}, \varphi q) a \to 0_{\mathcal{A}}$ as $n \to \infty$, and then $\lim_{n\to\infty} \varphi u_n = \psi q$. Lemma 2 yields that $\varphi q = \psi q$. If there is an element w in $\mathcal{X}$ such that $\psi w = \varphi w$, then Equation (2) yields

$$\sigma_b(\varphi q, \varphi q, \varphi w) = \sigma_b(\psi q, \psi q, \psi w) \preceq a^* \sigma_b(\varphi q, \varphi q, \varphi w) a,$$

In the same way as in Theorem 2, we obtain $\varphi q = \varphi w$ because

$$\begin{aligned}
&0 \leq ||\sigma_b(\varphi q, \varphi q, \varphi w)|| \leq ||a||^2 ||\sigma_b(\varphi q, \varphi q, \varphi w)|| \\
&\Rightarrow ||\sigma_b(\varphi q, \varphi q, \varphi w)|| = 0 \Rightarrow \sigma_b(\varphi q, \varphi q, \varphi w) = 0_{\mathcal{A}} \Rightarrow \varphi q = \varphi w.
\end{aligned}$$

Hence, ψ and φ have a unique point of coincidence in $\mathcal{X}$. Through Lemma 3, we conclude that ψ and φ have a unique common FP in $\mathcal{X}$. $\square$

Theorem 4. *Assume that $(\mathcal{X}, \mathcal{A}, \sigma_b)$ is a complete symmetric C*-AV-S_bM space and ψ, $\varphi : \mathcal{X} \to \mathcal{X}$ satisfies*

$$\sigma_b(\psi u, \psi u, \psi v) \preceq a\sigma_b(\psi u, \psi u, \varphi u) + a\sigma_b(\psi v, \psi v, \varphi v), \tag{3}$$

for all $u, v \in \mathcal{X}$, where $a \in \mathcal{A}'_+$ and $||a|| < \frac{1}{2}$. If $R(\psi)$, contained in $R(\varphi)$ and $R(\varphi)$, is complete in $\mathcal{X}$, then ψ and φ have a unique point of coincidence in $\mathcal{X}$. In addition, if ψ and φ are weakly compatible, then ψ and φ have a unique common FP in $\mathcal{X}$.

Proof. As in Theorem 3, we select $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ and set $\varphi u_n = \psi u_{n-1}$. Therefore, through Equation (3), we have

$$\begin{aligned}
\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) &= \sigma_b(\psi u_n, \psi u_n, \psi u_{n-1}) \\
&\preceq a\sigma_b(\psi u_n, \psi u_n, \varphi u_n) + a\sigma_b(\psi u_{n-1}, \psi u_{n-1}, \varphi u_{n-1}) \\
&= a\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) + a\sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1})
\end{aligned}$$

Thus, we obtain

$$(1-a)\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) \preceq a\sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1})$$

Since $||a|| < \frac{1}{2}$, then $1 - a$ is invertible, and $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$ which, together with $a \in \mathcal{A}'_+$, yields $(1-a)^{-1}a \in \mathcal{A}'_+$. Lemma 1's condition (2) leads to

$$\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) \preceq t\sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1}), \tag{4}$$

where $t = (1-a)^{-1}a \in \mathcal{A}'_+$ and $||t|| < 1$. Now, by induction and the use of Lemma 1's condition (2), we obtain

$$\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) \preceq t^n\sigma_b(\varphi u_1, \varphi u_1, \varphi u_0).$$

For each $m \geq 1$, $p \geq 1$, and $b \in \mathcal{A}'$ where $||b|| > 1$, we have

$$\begin{aligned}
\sigma_b(\varphi u_{m+p}, \varphi u_{m+p}, \varphi u_m) \preceq\ & b[\sigma_b(\varphi u_{m+p}, \varphi u_{m+p}, \varphi u_{m+p-1}) \\
& + \sigma_b(\varphi u_{m+p}, \varphi u_{m+p}, \varphi u_{m+p-1}) \\
& + \sigma_b(\varphi u_{m+p-1}, \varphi u_{m+p-1}, \varphi u_m)] \\
=\ & 2b\sigma_b(\varphi u_{m+p}, \varphi u_{m+p}, \varphi u_{m+p-1}) \\
& + \sigma_b(\varphi u_{m+p-1}, \varphi u_{m+p-1}, \varphi u_m) \\
\preceq\ & 2b\sigma_b(\varphi u_{m+p}, \varphi u_{m+p}, \varphi u_{m+p-1}) \\
& + 2b^2\sigma_b(\varphi u_{m+p-1}, \varphi u_{m+p-1}, \varphi u_{m+p-2}) \\
& + b^2\sigma_b(\varphi u_{m+p-2}, \varphi u_{m+p-2}, \varphi u_m) \\
& \vdots \\
\preceq\ & 2b\sigma_b(\varphi u_{m+p}, \varphi u_{m+p}, \varphi u_{m+p-1}) \\
& + 2b^2\sigma_b(\varphi u_{m+p-1}, \varphi u_{m+p-1}, \varphi u_{m+p-2}) \\
& + 2b^3\sigma_b(\varphi u_{m+p-2}, \varphi u_{m+p-2}, \varphi u_{m+p-3}) \\
& + \cdots + 2b^p\sigma_b(\varphi u_{m+1}, \varphi u_{m+1}, \varphi u_m)
\end{aligned}$$

$$
\begin{aligned}
&\preceq 2bt^{m+p-1}\sigma_b(\varphi u_1, \varphi u_1, \varphi u_0) \\
&+ 2b^2 t^{m+p-2}\sigma_b(\varphi u_1, \varphi u_1, \varphi u_0) \\
&+ 2b^3 t^{m+p-3}\sigma_b(\varphi u_1, \varphi u_1, \varphi u_0) \\
&+ \cdots + 2b^p t^m \sigma_b(\varphi u_1, \varphi u_1, \varphi u_0) \\
&= 2bt^{m+p-1}B_0 + 2b^2 t^{m+p-2}B_0 \\
&+ 2b^3 t^{m+p-3}B_0 + \cdots + 2b^p t^m B_0 \\
&= 2\sum_{k=1}^{p} b^k t^{m+p-k} B_0 \\
&= 2\sum_{k=1}^{p} |B_0^{\frac{1}{2}} t^{\frac{m+p-k}{2}} b^{\frac{k}{2}}|^2 \\
&\preceq 2||B_0|| \sum_{k=1}^{p} ||b||^k ||t||^{m+p-k} 1_{\mathcal{A}} \\
&\preceq 2||B_0|| \frac{||b||^p ||t||^{m+1}}{||t|| - ||b||} 1_{\mathcal{A}} \\
&\to 0, (m \to \infty),
\end{aligned}
$$

where $B_0 = \sigma_b(\varphi u_1, \varphi u_1, \varphi u_0)$. Hence, $\{\varphi u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $R(\varphi)$. Since $R(\varphi)$ is complete, there exists $q \in \mathcal{X}$ such that $\lim_{n\to\infty} \varphi u_n = \varphi q$. Again, according to Equation (4), we have

$$
\sigma_b(\varphi u_n, \varphi u_n, \psi q) = \sigma_b(\psi u_{n-1}, \psi u_{n-1}, \psi q) \preceq t\sigma_b(\varphi u_{n-1}, \varphi u_{n-1}, \varphi q)
$$

This implies that $\lim_{n\to\infty} \varphi u_n = \psi q$. Under Lemma 2, $\psi q = \varphi q$. Therefore, ψ and φ have a point of coincidence in $\mathcal{X}$. Here, we prove the uniqueness of points of coincidence. For this, let there be $p \in \mathcal{X}$ such that $\psi p = \varphi p$. By applying Equation (3), we have

$$
\sigma_b(\varphi p, \varphi p, \varphi q) = \sigma_b(\psi p, \psi p, \psi q) \preceq a\sigma_b(\psi p, \psi p, \varphi p) + a\sigma_b(\psi q, \psi q, \varphi q),
$$

This implies that $||\sigma_b(\varphi p, \varphi p, \varphi q)|| = 0$, and thus $\varphi p = \varphi q$. Therefore, under Lemma 3, ψ and φ have a unique common FP in $\mathcal{X}$. □

Theorem 5. *Assume that $(\mathcal{X}, \mathcal{A}, \sigma_b)$ is a complete symmetric C*-AV-S_bM space and ψ, $\varphi : \mathcal{X} \to \mathcal{X}$ satisfies*

$$
\sigma_b(\psi u, \psi u, \psi v) \preceq a\sigma_b(\psi u, \psi u, \varphi v) + a\sigma_b(\varphi u, \varphi u, \psi v), \tag{5}
$$

for every $u, v \in \mathcal{X}$, in which $a \in \mathcal{A}'_+$ and $||ab|| < \frac{1}{3}$. If $R(\psi)$, contained in $R(\varphi)$ and $R(\varphi)$, is complete in $\mathcal{X}$, then ψ and φ have a unique point of coincidence in $\mathcal{X}$. Additionally, if ψ and φ are weakly compatible, then ψ and φ have a unique common FP in $\mathcal{X}$.

Proof. As in Theorem 3, we select $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ and set $\varphi u_n = \psi u_{n-1}$. Therefore, under Equation (5), we have

$$
\begin{aligned}
\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) &= \sigma_b(\psi u_n, \psi u_n, \psi u_{n-1}) \\
&\preceq a\sigma_b(\psi u_n, \psi u_n, \varphi u_{n-1}) + a\sigma_b(\varphi u_n, \varphi u_n, \psi u_{n-1}) \\
&= a\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_{n-1}) + a\sigma_b(\varphi u_n, \varphi u_n, \varphi u_n) \\
&\preceq ab[2\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) + \sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1})] \\
&= 2ab\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) + ab\sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1})
\end{aligned}
$$

Thus, we obtain

$$(1-2ab)\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) \preceq ab\sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1}),$$

Therefore, we have

$$\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) \preceq (1-2ab)^{-1}ab\sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1}),$$

and consequently

$$\sigma_b(\varphi u_{n+1}, \varphi u_{n+1}, \varphi u_n) \preceq t\sigma_b(\varphi u_n, \varphi u_n, \varphi u_{n-1}),$$

where $t = (1-2ab)^{-1}ab \in \mathcal{A}'_+$ and $||t|| < 1$.

Similar to the process in Theorem 4, we find that ψ and φ have a point of coincidence ψq in $\mathcal{X}$. Here, we prove the uniqueness of the points of coincidence. For this, let there be $p \in \mathcal{X}$ such that $\psi p = \varphi p$. By applying Equation (5), we obtain

$$\begin{aligned}\sigma_b(\varphi p, \varphi p, \varphi q) &= \sigma_b(\psi p, \psi p, \psi q)\\ &\preceq a\sigma_b(\psi p, \psi p, \varphi q) + a\sigma_b(\varphi p, \varphi p, \psi q)\\ &= a\sigma_b(\varphi p, \varphi p, \varphi q) + a\sigma_b(\varphi p, \varphi p, \varphi q),\end{aligned}$$

In other words, we have

$$\sigma_b(\varphi p, \varphi p, \varphi q) \preceq (1-a)^{-1}a\sigma_b(\varphi p, \varphi p, \varphi q).$$

Since $||(1-a)^{-1}a|| < 1$, this implies that $||\sigma_b(\varphi p, \varphi p, \varphi q)|| = 0$, and thus $\varphi p = \varphi q$. Therefore, Lemma 3 implies that ψ and φ have a unique common FP in $\mathcal{X}$. □

If we choose $\varphi = id_{\mathcal{X}}$ in Theorem 5, then we obtain $R(\varphi) = \mathcal{X}$, and ψ is weakly compatible with φ. We also have the following result:

Corollary 4. *Suppose that $(\mathcal{X}, \mathcal{A}, \sigma_b)$ is a complete symmetric C*-AV-S_bM space and $\psi : \mathcal{X} \to \mathcal{X}$ satisfies*

$$\sigma_b(\psi u, \psi u, \psi v) \preceq a\sigma_b(\psi u, \psi u, v) + a\sigma_b(\psi v, \psi v, u),$$

for all $u, v \in \mathcal{X}$, where $a \in \mathcal{A}'_+$ and $||ab|| < \frac{1}{3}$. Hence, ψ has a unique FP in $\mathcal{X}$.

5. Application in Integral Equations

Let us use the following equations:

$$\begin{aligned} llx(m) &= \int_{\mathcal{E}}(T_1(m, n, x(n))dn + J(m),\ m \in \mathcal{E}\\ x(m) &= \int_{\mathcal{E}}(T_2(m, n, x(n))dn + J(m),\ m \in \mathcal{E}\end{aligned} \tag{6}$$

in which $\mathcal{E}$ is a Lebesgue measurable set where $m(\mathcal{E}) < \infty$.

In fact, we suppose that $\mathcal{X} = L^\infty(\mathcal{E})$ presents the class of essentially bounded measurable functions on $\mathcal{E}$, where $\mathcal{E}$ is a Lebesgue measurable set such that $m(\mathcal{E}) < \infty$.

One may consider the functions T_1, T_2, α, β to fulfill the following assumptions:

(i) $T_1, T_2 : \mathcal{E} \times \mathcal{E} \times \mathbb{R} \to \mathbb{R}$ are integrable. In addition, an integrable function α is from $\mathcal{E} \times \mathcal{E}$ to $\mathbb{R}^{\geq 0}$, and $J \in L^\infty(\mathcal{E})$.

(ii) There exists $\ell \in (0, 1)$ such that

$$|T_1(m, n, x) - T_2(m, n, y)| \leq \ell|\alpha(m, n)||x - y|,$$

for $m, n \in \mathcal{E}$ and $x, y \in \mathbb{R}$.

(iii) $\sup_{m\in\mathcal{E}}\int_{\mathcal{E}}|\alpha(m,n)|dn\le 1$.

Theorem 6. *Let assumptions (i–iii) hold. Hence, the integral in Equation (6) has a unique common solution in $L^\infty(\mathcal{E})$.*

Proof. Suppose that $\mathcal{X}=L^\infty(\mathcal{E})$ and $B(L^2(\mathcal{E}))$ is a set of bounded linear operators on a Hilbert space $L^2(\mathcal{E})$. We equip $\mathcal{X}$ with the S_b metric $\sigma_b:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to B(L^2(\mathcal{E}))$, which is ascertained by

$$\sigma_b(\alpha,\beta,\gamma)=M_{(|\alpha-\gamma|+|\beta-\gamma|)^p},$$

where $M_{(|\alpha-\gamma|+|\beta-\gamma|)^p}$ is the multiplication operator on $L^2(\mathcal{E})$ ascertained by

$$M_h(\alpha)=h.\alpha\,;\ \alpha\in L^2(\mathcal{E}).$$

Therefore, $(\mathcal{X},B(L^2(\mathcal{E})),\sigma_b)$ is a complete C*-AV-S_bM space. We can describe the self-mappings $\Psi,\Phi:\mathcal{X}\to\mathcal{X}$ as follows:

$$\Psi x(m)=\int_{\mathcal{E}}T_1(m,n,x(n))dn+J(m),$$

$$\Phi x(m)=\int_{\mathcal{E}}T_2(m,n,x(n))dn+J(m),$$

for each $m\in\mathcal{E}$. Therefore, we have

$$\sigma_b(\Psi x,\Psi x,\Phi y)=M_{(|\Psi x-\Phi y|+|\Psi x-\Phi y|)^p}.$$

We can obtain

$$\begin{aligned}
||\sigma_b(\Psi x,\Psi x,\Phi y)|| &= \sup_{||h||=1}\langle M_{(|\Psi x-\Phi y|+|\Psi x-\Phi y|)^p}h,h\rangle\\
&= \sup_{||h||=1}\langle M_{(2|\Psi x-\Phi y|)^p}h,h\rangle\\
&= \sup_{||h||=1}\langle 2^pM_{|\Psi x-\Phi y|^p}h,h\rangle\\
&= \sup_{||h||=1}\int_{\mathcal{E}}(2^p|\Psi x-\Phi y|^p)h(t)\overline{h(t)}dt\\
&\preceq 2^p\sup_{||h||=1}\int_{\mathcal{E}}[\int_{\mathcal{E}}|T_1(m,n,x(n))-T_2(m,n,y(n))|]^p|h(t)|^2dt\\
&\preceq 2^p\sup_{||h||=1}\int_{\mathcal{E}}[\int_{\mathcal{E}}\ell|\alpha(m,n)(x(n)-y(n))|dn]^p|h(t)|^2dt\\
&\preceq 2^p\ell^p\sup_{||h||=1}\int_{\mathcal{E}}[\int_{\mathcal{E}}|\alpha(m,n)|dn]^p|h(t)|^2dt.||x-y||_\infty^p\\
&\preceq \ell\sup_{m\in\mathcal{E}}\int_{\mathcal{E}}|\alpha(m,n)|dn.\sup_{||h||=1}\int_{\mathcal{E}}|h(t)|^2dt2^p||x-y||_\infty^p\\
&\preceq 2^p\ell||x-y||_\infty^p\\
&= \ell||2(x-y)||_\infty^p\\
&= \ell||M_{(|x-y|+|x-y|)^p}||\\
&= ||a||||\sigma_b(x,x,y)||
\end{aligned}$$

By setting $a=\ell 1_{B(L^2(\mathcal{E}))}$, then $a\in B(L^2(\mathcal{E}))$ and $||a||=\ell<1$. Therefore, Corollary 1 implies the result. □

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