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Hyperstability of Linear Feed-Forward Time-Invariant Systems Subject to Internal and External Point Delays and Impulsive Nonlinear Time-Varying Feedback Controls

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Abstract: This paper investigates the asymptotic hyperstability of a single-input–single-output closed-loop system whose controlled plant is time-invariant and possesses a strongly strictly positive real transfer function that is subject to internal and external point delays. There are, in general, two controls involved, namely, the internal one that stabilizes the system with linear state feedback independent of the delay sizes and the external one that belongs to an hyperstable class and satisfies a Popov's-type time integral inequality. Such a class of hyperstable controllers under consideration combines, in general, a regular impulse-free part with an impulsive part.

Keywords: Lyapunov's stability; hyperstability; impulsive controls; Popov's inequality; time-delay systems; positive real transfer functions

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1. Introduction

Time delays are very common in real dynamic processes. Typical examples in the real world are, for instance, sunflower dynamics, war/peace models, diffusion problems, economic models, etc. Delays can happen either in the dynamics itself (internal delays) or in the control actions or forcing functions (external delays). Very commonly, both kinds of delays appear together. Also, the delays can be point delays or distributed delays, or they can be time-varying or constant. In general, the modeling of internal delays becomes difficult since the resulting system is infinite-dimensional even if its delay-free counterpart is finite-dimensional [1,2]. It is well-known that the stability property is very relevant in dynamic systems for theoretical studies and real applications and, in many practical situations, the stabilization has to be performed with feedback in the case when the uncontrolled system is unstable. The stability problem has also received much attention related to the stability of functional equations in Banach spaces. For instance, the so-called Hyers-type stability related to the maintenance of the stability of functional equations under certain perturbations was initially studied in [3,4] and many other works that followed them later on, and, nowadays, the subject continues to receive relevant attention. The so-called absolute stability is a property of the global asymptotic stability in dynamic systems that is achievable under any feedback non-linear control device whose definition functions belong to a certain sector, for instance, the so-called Lur´e sectors or the so-called Popov sectors [5,6]. An extension of that property is the so-called asymptotic hypertability, that is, the global asymptotic stability under any non-linear and possibly time-varying member in the class of feedback controllers that satisfy a very general Popov´s integral inequality. See, for instance, refs. [7–12] and some of the references therein. A case of interest in the context of hyperstability studies is when the feed-forward system is linear and time-invariant, characterized by a positive real transfer function $G(s)$, that is, $Re\ G(s) \geq 0$ for $Re\ s \geq 0$, and the feedback controller is any member of a general class (in short, an hyperstable class of controllers) that satisfies a Popov's-type integral inequality for all time [7–9]. Since the transfer function of the uncontrolled system is positive real, another consequent property for those kinds of controllers is that the input–output energy in the feed-forward part of the whole closed-loop system, that is, that of the controlled system, is non-negative and bounded for all time. This translates into the fact that the closed-loop system is globally stable for any given finite initial conditions (in short, hyperstable). If, in addition, the transfer functions are strictly positive real (rather than just positive real), that is, $Re\ G(s) > 0$ for $Re\ s \geq 0$, then the generated controls using the hyperstable class of controllers are, furthermore, square-integrable and vanish asymptotically as time tends to infinity [9–12]. As a result, the resulting closed-loop system is globally asymptotically stable for any finite initial conditions (in short, asymptotically hyperstable). In [11], the hyperstability formulation was extended to the presence of point time delays in the feed-forward control loop, that is, in the controlled system and, in [12], to the case of hyperstable controllers that include impulsive actions through time. It can be pointed out that impulsive controls are useful to describe switching actions between configurations since impulses in the controls at certain time instants translate into finite jumps in the solution for the differential system, which is a useful mathematical tool to progress with the solution evolution under another potential parameterization, or configuration, of such a system at hand. This allows us to describe problems more tightly where the parameterizations change through time as, for instance, in certain industrial diffusion processes. Important effort was allocated to design stable adaptive control configurations based on hyperstability theory in theoretical approaches and in some practical implementations. See, for instance, refs. [9,13–16] and some of the references therein.

On the other hand, concerning the properties and use of impulsive controllers, some stabilization properties were discussed in [17] based on Lyapunov stability theory. In particular, mixed conditions on the characteristics of the controlled system and the frequency and gains of the impulsive actions were given explicitly to guarantee the stability of the closed-loop system. In [18], the existence of periodic solutions for a class of impulsive differential equations with piecewise constant arguments was investigated. In [19], the stability of a class of time-invariant linear impulsive neutral delay differential equations was studied. Further results on the stability of nonlinear impulsive stochastic systems and time-delay systems with delay-dependent stability were performed in [20,21], respectively. It can be pointed out that hyperstability theory generalizes that of absolute stability since the stability property holds under more general controller devices, not necessarily required to be static and time-invariant, which, in turn, generalizes Lyapunov´s stability theory [22–26]. The importance of the energetic transactions and the relevance of the consideration of the delays in modeling dynamics and controls have also been addressed in some applications. For instance, in [27], an indirect multi-energy transaction versus a collaborative energetic optimization in a local energy market while improving energy utilization was proposed. The delay consideration both in the dynamics and in the controllers and actuators is important in problems like, for instance, traffic networks partitioned into multiple regions [28] or appropriate controller designs for asymptotic tracking problems. See, for instance, ref. [29] and references therein. See also [2,17,19,21].

The main objective of this paper is to investigate the asymptotic hyperstability of a closed-loop single-input–single-output dynamic system of the nth-order when the linear feed-forward controlled system is time-invariant with a strong, strictly positive real transfer function, which is subject to internal and external constant point delays. The proposed class of hyperstable controllers can possess, in the general case, both an impulse-free regular part and an impulsive part and satisfies a Popov's-type integral inequality that, in fact, defines the class of hyperstable controllers. An internal control stabilizes, independent of the sizes of the delays, the controlled stabilizable system with state linear feedback control. This internal controller might be omitted if the given linear feed-forward controlled system is already stable. Also, an external control is generated using the hyperstable class, which is subject to output feedback, and it is, in general, non-linear and time-varying and of a mixed regular and impulsive nature under a Popov's integral inequality for all time. The main novelty of this paper is that it addresses the incorporation of the impulsive part into the class of asymptotically hyperstable controllers by extending such a class.

This paper is organized as follows. A subsection of this introductory section describes the main concepts involved as well as the main notation used. Section 2 describes the whole single-input–single-output uncontrolled (or open-loop) stabilizable system subject to known internal and external point constant delays subject to an internal input able to stabilize the system. There is an internal linear state feedback control that stabilizes the delayed system and an external output–feedback nonlinear and, in general, time-varying control generated by a class of hyperstable controllers according to a Popov's time integral inequality. The transfer function for the linear plant with respect with this second control is strongly strictly positive real. Such a positive realness property is achieved with stabilization from the internal control and the involvement of a sufficiently large positive interconnection gain from the external input to the output. This section also formulates the asymptotic hyperstability main result when the hyperstable class of controllers is impulsive-free. It is shown for the major property that the input–output energy under null initial conditions is non-negative (as a result of the positive realness of the transfer function) and bounded for all time (as result of Popov's-type integral inequality defining the hyperstable class of controllers). In addition, for a strongly positive real transfer function for the feed-forward loop, the energy is positive for all positive time instants or any control (being non-null over some time interval of nonzero measure) generated with the hyperstable class of controllers. This combined positivity and boundedness in the external input–output energy under zero initial conditions translates into the fact that the external control asymptotically vanishes, which leads to the closed-loop asymptotic hyperstability. In the case of non-zero finite initial conditions, the external input–output energy is not guaranteed to be positive for all time, but it is still bounded for all time, and this fact guarantees that the above properties still hold. Later on, Section 3 extends the results of Section 2 when the hyperstable class of controllers consists of a regular part and an impulsive part. The set of impulsive time instants can be finite or infinite. In this case, the hyperstable class of controllers also satisfies a Popov's-type integral inequality for all time. The performed analysis becomes more complicated than that the one in the former section for the impulse-free case since the output finite jumps, caused by the impulses in the time-derivative of the state dynamics, cause the initial conditions to restart after each jump for generating the output solution along the next inter-impulsive time interval, which happens even under null initial conditions. In practice, such a drawback translates into the need for a specific stability analysis taking into account the contribution of the unforced output to the external input–output energy, which is unnecessary and omitted in the impulse-free case. Finally, conclusions end this paper. Two appendixes are incorporated to describe, respectively, the jumps caused by the control impulses in the relevant signals and an auxiliary technical convergence and stability result based on Venter's theorem [27].

Notation and Nomenclature

$\boldsymbol{R}$ is the set of real numbers, $\boldsymbol{R}_+$ is the set of positive real numbers, and $\boldsymbol{R}_-$ is the set of negative real numbers, where $\boldsymbol{R}_{0+} = \boldsymbol{R}_+ \cup \{0\}$ and $\boldsymbol{R}_{-0} = \boldsymbol{R}_- \cup \{0\}$ and $\overline{n} = \{1, 2, \ldots, n\}$.

$\boldsymbol{Z}$ is the set of integer numbers, $\boldsymbol{Z}_+$ is the set of positive integer numbers, and $\boldsymbol{Z}_-$ is the set of negative integer numbers, where $\boldsymbol{Z}_{0+} = \boldsymbol{Z}_+ \cup \{0\}$ and $\boldsymbol{Z}_{-0} = \boldsymbol{Z}_- \cup \{0\}$.

$\boldsymbol{C}$ is the set of complex numbers, $\boldsymbol{C}_+$ is the set of complex numbers with positive real part, and $\boldsymbol{C}_-$ is the set of complex numbers with negative real part, where $\boldsymbol{C}_{0+} = \boldsymbol{C}_+ \cup \{0\}$ and $\boldsymbol{C}_{-0} = \boldsymbol{C}_- \cup \{0\}$.

If A is a n-th complex square matrix with spectrum $sp(A) = \{\lambda_i : i \in \overline{n}\}$, then its spectral matrix measure, or logarithmic norm, $\mu_2(A)$, is

$$\mu_2(A) = \lim_{h \to 0^+} \frac{\|I_n + hA\| - h}{h} = \frac{1}{2} \max_{1 \le i \le n} \lambda_i(A + A^*)$$

The spectral abscissa of A is $-\rho = -\rho(A) = \underset{1\le i\le n}{max}\, Re\lambda_1$, which is negative if A is a stability matrix, and $\|e^{At}\| \le Ke^{\rho t}$ for some norm-dependent real constant $K \ge 1$.

I_n is the n-th identity matrix.

$G(s)$, where "s" is the Laplace transform variable, is a transfer function corresponding to the impulse response $g(t)$. The transfer function $G(s)$ of order n is said to be strictly stable if all its poles are in the open left-hand-side plane. A state–space realization of $G(s)$ is of order n, that is, it has a state vector $x(t)$ of dimension n for any $t \in \boldsymbol{R}_{0+}$ provided that $G(s)$ has n poles.

The sets PR, SPR, and $SSPR$ denote, respectively, the sets of positive real transfer functions, strictly positive real transfer functions, and strongly strictly positive real transfer functions. In particular:

$G(s)$ is said to be positive real ($G \in PR$) if $ReG(s) \ge 0$ for $Res > 0$, which holds if all its poles are in the closed left half plane and $ReG(\boldsymbol{i}\omega) \ge 0; \forall \omega \in \boldsymbol{R}$, where $\boldsymbol{i} = \sqrt{-1}$ is the complex unit.

$G(s)$ is said to be strictly positive real ($G \in SPR$) if $ReG(s) > 0$ for $Res \ge 0$, that is, if $G_\varepsilon(\equiv G(s-\varepsilon)) \in PR$ for some real $\varepsilon > 0$, which holds if all its poles are in the open left half plane and $ReG(\boldsymbol{i}\omega) > 0; \forall\omega \in \boldsymbol{R}$.

$G(s)$ is said to be strongly strictly positive real ($G \in SSPR$) if $G \in SPR$ and $\underset{\omega\to\infty}{lim}\, \omega^2 Re\, G(\boldsymbol{i}\omega) > 0$. If $G(s) \in SPR$ but $G(s) \notin SSPR$, then $G(s)$ is said to be weakly strictly positive real. It turns out that the above sets satisfy the following set inclusion chain $SSPR \subset SPR \subset PR$. It also turns out that, under zero controls and for any finite initial conditions, $x(t) \to 0(\in \boldsymbol{R}^n)$ and $y(t) \to 0(\in \boldsymbol{R})$ as $t \to \infty$ if $x(t)$ and $y(t)$ are the state vector and output, respectively, of a state–space realization of any $G \in SPR$ of order n since strictly positive real transfer functions are strictly stable.

Let SI be the set of impulsive time instants, that is, the set of time instants at which external control impulses are injected. If $t_i \in SI$, then t_i^- denotes its left limit, where impulse is not still injected, and $t_i = t_i^+$ denotes its right limit, which is not denoted with an explicit "+" notation (in particular, $0 = 0^+$), where the impulse is already effective. According to that notation, $u(t) = u(t^-)$ if $t \notin SI$ and $u(t) \ne u(t^-)$ if $t \in SI$. The subsets SI_{t-} and SI_t of SI are defined for any given $t \in \boldsymbol{R}_{0+}$ as $SI_{t-} = \{t_i \in SI : t_i < t\} \subset SI_t$ and $SI_t = \{t_i \in SI : t_i \le t\}$ so that $SI_t = SI_{t-}$ if $t \notin ST$ and $SI_t = SI_{t-} \cup t$ if $t \in SI_t$, $SI_{0-} = \varnothing$, $SI_0 = \{\varnothing\}$ if $0 \notin SI$, and $SI_0 = \{0\}$ if $0 \in SI$.

$cardSI = \vartheta \le \chi_0$, where χ_0 is the infinity cardinal of a denumerable set; thus, $\vartheta \le \chi_0$ means that they can happen either finitely many or countable infinitely many impulses. Also, $card\ SI_t = \vartheta_t$ and $card\ SI_{t-} = \vartheta_{t-}; \forall t \in \boldsymbol{R}_{0+}$. It follows that

$$\vartheta = card\ SI \ge \vartheta_t = cardS\ I_t \ge \vartheta_{t^-} = card\ SI_{t^-};\ \forall t \in \boldsymbol{R}_{0+}$$

Consider a function $f : \boldsymbol{R} \to \boldsymbol{R}^n$ and $(t_1, t_2) \subset \boldsymbol{R}$. Then, $f_{t_1,t_2}(t)$ the truncated $f(t)$ on (t_1, t_2) is defined as $f_{t_1,t_2}(t) = f(t)$ for $t \in (t_1,\ t_2)$ and $f_{t_1,t_2}(t) = 0$ for $t \in (-\infty,\ t_1] \cup [t_2,\ +\infty)$. In particular, if $f : \boldsymbol{R}_{0+} \to \boldsymbol{R}^n$, then the truncation of f on $(0,\ t)$ can be denoted as $f_{0,\ t}(t)$.

$F(\boldsymbol{i}\omega)$ and $F_{t_1,t_2}(\boldsymbol{i}\omega)$ are, respectively, the Fourier transforms of $f(t)$ and $f_{t_1,t_2}(t)$, provided that they exist.

2. Controlled System Description and Some Preliminary Results

The following single-input–single-output dynamic system of n-th order subject to nonlinear feedback is considered:

$$\dot{x}(t) = Ax(t) + A_d x(t-h) + b_0 u_0(t) + bu(t) + b_d u(t-h') \tag{1}$$

$$y(t) = c^T x(t) + du(t) + y_{ic}(t) \tag{2}$$

$$u_0(t) = k^T x(t) + k_d^T x(t-h) \tag{3}$$

$$u(t) = -f(y(t)\,,t) \tag{4}$$

The system is subject to an absolutely continuous function of initial conditions on $[-h,\ 0]$ with eventual finite jumps, where $h \geq 0$ and $h' > 0$ are, respectively, the known internal (i.e., in the state) and the external (i.e., in the output) delays; $x : [-h,\ +\infty) \to \boldsymbol{R}^n$ is the state n-th vector function; $u_0 : [0,\ +\infty) \to \boldsymbol{R}$ and $u : [0,\ +\infty) \to \boldsymbol{R}$ are, respectively, the scalar stabilizing internal-loop with state linear feedback, and the scalar, eventually impulsive, hyperstabilizing external-loop controls with output feedback; $y : [0,\ +\infty) \to \boldsymbol{R}$ is the scalar output function; $y_{ic}(t)$ is an output impulsive compensation signal, which is identically zero if the control $u(t)$ is not impulsive and removes potential impulsive contributions to the output caused by the impulsive controls (see Assumption 4 below for further details); $A \in \boldsymbol{R}^{n\times n}$ and $A_d \in \boldsymbol{R}^{n\times n}$ are, respectively, the delay-free and the delayed matrices of dynamics; $b_0 \in \boldsymbol{R}^n$, $b \in \boldsymbol{R}^n$, and $b_d \in \boldsymbol{R}^n$ are, respectively, the internal, the external delay-free, and the external delayed control vectors; $c \in \boldsymbol{R}^n$ is the output vector; and $d \in \boldsymbol{R}$ is the input–output interconnection gain, and $f : \boldsymbol{R} \times ([-h',\ 0) \cup \boldsymbol{R}_{0+}) \to \boldsymbol{R}$ is the, in general, nonlinear and time-varying control device. The definition of the external control as a function $u : [0,\ +\infty) \to \boldsymbol{R}$ assumes, with no loss in generality, that $u(t) = 0$ for $t \in [-h',\ 0)$, and such an assumption is made to facilitate the presentation of some of the proofs.

The control u_0 is the stabilizing internal-loop control since it stabilizes the system in the global asymptotic sense independent of delay if the external control $u \equiv 0$. It can be zeroed if the unforced system is globally asymptotically stable. The external control u is a hyperstabilizing control since it is used to achieve asymptotic hyperstability in the closed-loop system under certain conditions to be fulfilled using the feed-forward part of systems (1)–(4); basically, its transfer function is *SSPR*.

The following reasonable assumptions are made:

Assumption 1. *The initial condition is defined by a bounded absolutely continuous vector function* $\varphi : [-h,\ 0] \to \boldsymbol{R}^n$ *except possibly at a set of bounded isolated discontinuities. It will be often denoted* $x_0 = x(0) = \varphi(0)$ *for the value of the function of initial conditions at* $t = 0$.

Assumption 2. *The pair* (A, b_0) *is stabilizable.*

Assumption 3. *Given a delay-free controller gain* $k \in \boldsymbol{R}^n$ *such that* $A + b_0k^T$ *is a stability matrix (such a gain k exists from A2), with spectral abscissa* $\left(-\rho\left(A + b_0k^T\right)\right) < 0$. *Then, there exists a delay controller gain* $k_d \in \boldsymbol{R}^n$ *such that:*

C1: either $\|A_d + b_0k_d^T\|_2 \leq \rho\left(A + b_0k^T\right) - \varepsilon$ *for some* $\varepsilon\left(< \rho\left(A + b_0k^T\right)\right) \in \boldsymbol{R}_{0+}$;

C2: or $\|A_d + b_0k_d^T\|_2 + \mu_2\left(A + b_0k^T\right) < 0$.

Assumption 4. $f : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$ *is of the form* $f(y(t), t) = f_0(y(t), t) + \sum_{i=1}^{\vartheta} K(y(t^-), t)\, y(t^-)\delta(t - t_i)$, *where:*

- *The regular controller function* $f_0 : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$ *is piecewise continuous.*
- *The impulsive controller distribution* $\sum_{i=1}^{\vartheta} K(y(t^-), t)y(t^-)\delta(t - t_i)$ *is such that:*
    - *a.* $\delta : [0,\ +\infty) \to \boldsymbol{R}$ *is the Dirac distribution, supported with the test bounded function,* $K : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$, *is identically zero on* $[-h\ ,\ 0)$, *with a support of zero Lebesgue measure. That is, it is identically zero except perhaps on a subset of* $[0,\ +\infty)$ *of numerable isolated strictly ordered impulsive time instants* $SI = \{t_i\}_{i=1}^{\vartheta}$ *such that* $((t_i\ , t_{i+1}) \cap \boldsymbol{R}) \cap SI = \varnothing$. *That is, there is no impulsive time instant between* t_i *and* t_{i+1}, *which are then two consecutive impulsive time instants, with* $\vartheta \leq \chi_0$ *(where* χ_0 *denotes the infinite cardinal of a set of countable infinitely many time instants) at which its left limits are null and its right limits are finite.*
    - *b.* $K(0\ , t) = 0; \forall t \in \boldsymbol{R}_{0+}$; $K(y(t^-), t) \neq 0$ *if* $t \in SI$.
    - *c. The output impulsive compensation in (3) is* $y_{ic}(t) = dK(y(t^-), t)y(t^-)\delta(0); \forall t \in \boldsymbol{R}_{0+}$.

Note that if $K(0\,,t) = 0$ or if $y(t^-) = 0$, then $t \notin SI$, that is, t cannot be chosen as candidate for impulsive time instant. Note also that if $t \notin SI$, then $y_{ic}(t) = 0$. In this way, the impulsive controls are reflected in the external input, but their effects are removed from the output with the output impulsive compensation signal.

Assumption 5. *The control law (4) is defined by a function* $f : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$ *, subject to Assumption 4, which satisfies the subsequent Popov's-type integral inequality for some finite* $\gamma \in \boldsymbol{R}_+$*:*

$$\eta(t) = \int_0^t f(y(\tau)\,,\tau)y(\tau)d\tau \geq -\gamma;\ \forall t \in \boldsymbol{R}_{0+} \tag{5}$$

such that $\eta : \boldsymbol{R}_{0+} \to \boldsymbol{R}$ *exists everywhere in* $\boldsymbol{R}_{0+}$*. Note that the assumption requires the necessary condition* $\liminf_{t\to+\infty} \int_0^t f(y(\tau)\,,\tau)y(\tau)d\tau > -\infty$*. Note also that (5) is equivalent to the uniform boundedness of the external input–output energy in the linear part of the system for all time, that is:*

$$E(t) = \int_0^t y(\tau)u(\tau)d\tau = -\int_0^t f(y(\tau)\,,\tau)y(\tau)d\tau = -\eta(t) \leq \gamma < +\infty;\ \forall t \in \boldsymbol{R}_{0+} \tag{6}$$

Assumptions 1 and 4 guarantee the uniqueness of the solutions $x : [-h\,,\,+\infty) \to \boldsymbol{R}^n$ and $\sigma : [0,\,+\infty) \to \boldsymbol{R}$ for each given vector function of initial conditions. Also, Assumption 3 guarantees that $k : [0, +\infty) \to \boldsymbol{R}$ and $k_d : [-h, +\infty) \to \boldsymbol{R}$ are time-differentiable almost everywhere in their definition domains.

Remark 1. *If the pair* (A, b_0) *is controllable, then it is also stabilizable, and Assumption 2 holds. Note that* (A, b_0) *is controllable if and only if* $rank(sI_n - A\,,\,b_0) = n;\ \forall s \in \boldsymbol{C}$*, which holds if and only if* $rank(sI_n - A\,,\,b_0) = n;\ \forall s \in sp(A)$*. Also,* (A, b_0) *is stabilizable if and only if* $rank(sI_n - A\,,\,b_0) = n;\ \forall s \in \boldsymbol{C}_{0+}$*, which holds if and only if* $rank(sI_n - A\,,\,b_0) = n;$ $\forall s \in sp(A) \cap \boldsymbol{C}_{0+}$*. The above properties are, respectively, referred to as controllability and stabilizability Popov–Belevitch–Hautus rank tests [26]. If* (A, b_0) *is controllable, then it is closed-loop spectrum assignable in the sense that the eigenvalues of the closed-loop matrix of dynamics to be obtained using the controller gain* $k \in \boldsymbol{R}^n$*, that is,* $\left(A + b_0k^T\right)$ *can be allocated in prescribed positions with the choice of k. If* (A, b_0) *is stabilizable, then there exists some* $k \in \boldsymbol{R}^n$ *such that the closed-loop eigenvalues can be allocated in* $\boldsymbol{C}_-$*, although not in prescribed positions in general. The above discussion concludes that a controllable linear time-invariant system can be always stabilized with an appropriate linear state feedback and that a stabilizable, but uncontrollable, linear time-invariant system can be stabilized with linear state- feedback but can have poor transient behavior since the closed-loop dynamics cannot be fully prescribed with the choice of the controller gain.*

Remark 2. *Note from Assumption 2 (see Remark 1) that there always exists a controller gain* $k \in \boldsymbol{R}^n$ *such that* $A + b_0k^T$ *is a stability matrix, so that the starting stipulation of Assumption 3 always holds.*

Remark 3. *If the pair* (A_d, b_0) *is controllable, then there always exists a delay controller gain* $k_d \in \boldsymbol{R}^n$ *such that* $A_d + b_0k_d^T$ *can be fixed, so that its eigenvalues can be allocated in arbitrary prescribed stable positions (see Remark 1), so that both conditions C1 and C2 of Assumption 3 are always feasible.*

Remark 4. *Under Assumptions 1–3, the following auxiliary delayed linear system:*

$$\dot{z}(t) = Az(t) + A_dz(t-h) + b_0u_0(t) = A_cz(t) + A_{dc}z(t-h) \tag{7}$$

is globally asymptotically stable, where $z(t) = \varphi(t)$ *for* $t \in [-h\,,\,0]$ *satisfies Assumption 1, and* $A_c = A + b_0k^T$ *and* $A_{dc} = A_d + b_0k_d^T$ *satisfy Assumption 3. Note, in particular, that the fundamental matrix* e^{A_ct} *of* $\dot{z}_L(t) = A_cz_L(t)$*, whose infinitesimal generator is* A_c *with spectral* *abscissa $-\rho(A_c) < 0$, satisfies $\|e^{A_c t}\| \leq Ke^{-\rho(A_c)t}$ for some (norm-dependent) real constant $K \geq 1$. Note that the solution of (5) satisfies:*

$$\begin{aligned} \|z(t)\| &\leq Ke^{-\rho(A_c)t}\left(\|x_0\| + \int_0^h e^{\rho(A_c)\tau}\|A_{dc}\|\|\varphi(\tau-h)\|d\tau + \int_h^t e^{\rho(A_c)\tau}\|A_{dc}\|\,\|z(\tau-h)\|d\tau\right) \\ &= Ke^{-\varepsilon t}e^{-(\rho(A_c)-\varepsilon)t}\left(\|x_0\| + \int_0^h e^{\rho(A_c)\tau}\|A_{dc}\|\|\varphi(\tau-h)\|d\tau + \|A_{dc}\|\int_h^t e^{\rho(A_c)\tau}\|z(\tau-h)\|d\tau\right) \\ &\leq Ke^{-\varepsilon t}e^{-(\rho(A_c)-\varepsilon)t}\left(\|x_0\| + \int_0^h e^{\rho(A_c)\tau}\|A_{dc}\|\|\varphi(\tau-h)\|d\tau + \|A_{dc}\|\int_h^t e^{(\rho(A_c)+\varepsilon)\tau}\|z(\tau-h)\|d\tau\right); \\ & t \in [h\,,+\infty) \end{aligned} \tag{8}$$

for $0 < \varepsilon < \rho(A_c)$ (Condition C1 of Assumption 3). Then, for $t_a = (1/\varepsilon)lnK \geq h$ and all $t \geq t_a$, $Ke^{-\varepsilon t} \leq 1$ so that

$$\|z(t)\| \leq Ke^{-\rho(A_c)(t-t_a)}\|x(t_a)\| + \int_{t_a}^{t} e^{(\rho(A_c)-\varepsilon)(t-t_a+\tau)}\|A_{dc}\|\|z(\tau-h)\|d\tau;\ t \in [t_a\,,+\infty) \tag{9}$$

and, for some $t_b \geq t_a$ and all $t \geq t_b$, since $\|A_{dc}\| < \rho$,

$$\|z(t)\| \leq \left(Ke^{-\rho(A_c)(t-t_b)} + \rho^{-1}\|A_{dc}\|\right) \sup_{t_b \leq \tau \leq t-h}\|z(\tau)\| < \sup_{t_b \leq \tau \leq t-h}\|z(\tau)\|;\ t \in [t_b\,,+\infty) \tag{10}$$

which implies that $z(t) \to 0$ as $t \to +\infty$, irrespective of the internal delay value. A similar conclusion arises under Condition C2 of Assumption 3 since $\mu_2(A_c) \geq \max_{1 \leq i \leq n} Re\,\lambda(A_c) = -\rho(A_c)$. Thus, both conditions C1 and C2 guarantee global asymptotic stability in the delayed system under a stabilizing internal state-feedback control $u_0(t)$ in the absence of the external feedback control $u(t)$. It can be pointed out that Condition C2 is more restrictive than condition C1 since the matrix measure of a stability matrix can be positive in some cases and the condition requires that it be negative for use.

Particular cases of interest for the above system include the scalar time-invariant case of a single constant point-delayed asymptotically stable unforced equation and its extended impulsive version. In the second example, an appropriate impulsive control can be used to stabilize a system whose free dynamics are unstable.

Example 1. *Consider the scalar homogeneous time-invariant differential functional equation with a constant point delay*:

$$\dot{x}(t) = -ax(t) + a_d x(t-h) \tag{11}$$

with initial conditions defined by $x(t) = \varphi(t)$; $t \in [-h\,,\,0]$, where $\varphi : [-h\,,\,0] \to \boldsymbol{R}$ is piecewise continuous with $x(0) = \varphi(0) = x_0$, $a \in \boldsymbol{R}_+$, $a_d \in \boldsymbol{R}$, and $|a_d| \leq \left(1 - e^{-ah}\right)a$. This system is a particular scalar (of unit order) unforced version of (1)–(4) with the output identical to the scalar state. The solution is:

$$x(t+\sigma) = e^{-a(h+\sigma)}\left(x(t-h) + a_d\int_{t+\sigma-2h}^{t+\sigma-h} e^{a\tau}x(\tau)d\tau\right) \tag{12}$$

for any real $\sigma \in [0\,,\,h)$. Then,

$$\begin{aligned} |x(t+\sigma)| \leq &\left(e^{-a(h+\sigma)} + \frac{|a_d|}{a}\right)\left(\sup_{t+\sigma-2h \leq \tau \leq t+\sigma-h}|x(\tau)|\right) \\ &\leq \left(e^{-ah} + \frac{|a_d|}{a}\right)\left(\sup_{t-2h \leq \tau \leq t+\sigma-h}|x(\tau)|\right) \end{aligned} \tag{13}$$

so that, since $e^{-ah} + |a_d|/a \leq 1$,

$$\begin{aligned}
\sup_{t\le\tau\le t+h}|x(\tau)| &\le \left(e^{-ah}+\frac{|a_d|}{a}\right)\left(\sup_{t-2h\le\tau\le t}|x(\tau)|\right)\le \sup_{t-2h\le\tau\le t}|x(\tau)| \\
&\le max\left(\sup_{t-2h\le\tau\le t-h}|x(\tau)|\,,\ \sup_{t-h\le\tau\le t}|x(\tau)|\right) \\
&\le max\left(\sup_{t-4h\le\tau\le t-2h}|x(\tau)|\,,\ \sup_{t-3h\le\tau\le t-h}|x(\tau)|\right) \\
&= \sup_{t-4h\le\tau\le t-h}|x(\tau)| \\
&\le \sup_{t-4h\le\tau\le t-h}|x(\tau)| \le \sup_{t-6h\le\tau\le t-2h}|x(\tau)| \le \sup_{t-8h\le\tau\le t-3h}|\varphi(\tau)| \\
&\le \ldots \le \sup_{-\infty<\tau\le 0}|\varphi_e(\tau)| = \sup_{-h\le t\le 0}|\varphi(t)|;\ \forall t\in \boldsymbol{R}_{0+}
\end{aligned} \tag{14}$$

where $\varphi_e : (-\infty\,,\,0]\to \boldsymbol{R}$ *is defined by* $\varphi_e(t) = 0$ *for* $-\infty < t < h$ *and* $\varphi_e(t) = \varphi(t)$ *for* $t\in[-h\,,\,0]$*. As a result,* $\sup_{-h\le t<+\infty}|x(t)| \le \sup_{-h\le t\le 0}|\varphi(t)|$*. Then,*

(*a*) *If* $\varphi : [-h\,,\,0]\to\boldsymbol{R}$ *is bounded and* $|a_d|\le\left(1-e^{-ah}\right)a$*, then differential Equation (3) is globally stable and its solution absolute value is upper-bounded for all time by the maximum absolute value of the function of initial conditions.*

(*b*) *In addition, if* $|a_d| < \left(1-e^{-ah}\right)a$*, then the inequalities of (6) become strict, and* $\sup_{t-h\le\tau\le t}|x(\tau)|$ *is strictly decreasing on* $[h\,,\,+\infty)$ *so that* $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty}\left(\sup_{t-h\le t\le t}|x(t)|\right) = 0$*. Then, differential Equation (3) is globally asymptotically stable with* $\sup_{-h\le t<+\infty}|x(t)| \le \sup_{-h\le t\le 0}|\varphi(t)|$*. Note that, in fact, under the weaker condition* $|a_d| < a$*, the differential equation is still globally asymptotically stable and globally exponentially stable. However, in general, it does not satisfy* $\sup_{-h\le t<+\infty}|x(t)| \le M \sup_{-h\le t\le 0}|\varphi(t)|$ *with* $M = 1$ *but with some real constant* $M > 1$*.*

Example 2. *Consider scalar differential Equation (11) modified with an impulsive forcing term*:

$$\dot{x}(t) = (-a + K(t)\delta(0))x(t) + a_d x(t-h) \tag{15}$$

where $a\,,a_d\in\boldsymbol{R}$*,* $\delta : [0,\,+\infty)\to\boldsymbol{R}$ *is the Dirac distribution supported with the test-bounded function* $K : [0,\,+\infty)\to\boldsymbol{R}$ *with a support of zero Lebesgue measure, that is, it is identically zero except perhaps on a subset of numerable isolated strictly ordered impulsive time instants* $SI = \{t_i\}_{i=1}^{\vartheta}$*. The initial conditions are defined by* $x(t) = \varphi(t);\ t\in[-h\,,\,0]$*, where* $\varphi : [-h\,,\,0]\to\boldsymbol{R}$ *is piecewise continuous with* $x(0) = \varphi(0) = x_0$ *and* $a > 0$*. It is assumed that* $T_i = t_{i+1} - t_i \ge T > 0$*;* $x\left(t_i^-\right)$ *and* $x(t_i) = x\left(t_i^+\right)$ *are the left and right limits, respectively, of* $x(t)$ *at* $t = t_i$*;* $\forall t_i\in SI$*. The solution of (15) is:*

$$x(t) = e^{-aT_i}\left[x(t_i) + a_d\int_0^{T_i} e^{a\tau}x\left(t_i^+ - h + \tau\right)d\tau\right];\ t\in[t_i,\ t_{i+1})\ \text{for}\ t_i\,,t_{i+1}\in SI \tag{16}$$

$$x(t_{i+1}) = (1 + K(t_{i+1}))x\left(t_{i+1}^-\right)\ \text{for}\ t_{i+1}\in SI \tag{17}$$

for any bounded function of initial conditions defined by $x(t) = \varphi(t);\ t\in[-h\,,\,0]$*, where* $\varphi : [-h\,,\,0]\to\boldsymbol{R}$ *is piecewise continuous with* $x(0) = \varphi(0) = x_0$*. One obtains, after combining (16) and (17):*

$$|x(t_{i+1}+\sigma)| \leq |1+K(t_{i+1})| \left(e^{-a(T_i+\sigma)} + \left|\frac{a_d}{a}\right| \left|1-e^{-a(T_i+\sigma)}\right| \right) \sup_{t_i^+ - h + \sigma \leq \tau \leq t_{i+1}^- - h + \sigma} |x(\tau)|,\ \sigma \in [0\,, T_{i+1}) \tag{18}$$

so that $|x(t_{i+1}+\sigma)| < \sup_{t_i^+ - h \leq \tau \leq t_{i+1}^- - h} |x(\tau)|$ *for* $\sigma \in [0\,, T_{i+1})$ *if*

$$|1+K(t_{i+1})| < \inf_{0 \leq \sigma < T_{i+1}} \frac{|a|}{|a|e^{-a(T_i+\sigma)} + |a_d|\left|1-e^{-a(T_i+\sigma)}\right|} \tag{19}$$

Note that $a > 0$ *(respectively,* $a < 0$*) implies that the delay-free dynamics are globally stable (respectively, unstable) in the absence of delays and impulsive controls. Consider the following cases concerning the presence of positive, negative, or null impulses combined with the sign of* a*:*

(a) $K(t_{i+1}) > 0$. *The condition (19) becomes:*

$$0 < K(t_{i+1}) < \inf_{0 \leq \sigma < T_{i+1}} \frac{|a|}{|a|e^{-a(T_i+\sigma)} + |a_d|\left|1-e^{-a(T_i+\sigma)}\right|} - 1 \tag{20}$$

which is satisfied if

$$(a1)\ a > 0,\ 0 < K(t_{i+1}) < \frac{a}{ae^{-aT_i} + |a_d|\left(1-e^{-a(T_i+T_{i+1})}\right)} - 1 \tag{21}$$

subject to $|a_d| < \frac{a\left(1-e^{-aT_i}\right)}{1-e^{-a(T_i+T_{i+1})}}$;

$$(a2)\ a < 0,\ 0 < K(t_{i+1}) < \frac{|a|}{|a|e^{|a|(T_i+T_{i+1})} + |a_d|\left(e^{|a|(T_i+T_{i+1})} - 1\right)} - 1 \tag{22}$$

which is impossible.

(b) *For* $-1 < K(t_{i+1}) < 0$. *The condition (19) becomes:*

$$|K(t_{i+1})| > 1 - \inf_{0 \leq \sigma < T_{i+1}} \frac{|a|}{|a|e^{-a(T_i+\sigma)} + |a_d|\left|1-e^{-a(T_i+\sigma)}\right|} \tag{23}$$

which is satisfied if

$$(b1)\ a > 0,\ 1 > |K(t_{i+1})| > 1 - \frac{a}{ae^{-aT_i} + |a_d|\left(1-e^{-a(T_i+T_{i+1})}\right)} \tag{24}$$

subject to $|a_d| > \frac{a\left(1-e^{-aT_i}\right)}{1-e^{-a(T_i+T_{i+1})}}$;

$$(b2)\ a < 0,\ 1 > |K(t_{i+1})| > 1 - \frac{|a|}{|a|e^{|a|(T_i+T_{i+1})} + |a_d|\left(e^{|a|(T_i+T_{i+1})} - 1\right)} \tag{25}$$

(c) $K(t_{i+1}) < -1$. *The condition (19) becomes:*

$$|K(t_{i+1})| > 1 + \inf_{0 \leq \sigma < T_{i+1}} \frac{|a|}{|a|e^{-a(T_i+\sigma)} + |a_d|\left|1-e^{-a(T_i+\sigma)}\right|} \tag{26}$$

which is satisfied if

$$(c1)\ a > 0,\ |K(t_{i+1})| > 1 + \frac{a}{ae^{-aT_i} + |a_d|\left(1-e^{-a(T_i+T_{i+1})}\right)} \tag{27}$$

$$(c2)\ a<0,\ |K(t_{i+1})|>1+\frac{|a|}{|a|e^{|a|(T_i+T_{i+1})}+|a_d|\left(e^{|a|(T_i+T_{i+1})}-1\right)} \tag{28}$$

(*d*) $K(t_{i+1})=0$. *The condition (24) becomes:*

$$1<\inf_{0\le\sigma<T_{i+1}}\frac{|a|}{|a|e^{-a(T_i+\sigma)}+|a_d|\left|1-e^{-a(T_i+\sigma)}\right|} \tag{29}$$

which is equivalent to $|a_d|<\frac{a\left(1-e^{-aT_i}\right)}{1-e^{-a(T_i+T_{i+1})}}$ *if* $a>0$, *and it is impossible if* $a<0$.

(*e*) $K(t_{i+1})=-1$. *The condition (19) becomes:*

$$\inf_{0\le\sigma<T_{i+1}}\frac{|a|}{|a|e^{-a(T_i+\sigma)}+|a_d|\left|1-e^{-a(T_i+\sigma)}\right|}>0 \tag{30}$$

which always holds if $a>0$ *and if* $a<0$ *becomes:*

$$\frac{|a|}{|a|e^{|a|(T_i+T_{i+1})}+|a_d|\left(e^{|a|(T_i+T_{i+1})}-1\right)}>0 \tag{31}$$

In both sub-cases, $x\left(t_{i+1}^{+}\right)=0$.

Remark 5. *Note that Popov's inequality* $\eta(t)\ge-\gamma>-\infty;\ \forall t\in\mathbf{R}_{0+}$ *(Assumption 5) implies that*

$$-\eta_0(t)=-\int_0^t f_0(y(\tau),\tau)y(\tau)d\tau\le\gamma+\sum_{i=1}^{\vartheta_t}K\left(y\left(t_i^-\right),t_i\right)y^2\left(t_i^-\right);\ t\in[t_i\ ,\ t_{i+1}),\ \forall t_i\in SI \tag{32}$$

where $\vartheta_i=card(SI_i)$, $SI_i=\{t_j\}_{j=1}^{i}\subset SI;\ i\in\overline{\vartheta}$.

Remark 6. *The internal control* $u_0:[0,\ +\infty)\to\mathbf{R}$ *has the role of stabilizing* A *by fixing* A_c *to a stability matrix with linear state-feedback. In this way, the auxiliary linear system with internal delay (7) becomes globally asymptotically stable independent of the internal delay for any admissible function of initial conditions according to Assumption 1. The role of the external stabilizing control* $u:[0,\ +\infty)\to\mathbf{R}$ *is that of stabilizing the whole closed-loop system with non-linear output feedback using a control law (4) satisfying Assumptions 4 and 5 for a complete class of controllers (referred to as a hyperstable class of controllers) rather than for an individual controller.*

The following two simple preliminary technical results will be useful in the sequel. The first one relies on the fact that the strict stability of the transfer function from the external input to the output guarantees its strongly positive realness if the external input-output interconnection gain exceeds a certain finite positive threshold. Note that the necessary strict stability is achievable easily with the appropriate design of the internal control under Assumptions 1 and 2. The second result establishes that, at impulsive time instants, the sign of the instantaneous external input–output power can be changed by selecting the impulsive gains. This translates into an associate increase or decrease in the levels of the Popov's integral inequality or the energy (in a reversed sense) over a certain later time interval.

Proposition 1. *Assume that the controller gains* k *and* k_d *are selected under Assumption 2 such that Assumption 3 holds. Then, the following properties hold:*

(*i*) *The transfer Function*

$$G(s) = G_0(s) + d = c^T\left(sI_n - A_c - A_{dc}e^{-hs}\right)^{-1}\left(b + b_d e^{-h's}\right) + d \tag{33}$$

from the external input $u(t)$ to the output $y(t)$ is strictly stable irrespective of the internal and external delays.

(*ii*) *There exist a minimum interconnection external input–output gain finite threshold $d_{min} > 0$ such that $G(s)$ is SSPR, irrespective of the internal and external delays, if $d > d_{min}$.*

Proof. If the internal controller gains satisfy Assumption 3, then the auxiliary system (7) is globally asymptotically stable irrespective of the delays and the transfer function $G(s)$ is strictly stable irrespective of the delays h and h'. Furthermore, note that

$$\begin{aligned} Re\ G(s) = Re\ G_0(s) + d &\geq -\underset{s\in \boldsymbol{C}}{Inf}|Re\ G_0(s)| + d \\ &\geq -\|G_0(s)\|_\infty + d = d - \underset{\omega\in \boldsymbol{R}}{sup}|G_0(\boldsymbol{i}\omega)| > 0 \end{aligned} \tag{34}$$

if $d > d_{min} = \underset{\omega\in \boldsymbol{R}}{sup}|G_0(\boldsymbol{i}\omega)|$ and d_{min} is finite, since $G_0(s)$ has no poles on the imaginary complex axis, because it is strictly stable and of relative degree unity so that its H_∞-norm exists, i.e., it is finite. □

Proposition 2. *Assume that $T_{min} = min(t_{i+1} - t_i : t_i, t_{i+1} \in SI) > h'$. If the instantaneous external input–output power satisfies $P(t^-) = u(t^-)y(t^-) \neq 0$ for any given $t \in SI$ then:*

(*i*) *$\eta(\tau) > \eta(t^-)$, or equivalently, $E(\tau) < E(t^-)$, for $\tau \in [t, t+\varepsilon)$ for some $\varepsilon > 0$ if $K(y(t^-), t) < 0$.*

(*ii*) *$\eta(\tau) < \eta(t^-)$, or equivalently, $E(\tau) > E(t^-)$, for $\tau \in [t, t+\varepsilon)$ for some $\varepsilon > 0$ if $K(y(t^-), t) > 0$.*

(*iii*) *If the controller impulsive gain $K(y(t^-), t)$ of the external output feedback law is modified to the form $K(y(t^-), t) = K_a(y(t^-), t)/y(t^-)$, then:*

$\eta(\tau) > \eta(t^-)$, or equivalently, $E(\tau) < E(t^-)$, for $\tau \in [t, t+\varepsilon)$ for some $\varepsilon > 0$ if $sgn(K(y(t^-), t)) \neq sgn(y(t^-))$;

$\eta(\tau) < \eta(t^-)$, or equivalently, $E(\tau) > E(t^-)$, for $\tau \in [t, t+\varepsilon)$ for some $\varepsilon > 0$ if $sgn(K(y(t^-), t)) = sgn(y(t^-))$.

Proof. Note that the instantaneous external input–output power satisfies the relations:

$$P(t) = \dot{E}(t) = -\dot{\eta}(t) = y(t)u(t) = -y(t^-)\left(f_0\left(y(t^-), t\right) + K\left(y(t^-), t\right)y(t^-)\delta(0)\right) \tag{35}$$

$$P(t) - P\left(t^-\right) = \dot{\eta}\left(t^-\right) - \dot{\eta}(t) = -y^2\left(t^-\right)K\left(y\left(t^-\right), t\right)\delta(0) \tag{36}$$

since $T_{min} = min(t_{i+1} - t_i : t_i . t_{i+1} \in SI) > h'$ so that if $t \in SI$, then $(t - h') \notin SI$, then $K(y(t^- - h'), t - h')y(t^- - h') = 0$ (see Appendix A). Now, if $P(t^-) \neq 0$, then $y(t^-) \neq 0$ and $u(t^-) \neq 0$. If $t \in SI$, then $y(t^-) \neq 0$ and $K(y(t^-), t) \neq 0$ so that $P(t) \neq P(t^-)$, and if $t \in SI$

$$\mathrm{sgn}\left(P(t) - P\left(t^-\right)\right) = \mathrm{sgn}\left(\dot{\eta}\left(t^-\right) - \dot{\eta}(t)\right) = -\mathrm{sgn}\left(K\left(y\left(t^-\right), t\right)\right) \tag{37}$$

which leads directly to the claimed result since $\eta(t)$ and $E(t)$ are continuous functions on the time intervals $[t_i\ ,\ t_{i+1})$ where $t_i\ , t_{i+1}(> t_i)$ are consecutive impulsive time instants. Properties ((i)–(ii)) are thus proved. Property (iii) follows directly from Properties ((i)–(ii)) for an impulsive controller gain of the form $K(y(t^-), t)y(t^-) = K_a(y(t^-), t)$. Then, $K_a(y(t^-), t) < 0$ if $\mathrm{sgn}(K(y(t^-), t)) \neq \mathrm{sgn}(y(t^-))$ and $K_a(y(t^-), t) > 0$, otherwise. □

In the sequel, the subscripts for matrix norms and measures and associated constants are deleted in the notation for the sake of simplicity while assuming that ℓ_2 (or spectral)-norm and its associated measure are used.

3. Closed-Loop Asymptotic Hyperstability for a Class of Hyperstable Impulse-Free Controllers

The concept of hyperstability relies on global stability for each controller belonging to a class of controllers defined by an integral Popov's inequality (Assumption 5). See also Remark 5. Thus, the relevance of hyperstability is that global stability is achieved for a whole class of controllers rather than for an individual one. On the other hand, it generalizes in parallel the concept of absolute stability since the class of hyperstable controllers includes time-varying members that satisfy a Popov's-type inequality [6,7,12]. From an energy point of view, the input–output energy in the feed-forward part of an asymptotically hyperstable system is jointly positive and bounded for all time. The technical reason for this is that the feed-forward loop is defined by a strictly positive real transfer function. Through this section, it is assumed that the hyperstable controller class is impulse-free.

Definition 1. *Consider an impulse-free controller class (4), which satisfies Assumption 4 with* $f(y(t),t) = f_0(y(t),t)$ *and Assumption 5 for some* $\gamma \in \boldsymbol{R}_+$. *Such a class is said to be the class of* γ*-hyperstable controllers.*

It turns out that the class of hyperstable controllers contains the classes of γ-hyperstable controllers for any $\gamma \in \boldsymbol{R}_+$. Note also that the class of γ-hyperstable controllers is contained in that of γ_1-hyperstable controllers for any real $\gamma_1 \geq \gamma$.

Definition 2. *The system (1)–(4) is said to be hyperstable if it is globally stable for any function of initial conditions satisfying Assumption 1 for any possibly nonlinear and eventually time-varying control* $u(t)$ *in the class of Assumption 4, with* $f(y(t),t) = f_0(y(t),t)$ *satisfying Popov's integral inequality of Assumption 5 for some* $\gamma \in \boldsymbol{R}_+$. *The system (1)–(4) is said to be asymptotically hyperstable if it is globally asymptotically stable under any hyperstable controller class.*

Definition 3. *The class of controllers that generate controls* $u(t) = -f_0(y(t),t)$ *under Assumptions 4–5 for some finite* $\gamma \in \boldsymbol{R}_+$ *is said to be the* γ*-hyperstabilizing class of impulse controllers.*

The following result is obvious from Assumptions 4 and 5 and Remark 5, Equation (32):

Assertion 1. Assume that a controller is defined by $f : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$, satisfying Assumption 4, and that its impulse-free part $f_0 : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$ is γ_0-hyperstable for some $\gamma_0 \in \boldsymbol{R}_+$. If its impulsive part has impulsive time instants and associated impulsive gains, which satisfy:

$$\sum_{i=1}^{\vartheta_t} K\left(y\left(t_i^-\right), t_i\right) y^2\left(t_i^-\right) \geq \gamma_0 - \gamma;\ t \in [t_i\,,\ t_{i+1});\ \forall t_i \in SI$$

for some $\gamma \in \boldsymbol{R}_+$, then the whole controller $f : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$ is γ-hyperstable if $\gamma \geq \gamma_0$ and γ_0-hyperstable if $\gamma_0 \geq \gamma$. □

The following conclusion from Assertion 1 is also direct:

Assertion 2. Assume that $f_0 \equiv 0$ and that the impulsive controller satisfies $\sum_{i=1}^{\vartheta_t} K\left(y\left(t_i^-\right), t_i\right) y^2\left(t_i^-\right) \geq -\gamma;\ t \in [t_i\,,\ t_{i+1});\ \forall t_i \in SI$ for some $\gamma \in \boldsymbol{R}_+$. Then, the whole controller $f : \boldsymbol{R} \times \boldsymbol{R}_{0+} \to \boldsymbol{R}$ is γ-hyperstable. □

Note that Assertion 1 and the fact that the regular controller defined by an identically zero f_0 is γ-hyperstable for any $\gamma \in \boldsymbol{R}_+$ directly yields Assertion 2.

Classically, the following terminology is used to characterize the closed-loop stability under, in general, nonlinear and time-varying controllers satisfying Popov's-type integral inequalities:

(a) The class of positive real transfer functions (respectively, that of strongly strictly positive real transfer functions) is referred to as the hyperstable (respectively, asymptotically hyperstable) class of linear feed-forward, time-invariant systems.
(b) The class of negative feedback nonlinear and possibly time-varying controllers satisfying Assumption 5 for some finite $\gamma \in \boldsymbol{R}_+$ is referred to as the class of γ-hyperstable controllers. That is, the γ-hyperstabilizing class of impulsive controllers (Definition 2) belongs to the class of hyperstable controllers.
(c) The closed- loop system is said to be hyperstable (respectively, asymptotically hyperstable) if the feed-forward, time-invariant system characterized by a positive real (respectively, a strongly strictly positive real) transfer function is globally stable (respectively, globally asymptotically stable) under any feedback controller belonging to the hyperstable class of controllers.

The hyperstability of the closed-loop system for a class of controllers $u(t) = -f_0(y(t), t)$ under Assumptions 4–5 requires the transfer function of the time-invariant, feed-forward system to be positive real, while its asymptotic hyperstability requires the necessary condition that the feed-forward transfer function be strongly strictly positive real.

The following result establishes that any closed-loop system whose linear time-invariant, feed-forward loop is given by transfer Function (33) related to the external control, which is strongly strictly positive real, and the nonlinear and, eventually, time-varying nonlinear controller satisfies Assumption 4, with no impulsive actions, and Assumption 5 has two important properties, namely:

(a) The external input–output energy is finite for all time and the external control input is square-integrable and essentially bounded on $\boldsymbol{R}_{0+}$;
(b) Under zero initial conditions, the external input–output energy is, furthermore, non-negative for all time.

The above properties are the basis for the closed-loop asymptotic stability, which we remember that, roughly speaking, means the global asymptotic stability for any function of initial conditions that fulfills Assumption 1 for any controller satisfying Assumption 5, provided that the feed-forward system is given by a strongly strictly positive real transfer function.

Theorem 1. *Assume that the transfer function of the feed-forward system is $G \in SSPR$, given by (33), then Assumption 4 holds free of impulses, that is, $K(y(t),\ t) \equiv 0$ so that $u(t) = -f_0(y(t),\ t)$, and Assumption 5 holds. Then, the following properties hold:*

(*i*) *The external input–output energy is non-negative and bounded for all time under zero initial conditions and is bounded for any function of initial conditions which satisfies Assumption 1.*
(*ii*) *Any external control $u(t)$ is always square-integrable on $\boldsymbol{R}_{0+}$.*
(*iii*) *The closed-loop system is asymptotically hyperstable.*

Proof. The impulse response of the feed-forward linear part is the inverse Laplace transform of the transfer function $G(s)$, i.e., $g(t) = L^{-1}G(s);\ \forall t \in \boldsymbol{R}_{0+}$. The Fourier transform of the impulse response $g(t)$, usually known in engineering as the frequency response of the linear part, is

$$G(\boldsymbol{i\omega}) = G_0(\boldsymbol{i\omega}) + d = c^T\left(\boldsymbol{i}\omega I_n - A_c - A_{dc}e^{-\boldsymbol{i}h\omega}\right)^{-1}\left(b + b_d e^{-\boldsymbol{i}h'\omega}\right) + d \tag{38}$$

where $\omega = 2\pi f$ is the frequency in rad/sec and f is the frequency in Hertz. Note that (6) can be expressed equivalent using truncated time functions and Parseval's theorem in frequency terms as follows:

$$\begin{aligned} E(t) &= \int_0^t y(\tau)u(\tau)d\tau = \int_{-\infty}^{\infty} y_{0,t}(\tau)u_{0,t}(\tau)d\tau \\ &= \int_{-\infty}^{\infty} y_{0,t}(\tau)u_{0,t}(\tau)d\tau = \int_{-\infty}^{\infty} y(\tau)u_{0,t}(\tau)d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y_{0,\ t}(\boldsymbol{i}\omega)U_{0,t}(-\boldsymbol{i}\omega)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\boldsymbol{i}\omega)U_{0,t}(-\boldsymbol{i}\omega)d\tau';\ \forall t \in \boldsymbol{R}_{0+} \end{aligned} \tag{39}$$

If the function of initial conditions φ is identically zero on $[-h\ ,\ 0]$ and $G \in SSPR$, then the input–output energy in the forced solution of the feed-forward part of the system satisfies the following set of relations:

$$
\begin{aligned}
+\infty \geq \gamma > E(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y_{0\ ,\ t}(\boldsymbol{i}\omega)U_{0,t}(-\boldsymbol{i}\omega)d\sigma \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\boldsymbol{i}\omega)U_{0,t}(\boldsymbol{i}\omega)U_{0,t}(-\boldsymbol{i}\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\boldsymbol{i}\omega)|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} Re\ G(\boldsymbol{i}\omega)|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega + \mathrm{Im}G(\boldsymbol{i}\omega)|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} Re\ G(\boldsymbol{i}\omega)|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega\right) + 0 \\
&\geq \underset{\omega\in\boldsymbol{R}}{Inf}\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} Re\ G(\boldsymbol{i}\omega)|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega\right) \\
&\geq \frac{1}{2\pi}\underset{\omega\in\boldsymbol{R}}{Inf}Re\ G(\boldsymbol{i}\omega)\left(\int_{-\infty}^{\infty}|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega\right) \\
&= \frac{1}{2\pi}\underset{\omega\in\boldsymbol{R}_0^+}{Inf}\ ReG(\boldsymbol{i}\omega)\left(\int_{-\infty}^{\infty}|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega\right) \\
&= \frac{d}{2\pi}\int_{-\infty}^{\infty}|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega = d\int_{-\infty}^{\infty} u_{0,t}^2(\tau)d\tau = d\int_0^t u^2(\tau)d\tau \geq 0; \\
&\qquad \forall t \in \boldsymbol{R}_{0+}
\end{aligned} \tag{40}
$$

where it has been taken into account that $\mathrm{Im}G(\boldsymbol{i}\omega) = -\mathrm{Im}G(-\boldsymbol{i}\omega)$ and $Re\ G(\boldsymbol{i}\omega) = Re\ G(-\boldsymbol{i}\omega)$ for $\omega \in \boldsymbol{R}$ and that $Re\ G(\boldsymbol{i}\omega)$ for $\omega \in \boldsymbol{R}_+$ and $\underset{|\omega|\to\infty}{lim}\ Re\ G(\boldsymbol{i}\omega) \geq d$.

Equation (40) is only fulfilled using square-integrable controls on $\boldsymbol{R}_{0+}$, which, in addition, fulfil $u(t) \to 0$ as $t \to +\infty$ as a result. If the function of initial conditions is nonzero but satisfies Assumption 1, then the output from (39) has a contribution to the input $y_f(t)$, which satisfies the relations in (39), and another one due to the nonzero initial conditions, which satisfy Assumption 1, $y_{uf}(t)$. Thus, since if $G \in SSPR$, then it is also strictly stable (that is, all its poles are in Re s < 0), and then the output solution to non-zero finite conditions vanishes exponentially as time tends to infinity, and its time integral is then absolutely integrable. As a result, $E(t)$ is still bounded for all time for any function of initial conditions that satisfies Assumption 1. For any such a function, the unforced response asymptotically vanishes since $G(s)$ is strictly stable.

$$
\begin{aligned}
\gamma \geq -\eta(t) = \quad & E(t) = \int_0^t y_f(\tau)u(\tau)d\tau + \int_0^t y_{uf}(\tau)u(\tau)d\tau \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} Re\ G(\boldsymbol{i}\omega)|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega\right) + \int_0^t y_{uf}(\tau)u(\tau)d\tau \\
&\geq \left|\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} Re\ G(\boldsymbol{i}\omega)|U_{0,t}(\boldsymbol{i}\omega)|^2 d\omega\right) - \left|\int_0^t y_{uf}(\tau)u(\tau)d\tau\right|\right| \\
&\geq d\int_0^t u^2(\tau)d\tau - \underset{0\leq\tau\leq t}{ess\ sup}\ |u(\tau)|\gamma_{uf}
\end{aligned} \tag{41}
$$

where $\gamma_{uf} = \left|\int_0^t y_{uf}(\tau)d\tau\right| < +\infty$, which leads to

$$
\int_0^t \overline{u}^2(\tau)d\tau \leq \frac{1}{d}\left(\frac{\gamma}{\underset{0\leq\tau\leq t}{ess\ sup}\ |u(\tau)|} + \gamma_{uf}\right) \tag{42}
$$

where $\overline{u}(t) = |u(t)|/\sqrt{\underset{0\leq\tau\leq t}{ess\ sup}\ |u(\tau)|}$. Now assume two cases:

Case a: $\underset{t\to+\infty}{lim}\ \underset{0\leq\tau\leq t}{sup}\ |u(\tau)| = +\infty$. Then, $+\infty = +\left(\sqrt{\infty}/\infty\right) = \int_0^{\infty}\overline{u}^2(\tau)d\tau \leq \frac{\gamma_{uf}}{d} < +\infty$ is a contradiction.

Case b: the external control input has a discontinuity of the second class at finite time $t \in \boldsymbol{R}_{0+}$, i.e., either $\underset{\tau\downarrow t}{lim}|u(\tau)| = +\infty$ or $\underset{\tau\uparrow t}{lim}|u(\tau)| = +\infty$. Then, for some $\varepsilon \in \boldsymbol{R}_+$,

$M : \boldsymbol{R}_{0+} \cap [-2\varepsilon\ ,\ 0) \to \boldsymbol{R}_{0+}$ defined by $M(t) = \int_0^{t-\varepsilon} \overline{u}^2(\varsigma)d\varsigma$ if $t \geq \varepsilon$ and $M(t) = 0$ otherwise, either

$$\begin{aligned} +\infty = \quad & +\infty - \frac{\gamma_{uf}}{d} < M(t-\varepsilon) + \infty - 0 - \frac{\gamma_{uf}}{d} \\ & = \underset{\varepsilon \to 0^+}{lim\ sup} \left(\int_0^{t-\varepsilon} \overline{u}^2(\varsigma)d\varsigma + \int_{t-\varepsilon}^{t+\varepsilon} \overline{u}^2(\varsigma)d\varsigma - \frac{\gamma}{\underset{0 \leq \tau \leq t+\varepsilon}{ess\ \ sup}\ |u(\tau)|} - \frac{\gamma_{uf}}{d} \right) \leq 0 \end{aligned} \tag{43}$$

or

$$\begin{aligned} +\infty = \quad & +\infty - \frac{\gamma_{uf}}{d} < M(t-2\varepsilon) + \infty - 0 - \frac{\gamma_{uf}}{d} \\ & = \underset{\varepsilon \to 0^+}{lim\ sup} \left(\int_0^{t-2\varepsilon} \overline{u}^2(\varsigma)d\varsigma + \int_{t-2\varepsilon}^{t-\varepsilon} \overline{u}^2(\varsigma)d\varsigma - \frac{\gamma}{\underset{0 \leq \tau \leq t-\varepsilon}{ess\ \ sup}\ |u(\tau)|} - \frac{\gamma_{uf}}{d} \right) \leq 0 \end{aligned} \tag{44}$$

which are both impossible since they lead to contradictions. Then, (42) implies that $\underset{t \in \boldsymbol{R}_{0+}}{ess\ sup}\ |u(t)| < +\infty$ and, for any non-identically zero control on some time interval of finite non-zero measure, $\underset{t \in \boldsymbol{R}_{0+}}{sup}\ |u(t)| > 0$. Then, again from (42), $\underset{t \to +\infty}{lim} \int_0^t \overline{u}^2(\tau)d\tau < +\infty$, and then $\overline{u}(t) \to 0$ and $u(t) \to 0$ as $t \to +\infty$, where $\overline{u}(t)$ is square-integrable on $\boldsymbol{R}_{0+}$, and, since $\underset{t \in \boldsymbol{R}_{0+}}{ess\ sup}\ |u(t)| < +\infty$, then $u(t)$ is square-integrable on $\boldsymbol{R}_{0+}$ as well. The proofs for Properties (i) and (ii) are thus complete. Property (iii) follows directly since $G \in SSPR$, then it is strictly stable so that any state space realization under a hyperstable class of controllers fulfils that its state and output converge asymptotically to zero as time tends to infinity for any function of initial conditions subject to Assumption 1. Thus, any such state–space realization is globally asymptotically stable. □

4. Closed-Loop Asymptotic Hyperstability for a Class of Hyperstable Eventually Impulsive Controllers

Now, we discuss the counterpart of the above result under impulsive external controls. This section presents the incorporation of impulsive controllers to an extended class of hyperstable ones. In this way, the new extended class of hyperstable controllers includes static and eventually time-varying ones with possible impulsive actions. The usefulness of impulsive controls relies on allowing finite jumps in the state and/or output. See, for instance, refs. [17–20] and references therein. Another usefulness is approximating great control efforts over short periods of time, as, for instance, in the case of impulsive vaccination. This is of interest, for instance, when considering several existing configurations of the controlled system and eventual jumps from some existing configuration to another one with a different parameterization. The basic supporting idea is that the time-derivative of a finite state/output jump (this reflects switching from a configuration to another one) is a Dirac delta (that is, an impulse). Such an impulse can be generated with the control law. Therefore, a natural extension of the class of asymptotically hyperstable controllers is to another more general one, which considers the inclusion of impulsive elements for controlled system possessing, in general, known delays in the dynamics (internal delays) and in the inputs or outputs (external delays).

The output response to the external control is denoted by $y_f(.)$, while that to the initial conditions is denoted by $y_{uf}(.)$ taken at the initial time instant $t_0 = 0$ and successively later on to any impulsive time instants. A main difference compared to the former impulse-free case is that now the contribution of the initial conditions to the external input–output power integral has to be considered explicitly for each impulsive time instant. Such initial conditions cannot destabilize the closed-loop system or generate an unbounded external input–output energy if the feed-forward system is given by a strongly strictly positive real transfer function. This is not always the case for impulsive controllers. The reason is that there is a jump at each of these impulsive instants in the output level caused by the input impulse.

In other words, the main difference between the impulsive class of controllers and the impulse-free one is that the influence of the initial conditions within the inter-impulse time intervals updated after the jumps at impulsive time instants have to be considered. However, in the impulse-free case, it is known that the contribution of finite initial conditions to the power is bounded for all time since the transfer function of the linear feed-forward block is strictly stable as a result of being strongly strictly positive real.

Now, consider the sets of impulsive time instants $SI(\neq \varnothing) = \{t_i\}_{i=1}^{\vartheta} \subset \boldsymbol{R}_{0+}$, of cardinal $\vartheta \leq \chi_0$, and $SI_{t^-} = \{t_i\}_{i=1}^{\vartheta_{t^-}} \subset SI \cap [0,\, t)$, $SI_t = \{t_i\}_{i=1}^{\vartheta_t} = SI_{t^-} \cup \{t\} \subset SI \cap [0\,,\, t]$ if $t \in SI$ and $SI_t = SI_{t^-}$ if $t \notin SI$; $\forall t \in \boldsymbol{R}_{0+}$. Note that the output $y(t) = y_{uf}(t) + y_f(t)$ is contributed by its unforced output component $y_u(t)$, depending on initial conditions, and the forced component $y_f(t)$, which depends on the control. $E_{uf}(t)$ is the external input–output energy, depending on the unforced output, and $E_f(t)$ is that depending on the forced output. Thus, if Assumption 4 holds, then, in the presence of mixed impulsive and regular controls, it follows that

$$\begin{gathered}\int_0^t y(\tau)u(\tau)d\tau = \int_0^{t^-} y(\tau)u(\tau)d\tau + K(y(t^-),\, t)y^2(t^-);\\ \sum_{i\in\overline{\vartheta}_t} K\left(y\left(t_i^-\right),\, t_i\right)y^2\left(t_i^-\right) = \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right),\, t_i\right)y^2\left(t_i^-\right) + K(y(t^-),\, t)y^2(t^-)\end{gathered}$$

for all $t \in \boldsymbol{R}_{0+}$, so that

$$\int_0^t y(\tau)u(\tau)d\tau = \int_0^{t^-} y(\tau)u(\tau)d\tau;\ \sum_{i\in\overline{\vartheta}_t} K\left(y\left(t_i^-\right),\, t_i\right)y^2\left(t_i^-\right) = \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t^-\right),\, t\right)y^2\left(t^-\right)$$

if and only if $K(y(t^-),\, t)y^2(t^-) = 0$, that is, if and only if either $K(y(t^-),\, t) = 0$ because $t \notin SI$ or if $y(t^-) = 0$.

Then, for all $t \in \boldsymbol{R}_{0+}$, and since $d > 0$ since $G \in SSPR$, one obtains for the external input-output energy $E(t^-)$ and $E(t) = E(t^-) - K(y(t^-),\, t)y^2(t^-) \geq \underline{E}(t)$, the following relations (see Appendix B):

$$\begin{aligned}
E(t^-) &= \int_0^{t^-} y(\tau)u(\tau)d\tau \\
&= -\int_0^{t_1^-} y(\tau)f_0(y(\tau),\, \tau)d\tau - \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right),\, t_i\right)y^2\left(t_i^-\right) \\
&\quad -\sum_{i\in\overline{\vartheta}_{t^-}} \int_{t_i}^{t_{i+1}^-} y(\tau)f_0(y(\tau),\, u(\tau))d\tau - \int_{t_{\vartheta_t}}^{t^-} y(\tau)f_0(y(\tau),\, \tau)d\tau \leq \gamma < +\infty \\
&= \frac{1}{2\pi}\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}}\left(\int_{-\infty}^{\infty} Re\, G(\boldsymbol{i}\omega)U_{t_i,t_{i+1}}(\boldsymbol{i}\omega)d\omega\right) - \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right), t_i\right)y^2\left(t_i^-\right) \\
&\quad +\sum_{i\in\overline{\vartheta}_{t^-}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)u(\tau)d\tau \\
&\geq \frac{d}{2\pi}\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}}\left(\int_{-\infty}^{\infty}\left|U_{t_i,t_{i+1}}(\boldsymbol{i}\omega)\right|^2 d\omega\right) - \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right), t_i\right)y^2\left(t_i^-\right) \\
&\quad -\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)f_0(y(\tau),\, \tau)d\tau \\
&\geq \frac{d}{2\pi}\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}}\left(\int_{-\infty}^{\infty}\left|U_{t_i,t_{i+1}}(\boldsymbol{i}\omega)\right|^2 d\omega\right) - \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right), t_i\right)y^2\left(t_i^-\right) \\
&\quad -\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)f_0(y(\tau),\, \tau)d\tau \\
&= \frac{d}{2\pi}\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}}\left(\int_{-\infty}^{\infty} u_{t_1,t_2}^2(\tau)d\tau\right) - \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right), t_i\right)y^2\left(t_i^-\right) \\
&\quad -\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)f_0(y(\tau),\, \tau)d\tau \\
&= d\int_0^{t^-} u^2(\tau)d\tau - \sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right), t_i\right)y^2\left(t_i^-\right) \\
&\quad -\sum_{i\in\overline{\vartheta}_{t^-}\cup\{0\}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)f_0(y(\tau),\, \tau)d\tau
\end{aligned} \tag{45}$$

and

$$
\begin{aligned}
\underline{E}(t) = \ & d\int_0^t u^2(\tau)d\tau - \sum_{i\in\overline{\vartheta}_{t-}} K\big(y\big(t_i^-\big), t_i\big)y^2\big(t_i^-\big) \\
& -\sum_{i\in\overline{\vartheta}_{t-}\cup\{0\}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)f_0(y(\tau),\ \tau)d\tau - K(y(t^-),\ t)y^2(t^-) \\
& \leq E(t) = \int_0^t y(\tau)u(\tau)d\tau = E_f(t) + E_{uf}(t) = E(t^-) - K(y(t^-),\ t)y^2(t^-) \\
& = \int_0^{t^-} y(\tau)u(\tau)d\tau - K(y(t^-),\ t)y^2(t^-) \\
& = -\int_0^{t_1^-} y(\tau)f_0(y(\tau),\tau)d\tau - \sum_{i\in\overline{\vartheta}_{t-}} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big) - K(y(t^-),t\,)y^2(t^-) \\
& -\sum_{i\in\overline{\vartheta}_{t-}} \int_{t_i}^{t_{i+1}^-} y(\tau)f_0(y(\tau)\,,\,u(\tau))d\tau - \int_{t_{\vartheta_t}}^{t^-} y(\tau)f_0(y(\tau)\,,\tau)d\tau \leq \gamma < +\infty
\end{aligned} \tag{46}
$$

Furthermore, since $t = t_{\vartheta_t}$ if $t \in SI$ and $K(y(t^-),\ t) = 0$, if $t \notin SI$,

$$
\begin{aligned}
0 \leq d\int_0^{t^-} u^2(\tau)d\tau \leq d\int_0^t u^2(\tau)d\tau & = d\Big(\int_0^{t^-} u^2(\tau)d\tau + \int_{t^-}^t K^2(y(\tau^-),\tau)y^2(\tau^-)\delta(\tau - t^-)d\tau\Big) \\
& = d\Big(\int_0^{t^-} u^2(\tau)d\tau + K^2(y(t^-),t)\Big) \\
& = d\Big[\int_0^{t_1^-} u^2(\tau)d\tau + \sum_{i\in\overline{\vartheta_t - 1}} \int_{t_i}^{t_{i+1}^-} u^2(\tau)d\tau + \int_{t_{\vartheta_t}}^t u^2(\tau)d\tau + \sum_{i\in\overline{\vartheta}_t} K^2\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big)\Big] \\
& \leq \gamma + \int_0^{t^-} y_{uf}(\tau)f_0(y(\tau),\ \tau)d\tau + \sum_{i\in\overline{\vartheta}_{t-}} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big) + K(y(t^-),t\,)y^2(t^-) \\
& = \gamma + \Big(\sum_{i,\ (i+1)\in\overline{\vartheta}_{t-}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)f_0(y(\tau),\ \tau)d\tau + \int_0^{t_1^-} y_{uf}(\tau)f_0(y(\tau),\ \tau)d\tau + \int_{t_{\vartheta_t}}^{t^-} y_{uf}(\tau)f_0(y(\tau),\ \tau)\Big)d\tau \\
& +\sum_{i\in\overline{\vartheta}_t} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big) \\
& \leq \gamma + \sum_{i\in\overline{\vartheta}_{t-}} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big) + K(y(t^-),\ t)y^2(t^-) + \tfrac{\|c\|K}{\rho} \\
& \times\Big(\sum_{i,\ (i+1)\in\overline{\vartheta}_{t-}} \big|(1 - K\big(y\big(t_i^-\big),t_i\big))y^2\big(t_i^-\big)\big|e^{-\rho t_i}\Big(1 - e^{-\rho(t_{i+1}-t_i)}\Big) \sup_{t_i\leq\tau\leq t_{i+1}} |u(\tau)| + |y(0^-)|\big(1 - e^{-\rho t_1}\big) \sup_{0\leq\tau\leq t_1} |u(\tau)| \\
& +\big|(1 - K(y(t^-),\ t))y^2(t^-)\big|e^{-\rho\vartheta_t}\Big(1 - e^{-\rho(t-\vartheta_t)}\Big) \sup_{\vartheta_t\leq\tau\leq t} |u(\tau)|\Big) \\
& \leq \gamma + \sum_{i\in\overline{\vartheta}_t} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big) + \tfrac{K\|c\|}{\rho}\Big(\sup_{0\leq\tau\leq t} |u(\tau)|\Big) \\
& \times\Big(|y(0^-)|\big(1 - e^{-\rho t_1}\big) + \Big|\Big(1 - K\Big(y\Big(t_{\vartheta_t}^-\Big),\ t_{\vartheta_t}\Big)\Big)y^2\Big(t_{\vartheta_t}^-\Big)e^{-\rho\vartheta_t}\Big|\Big(1 - e^{-\rho(t-\vartheta_t)}\Big) + \Big(\sum_{i,\ (i+1)\in\overline{\vartheta}_{t-}} \big|(1 - K\big(y\big(t_i^-\big),\ t_i\big))y^2\big(t_i^-\big)\big|\Big)e^{-\rho t_i}\Big(1 - e^{-\rho(t_{i+1}-t_i)}\Big)\Big) \\
& ;\ \forall t \in \boldsymbol{R}_{0+}
\end{aligned} \tag{47}
$$

since $y(t_i) = \big(1 - K\big(y\big(t_i^-\big),t_i\big)\big)y\big(t_i^-\big);\ \forall t_i \in SI$ with $t_{\vartheta_t} = \{max\,\tau \in \boldsymbol{R}_{0+}{:}\tau \leq t\};\ \forall t \in \boldsymbol{R}_{0+}$, where $(-\rho) < 0$ is the stability abscissa of the linear auxiliary system, $T_{min} = inf(t_{i+1} - t_i : t_i \in SI)$, and $K(\geq 1)$ is a norm-dependent constant. The above equation implies that

$$
\begin{aligned}
& \int_0^t|u(\tau)|d\tau = \int_0^{t^-} |u(\tau)|d\tau + |K(y(t^-),\ t)y(t^-)| \\
& \leq \frac{1}{\sup\limits_{0\leq\tau\leq t} |u(\tau)|}\int_0^t u^2(\tau)d\tau \leq \frac{\gamma}{d\sup\limits_{0\leq\tau\leq t} |u(\tau)|} + \frac{1}{d\sup\limits_{0\leq\tau\leq t} |u(\tau)|}\Big(\sum_{i\in\overline{\vartheta}_t} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big)\Big) + \frac{K\|c\|}{\rho} \\
& \times\Big(|y(0^-)|\big(1 - e^{-\rho t_1}\big) + \Big|\Big(1 - K\Big(y\Big(t_{\vartheta_t}^-\Big),\ t_{\vartheta_t}\Big)\Big)y^2\Big(t_{\vartheta_t}^-\Big)\Big|e^{-\rho\vartheta_t}\Big(1 - e^{-\rho(t-\vartheta_t)}\Big) + \Big(\sum_{i,\ (i+1)\in\overline{\vartheta}_{t-}} \big|(1 - K\big(y\big(t_i^-\big),\ t_i\big))y^2\big(t_i^-\big)\big|\Big)e^{-\rho t_i}\Big(1 - e^{-\rho(t_{i+1}-t_i)}\Big)\Big) \\
& ;\forall t \in \boldsymbol{R}_{0+}
\end{aligned} \tag{48}
$$

Thus, the following theorem is stated directly as a result followed from the above derivations.

Theorem 2. *Assume that transfer Function (33) of the feed-forward system is $G \in SSPR$ and that Assumptions 1 and 4 hold. Assume also that the impulsive control gains satisfy the subsequent constraints:*

$$
\sum_{i\in\overline{\vartheta}_t} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big) > -\gamma - \int_0^t y(\tau)f_0(y(\tau),\ \tau)d\tau;\ \forall t \in \boldsymbol{R}_{0+}\ \text{(Assumption 5)} \tag{49}
$$

and

$$
\begin{aligned}
& \sum_{i\in\overline{\vartheta}_t} K\big(y\big(t_i^-\big),\ t_i\big)y^2\big(t_i^-\big) + \Big|\Big(1 - K\Big(y\Big(t_{\vartheta_t}^-\Big),\ t_{\vartheta_t}\Big)\Big)y^2\Big(t_{\vartheta_t}^-\Big)e^{-\rho\vartheta_t}\Big|\Big(1 - e^{-\rho(t-\vartheta_t)}\Big) \\
& +\Big(\sum_{i,\ (i+1)\in\overline{\vartheta}_{t-}} \big|(1 - K\big(y\big(t_i^-\big),\ t_i\big))y^2\big(t_i^-\big)\big|e^{-\rho t_i}\Big)\Big(1 - e^{-\rho(t_{i+1}-t_i)}\Big) = O\Big(esssup_{0\leq\tau\leq t}|u(\tau)|\Big); \\
& \forall t \in \boldsymbol{R}_{0+}
\end{aligned} \tag{50}
$$

Then, the following properties hold:

(*i*) *The external input–output energy is bounded for all time.*
(*ii*) *Any external control $u(t)$ is always bounded and square-integrable on $\boldsymbol{R}_{0+}$ so that $u(t) \to 0$ for $t \in (t_i,\ t_{i+1}]$; $\forall t_i, t_{i+1} \in SI$ as $t_i \to \infty$ if $\vartheta = \chi_0$ and $u(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$ if $\vartheta < \chi_0$.*
(*iii*) *The closed-loop system is asymptotically hyperstable.*

Remark 7. *Note that the first constraint of Theorem 2, Equation (49), is Popov's integral inequality (Assumption 5) for the case of mixed impulsive and regular controls.*

Such a constraint guarantees that the whole controller is γ-hyperstable so that the first condition of Theorem 2 holds. This is obvious since

$$\begin{aligned}\eta(t) &= -E(t) = \int_0^t f(y(\tau),\ \tau)y(\tau)d\tau \\ &= \int_0^t f_0(y(\tau),\ \tau)y(\tau)d\tau + \sum_{i=1}^{\vartheta_t} K\left(y\left(t_i^-\right),\ t_i\right)y\left(t_i^-\right) \geq -\gamma\ ;\ \forall t_i \in SI,\ \forall t \in \boldsymbol{R}_{0+}\end{aligned} \tag{51}$$

$$-\eta_0(t) = -\int_0^t f_0(y(\tau)\,,\tau)y(\tau)d\tau \leq \gamma_0 \leq \gamma + \sum_{i=1}^{\vartheta_t} K\left(y\left(t_i^-\right)\,,t_i\right)y\left(t_i^-\right);\ \forall t \in \boldsymbol{R}_{0+} \tag{52}$$

However, such a condition is necessary, but not sufficient, for the impulsive controller to be in the hyperstable class. Another sufficiency-type constraint, Equation (50), is incorporated, which quantifies a trade-off between the impulsive gains and the sequence of impulsive time instants so as to achieve the global closed-loop stability.

Note also that if the impulse-free controller part is γ_0-hyperstable, according to Assumption 5, and the whole controller is γ-hyperstable, both of them according to Assumption 5, then such a first condition on the impulsive controls of Theorem 2 holds if

$$\sum_{i=1}^{\vartheta_t} K\left(y\left(t_i^-\right),\ t_i\right)y\left(t_i^-\right) \geq \gamma_0 - \gamma;\ \forall t \in \boldsymbol{R}_{0+} \tag{53}$$

The following sufficient further condition for fulfillment of Theorem 2 is directly obtained if trade-offs between the impulsive gains and a minimum and a maximum time interval between consecutive impulsive time instants are prefixed.

Corollary 1. *Theorem 2 holds if the second constraint (50) is replaced with*

$$\begin{aligned}&\sum_{i\in\overline{\vartheta}_t} K\left(y\left(t_i^-\right),\ t_i\right)y^2\left(t_i^-\right) + \left|\left(1 - K\left(y\left(t_{\vartheta_t}^-\right),\ t_{\vartheta_t}\right)\right)y^2\left(t_{\vartheta_t}^-\right)\right|e^{-\vartheta_t T_{min}}\left(1 - e^{-\rho T_{max}}\right) \\ &+\left(\sum_{i,\ (i+1)\in\overline{\vartheta}_{t^-}} \left|\left(1 - K\left(y\left(t_i^-\right),\ t_i\right)\right)y^2\left(t_i^-\right)\right|e^{-iT_{min}}\right)\left(1 - e^{-\rho T_{max}}\right) = O\left(esssup_{0\leq\tau\leq t}|u(\tau)|\right);\ \forall t \in \boldsymbol{R}_{0+}\end{aligned} \tag{54}$$

where $T_{min} = \inf_{t_i\in SI}\,(t_{i+1} - t_i) > 0$ *and* $T_{max} = \sup_{t_i\in SI}\,(t_{i+1} - t_i) \geq T_{min}$. □

Remark 8. *Note that (49) implies*

$$\eta(t) = \theta\left(t^-\right) + K\left(y\left(t^-\right),\ t\right)y^2\left(t^-\right) \geq -\gamma > -\infty,\ \forall t \in \boldsymbol{R}_{0+} \tag{55}$$

where

$$\begin{aligned}\eta\left(t^-\right) = &\left(\sum_{i,\ (i+1)\in\overline{\vartheta}_{t^-}} \int_{t_i}^{t_{i+1}^-} y_{uf}(\tau)f_0(y(\tau),\ \tau)d\tau + \int_0^{t_1^-} y_{uf}(\tau)f_0(y(\tau),\ \tau)d\tau + \int_{t_{\vartheta_t}}^{t^-} y_{uf}(\tau)f_0(y(\tau)\,,\tau)\right)d\tau \\ &+\sum_{i\in\overline{\vartheta}_{t^-}} K\left(y\left(t_i^-\right),\ t_i\right)y^2\left(t_i^-\right);\ \forall t \in \boldsymbol{R}_{0+}\end{aligned} \tag{56}$$

irrespective of whether Theorem 2 holds or not. However, it is not directly guaranteed from (55) that $\theta(t)$ is finitely upper bounded for all time if Theorem 2 does not hold.

Now, we propose a mechanism for choosing impulsive time instants such that Theorem 2 holds.

According to Theorem 2, $u(t)$ is bounded and converges asymptotically to zero as $t \in \cup_{t_i \in SI}(t_i,\ t_{i+1})$ tends to infinity. If $cardSI < \chi_0$, then the control converges to zero as $t \to \infty$. According to (50), note that a sufficient condition for the above property to hold is that the following amount be bounded for all time:

$$\begin{aligned}\Omega(t) = {} & \sum_{i \in \overline{\vartheta}_t^-} K\left(y\left(t_i^-\right),\ t_i\right) y^2\left(t_i^-\right) + K\left(y\left(t_{\vartheta_t}^-\right),\ t_{\vartheta_t}\right) y^2\left(t_{\vartheta_t}^-\right) \\ & + \left|\left(1 - K\left(y\left(t_{\vartheta_t}^-\right),\ t_{\vartheta_t}\right)\right) y^2\left(t_{\vartheta_t}^-\right)\right| \left(e^{-\rho\vartheta_t} - e^{-\rho t}\right) + \left(\sum_{i,\ (i+1) \in \overline{\vartheta}_{t^-}} \left|\left(1 - K\left(y\left(t_i^-\right),\ t_i\right)\right) y^2\left(t_i^-\right)\right|\right) e^{-\rho t_i}\left(1 - e^{-\rho(t_{i+1} - t_i)}\right) \\ & ;\ \forall t \in [\vartheta_t,\ t_{\vartheta_t+1})\end{aligned} \tag{57}$$

where $t_{\vartheta_t+1} \in SI$ and $t_{\vartheta_t} = (maxt_i \in SI:\ t \leq \vartheta_t)$. Since $t_{\vartheta_t} \in SI$, $y\left(t_{\vartheta_t}^-\right) \neq 0$. Assume that $\vartheta_t \geq 2$ so that the last impulsive time instant previous to t was not less than 2. Thus, note that

$$\Omega(t) = \Omega(t_{\vartheta_t}) + \left|\left(1 - K\left(y\left(t_{\vartheta_t}^-\right),\ t_{\vartheta_t}\right)\right) y^2\left(t_{\vartheta_t}^-\right)\right| \left(e^{-\rho\vartheta_t} - e^{-\rho t}\right);\ \forall t \in [t_{\vartheta_t},\ t_{\vartheta_t+1}) \tag{58}$$

where for any given $\omega(t_{\vartheta_t-1}) \in \boldsymbol{R}$, and since $t_{\vartheta_t-1} \in SI$ with $t_{\vartheta_t-1} = (maxt_i \in SI:\ t \leq \vartheta_t - 1)$. Since $y\left(t_{\vartheta_t-1}^-\right) \neq 0$, one obtains from (57) that:

$$\begin{aligned}\Omega(t_{\vartheta_t}) = {} & \Omega(t_{\vartheta_t-1}) + K\left(y\left(t_{\vartheta_t}^-\right),\ t_{\vartheta_t}\right) y^2\left(t_{\vartheta_t}^-\right) + \left|\left(1 - K\left(y\left(t_{\vartheta_t-1}^-\right),\ t_{\vartheta_t-1}\right)\right) y^2\left(t_{\vartheta_t-1}^-\right)\right| \left(e^{-\rho\vartheta_t-1} - e^{-\rho\vartheta_t}\right) \\ & = (1 - \omega(t_{\vartheta_t-1}))(\Omega(t_{\vartheta_t-1}) - \xi(t_{\vartheta_t-1})) + \varsigma(t_{\vartheta_t-1}) + \xi(t_{\vartheta_t})\end{aligned} \tag{59}$$

where

$$\begin{aligned}& -\omega(t_{\vartheta_t-1})(\Omega(t_{\vartheta_t-1}) - \xi(t_{\vartheta_t-1})) + \varsigma(t_{\vartheta_t-1}) + \xi(t_{\vartheta_t}) - \xi\left(t_{\vartheta_{t-1}}\right) \\ & = K\left(y\left(t_{\vartheta_t}^-\right),\ t_{\vartheta_t}\right) y^2\left(t_{\vartheta_t}^-\right) + \left|\left(1 - K\left(y\left(t_{\vartheta_t-1}^-\right),\ t_{\vartheta_t-1}\right)\right) y^2\left(t_{\vartheta_t-1}^-\right)\right| \left(e^{-\rho\vartheta_t-1} - e^{-\rho\vartheta_t}\right)\end{aligned} \tag{60}$$

after an appropriate definition of the real non-unique sequences $\omega(t_{\vartheta_t-1})$, $\xi(t_{\vartheta_t})$, and $\varsigma(t_{\vartheta_t-1})$ such that the above identity (60) holds.

The following result holds:

Theorem 3. *Assume that transfer Function (33) of the feed-forward system is $G \in SSPR$ and that Assumptions 1 and 4 hold. Assume also that $T_{min} = min(t_{i+1} - t_i : t_i,\ t_{i+1} \in SI) > h'$. Then, the following properties hold:*

(*i*) *If $cardSI < \chi_0$, then $\{\Omega(t_i)\}_{t_i \in SI}$ is bounded irrespective of the impulsive set of time instants SI.*

(*ii*) *Assume that $cardSI = \chi_0$ and that the impulsive time instant $t_{i+1} \in SI$ is selected, provided that $y\left(t_{i+1}^-\right) \neq 0$, with the impulsive control gain:*

$$\begin{aligned}& K\left(y\left(t_{i+1}^-\right),\ t_{i+1}\right) = \frac{1}{y^2\left(t_{i+1}^-\right)} \\ & \times \left(\varsigma(t_i) + \xi(t_{i+1}) - \xi(t_i) - \omega(t_i)(\Omega(t_i) - \xi(t_i)) - \left|\left(1 - K\left(y\left(t_i^-\right),\ t_i\right)\right) y^2\left(t_i^-\right)\right| \left(e^{-\rho t_i} - e^{-\rho t_{i+1}}\right)\right)\end{aligned} \tag{61}$$

For the given preceding impulsive time instants accumulated in the impulsive set of time instants $SI_{t_{i+1}^-} = \{t_1,\ t_2, \ldots, t_i\}$, where the parameterizing sequences are arbitrary except that they are subject to the constraints $\{\omega(t_i)\}_{i=0}^{\chi_0} \subset (0,\ 1)$, $\sum_{i \in SI} \omega(t_i) = \sum_{i=0}^{\chi_0} \omega(t_i) = +\infty$; $\{\varsigma(t_i)\}_{t_i \in SI} \subset \boldsymbol{R}_{0+}$; $\sum_{i \in SI} \varsigma(t_i) = \sum_{i=0}^{\chi_0} \varsigma(t_i) < +\infty$ (then $\varsigma(t_i) \to 0$ as $t_i (\in SI) \to +\infty$); $\{\xi(t_i)\}_{i=1}^{\infty} (\subset (0,\ M_\xi) \cap \boldsymbol{R})$; $\xi(t_{i+1}) \geq (1 - \omega(t_i))(\xi(t_i) - \Omega(t_i)) - \varsigma(t_i)$; $\forall t_i \in SI$ with the initial $\xi(0)$ being fixed such that $(\Omega(0) - \xi(0)) \in \boldsymbol{R}_{0+}$.

Then the following properties hold:

(*ii.1*) *$\{\Omega(t_i) - \xi(t_i)\}_{t_i \in SI} \to 0$ and it is bounded, $\{\Omega(t_i)\}_{t_i \in SI}$ is bounded, and $\{\Omega(t_i)\}_{t_i \in SI}$ converges if $\{\xi(t_i)\}_{t_i \in I}$ converges, provided that the parameterizing sequences satisfy the additional constraints*

$$\{\omega(t_i)\}_{t_i \in SI} (\subset ((0,\ 1) \cap \boldsymbol{R})) \to 0,\ \sum_{t_i \in SI} \omega(t_i) = +\infty, \{\varsigma(t_i)\}_{t_i \in SI} \subset \boldsymbol{R}_{0+}.$$

(*ii.2*) *Property (i) also holds if*

$$\{\omega(t_i)\}_{t_i \in SI}(\subset (0\ ,\ C) \cap \boldsymbol{R}),\ 0 < \underset{t_i(\in SI)\to\infty}{lim\ inf}\ \omega(t_i) \leq \underset{t_i(\in SI)\to\infty}{lim\ sup}\ \omega(t_i) < 1 \text{ and } \{\varsigma(t_i)\}_{t_i \in SI} \equiv 0.$$

(*ii.3*) *If* $\{\omega(t_i)\}_{t_i \in SI}(\subset (0\ ,\ C) \cap \boldsymbol{R})$, $\{\varsigma(t_i)\}_{t_i \in SI} \equiv 0$ *and there exists a finite positive integer* $N_k = N_k(k) \leq \hat{N}$ *for each* $k \in \boldsymbol{Z}_{0+}$, *such that* $\prod_{j=k}^{k+N_k-1}\left[(1-\omega_j)\right] \leq 1$, *then* $\{\Omega(t_i)\}_{t_i \in SI}$ *is bounded.*

(*iii*) *Assume that either* $cardSI < \chi_0$ *or* $cardSI = \chi_0$ *and that the impulsive time instant* $t_{i+1} \in SI$ *is selected, provided that* $y\left(t_{i+1}^-\right) \neq 0$, *with the impulsive control gain generated with (61). Assume also that transfer Function (33) of the feed-forward system is* $G \in SSPR$ *and that Assumptions 1 and 4 hold.*

Then, the external input–output energy is bounded for all time; any external control $u(t)$ *is always bounded and square-integrable on* $\boldsymbol{R}_{0+}$, *and, as a result, the closed-loop system is asymptotically hyperstable.*

Proof. Property (i) refers to the case of a finite number of impulsive actions. Its proof is direct since the impulsive gains are a finite number of bounded impulsive controls. The proof for Property (ii) is directly from the auxiliary result that Theorem A1 stated and proved in Appendix B for the case of infinitely many impulsive time instants.

To prove Property (iii), first note that if $\Omega(t)$ is bounded for all $t \in SI$ (Properties (i)–(ii)), then $\Omega(t)$ is bounded for all time since the function of initial conditions is bounded, $t_1 \in SI$ is finite, and since $\Omega(t_i)$ is bounded for all $t_i \in SI$, then it cannot be unbounded for $t \in (t_i\ ,\ t_{i+1})$ from (57)–(60) since $T_i = t_{i+1} - t_i \leq T_{max} < +\infty$. Thus, $\Omega(t)$ is bounded on the time interval $[0\ ,\ t_\vartheta]$ irrespective of t_ϑ being finite ($cardSI < \chi_0$) or infinite ($cardSI = \chi_0$). This implies as a result that $\Omega(t)$ is bounded if $cardSI = \chi_0$; otherwise, if $cardSI < \chi_0$, then $\Omega(t)$ is also bounded on $[t_\vartheta\ ,\ +\infty)$ from (57)–(60) since $K(y(t^-)\ ,\ t) = 0;\ \forall t > t_\vartheta$. As a result, $\Omega(t)$ is bounded on $\boldsymbol{R}_{0+}$ for any set SI with $card\ SI \leq \chi_0$. Now, one obtains from (47) that, since $\Omega(t)$ is bounded on $\boldsymbol{R}_{0+}$, then $\int_0^t |u(\tau)| d\tau < +\infty;\ \forall t \in \boldsymbol{R}_{0+}$. As a result, Property (iii) follows directly from Theorem 2. □

5. Conclusions

The asymptotic hyperstability of a closed-loop system was investigated, whose controlled plant is time-invariant with constant internal and external point delays such that its transfer function is strongly strictly positive real. The proposed hyperstable controller class combines a regular impulse-free part with an impulsive part. There are, in general, two control laws involved, namely, the internal one, which stabilizes the system with linear state feedback independent of the delays sizes, and the external one, which belongs to a hyperstable class and satisfies a Popov's-type time-integral inequality. Such a class combines, in general, an impulse-free part and an impulsive part. The transfer function of the linear plant with respect with this second control is assumed to be strongly strictly positive real with a minimum positive external input–output interconnection gain. The general hyperstable class of controllers consists of a regular impulse-free part and an impulsive part and also satisfies a Popov's-type integral inequality for all time that, in fact, defines the class of hyperstable controllers. The performed analysis becomes significantly more involved in the impulsive case than in the impulse-free case since the output finite jumps caused by the impulses in the time-derivative of the state dynamics cause the re-starting of initial conditions after each solution jump cause by a control impulse. This problem translates into the need for taking into account the contribution of the unforced output to the external input–output energy, which is unnecessary and then omitted in the impulse-free case.

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Appendix A. Main Impulsive Contributions

If $t \in SI$, then $y(t) = y(t^-) - K(y(t^-))$ for the given hyperstable controller class (Assumption 4)

$$u(t) = -f(y(t), t) = -f_0(y(t), t) - \sum_{i=1}^{\vartheta} K\left(y\left(t^-\right),\ t\right) y\left(t^-\right) \delta(0) \tag{A1}$$

Then,

$$u(t) - u\left(t^-\right) = -K\left(y\left(t^-\right), t\right) y\left(t^\cdot\right) \delta(0) \tag{A2}$$

$$x(t) - x\left(t^-\right) = -bK\left(y\left(t^-\right), t\right) y\left(t^-\right) \tag{A3}$$

$$y(t) - y\left(t^-\right) = c^T\left(x(t) - x\left(t^-\right)\right) + d\left(u(t) - u\left(t^-\right)\right) + \left(y_{ic}(t) - y_{ic}\left(t^-\right)\right) \tag{A4}$$

then,

$$\begin{aligned} y(t) - y(t^-) &= -\left(c^T b + d\delta(0)\right) K(y(t^-), t) y(t^-) - c^T b_d K(y(t^- - h'), t) y(t^- - h') + y_{ic}(t) \\ &= -\left(c^T b + d\delta(0)\right) K(y(t^-), t) y(t^-) + d\delta(0) K(y(t^-), t) y(t^-) - c^T b_d K(y(t^- - h'), t) y(t^- - h') \\ &= -c^T (bK(y(t^-), t) y(t^-) + b_d K(y(t^- - h'), t) y(t^- - h')) \end{aligned} \tag{A5}$$

with

$$y_{ic}\left(t^-\right) = 0; y_{ic}(t) = dK\left(y\left(t^-\right), t\right) y\left(t^-\right) \delta(0) \tag{A6}$$

If $(t - h') \notin SI$, which is guaranteed for $t \in SI$ under the constraint $T_{min} = min$ $(t_{i+1} - t_i : t_i, t_{i+1} \in SI) > h'$, then, so that $K\left(y\left(t^{-\prime}\right), t\right) \neq 0$ and $K(y(t^- - h'), t) = 0$, (A6) becomes:

$$y(t) = \left[1 - c^T\left(bK\left(y\left(t^-\right), t\right)\right)\right] y\left(t^-\right) \tag{A7}$$

$$\begin{aligned} E(t) - E(t^-) &= \int_{t^-}^{t} y(\tau) u(\tau) d\tau \\ &= y(t^-) \int_{t^-}^{t} K(y(\tau^-), \tau) y(\tau) \delta(\tau - t) d\tau \\ &= K(y(t^-), t) y^2(t^-) \end{aligned} \tag{A8}$$

If $\left(t + \widetilde{t}\right) \notin SI$ for $\widetilde{t} \in (0, \sigma]$, then for some real constants $\delta_y > 0$, $K \geq 1$, $\rho > 0$ and, since $G(s)$ is strictly stable,

$$\begin{aligned} \left|E_{uf}(t+\sigma) - E_{uf}(t)\right| &= \left|\int_t^{t+\sigma} y_{uf}(\tau) u(\tau) d\tau\right| \\ &\leq \left(\underset{t \leq \tau \leq \sigma}{esssup} |u(\tau)|\right) \left|\int_t^{t+\sigma} y_{uf}(\tau) d\tau\right| \\ &= K\delta_y \left(\underset{t \leq \tau \leq \sigma}{esssup} |u(\tau)|\right) \left|y_{uf}(t)\right| \left(\int_t^{t+\sigma} e^{-\rho\tau} d\tau\right) \\ &= \frac{K\delta_y}{\rho} \left(\underset{t \leq \tau \leq \sigma}{esssup} |u(\tau)|\right) \left|y_{uf}(t)\right| e^{-\rho t} \left(1 - e^{-\rho\sigma}\right) \end{aligned} \tag{A9}$$

Appendix B. Auxiliary Technical Stability Result

Theorem A1. *Consider the discrete real evolution equation:*

$$x_{k+1} = c_k(x_k - m_k) + b_k + m_{k+1};\ \forall k \in \mathbf{Z}_{0+} \tag{A10}$$

with $x_0 \in \mathbf{R}_{0+}$*, where* $\{m_k\}_{k=0}^{\infty}(\subset (0,\ M) \cap \mathbf{R})$*,* $m_{k+1} \geq c_k(m_k - x_k) - b_k$*;* $\forall k \in \mathbf{Z}_{0+}$*, and* $(x_0 - m_0) \in \mathbf{R}_{0+}$*.*

Then, the following properties hold:

(*i*) $\{x_k - m_k\}_{k=0}^{\infty} \to 0$*, and it is bounded,* $\{x_k\}_{k=0}^{\infty}$ *is bounded, and* $\{x_k\}_{k=0}^{\infty}$ *converges if* $\{m_k\}_{k=0}^{\infty}$ *converges provided that*

$$\{c_k\}_{k=0}^{\infty}(\subset ((0,\ 1) \cap \mathbf{R})) \to 1,\ \sum_{k=0}^{\infty}(1 - c_k) = +\infty,\ \{b_k\}_{k=0}^{\infty} \subset \mathbf{R}_{0+},\ \sum_{k=0}^{\infty} b_k < +\infty.$$

(*ii*) *Property (i) also holds if*

$$\{c_k\}_{k=0}^{\infty}(\subset (0\ ,\ C) \cap \mathbf{R}),\ \limsup_{k\to\infty} c_k < 1 \text{ and } \{b_k\}_{k=0}^{\infty} \equiv 0.$$

(*iii*) *If* $\{c_k\}_{k=0}^{\infty}(\subset (0\ ,\ C) \cap \mathbf{R})$*,* $\{b_k\}_{k=0}^{\infty} \equiv 0$*, and there exists a finite positive integer* $N_k = N_k(k) \leq \hat{N}$ *for each* $k \in \mathbf{Z}_{0+}$ *such that* $\prod_{j=k}^{k+N_k-1}[c_j] \leq 1$*, then* $\{x_k\}_{k=0}^{\infty}$ *is bounded.*

Proof. Define the sequence $\{\widetilde{x}_k\}_{k=0}^{\infty}$ by $\widetilde{x}_k = x_k - m_k$; $\forall k \in \mathbf{Z}_{0+}$. Thus, from (A10):

$$\widetilde{x}_{k+1} = c_k\widetilde{x}_k + b_k;\ \forall k \in \mathbf{Z}_{0+} \tag{A11}$$

with $\widetilde{x}_0 = x_0 - m_0 \geq 0$. Since $\{c_k\}_{k=0}^{\infty}(\subset (0,\ 1)) \to 1$, $\sum_{k=0}^{\infty}(1 - c_k) = +\infty$, $\sum_{k=0}^{\infty} b_k < +\infty\left(\Rightarrow \{b_k\}_{k=0}^{\infty} \to 0\right)$, one has from Venter's theorem [30] that $\{\widetilde{x}_k = x_k - m_k\}_{k=0}^{\infty} \to 0$, and $\{\widetilde{x}_k\}_{k=0}^{\infty}$ is bounded. Since $\{m_k\}_{k=0}^{\infty}$ is bounded, then $\{x_k\}_{k=0}^{\infty}$ is bounded as well, and $\{x_k\}_{k=0}^{\infty} \subset \mathbf{R}_{0+}$ since $x_0 \in \mathbf{R}_{0+}$, $m_0 \leq x_0$ and $m_{k+1} \geq c_k(m_k - x_k) - b_k$; $\forall k \in \mathbf{Z}_{0+}$. If $\{m_k\}_{k=0}^{\infty} \to m$, then $\{x_k\}_{k=0}^{\infty} \to m$. Property (i) is thus proven.

If $b_k = 0$; $\forall k \in \mathbf{Z}_{0+}$ then $\limsup_{k\to\infty}\widetilde{x}_{k+1} = \widetilde{x}_0 \limsup_{k\to\infty}\left(\prod_{j=0}^{k}[c_j]\right) = 0$, provided that $\limsup_{k\to\infty} c_k < 1$, so that $\lim_{k\to\infty} \widetilde{x}_k = 0$. The remainder of the proof for Property (ii) follows in a similar way as that of its counterpart for Property (i). Property (iii) follows since the sequence $\left\{x_{\sum_{j=0}^{k} N_j}\right\}_{k=0}^{\infty}$ is non-negative and non-increasing then bounded. Since $0 < N_{k+1} = \sum_{j=0}^{k+1} N_j - \sum_{j=0}^{k} N_j \leq \hat{N} < +\infty$; $\forall k \in \mathbf{Z}_{0+}$ and $x_{\sum_{j=0}^{k} N_j}$ is finite, then $x_{\sum_{j=0}^{k} N_j + j}$ is also finite for all $j \in \overline{N_k} - 1$; $\forall k \in \mathbf{Z}_{0+}$ and then $\{x_k\}_{k=0}^{\infty}$ is bounded as claimed. □

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