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A New Approach to Involution in Fuzzy C*-Algebra via Functional Inequality and Python Implementation

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Abstract: This article explores the stability of involution in fuzzy C*-algebras through the use of a functional inequality. We present an approach to obtaining an approximate involution in fuzzy C*-algebras by utilizing a fixed-point method. Moreover, for greater clarity, we implemented Python code for the main theorem.

Keywords: functional inequality; Hyers–Ulam stability; fixed-point theorem; Python programming

MSC: 39B05; 39B82; 46S40; 47H106

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1. Introduction

Hyers–Ulam stability is a fundamental concept in the field of functional equations. The issue of the stability of functional equations arises from the question: "Is a function that approximately satisfies a functional equation also close to a function that exactly satisfies the same functional equation". In the field of mathematics, the Hyers–Ulam theorem has been a topic of great interest and discussion among scholars. This concept was first introduced in 1940 by Stanislaw Ulam in his famous speech at the University of Wisconsin, where he presented a problem on the stability of functional equations [1]. This problem was solved by Donald Hyers in 1941 for additive mappings [2]. Hyers's theorem was as follows:

Theorem 1 ([2]). *Assume that U and V are Banach spaces, and let f be a mapping from U to V that satisfies the inequality*

$$\|f(u+v) - f(u) - f(v)\| \leq \delta$$

for some $\delta > 0$ and all $u, v \in U$. Then, there exists a unique additive mapping $T : U \to V$ such that

$$\|T(u) - f(u)\| \leq \delta$$

holds for every $u \in U$.

Ten years later, Takashi Aoki had a major impact on the development of the Hyers–Ulam stability theorem by introducing a modification to the control function, replacing δ with $K(\|x\|^p + \|y\|^p)$ for $K > 0$ and $0 \leq p < 1$ [3]. In a later development, Rassias extended the Hyers–Ulam stability theorem and Aoki's results to linear mappings with different control functions [4].

Today, Hyers–Ulam stability has become an important topic of research in mathematics, as it has applications in several fields, including physics, engineering, economics, and computer science. The results of Hyers–Ulam stability have been extended to various types of functional equations, including quadratic, cubic, Jensen, differential, and integral equations, among others.

Many mathematicians have contributed to the development of the theory of Hyers–Ulam stability by proposing and proving new theorems. They have changed the type of functional equation, control function, and space in the Hyers–Ulam stability theorem to investigate and prove new conditions [5–9]. The Hyers–Ulam stability theorem has been used to prove many other results in different branches of mathematics.

In 1984, Katrasas introduced the concept of fuzzy norm spaces [10], where a fuzzy norm is defined in a linear space and the topological structure of the fuzzy vector is established. Several mathematicians have investigated fuzzy norms on linear spaces from a variety of perspectives over time. One important study [11] in 2003 added a fuzzy norm and created a fuzzy measure in the concept of Kramosil and Michalek [12]. They also formulated a theorem on decomposing a fuzzy norm into a set of crisp norms and examined certain features of fuzzy-normed spaces.

In recent times, there has been a surge in the study of functional equations stability in fuzzy-normed spaces. Researchers have explored several fuzzy stability outcomes relating to Cauchy, Jensen, simple quadratic, and cubic functional equations. Additionally, the concept of intuitionistic fuzzy-normed spaces has been introduced, and stability results in this area have also been examined.

An investigation into the relationship between Ulam's stability and self-testing correcting programs has recently been carried out in [13]. Based on this research, we aim to implement Hyers–Ulam stability in fuzzy $C^\star$ algebra by using the Python programming language.

Python is a high-level programming language that can be used in various ways. It has access to powerful libraries, such as Numpy, which enable efficient calculations in mathematics and science. Travis Oliphant founded the open-source Numpy project in 2005 [14] and a huge group of collaborators actively maintain and develop it today.

We will attempt to write Python code to implement the Hyers–Ulam stability concept. This explains Hyers–Ulam stability for specialists in the computer science and mathematics fields.

Definition 1. *A function $d : U \times U \to [0, \infty]$ is said to be a generalized metric on the set U if it satisfies the following properties:*
(M_1) $d(u, v) = 0$ *if and only if* $u = v$
(M_2) $d(u, v) = d(v, u)$ *for all* $u, v \in U$
(M_3) $d(u, w) \leq d(u, v) + d(v, w)$ *for all* $u, v, w \in U$.

We now introduce one of the fundamental results of the fixed-point theory.

Theorem 2 ([15,16])**.** *Suppose (U, d) is a generalized complete metric space and let $G : U \to U$ be a strictly contractive operator with the Lipschitz constant $L < 1$. Assume there exists a non-negative integer n_0 such that $d\big(G^{n_0+1}u, G^{n_0}u\big) < \infty$ for some $u \in U$. Then, the following conclusions hold:*
(i) *The sequence $\{G^n u\}$ converges to a fixed point u_0 of G,*
(ii) *The fixed point u_0 is unique in $V = \{v \in U \mid d(G^{n_0}u, v) < \infty\}$,*
(iii) *For any $v \in V$, the inequality*

$$d(v, u_0) \leq \frac{1}{1-L} d(Gv, v),$$

holds.

Definition 2 ([11])**.** *A fuzzy norm on a real vector space U is a function $N : U \times \mathbb{R} \to [0, 1]$ that satisfies the following conditions for any $u, v \in U$ and $s, t \in \mathbb{R}$:*
(N_1) $N(u, t) = 0$ *for all* $t \leq 0$;
(N_2) $u = 0$ *if and only if* $N(u, t) = 1$ *for all* $t > 0$;
(N_3) $N(cu, t) = N(u, \frac{t}{|c|})$ *if* $c \neq 0$;
(N_4) $N(u + v, s + t) \geq \min\{N(u, s), N(v, t)\}$;

(N_5) *$N(u,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(u,t) = 1$;*
(N_6) *For $u \neq 0$, $N(u,.)$ is continuous on $\mathbb{R}$.*
The pair (U,N) is then referred to as a fuzzy-normed vector space.

Definition 3. *Consider a fuzzy-normed vector space (U,N).*
(1) A sequence $\{u_n\}$ in U is said to be convergent if there exists $u \in U$ such that for all $t > 0$, $\lim_{n\to\infty} N(u_n - u, t) = 1$. The limit of $\{u_n\}$ is denoted as N-limit, i.e., $N - \lim_{n\to\infty} u_n = u$.
(2) A sequence $\{u_n\}$ in U is called Cauchy if for every $\epsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $p > 0$, $N\big(u_{n+p} - u_n, t\big) > 1 - \epsilon$.

It is a well-known fact that in a fuzzy-normed space, every convergent sequence is also Cauchy. If every Cauchy sequence in a fuzzy-normed space converges, then the fuzzy norm is said to be complete, and the fuzzy-normed space is referred to as a fuzzy Banach space. A mapping of $f : U \to V$ between two fuzzy-normed vector spaces U and V is said to be continuous at a point $u_0 \in U$ if, for every sequence $\{u_n\}$ converging to u_0 in U, the sequence $\{f(u_n)\}$ converges to $f(u_0)$. If $f : U \to V$ is continuous at every point $u_0 \in U$, then $f : U \to V$ is said to be continuous on U.

Definition 4 ([17])**.** *Let U be an algebra and (U,N) a fuzzy-normed space. The fuzzy-normed space (U,N) is called a fuzzy-normed algebra if*

$$N\big(uu', st\big) \geq N(u,s)N\big(u',t\big) \quad \forall u,u' \in U, s,t \in \mathbb{R}^+.$$

Complete fuzzy-normed algebra is called a fuzzy Banach algebra.

Example 1. *Every normed algebra $(U, \|.\|)$ defines a fuzzy-normed algebra (U,N), where N is defined by*

$$N(u,t) = \frac{t}{t + \|u\|} \quad \forall u \in U, \forall t > 0.$$

This space is called the induced fuzzy-normed algebra.

Definition 5. *Let U be a complex algebra. An involution on U is a function $\star : U \to U$ defined by $a \mapsto a^\star$, satisfying the following properties:*
(i) $(\alpha a + \beta b)^\star = \bar{\alpha} a^\star + \bar{\beta} b^\star$ for all $a, b \in U$ and $\alpha, \beta \in \mathbb{C}$;
(ii) $(ab)^\star = b^\star a^\star$ for all $a, b \in U$;
(iii) $a^{\star\star} = a$ for all $a \in U$.
If U is a complex algebra with an involution, then it is called a $\star$-algebra. A $C^\star$-algebra is a Banach algebra with an involution $\star$ such that $\|a^\star a\| = ||a||^2$.

Definition 6 ([18])**.** *Let U be an $\star$-algebra and (U,N) a fuzzy-normed algebra. The fuzzy-normed algebra (U,N) is called a fuzzy-normed $\star$-algebra if*

$$N(a^\star, t) = N(a,t) \quad \forall a \in U, \forall t \in \mathbb{R}^+.$$

A complete fuzzy-normed $\star$-algebra is called a fuzzy Banach $\star$-algebra.

Definition 7 ([18])**.** *Let (U,N) be a fuzzy Banach $\star$-algebra. Then (U,N) is called a fuzzy $C^\star$-algebra if*

$$N(a^\star a, st) = N(a^\star, s)N(a,t) \quad \forall a \in U, \forall s,t \in \mathbb{R}^+.$$

2. Results

In this section, we will use the fixed-point theorem and functional inequalities to prove the existence of a unique involution for the fuzzy Banach algebra (U,N). We will also demonstrate under what conditions this fuzzy Banach algebra can be transformed into a

$C^{\star}$-algebra. For this purpose, we will first prove two simple lemmas and then proceed to prove the main theorems.

Lemma 1. *Let $F : U \to U$ be a mapping satisfying the following*

$$N(F(2u) + F(2v) + 2F(w), t) \geq N\left(F(u + v + w), \frac{t}{3}\right), \tag{1}$$

for all $u, v, w \in U$. Then, F is additive, i. e., $F(u + v) = F(u) + F(v)$, for all $u, v \in U$.

Proof. Putting $u = v = w = 0$ in (1), we obtain

$$N(4F(0), t) \geq N\left(F(0), \frac{t}{3}\right) \Longrightarrow N\left(F(0), \frac{t}{4}\right) \geq N\left(F(0), \frac{t}{3}\right),$$

for all $t > 0$. After applying $N(5)$ and $N(6)$, it can be deduced that the value of $N(F(0), t)$ is 1. It means that $F(0) = 0$. Letting $v := -u$ and $w = 0$ in (1) so we have

$$N(F(2u) + F(-2u), t) \geq N\left(F(0), \frac{t}{3}\right) = 1,$$

thus $F(-u) = -F(u)$. Next, putting $w := -u$ and $v = 0$ in (1), we get

$$N(F(2u) + 2F(-u), t) \geq N\left(F(0), \frac{t}{3}\right) = 1,$$

therefore $F(2u) = 2F(u)$. Finally, replacing w by $-u - v$ in (1), we have

$$N(F(2u) + F(2v) + 2F(-(u + v), t) \geq N\left(2F(u + v - u - v), \frac{t}{3}\right) = 1,$$

so, $F(2u) + F(2v) = -2F(-(u + v))$, i. e., F is additive. □

Lemma 2. *Let $F : U \to U$ be a mapping satisfying*

$$N(F(2u) + \bar{\mu}F(2v) + 2F(w), t) \geq N\left(F(u + \mu v + w), \frac{t}{3}\right), \tag{2}$$

for all $\mu \in \mathbb{T}^1 := \{z \in \mathbb{C} : |z| = 1\}$, and for every $u, v, w \in U$, then F is $\mathbb{C}-$linear.

Proof. We prove that $F(\alpha u) = \bar{\alpha}F(u)$ for all $\alpha \in \mathbb{C}$. If $\mu = 1$ in (2), then F is additive by Lemma 1. Substituting u, v, w by $\mu u, -v, 0$ in (2), respectively. We have

$$N(F(2\mu u) + \bar{\mu}F(-2u), t) \geq N\left(F(\mu u - \mu u + 0), \frac{t}{3}\right) = 1,$$

therefore,

$$F(2\mu u) + \bar{\mu}F(-2u) = 0 \longrightarrow F(2\mu u) = -\bar{\mu}F(-2u),$$

that is, $F(\mu u) = \bar{\mu}F(u)$ for all $\mu \in \mathbb{T}^1$ and $u \in U$. We know that

$$|\mu| = |\bar{\mu}| = 1, \qquad \frac{\mu + \bar{\mu}}{2} = t \ \text{(real part of } \mu),$$

so $|t| = \left|\frac{\mu+\bar{\mu}}{2}\right| \leq 1$. Thus, for all real number t with $|t| \leq 1$ and $u \in U$, we get

$$F(2tu) = F(\mu u + \bar{\mu}u) = F(\mu u) + F(\bar{\mu}u) = \mu F(u) + \bar{\mu}F(u) = 2tF(u)$$

for all $\mu \in \mathbb{T}^1$ and $u \in U$. So, $F(tu) = tF(u)$ for all $u \in U$ and $|t| \leq 1$. Moreover, It is a known fact that if $F(2u) = 2F(u)$, then it follows that $F(2^n u) = 2^n F(u)$ for any positive integer n. Additionally, by utilizing the Archimedean property, we can determine that, for any real number t, there exists a positive integer n such that $|t| \leq 2^n$. Hence,

$$F(tu) = F\left(2^n.\frac{t}{2^n}u\right) = 2^n F\left(\frac{t}{2^n}u\right) = 2^n.\frac{t}{2^n}F(u) = tF(u),$$

for all $t \in \mathbb{R}$. Suppose that $\alpha \in \mathbb{C}$ be an arbitrary complex number. Therefore, $\alpha = t + is$ for some real numbers t, s. Since $|i| = 1$, so $F(iu) = iF(u)$ for all $u \in U$. Finally, we get

$$\begin{aligned} F(\alpha u) = F((t+is)u) = F(tu + isu) = F(tu) + F(isu) \;\; (F \text{ is additive}) \\ = tF(u) + sF(iu) = tF(u) + \bar{i}sF(u) = tF(u) - isF(u) = \bar{\alpha}F(u), \end{aligned}$$

Then, F is $\mathbb{C}-$linear. □

Theorem 3. *Let $\varphi : U^3 \to (0, \infty)$ and $\Phi : U^2 \to [0, \infty)$ be functions such that there exists an $L < 1$ with*

$$\varphi(u, v, w) \leq \frac{L}{2}\varphi(2u, 2v, 2w), \tag{3}$$

and

$$\lim_{n\to\infty} 4^n \Phi\left(\frac{u}{2^n}, \frac{v}{2^n}\right) = 0, \tag{4}$$

for all $u, v, w \in U$. In addition, assume $F : U \to U$ to be an odd function satisfying

$$N(F(2u) + \bar{\mu}F(2v) + 2F(w), t) \geq \min\left\{N\left(2F(u + \mu v + w), \frac{t}{3}\right), \frac{t}{t + \varphi(u, v, w)}\right\}, \tag{5}$$

for all $u, v, w \in U$ and $t > 0$. Moreover, suppose that

$$N(F(uv) - F(v)F(u), t) \geq \frac{t}{t + \Phi(u, v)}, \tag{6}$$

and

$$N - \lim_{k\to\infty} 2^k F\left(2^{-k}(N - \lim_{m\to\infty} 2^m F(2^{-m}u))\right) = u \tag{7}$$

for all $u, v \in U$ and $t > 0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique involution $A : U \to U$ such that

$$A(u) := N - \lim_{k\to\infty} 2^k F\left(\frac{u}{2^k}\right) \tag{8}$$

and

$$N(F(u) - A(u), t) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(0, u, -u)} \tag{9}$$

for all $u \in U$ and $t > 0$.

Theorem 4. *Under the assumptions of Theorem (3), if*

$$N([N(F(u), t) - N(u, t)]u, t) \geq \frac{t}{t + \varphi(u, u, 0)} \tag{10}$$

Then, (U, N) is fuzzy Banach $\star-$algebra. Moreover, if

$$N([N(F(u).u, st) - N(F(u), s).N(u, t)]u, t) \geq \frac{t}{t + \varphi(u, u, 0)} \tag{11}$$

Then (U, N) is fuzzy $C^\star$-algebra with involution $u^\star = A(u)$ for all $u \in U$. Furthermore, if

$$N(F(u) - u, t) \geq \frac{t}{t + \Phi(u, u)}, \tag{12}$$

then u is self-adjoint element of U.

Proof of Theorem 3. Replacing μ, u, v, w by $1, 0, 2u, -2u$ in (5), we have

$$\begin{aligned} N(F(0)) + F(4u) + 2F(-2u), t) &\geq \min\left\{ N(2F(0), \frac{t}{3}), \frac{t}{t + \varphi(0, 2u, -2u)} \right\} \\ &= \frac{t}{t + \varphi(0, 2u, -2u)}, \end{aligned}$$

therefore,

$$N(F(4u) - 2F(2u), t) \geq \frac{t}{t + \varphi(0, 2u, -2u)}, \tag{13}$$

for all $u \in U$ and $t > 0$. So

$$N(F(2u) - 2F(u), t) \geq \frac{t}{t + \varphi(0, u, -u)},$$

for all $u \in U$ and $t > 0$. We define the set $\mathcal{S}$ and introduce the function $d : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ as follows.

$$\begin{aligned} S &: \{G : U \to U \mid G(0) = 0\} \\ d(G, H) &= \inf\left\{ \lambda \in \mathbb{R}^+ : \ N(G(u) - H(u), \lambda t) \geq \frac{t}{t + \varphi(0, u, -u)} \right\}, \end{aligned} \tag{14}$$

where, as usual, $\inf \emptyset = +\infty$. The proof that $(\mathcal{S}, d)$ is a generalized complete metric space has been investigated in ([19]). Now we consider the linear mapping $\Pi : \mathcal{S} \to \mathcal{S}$ such that

$$\Pi G(u) := 2G\left(\frac{u}{2}\right)$$

for all $u \in U$. We prove that $\Pi : G \to G$ is a strictly contractive mapping with Lipschitz constant L. For this purpose, Let $G, H \in \mathcal{S}$ be given such that $d(G, H) = \epsilon$. So, according to the definition of metric d in (14), we have

$$N(G(u) - H(u), \epsilon t) \geq \frac{t}{t + \varphi(0, u, -u)}$$

for all $u \in U$ and $t > 0$. Therefore,

$$\begin{aligned} N(\Pi G(u) - \Pi H(u), L\epsilon t) &= N\left(2G\left(\frac{u}{2}\right) - 2H\left(\frac{u}{2}\right), L\epsilon t\right) \\ &= N\left(G\left(\frac{u}{2}\right) - H\left(\frac{u}{2}\right), \frac{L}{2}\epsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi(0, \frac{u}{2}, -\frac{u}{2})} \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(0, u, -u)} \quad \text{(by definition of } \varphi) \end{aligned}$$

for all $u \in U$ and $t > 0$. Then, assuming $d(G, H) = \epsilon$, we have proved that $d(\Pi G, \Pi H) = L\epsilon$. This means that

$$d(\Pi G, \Pi H) \leq L d(G, H),$$

for all $G, H \in \mathcal{S}$. Therefore Π is a strictly contractive mapping with Lipschitz constant L. Replacing u by $\frac{u}{4}$ in (13) and applying (3), we obtain

$$N\left(F(u) - 2F\left(\frac{U}{2}\right), t\right) \geq \frac{t}{t + \varphi(0, \frac{u}{2}, \frac{-u}{2})} \geq \frac{t}{t + \frac{L}{2}\varphi(0, u, -u)}$$

Substituting $\frac{Lt}{2}$ by t in the above inequality, we obtain

$$N\left(F(u) - 2F\left(\frac{U}{2}\right), \frac{Lt}{2}\right) \geq \frac{t}{t + \varphi(0, u, -u)}$$

for all $u \in U$ and $t > 0$. Therefore,

$$d(F, \Pi F) < \frac{L}{2}. \tag{15}$$

The conditions of Theorem (2) are satisfied. Hence,
(**I**) There exists a mapping $A : U \to U$ such that it is a fixed point Π. This means that

$$A\left(\frac{u}{2}\right) = \frac{1}{2}A(u), \tag{16}$$

for all $u \in U$. The mapping A is a unique fixed point of Π in the set

$$\Gamma : \{G \in \mathcal{S} : d(F, G) < \infty\}.$$

This suggests that $A : U \to U$ is a unique mapping satisfying (16), furthermore, there exists a $\lambda \in (0, \infty)$ satisfying

$$N(F(u) - A(u), \lambda t) \geq \frac{t}{t + \varphi(0, u, -u)}$$

for all $u \in U$.
(**II**) $d(\Pi^n F, A) \to 0$ as $n \to \infty$. This suggests that

$$A(u) = \lim_{n \to \infty} 2^n F(2^{-n} u) \tag{17}$$

for every $u \in U$.
(**III**) $d(F, A) \leq \frac{d(F, \Pi F)}{1 - L}$. By (15), we have

$$d(F, A) \leq \frac{L}{2 - 2L}.$$

This suggests that (9) is held. Replacing u, v, w by $\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}$, respectively, in (5); therefore,

$$\begin{aligned} &N\left(2^n F(\frac{2u}{2^n}) + \bar{\mu} 2^n F(\frac{2v}{2^n}) + 2^n 2F(\frac{w}{2^n}), 2^n t\right) \\ &\quad \geq \min\left\{N(2^{n+1} F(\frac{u}{2^n} + \mu \frac{v}{2^n} + \frac{w}{2^n}), \frac{2^n t}{3}), \frac{t}{t + \varphi(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n})}\right\}, \end{aligned}$$

for all $u, v, w \in U$ and $t > 0$. Replacing t by $\frac{t}{2^n}$, we thus obtain

$$\begin{aligned} &N\left(2^n F(\frac{2u}{2^n}) + \bar{\mu} 2^n F(\frac{2v}{2^n}) + 2^n 2F(\frac{w}{2^n}), t\right) \\ &\quad \geq \min\left\{N(2^{n+1} F(\frac{u}{2^n} + \mu \frac{v}{2^n} + \frac{w}{2^n}), \frac{t}{3}), \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \varphi(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n})}\right\}, \end{aligned} \tag{18}$$

for all $u, v, w \in U$ and $t > 0$. Note that, by using (3), we obtain

$$\varphi\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) \leq \frac{L}{2}\varphi(u, v, w).$$

Putting u, v, w by $\frac{u}{2}, \frac{v}{2}, \frac{w}{2}$ in the above inequality, such that

$$\varphi\left(\frac{u}{2^2}, \frac{v}{2^2}, \frac{w}{2^2}\right) \leq \frac{L}{2}\varphi\left(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}\right) \leq \frac{L^2}{2^2}\varphi(u, v, w).$$

By continuing this process, we have

$$\varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) \leq \frac{L^n}{2^n}\varphi(u, v, w).$$

Therefore

$$\frac{\frac{t}{2^n}}{\frac{t}{2^n} + \varphi(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n})} \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(u, v, w)} = \frac{t}{t + L^n\varphi(u, v, w)}.$$

Thus,

$$\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \varphi(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n})} > 1.$$

By passing $n \to \infty$ in (18) and using (17), we obtain

$$N(A(2u) + \bar{\mu}A(2v) + 2A(w), t) \geq N(2A(u + \mu v + w), \frac{t}{3}),$$

for all $u, v, w \in U$, $t > 0$ and $\mu \in \mathbb{T}^1$. By Lemma (2), the mapping $A : U \to U$ is $\mathbb{C}$-linear. We replace $\frac{u}{2^n}, \frac{v}{2^n}$ with u, v, respectively, in (6). Hence,

$$N\left(F(\frac{u}{2^n}\frac{v}{2^n}) - F(\frac{v}{2^n})F(\frac{u}{2^n}), t\right) \geq \frac{t}{t + \Phi(\frac{u}{2^n}, \frac{v}{2^n})},$$

therefore,

$$N\left(4^n F(\frac{u}{2^n}.\frac{v}{2^n}) - 2^n F(\frac{v}{2^n}).2^n F(\frac{u}{2^n}), 4^n t\right) \geq \frac{t}{t + \Phi(\frac{u}{2^n}, \frac{v}{2^n})},$$

so,

$$N\left(F(\frac{u}{2^n}\frac{v}{2^n}) - F(\frac{v}{2^n})F(\frac{u}{2^n}), t\right) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \Phi(\frac{u}{2^n}, \frac{v}{2^n})}.$$

Passing $n \to \infty$ and applying (4) and (17), we get

$$N(A(u.v) - A(v)A(u), t) \geq 1 \longrightarrow A(u.v) = A(v)A(u).$$

Furthermore, by (7) we have

$$A(A(u)) = N - \lim_{k\to\infty} 2^k f(2^{-k}(N - \lim_{m\to\infty} 2^m f(2^{-m}u))) = u.$$

We proved that $A : U \to U$ is $\mathbb{C}-$linear, $A(u.v) = A(v).A(u)$ and $A(A(u)) = u$. These mean that A is involution. $\square$

Proof of Theorem 4. Putting $u = \frac{u}{2^n}$ in (10), we obtain

$$N\Big(\Big[N(2^nF(\frac{u}{2^n}),2^nt) - N(\frac{u}{2^n},t)\Big]\frac{u}{2^n},t\Big) \geq \frac{t}{t+\varphi(\frac{u}{2^n},\frac{u}{2^n},0)}, \text{ for all } t>0, u\in U$$

therefore,

$$N\Big(\Big[N(2^nF(\frac{u}{2^n}),2^nt) - N(u,2^nt)\Big]u,2^nt\Big) \geq \frac{t}{t+\varphi(\frac{u}{2^n},\frac{u}{2^n},0)},$$

so,

$$\begin{aligned} N\Big(\Big[N(2^nF(\frac{u}{2^n}),t) - N(u,t)\Big]u,t\Big) &\geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\varphi(\frac{u}{2^n},\frac{u}{2^n},0)} \\ &\geq \frac{t}{t+L^n\varphi(u,u,0)}, \end{aligned}$$

by passing $n \to \infty$, we have

$$N([N(A(u),t) - N(u,t)]u,t) \geq 1 \longrightarrow N(A(u),t) - N(u,t)]u = 0,$$

for all $u \in U$ and $t > 0$. This suggests that $N(A(u),t) = N(u,t)$. Then, (U,N) is fuzzy Banach $\star-$algebra.

Next, we suppose that A satisfies in (11), so we obtain

$$N\Big(\Big[[N(2^nF(\frac{u}{2^n}).u,2^{2n}st) - N(2^nF(\frac{u}{2^n}),2^ns).N(u,2^nt)\Big]u,2^nt\Big) \geq \frac{t}{t+\varphi(\frac{u}{2^n},\frac{u}{2^n},0)},$$

for all $u \in U$ and $t,s > 0$. Thus,

$$\begin{aligned} N\Big(\Big[N(2^nF(\frac{u}{2^n}).u,st) - N(2^nF(\frac{u}{2^n}),s).N(u,t)\Big]u,t\Big) &\geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\varphi(\frac{u}{2^n},\frac{u}{2^n},0)}, \\ &\geq \frac{t}{t+L^n\varphi(u,u,0)}, \end{aligned}$$

for every $u \in U$ and $t,s > 0$. By passing $n \to \infty$, we have

$$N([N(A(u).u,st) - N(A(u),s).N(u,t)]u,t) \geq 1.$$

This means that

$$N(A(u).u,st) = N(A(u),s).N(u,t)$$

for all $u \in U$ and $t > 0$. Therefore, U is a $C^\star$-algebra with involution $u^\star = A(u)$ for all $u \in U$. Moreover, replacing u with $\frac{u}{2^n}$ in (12), we obtain

$$N\Big(2^nF(\frac{u}{2^n}) - u,2^nt\Big) \geq \frac{t}{t+\Phi(\frac{u}{2^n},\frac{u}{2^n})},$$

therefore

$$N\Big(2^nF(\frac{u}{2^n}) - u,t\Big) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n}+\Phi(\frac{u}{2^n},\frac{u}{2^n})},$$

by passing $n \to \infty$ and using (4), we have

$$N(A(u) - u,t) = 1 \longrightarrow u^\star = A(u) = u$$

it means that u is self-adjoint element of U. □

Corollary 1. *Let $p > 1, q > 2$. Suppose that $\eta, \eta' : U \to [0, \infty)$ are functions such that*

$$\eta\left(\frac{u}{2}\right) = \left(\frac{1}{2}\right)^p \eta(u) \tag{19}$$

$$\eta'\left(\frac{u}{2}\right) = \left(\frac{1}{2}\right)^q \eta'(u) \tag{20}$$

for all $u \in U$. Suppose that $F : U \to U$ is a odd function that satisfies

$$N(F(2u) + \bar{\mu}F(2v) + 2F(w), t) \geq \frac{t}{t + \eta(u) + \eta(v) + \eta(w)} \tag{21}$$

for all $u, v, w \in U$. Moreover, assume that

$$N(F(uv) - F(v)F(u), t) \geq \frac{t}{t + \eta'(u) + \eta'(u)}, \tag{22}$$

and

$$N - \lim_{k \to \infty} 2^k F(2^{-k}(N - \lim_{m \to \infty} 2^m F(2^{-m}u))) = u \tag{23}$$

for all $u, v \in U$ and $t > 0$ and all $\mu \in \mathbb{T}^1$. Then, there exists a unique involution $A : U \to U$ such that

$$A(x) := N - \lim_{k \to \infty} 2^k F\left(\frac{x}{2^k}\right) \tag{24}$$

and

$$N(F(u) - A(u), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + \eta(2u) + \eta(-2u)} \tag{25}$$

for all $u \in U$ and $t > 0$.

Proof. We define $\varphi : U^3 \to (0, \infty)$ and $\Phi : U^2 \to [0, \infty)$ by

$$\varphi(u, v, w) = \eta(u) + \eta(v) + \eta(w) \quad \text{for all } u, v, w \in U,$$
$$\Phi(u, v) = \eta'(u) + \eta'(v) \quad \text{for all } u, v \in U.$$

From (19), we have

$$\begin{aligned} \varphi(2u, 2v, 2w) &= \eta(2u) + \eta(2v) + \eta(2w) \\ &= 2^p \Big[\eta(u) + \eta(v) + \eta(w)\Big] \\ &= 2^p \varphi(u, v, w), \end{aligned}$$

for all $u, v, w \in U$. Therefore,

$$\begin{aligned} \varphi(u, v, w) &= \frac{1}{2^p} \varphi(2u, 2v, 2w) \\ &= \frac{2^{1-p}}{2} \varphi(2u, 2v, 2w), \end{aligned}$$

we put $L := 2^{1-p}$, since $p > 1$, so $L < 1$. On the other

$$\begin{aligned}\lim_{n\to\infty} 4^n\Phi(\frac{u}{2^n}, \frac{v}{2^n}) &= \lim_{n\to\infty} 4^n\left[\eta'(\frac{u}{2^n}) + \eta'(\frac{v}{2^n})\right] \\ &= \lim_{n\to\infty} 4^n\left[(\frac{1}{2})^{nq}\left[\eta'(u) + \eta'(v)\right]\right] \\ &= \lim_{n\to\infty} 2^{n(2-q)}\left[\eta'(u) + \eta'(v)\right] = 0, \quad \text{since } q > 2.\end{aligned}$$

By Theorem (3), there exists a unique involution $A : U \to U$ such that

$$\begin{aligned}N(f(u) - A(u), t) &\geq \frac{(2 - 2.2^{1-p})t}{(2 - 2.2^{1-p})t + 2^p\big(\eta(0) + \eta(u) + \eta(-u)\big)} \\ &= \frac{(2 - 2^{2-p})t}{(2 - 2^{2-p})t + 2^p\big(\eta(u) + \eta(-u)\big)} \\ &= \frac{(2^p - 4)t}{(2^p - 4)t + \eta(2u) + \eta(-2u)}, \quad \text{for all } u \in U, t > 0\end{aligned}$$

Note that $\eta(2u) = 2^p\eta(u)$. $\square$

3. Implement Python Code for Theorem 3

In this part, the Python code of Theorem (3) is provided. The purpose of writing the code is to first increase the clarity of the Theorem (3) by providing a practical representation of the assumptions and structure of the theorem. Secondly, it is to make a connection between the two topics of Hyers–Ulam stability and the field of computer science. In this code, the user is asked to enter N, F, φ, and Φ functions.

```
import numpy as np

def N(u,t):
    return # The user defines the fuzzy-normed algebra N.

def φ(u,v,w):
    return # The user defines the φ function, for example: return (u + v + w)**2 - 4*(u*v + v*w + w*u).

def Φ(u,v):
    return # The user defines the Φ function, for example: return np.exp(u) * np.exp(2*v).

def F(u):
    return # The user defines the F function, for example: return np.sin(u).

def check_limit(Φ,u,v):
    n = 1
    while True:
        term = 4**n * Φ(u/(2**n), v/(2**n))
        if np.abs(term) < 1e-10:
            return True
        if n > 1e5:
            return False
        n =n+ 1
```

```
def inequality5(u, v, w, t, μ = None):   # the execution of inequality (5)
    return # True if inequality (5) holds, False otherwise

def inequality6(u, v, t):   # the execution of inequality (6)
    return # True if inequality (6) holds, False otherwise

def Equation (7)(u):   # the execution of Equation (7)
    return # True if Equation (7) holds, False otherwise

def inequality9(u):   # the execution of inequality (9)
    return # True if inequality (9) holds, False otherwise

def check_assumptions(L, U, 𝕋¹): # Check if the assumptions of the theorem are satisfied,
    for u, v, w ∈ U:
        if φ(u, v, w) > (L/2) * φ(2 * u, 2 * v, 2 * w):
            return False
    for u, v ∈ U:
        if not check_limit(Φ, u, v):
            return False
    for u, v, w ∈ U:
        t = # The user defines the number t
        for μ ∈ 𝕋¹:
            if not inequality5 (u, v, w, t, μ):
                return False
    for u, v ∈ U:
        t = # The user defines the number t
        if not inequality(6) (u, v, t):
            return False
    for u ∈ U:
        if not Equation (7)(u):
            return False
    return True

def A(u,t): #Define the function A
    k = 0
    tolerance= # The user enter tolerance
    while True:
        A_k = 2^k * F(u/(2^k))
        A_k_next = 2^(k+1) * F(u/(2^(k+1)))
        if N(A_k - A_k_next, t) > tolerance:
            break
        k = k+1
    return A_k

L = # The user defines the number L
U = # define the set of values to check the assumptions
𝕋¹ = # define the 𝕋¹
t= # The user enter any t>0

if check_assumptions (L, U, 𝕋¹):
    print(f"{A(u,t)} is a unique involution")
    print(f" A satisfies in {inequality(9)(u)}")
else:
```

print("The conditions of the Theorem are not upheld")

In this code, we first import the Numpy library so that we can use mathematical functions throughout the code. Now, we define the function of two variables N, the function of three variables φ, the function of two variables Φ, and the function F. These functions are given to the system by the user. To check assumption (4), we define the function **"check_limit**(Φ, u, v)**"**. This function has three variables and checks the limit of the function Φ at infinity. In the following, inequality(5) is defined. It takes five arguments u, v, w, t, and an optional μ argument. Then, the inequalities (6), (9), and Equation (7) have been defined by the functions **"def inequality(6)**(u, v, t)**"**, **"def Equation (7)**(u)**"** and **"def inequality(9)**(u)**"**, respectively.

Now, we will check whether the assumptions of the Theorem (3) are valid. For this purpose, we define the function **"def check_assumptions**$(L, U, \mathbb{T}^1)$**"**. The task of this function is to check the validity of all five assumptions of the Theorem (3).

In **"**$def\, A(u,t)$**"**, we define the function A and check whether A is a Cauchy sequence. Note that since U is Banach space the Cauchyness of the sequence guarantees its convergence. We have defined the necessary functions. Now, to run the code, we first define the values of L, U, T, and t and call the function **"check_assumptions"** $(L, U, \mathbb{T}^1)$. If this function is true, A is declared as an involution that applies to the inequality (9).

4. Conclusions

In this article, we first investigated the stability of Hyers–Ulam involution in fuzzy $C^\star$-algebras using functional inequality and the fixed-point method. We then implemented the code of the main theorem in Python with the aim of making the theorem clearer and making the connection between Hyers–Ulam stability and computer science. We hope to make Hyers–Ulam stability more accessible to researchers in mathematics and computer science and to encourage further research on the connection between these two fields.

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