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Research Article

On the Extensions of Krasnoselskii-Type Theorems to p-Cyclic Self-Mappings in Banach Spaces

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A set of $np(\geq 2)$-cyclic and either continuous or contractive self-mappings, with at least one of them being contractive, which are defined on a set of subsets of a Banach space, are considered to build a composed self-mapping of interest. The existence and uniqueness of fixed points and the existence of best proximity points, in the case that the subsets do not intersect, of such composed mappings are investigated by stating and proving ad hoc extensions of several Krasnoselskii-type theorems.

1. Introduction

In the last years, important attention is being devoted to extend the fixed point theory by weakening the conditions on both the maps and the sets where those maps operate [1, 2]. For instance, every nonexpansive self-mappings on weakly compact subsets of a metric space has fixed points if the weak fixed point property holds [1]. It has also to be pointed out the relevance of fixed point theory in the stability of complex continuous-time and discrete-time dynamic systems [3–5]. On the other hand, Meir-Keeler self-mappings have received important attention in the context of fixed point theory perhaps due to the associated relaxing in the required conditions for the existence of fixed points compared with the usual contractive mappings [6–10]. Another interest of such maps is their usefulness as formal tool for the study of p-cyclic contractions even if the involved subsets of the metric space under study do not intersect [6]. The underlying idea is that the best proximity points are fixed points if such subsets intersect while they play a close role to fixed points otherwise. It has to be pointed out that there are close links between contractive self-mappings and Kannan self-mappings [2, 10–13].

A rich research is being devoted to the existence and uniqueness of best proximity points of cyclic mappings under different assumptions on the vector space to which the subsets involved in the cyclic mapping belong. For instance, in [14], the concept of 2-cyclic self-mappings is extended to $p(\geq 2)$-cyclic self-mappings and results about fixed points are derived in the case that the subsets of the considered complete metric space have a nonempty intersection. Some of the ideas in such a manuscript inspired the definition of p-cyclic self-mappings from X to X with X being the union of the subsets involved in the cyclic representation. On the other hand, the existence and uniqueness of best proximity points of $p(\geq 2)$-cyclic φ-contractive self-mappings are investigated in [15] borrowing the previous scenario investigated in [16, 17] for 2-cyclic φ-contractive self-mappings. Generally speaking, the so-called cyclic φ-contractive self-mappings, which are associated with some strictly increasing unbounded map, are based on a concept of weak contractiveness contrarily to the standard, and commonly used, (strict) contractive concept being inspired in the well-known Banach contraction principle. In those papers, the Banach space X under consideration is also assumed to be reflexive and strictly convex. These joint assumptions, which are less restrictive than the assumption that the space is uniformly convex (since uniformly convex Banach spaces are reflexive and strictly convex but the converse is not true in general), are proven to keep intact the essential properties of existence and uniqueness of best proximity points for cyclic φ-contractive self-mappings if the involved subsets are nonempty, weakly closed, and convex. The above formalism is revisited in [18] for cyclic φ-contractive self-mappings in the framework of ordered metric spaces. In [19], characterization of best proximity points is studied for non-self-mappings $S, T : A \to B$, where A and B are nonempty subsets of a metric space. In general, best proximity points do not fulfil in this context the usual condition $x = Sx = Tx$. However, they jointly globally optimize the mappings from x to the distances $d(x, Tx)$ and $d(x, Sx)$.

In this manuscript, (X, d) is a complete metric space and is considered associated to a Banach space X endowed with translation-invariant and homogeneous metric $d : X \times X \to \mathbf{R}_{0+}$ and $S_X := \{A_i \subseteq X : A_{j+p} \equiv A_j;\ \forall i \in \overline{p},\ \forall j \in \mathbf{Z}_{0+}\}$ is a set of p subsets of X. $\widehat{T} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic self-mapping if it satisfies $\widehat{T}(A_i) \subseteq A_{i+1}$; for all $i \in \overline{p}$. A valid metric is the norm of the Banach space X. If the p-cyclic self-mapping $\widehat{T} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is nonexpansive (resp., contractive—also referred to as a 2-cyclic contraction) then there exists a real constant $k \in [0, 1]$ (resp., $k \in [0, 1)$) such that

$$d\left(\widehat{T}x, \widehat{T}y\right) \leq kd(x, y) + (1 - k)\operatorname{dist}(A, B), \quad \forall x \in A,\ \forall y \in B. \tag{1.1}$$

The self-mapping $T : A \cup B \to A \cup B$ is said to be a 2-cyclic large contraction if A and B are two given intersecting subsets of X if

$$\begin{gathered} d(Tx, Ty) < d(x, y), \quad \forall x \in A,\ y(\neq x) \in B, \\ \left[\forall \varepsilon \in \mathbf{R}_+,\ \forall x \in A,\ y \in B : d(Tx, Ty) \geq \varepsilon\right] \Longrightarrow \left[\exists \delta \in [0, 1) : d(Tx, Ty) \leq \delta d(x, y)\right]. \end{gathered} \tag{1.2}$$

This concept of 2-cyclic large contraction on intersecting subsets extends that of large contraction [20], and both concepts extend to p-cyclic contractions. A p-cyclic large contraction satisfies also (1.2) by replacing $A \to A_i$, $B \to A_{i+1}$; for all $i \in \overline{p}$ provided that $\bigcap_{i\in\overline{p}} A_i \neq \emptyset$. In Section 2 of this paper, we consider a mapping $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$

defined by $T(x,y) = T_1x + T_2y$ such that $T^j(x,y) = T(T^{j-1}(x,y)) = T_1^jx + T_2^jy$; for all $j(\geq 2) \in \mathbf{Z}_+$; for all $x,y \in A_i$, for all $i \in \overline{2}$ and then $T(x,y) \in A_{i+1}$ where $T_i : A_1 \cup A_2 \to A_1 \cup A_2$; $i = 1,2$ are 2-cyclic self-mappings. It is not required for most of the obtained results that $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to A_1 \cup A_2$ so that such a map is not required to be 2-cyclic either. For the obtained results related to boundedness of distances between iterates through T, it is not required for the set of subsets of X to be either closed or convex. For the obtained results concerning fixed points and best proximity point, the sets A_1 and A_2 are required to be convex but they are not necessarily closed if the self-mapping T can be defined on the union of the closures of the sets A_1 and A_2. Also, concerning best proximity points in the case that the subsets do not intersect, it is assumed that the space is restricted to be a uniformly convex Banach space [6–9]. It turns out that since uniformly convex Banach spaces are also strictly convex, the results also hold under this more general assumption used in several papers (see, for instance [15–18]). In Section 3, the results of Section 2 are extended to mappings built in a close way via $p(\geq 2)$-cyclic self-mappings on a set of p subsets A_i $(i \in \overline{p})$ of X.

1.1. Notation

$$\mathbf{R}_{0+} := \mathbf{R}_+ \cup \{0\}, \qquad \mathbf{Z}_{0+} := \mathbf{Z}_+ \cup \{0\}, \qquad \overline{p} := \{1,2,\ldots,p\} \subset \mathbf{Z}_+. \tag{1.3}$$

Superscript $\top$ denotes vector or matrix transpose, $\mathrm{Fix}(T)$ is the set of fixed points of a self-mapping T on some nonempty convex closed subset A of a metric space (X,d), $\mathrm{cl}\,A$ denotes the closure of a subset A of X, $\mathrm{Dom}(T)$ and $\mathrm{Im}(T)$ denote, respectively, the domain and image of the self-mapping T and 2^X is the family of subsets of X, $\mathrm{dist}(A,B) = d_{AB}$ denotes the distance between the sets A and B for a 2-cyclic self-mapping $T : A \cup B \to A \cup B$ what is simplified as $\mathrm{dist}(A_i, A_{i+1}) = d_{A_iA_{i+1}} = d_i$; for all $i \in \overline{p}$ for distances between adjacent subsets of p-cyclic self-mappings T on $\bigcup_{i=1}^p A_i$ where A_i $(i \in \overline{p})$ are subsets of X.

$\mathrm{BP}_i(T)$ is the set of best proximity points on a subset A_i of a metric space (X,d) of a p-cyclic self-mapping T on $\bigcup_{i=1}^p A_i$, the union of a collection of nonempty subsets of (X,d) which do not intersect.

2. Results for Mappings Defined by 2-Cyclic Self-Mappings

The following result is concerned with the above-defined map $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ constructed with nonexpansive or contractive 2-cyclic self-mappings $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ for $i = 1,2$.

Theorem 2.1. *The following properties hold.*

(i) *If $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ are both 2-cyclic nonexpansive self-mappings then*

$$\begin{aligned} d\left(T^j(x,y), T^j(x',y')\right) &\leq d\left(T_1^jx, T_1^jx'\right) + d\left(T_2^jy, T_2^jy'\right) \leq d(x,x') + d(y,y') \\ &\leq \mathrm{diam}(A_1) + \mathrm{diam}(A_2) + \mathrm{dist}(A_1,A_2), \\ &\qquad \forall x,y \in A_1,\ \forall x',y' \in A_2,\ \forall j \in \mathbf{Z}_+, \end{aligned} \tag{2.1}$$

$$\limsup_{j\to\infty} d\left(T^j(x,y),T^j(x',y')\right) \le \operatorname{diam}(A_1)+\operatorname{diam}(A_2)+\operatorname{dist}(A_1,A_2),$$
$$\forall x,y\in A_1,\ \forall x',y'\in A_2. \tag{2.2}$$

(ii) *If $T_i : A_1\cup A_2 \to A_1\cup A_2$ for $i=1,2$ are both 2-cyclic nonexpansive self-mappings and at least one of them is contractive then Property (i) holds and, furthermore,*

$$d\left(T^j(x,y),T^j(x',y')\right) \le \max(\operatorname{diam}(A_1),\operatorname{diam}(A_2))+(1-\min(k_1,k_2))\operatorname{dist}(A_1,A_2),$$
$$\forall x,y\in A_1,\ \forall x',y'\in A_2,\ \forall j\in \mathbf{Z}_+, \tag{2.3}$$

$$\limsup_{j\to\infty} d\left(T^j(x,y),T^j(x',y')\right) \le \max(\operatorname{diam}(A_1),\operatorname{diam}(A_2))+(1-\min(k_1,k_2))\operatorname{dist}(A_1,A_2),$$
$$\forall x,y\in A_1,\ \forall x',y'\in A_2. \tag{2.4}$$

(iii) *If $T_i : A_1\cup A_2 \to A_1\cup A_2$ for $i=1,2$ are both 2-cyclic contractive self-mappings then Properties (i), (ii) hold and, furthermore,*

$$d\left(T^j(x,y),T^j(x',y')\right) \le k_1^j d(x,x')+k_2^j d(y,y')+\left(2-k_1^j-k_2^j\right)\operatorname{dist}(A_1,A_2) \tag{2.5a}$$

$$\le k_1 d(x,x')+k_2 d(y,y')+(2-k_1-k_2)\operatorname{dist}(A_1,A_2),$$
$$\forall x,y\in A_1,\ \forall x',y'\in A_2,\ \forall j\in \mathbf{Z}_+, \tag{2.5b}$$

$$\limsup_{j\to\infty} d\left(T^j(x,y),T^j(x',y')\right) \le 2\operatorname{dist}(A_1,A_2),\quad \forall x,y\in A_1,\ \forall x',y'\in A_2, \tag{2.6}$$

$$\exists \lim_{j\to\infty} d\left(T^j(x,y),T^j(x',y')\right)=0 \quad \textit{if}\ \operatorname{dist}(A_1,A_2)=0,\ \forall x,y\in A_1,\ \forall x',y'\in A_2, \tag{2.7}$$

(i.e., A_1 and A_2 are closed and intersect or at least one of them is open while their boundaries intersect).

Proof. Take $x,y\in A_1$ and $x',y'\in A_2$. Direct calculation yields by taking into account that $T_i : A_1\cup A_2 \to A_1\cup A_2$; $i=1,2$ are 2-cyclic, then both self-mappings satisfy $T_i(A_1)\subseteq A_2$, $T_i(A_2)\subseteq A_1$ for $i=1,2$, and (1.1) with respective contraction constants k_1 and k_2, and that the metric is translation-invariant and homogeneous:

$$\begin{aligned}
d(T(x,y),T(x',y')) &= d(T_1x+T_2y,T_1x'+T_2y') \\
&= d(T_1x,T_1x'+T_2y'-T_2y) \\
&\le d(T_1x,T_1x')+d(T_1x',T_1x'+T_2y'-T_2y) \\
&\le d(T_1x,T_1x')+d(T_2y,T_2y')
\end{aligned}$$

$$\leq k_1 d(x,x') + k_2 d(y,y') + (1-k_1)\operatorname{dist}(A_1,A_2) + (1-k_2)\operatorname{dist}(A_1,A_2),$$
$$\forall x,y \in A_1,\ \forall x',y' \in A_2,\ \forall j \in \mathbf{Z}_+. \tag{2.8}$$

If $k_1 = k_2 = 1$ then one gets from (2.8)

$$d\Big(T^j(x,y),T^j(x',y')\Big) \leq d\Big(T_1^j x, T_1^j x'\Big) + d\Big(T_2^j y, T_2^j y'\Big), \quad \forall x,y \in A_1,\ \forall x',y' \in A_2,\ \forall j \in \mathbf{Z}_+$$
$$\leq k_1^j d(x,x') + k_2^j d(y,y') + \sum_{\ell=1}^{j} k_1^{\ell}(1-k_1)\operatorname{dist}(A_1,A_2)$$
$$+ k_2^{\ell}(1-k_2)\operatorname{dist}(A_1,A_2),\ \forall x,y \in A_1,\ \forall x',y' \in A_2,\ \forall j \in \mathbf{Z}_+. \tag{2.9}$$

If $k_1 = k_2 = 1$; that is, if both 2-cyclic self-mappings $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ $(i = 1,2)$ are nonexpansive. Thus, Property (i) follows from (2.9). If only one of them is nonexpansive and the other is contractive then $\max(k_1,k_2) = 1$ and $0 \leq \min(k_1,k_2) < 1$ so that (2.3) follow from (2.9) and Property (ii) is proven. For real constants for some real constants $k_i \in [0,1)$, $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ $(i = 1,2)$ are both contractive and one gets from (2.9)

$$d\Big(T^j(x,y),T^j(x',y')\Big) \leq k_1^j d(x,x') + k_2^j d(y,y')$$
$$+ \sum_{\ell=0}^{j-1} k_1^{\ell}\Big[(1-k_1)\operatorname{dist}(A_1,A_2) + k_2^{\ell}(1-k_2)\operatorname{dist}(A_1,A_2)\Big], \tag{2.10}$$
$$\forall x,y \in A_1,\ \forall x',y' \in A_2,\ \forall j \in \mathbf{Z}_+$$

what leads to (2.5a), (2.5b), and to (2.6), (2.7) by taking $j \to \infty$. Property (iii) has been proven. □

Corollary 2.2. *If $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ are both 2-cyclic asymptotically contractive self-mappings, that is, they are nonexpansive with time-varying real sequences $\{k_{ij}\}_{j\in\mathbf{Z}_+}$ in $[0,1]$ for $i = 1,2$ which converge asymptotically to two respective real numbers k_1 and k_2 in $[0,1)$ as $j \to \infty$. Assume that the sequences $\{k_{ij}\}_{j\in\mathbf{Z}_+}$ for $i = 1,2$ are such that*

$$\limsup_{j\to\infty}\left(\sum_{\ell=1}^{j}\left[\left(\prod_{i=\ell+1}^{j}[k_{1i}]\right)(1-k_{1\ell}) + \left(\prod_{i=\ell+1}^{j}[k_{2i}]\right)(1-k_{2\ell})\right]\right) \leq K < \infty \tag{2.11}$$

is satisfied for some real $K \in \mathbf{R}_+$. Then, (2.1) holds and, furthermore, the following properties hold:

(i)

$$d\Big(T^j(x,y),T^j(x',y')\Big) \leq 2\varepsilon_j + K\operatorname{dist}(A_1,A_2) \leq 2\varepsilon + K_j\operatorname{dist}(A_1,A_2), \quad \forall j(\geq j_0) \in \mathbf{Z}_+,$$
$$\forall x,y \in A_1,\ \forall x',y' \in A_2 \tag{2.12}$$

for any bounded positive decreasing real sequence ε_j*, with arbitrary prescribed upper-bound* ε*, which converges asymptotically to zero as* $j \to \infty$ *for some finite* $j_0 = j_0(\varepsilon) \in \mathbf{Z}_+$ *and some bounded nondecreasing positive real sequence* K_j *of upper-bound* K*.*

(ii)

$$\begin{gathered}\limsup_{j\to\infty} d\left(T^j(x,y),T^j(x',y')\right), \quad \forall x,y \in A_1,\ \forall x',y' \in A_2,\\ \exists \lim_{j\to\infty} d\left(T^j(x,y),T^j(x',y')\right) = 0, \quad \text{if } \operatorname{dist}(A_1,A_2)=0,\ \forall x,y \in A_1,\ \forall x',y' \in A_2.\end{gathered} \tag{2.13}$$

Proof. Note that

$$\begin{aligned} d\left(T^j(x,y),T^j(x',y')\right) \le{} & \left(\prod_{\ell=1}^{j}[k_{1\ell}]\right)d(x,x') + \left(\prod_{\ell=1}^{j}[k_{2\ell}]\right)d(y,y')\\ & + \sum_{\ell=1}^{j}\left[\left(\prod_{i=\ell+1}^{j}[k_{1i}]\right)(1-k_{1\ell}) + \left(\prod_{i=\ell+1}^{j}[k_{2i}]\right)(1-k_{2\ell})\right]\operatorname{dist}(A_1,A_2),\\ & \forall x,y\in A_1,\ \forall x',y' \in A_2,\ \forall j \in \mathbf{Z}_+. \end{aligned} \tag{2.14}$$

Since the terms in the sum the summation term of the right-hand-side of the above equation are all nonnegative, it follows that:

$$\begin{aligned} &\limsup_{j\to\infty}\left(\sum_{\ell=1}^{j}\left[\left(\prod_{i=\ell+1}^{j}[k_{1i}]\right)(1-k_{1\ell}) + \left(\prod_{i=\ell+1}^{j}[k_{2i}]\right)(1-k_{2\ell})\right]\right)\\ &\quad \ge \sum_{\ell=1}^{j}\left[\left(\prod_{i=\ell+1}^{j}[k_{1i}]\right)(1-k_{1\ell}) + \left(\prod_{i=\ell+1}^{j}[k_{2i}]\right)(1-k_{2\ell})\right], \quad \forall j \in \mathbf{Z}_+. \end{aligned} \tag{2.15}$$

Note also that $\prod_{\ell=1}^{j}[k_{i\ell}] \le 1$ for $i = 1,2$, for all $j \in \mathbf{Z}_+$, $\exists j_0 = j_0(\varepsilon) \in \mathbf{Z}_+$ such that $\prod_{\ell=1}^{j}[k_{i\ell}] \le \varepsilon < 1$; for all $j \ge j_0$ for any given real constant $\varepsilon \in (0,1)$, an infinite subsequence $\{\prod_{\ell=1}^{j_i}[k_{i\ell}]\}_{j_i \in \mathbf{Z}_+}$ of $\{\prod_{\ell=1}^{j}[k_{i\ell}]\}_{j \in \mathbf{Z}_+}$ which is monotone decreasing and $\lim_{j\to\infty}\prod_{\ell=1}^{j}[k_{i\ell}] = 0$ for $i = 1,2$. Thus, if the sequences $\{k_{ij}\}$ in $[0,1]$ for $i = 1,2$ are such that there is some finite $K \in \mathbf{R}_+$ for which

$$\begin{aligned} &\sum_{\ell=1}^{j}\left[\left(\prod_{i=\ell+1}^{j}[k_{1i}]\right)(1-k_{1\ell}) + \left(\prod_{i=\ell+1}^{j}[k_{2i}]\right)(1-k_{2\ell})\right]\\ &\quad \le \limsup_{j\to\infty}\left(\sum_{\ell=1}^{j}\left[\left(\prod_{i=\ell+1}^{j}[k_{1i}]\right)(1-k_{1\ell}) + \left(\prod_{i=\ell+1}^{j}[k_{2i}]\right)(1-k_{2\ell})\right]\right) \le K, \quad \forall j \in \mathbf{Z}_+ \end{aligned} \tag{2.16}$$

according to (2.15) then if (2.11) holds, one gets from (2.11) that (2.12) holds for any given $\varepsilon \in \mathbf{R}_+$ and some positive decreasing real sequence $\{\varepsilon_j\}_{j\in\mathbf{Z}_+}$ and some nondecreasing positive real sequence $\{K_j\}_{j\in\mathbf{Z}_+}$ subject to $K_j \le K_{j+1} \le K$, so that $\limsup_{j\to\infty} K_j \le K$, and $\varepsilon_{j+1} \le \varepsilon_j \le \varepsilon$ which converges then to zero as $j \to \infty$ since it has an infinite monotone decreasing subsequence $\{\varepsilon_{j_i}\}_{j_i(j)\in\mathbf{Z}_+}$ for each $j \in \mathbf{Z}_+$, satisfying to $0 < j_{i+1} - j_i < \infty$ for any $i, j \in \mathbf{Z}_+$, then $\lim_{j_i\to\infty}\varepsilon_{j_i} = \lim_{j\to\infty}\varepsilon_j = 0$. As a result, (2.13) follows, whose upper-bound is zero if $\operatorname{dist}(A_1, A_2) = 0$. □

Remark 2.3. Note that a simple comparison between (2.6) in Theorem 2.1 and (2.12) in Corollary 2.2 concludes that K in (2.12) can be taken as small as 2 if the sequences $\{k_{ij}\}_{j\in\mathbf{Z}_+}$ in Corollary 2.2 are constant and equal to $k_i \in (0,1)$ for $i = 1,2$. Therefore, a logic procedure of accomplishing with (2.11) is to check for the existence of valid constants K possessing a lower-bound 2.

If the constants $(k_1 + k_2)$ $(i = 1,2)$ in Theorem 2.1, or the time-varying sequences $\{k_{1j} + k_{2j}\}_{j\in\mathbf{Z}_+}$ for $i = 1,2$ of Corollary 2.2, are less than unity then the following result holds.

Corollary 2.4. *The following properties hold.*

(i) *$T_i : A_1 \cup A_2 \to A_1 \cup A_2$ for $i = 1,2$ are both 2-cyclic contractive self-mappings with time-varying real sequences $\{k_{ij}\}_{j\in\mathbf{Z}_+}$ $(i = 1,2)$ such that any element of the sum sequence $\{k_{1j} + k_{2j}\}_{j\in\mathbf{Z}_+}$ is in $[0,1]$ for $i = 1,2$ with $k \le \max_{j\in\widehat{\mathbf{Z}}_+}(k_{1j} + k_{2j}) \in [0,1)$ and $k' := \min_{j\in\mathbf{Z}_+}(k_{1j} + k_{2j}) \ge 0$ where $\widehat{\mathbf{Z}}_+$ is some infinite subset of $\mathbf{Z}_+$. Then,*

$$d\left(T^j(x,y), T^j(x',y')\right) \le k^j \max\left(d(x,x') + d(y,y')\right) + \frac{2-\rho k}{1-k}\operatorname{dist}(A_1,A_2), \quad \forall x,y \in A_1,\ \forall x',y' \in A_2,\ \forall j \in \mathbf{Z}_+, \tag{2.17}$$

where $\rho := k'/k$ if $kk' \ne 0$, $\rho = 1$ if $k = k' = 0$, and $\rho = 0$ if $k \ne 0$ and $k' = 0$

$$\limsup_{j\to\infty} d\left(T^j(x,y), T^j(x',y')\right) \le \frac{2-\rho k}{1-k}\operatorname{dist}(A_1,A_2), \quad \forall x,y \in A_1,\ \forall x',y' \in A_2. \tag{2.18}$$

(ii) *Let $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ be both 2-cyclic nonexpansive and asymptotically contractive self-mappings for $i = 1,2$ with the sequences $\{k_{ij}\}_{j\in\mathbf{Z}_+}$ in $[0,1]$ for $i = 1,2$ and having limits $\overline{k}_i$ for $i = 1,2$ satisfying $\overline{k} := \max(\overline{k}_1, \overline{k}_2) < 1/2$, $\overline{k'} := \min(\overline{k}_1, \overline{k}_2) \ge 0$. Then,*

$$\limsup_{j\to\infty} d\left(T^{j+i}(x,y), T^{j+i}(x',y')\right) \le \frac{2\left(1-\overline{\rho}\overline{k}\right)}{1-2\overline{k}}\operatorname{dist}(A_1,A_2); \quad \forall x,y \in A_1,\ \forall x',y' \in A_2, \tag{2.19}$$

where $\overline{\rho} := \min(\overline{k}_1, \overline{k}_2)/\max(\overline{k}_1, \overline{k}_2) \in [0,1]$.

Proof. (i) First, note that the integer subset $\widehat{\mathbf{Z}}_+$ exists since the time-varying contraction sequences converge to real constants being less than unity. One gets from (2.5a) for $k' := \min(k_1, k_2) = \rho k \le k < 1/2$ where $\rho = k'/k \le \min(k_1, k_2)/\max(k_1, k_2) \in [0, 1]$

$$\begin{aligned} d\left(T^j(x,y), T^j(x',y')\right) &\le k^j \max(d(x,x') + d(y,y')) + (2-\rho k)\operatorname{dist}(A_1, A_2)\left(\sum_{i=0}^{j-1} k^i\right) \\ &\le k^j \max(d(x,x') + d(y,y')) + \frac{(2-\rho k)(1-k^j)}{1-k}\operatorname{dist}(A_1, A_2) \\ &\le k^j \max(d(x,x') + d(y,y')) + \frac{2-\rho k}{1-k}\operatorname{dist}(A_1, A_2), \\ &\qquad \forall x, y \in A_1,\ \forall x', y' \in A_2,\ \forall j \in \mathbf{Z}_+ \end{aligned} \tag{2.20}$$

and (2.17) holds. Property (i) has been proven.

(ii) The proof of Property (ii) is close to that of Property (i) by using a close method to that of the proof of Corollary 2.2(ii):

$$\begin{aligned} d\left(T^j(x,y), T^j(x',y')\right) &\le \left(2\overline{k} + \varepsilon_{1j}\right)^j \max(d(x,x') + d(y,y')) \\ &\quad + 2\left(1-\overline{\rho k}\right)\operatorname{dist}(A_1, A_2)\left(\sum_{i=1}^{j}\prod_{\ell=i+1}^{j}\left[\left(2\overline{k} + \varepsilon_{ij}\right)^{\ell}\right]\right) \\ &\le \left(2\overline{k} + \varepsilon_{1j}\right)^j \max(d(x,x') + d(y,y')) \\ &\quad + \frac{2\left(1-\overline{\rho k}\right)\left(1-\left(2\overline{k}\right)^j\right)}{1-2\overline{k}}\operatorname{dist}(A_1, A_2) + K(j,0) \\ &\le \left(2\overline{k} + \varepsilon_{0j}\right)^j \max(d(x,x') + d(y,y')) \\ &\quad + \frac{2\left(1-\overline{\rho k}\right)}{1-2\overline{k}}\operatorname{dist}(A_1, A_2) + K(j,0), \end{aligned} \tag{2.21}$$

for all $x, y \in A_1$, for all $x', y' \in A_2$, where $\varepsilon_{ij} := \left(\prod_{\ell=i+1}^{j}[k_{1j} + k_{2j} - 2\overline{k}]\right)^{1/(j-i)} (\in \mathbf{R}) \to 0$ as $j(>i) \to \infty$, for all $i \in \mathbf{Z}_{0+}$, and $\{K(j,i)\}_{j(>i)\in\mathbf{Z}_+}$; are sequences of positive finite real constants whose integer arguments j and i are related to the iterate of the left and right hand sides of (2.20), respectively, defined by

$$\begin{aligned} &K(j+i, i) \\ &\quad := \left(2\left(1-\overline{\rho k}\right)\left(\sum_{n=i}^{j}\prod_{\ell=n+1}^{j}\left[\left(2\overline{k} + \varepsilon_{nj}\right)^{\ell}\right]\right) - \frac{2\left(1-\overline{\rho k}\right)\left(1-\left(2\overline{k}\right)^j\right)}{1-2\overline{k}}\right)\operatorname{dist}(A_1, A_2). \end{aligned} \tag{2.22}$$

Since $0 \leq k_{1j} + k_{2j} \leq 1$; for all $j \in \mathbf{Z}_{0+}$ then $0 \leq 2\overline{k} + \varepsilon_{ij} \leq 1$; for all $i, j \in \mathbf{Z}_+$ and since the limit $k_{1j} + k_{2j} \to \overline{k} := \max(\overline{k}_1, \overline{k}_2) < 1/2$ as $j \to \infty$, $\{K(j,i)\}_{j(>i)\in\mathbf{Z}_+}$ is uniformly bounded and $\lim_{i\to\infty} K(j+i,i) = 0$; for all $j \in \mathbf{Z}_+$. Thus, combining (2.20) and (2.21), one gets

$$\begin{aligned} &d\Big(T^{j+i}(x,y), T^{j+i}(x',y')\Big) \\ &\quad\leq \Big(2\overline{k} + \varepsilon_{0j}\Big)^j \max\Big(d\Big(T^i x, T^i x'\Big) + d\Big(T^i y, T^i y'\Big)\Big) + \frac{2\Big(1-\overline{\rho}\overline{k}\Big)}{1-2\overline{k}} \operatorname{dist}(A_1, A_2) + K(j+i,i), \\ &\qquad\qquad \forall x, y \in A_1,\ \forall x', y' \in A_2, \end{aligned} \tag{2.23}$$

and then

$$\begin{aligned} \limsup_{j(>i)\to\infty,\ i\to\infty} & d\Big(T^{j+i}(x,y), T^{j+i}(x',y')\Big) \\ &\leq \Big(2\overline{k} + \varepsilon_{0j}\Big)^j \limsup_{i\to\infty} \max\Big(d\Big(T^i x, T^i x'\Big) + d\Big(T^i y, T^i y'\Big)\Big) \\ &\quad + \frac{2\Big(1-\overline{\rho}\overline{k}\Big)}{1-2\overline{k}} \operatorname{dist}(A_1, A_2), \\ \limsup_{j(>i)\to\infty,\ i\to\infty} & d\Big(T^{j+i}(x,y), T^{j+i}(x',y')\Big) \\ &= \limsup_{j\to\infty} d\Big(T^{j+i}(x,y), T^{j+i}(x',y')\Big) \\ &\leq \frac{2\Big(1-\overline{\rho}\overline{k}\Big)}{1-2\overline{k}} \operatorname{dist}(A_1, A_2), \quad \forall x, y \in A_1,\ \forall x', y' \in A_2. \end{aligned} \tag{2.24}$$

□

Some Krasnoselskii-type fixed point results follow for the map $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ defined trough 2-cyclic binary self-mappings for the case when A_1 and A_2 intersect.

Theorem 2.5. *Assume that X is a Banach space which has an associate complete metric space (X, d) with the metric $d : X \times X \to \mathbf{R}_{0+}$ being translation-invariant and homogeneous. Assume that A_1 and A_2 are nonempty, convex, and closed subsets of X which intersect. Assume also that $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ ($i = 1, 2$) are both 2-cyclic contractive self-mappings and $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ fulfilling $T((A_1 \cap A_2) \times (A_1 \cap A_2)) \subseteq A_1 \cap A_2$. Then, there is a unique fixed point $z = z_1 + z_2$ of $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ in $A_1 \cap A_2$ which satisfies $T_1 z + T_2 z = z$, where $z_i \in \operatorname{Fix}(T_i) \subseteq A_1 \cap A_2$ are also the respective unique fixed points of $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ ($i = 1, 2$).*

Proof. One gets from Theorem 2.1(iii), (2.9)

$$\begin{aligned} &\Big[0 \longleftarrow d\Big(T^j(x,y), T^j(x',y')\Big) \leq d\Big(T_1^j x, T_1^j x'\Big) + d\Big(T_2^j y, T_2^j y'\Big) \longrightarrow 0\Big] \\ &\quad\Longrightarrow \Big[d\Big(T_1^j x, T_1^j x'\Big) \longrightarrow 0;\ d\Big(T_2^j y, T_2^j y'\Big) \longrightarrow 0\Big], \quad \forall x, y \in A_1,\ \forall x', y' \in A_2 \text{ as } j \to \infty, \end{aligned} \tag{2.25}$$

since $A_1 \cap A_2 \neq \emptyset$ implies $\operatorname{dist}(A_1, A_2) = 0$. Since X is a Banach space then (X, d) is a complete metric space, since $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ $(i = 1, 2)$ are 2-cyclic contractive self-mappings then satisfying (1.1) with contraction constants $k_i \in [0, 1)$ and $\operatorname{dist}(A_1, A_2) = 0$ (since $A_1 \cap A_2 \neq \emptyset$) and subject $T_i(A_j) \subseteq A_\ell$ for $j, \ell(\neq j), i = 1, 2$, and since $T((A_1 \cap A_2) \times (A_1 \cap A_2)) \subseteq A_1 \cap A_2 \neq \emptyset$, the unique limits below for Cauchy sequences exist

$$\begin{gathered} T^j(x, y) \longrightarrow T(z, z) = T_1^i z + T_2^i z = z \in A_1 \cap A_2, \quad \forall i \in \mathbf{Z}_+ \text{ as } j \to \infty, \\ T_1^j x' \longrightarrow T_1^j x \longrightarrow T_1^i z_1 = z_1 \in A_1 \cap A_2, \qquad T_2^j y' \longrightarrow T_2^j x \longrightarrow T_1^i = z_2 \in A_1 \cap A_2, \\ \forall i \in \mathbf{Z}_+ \text{ as } j \to \infty,\ \forall x, y \in A_1;\ \forall x', y' \in A_2, \end{gathered} \tag{2.26}$$

where $z_i \in \operatorname{Fix}(T_i) \subseteq A_1 \cap A_2$ $(i = 1, 2)$ are unique in $A_1 \cap A_2$ since A_1, A_2 (and then $A_1 \cap A_2$) are nonempty, convex, and closed. It follows that $z = z_1 + z_2$ by making $i \to \infty$ in $T(z, z) = T_1^i z + T_2^i z = z \in \operatorname{Fix}(T) \subseteq A_1 \cap A_2$, for all $i \in \mathbf{Z}_+$. Since z_1 and z_2 then z is unique so that $\operatorname{Fix}(T) = \{z\}$. □

The following result follows from Corollary 2.2 and Theorem 2.5.

Corollary 2.6. *Theorem 2.5 also holds if $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ $(i = 1, 2)$ are both 2-cyclic asymptotically contractive self-mappings with respective time-varying contraction sequences being in $[0, 1]$ with limits in $[0, 1)$.*

Theorem 2.7. *Assume that X is a Banach space which has an associate complete metric space (X, d) with the metric $d : X \times X \to \mathbf{R}_{0+}$ being translation-invariant and homogeneous. Assume that A_1 and A_2 are nonempty, convex, and closed subsets of X, which intersect and have a convex union. Assume that $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ $(i = 1, 2)$ are both 2-cyclic self-mappings, the first one being continuous, the second one being contractive, and, furthermore, the mapping $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ satisfies*

$$T((A_1 \cup A_2) \times (A_1 \cup A_2)) \subseteq A_1 \cup A_2, \qquad T((A_1 \cap A_2) \times (A_1 \cap A_2)) \subseteq A_1 \cap A_2. \tag{2.27}$$

Then, there is a (in general, nonunique) fixed point $z \in A_1 \cup A_2$ of $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ which has the properties below:

$$T_1^i z + T_2^i z = z, \quad \forall i \in \mathbf{Z}_+, \qquad \exists \lim_{j \to \infty} T_1^j z = z - y = T_1 z + T_2 z - y, \tag{2.28}$$

where $y (= T_2^i y) \in \operatorname{Fix}(T_2) \equiv \{y\} \subseteq A_1 \cap A_2$, for all $i \in \mathbf{Z}_+$, is the unique fixed point of $T_2 : A_1 \cup A_2 \to A_1 \cup A_2$.

Proof. Note that the following properties hold

(1) By hypothesis, $A_1 \cup A_2$ is a nonempty convex set which is, furthermore, closed since A_1 and A_2 are both nonempty and closed.

(2) $A_1 \cap A_2$ is nonempty, convex, and closed since A_1 and A_2 are nonempty, closed, and convex with nonempty intersection.

(3) $(A_1 \cup A_2) \times (A_1 \cup A_2)$ is invariant through the mapping $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ by hypothesis.

(4) $A_1 \cup A_2$ is a nonempty compact set which is trivially invariant under the 2-cyclic continuous self-mapping $T_1 : A_1 \cup A_2 \to A_1 \cup A_2$.

(5) $T_2 : A_1 \cup A_2 \to A_1 \cup A_2$ is a 2-cyclic contraction which has then a unique fixed point $y \in \mathrm{Fix}(T_2) \equiv \{y\} \subseteq A_1 \cap A_2$ from Theorem 2.1 since $A_1 \cap A_2$ is nonempty, convex, and closed.

Thus, from the standard Krasnoselskii fixed point theorem [20, 21], under the above Properties (1)–(4), it exists $z \in \mathrm{Fix}(T) \subseteq A_1 \cup A_2$ which satisfies $T_1^i z + T_2^i z = z = T(z, z)$; for all $i \in \mathbf{Z}_+$. Also, one gets from Property (5) that $\lim_{j \to \infty} T_2^j z = T_2 y = y \in \mathrm{Fix}(T_2) \equiv \{y\} \subseteq A_1 \cap A_2$ are limits of Cauchy sequences so that $T_1^j z + T_2^j z = z = T(z, z) \to T_1^j z + y$ as $j \to \infty$ leading to $\exists \lim_{j \to \infty} T_1^j z = z - y = T_1 z + T_2 z - y$. □

The following results follow from Krasnoselskii fixed point theorem extended to 2-cyclic self-mappings since $A_1 \cup A_2$ is a compact set [20].

Theorem 2.8. *Theorem 2.7 holds if $T_2 : A_1 \cup A_2 \to A_1 \cup A_2$ is a 2-cyclic large contraction instead of being a contraction.*

Theorem 2.9. *Theorem 2.7 holds if the hypothesis $T((A_1 \cup A_2) \times (A_1 \cup A_2)) \subseteq A_1 \cup A_2$ is replaced by $(x = T_1 x + T_2 y;\ \forall y \in A_1 \cup A_2) \Rightarrow x \in A_1 \cup A_2$.*

Corollary 2.10. *Theorem 2.7 also follows with the replacement of the 2-cyclic contractive self-mapping $T_2 : A_1 \cup A_2 \to A_1 \cup A_2$ by a 2-cyclic asymptotic contractive self-mapping $T_2 : A_1 \cup A_2 \to A_1 \cup A_2$ for $i = 1, 2$ with time-varying contraction sequences in $[0, 1]$ with limits in $[0, 1)$.*

Corollary 2.10 follows by using Corollary 2.2 instead of Theorem 2.1. Also, Theorems 2.8 and 2.9 can be extended for $T_2 : A_1 \cup A_2 \to A_1 \cup A_2$ being asymptotically contractive closely to the extension Corollary 2.10 to Theorem 2.7.

On the other hand, note that if $A_1 \cap A_2 = \emptyset$ then the convergence through the mapping $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$ to a fixed point under the conditions of Theorem 2.5, or those of Corollary 2.6, cannot be achieved since fixed points, if any, are in $A_1 \cap A_2$ since $A_1 \cap A_2 = \emptyset$. See Theorem 2.1, (2.6), Corollary 2.2, (2.13), Corollary 2.4, (2.18), or (2.19). It is not either guaranteed the convergence to best proximity points because the guaranteed upper-bounds for the limit superiors of distances of the iterates exceed the distance between the adjacent subsets A_1 and A_2 since the lower upper-bounds of the above respective referred to limit superiors are of the form $\lambda\, \mathrm{dist}(A_1, A_2)$ for some real constant $\lambda > 1$ defined directly by inspection of (2.6), (2.13), or (2.18), or (2.19). Then, the following result follows instead of Theorem 2.5.

Theorem 2.11. *Assume that X is a Banach space which has an associate complete metric space (X, d) with the metric $d : X \times X \to \mathbf{R}_{0+}$ being translation-invariant and homogeneous. Assume also that A_1 and A_2 are nonempty subsets of X, which do not intersect. Assume that $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ $(i = 1, 2)$ are both 2-cyclic either contractive under Theorem 2.1 or under Corollary 2.4* *(i) (or asymptotically contractive under Corollary 2.2 or under Corollary 2.4(ii)) self-mappings, and, furthermore,*

$$T : (A_1 \cup A_2) \times (A_1 \cup A_2) \longrightarrow A_1 \cup A_2. \tag{2.29}$$

Then, $T^j(x, y)$ is asymptotically permanent as $j \to \infty$; for all $x \in A_1$, for all $y \in A_2$ entering a compact subset $\widehat{A}_1 \cup \widehat{A}_2$ of $\mathrm{cl}(A_1 \cup A_2)$, where:

$$\begin{aligned}
\widehat{A}_1 &:= \{z \in A_1 : \mathrm{dist}(A_1, A_2) \le \mathrm{dist}(z, A_2) \le \lambda\, \mathrm{dist}(A_1, A_2)\}; \\
\widehat{A}_2 &:= \{z \in A_2 : \mathrm{dist}(A_1, A_2) \le \mathrm{dist}(z, A_1) \le \lambda\, \mathrm{dist}(A_1, A_2)\},
\end{aligned} \tag{2.30}$$

and $\lambda > 1$ is defined directly from (2.6)*, under Theorem 2.1; from* (2.13)*, under Corollary 2.2, or from either* (2.18) *or* (2.19)*, under Corollary 2.4.*

Proof. It follows from either Theorem 2.1, Corollary 2.2 or Corollary 2.4 since (2.29) holds. □

Note that (2.29) which restricts the image of $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \to X$, used in Theorem 2.5 and in Corollary 2.6 is essential for the existence of the residual set where the iterates $T^j(x, y)$ enter asymptotically as $j \to \infty$. It is now proven how Corollary 2.4 may be improved to obtain $\lambda = 1$ in Theorem 2.11 so that the convergence of the iterates $T^j : ((A_1 \cup A_2) \times (A_1 \cup A_2)) \to A_1 \cup A_2$ converge to a best proximity point if A_1 and A_2 are nonempty, disjoint, convex, and closed. The part of Theorem 2.11 referred to the fulfilment of Corollary 2.4 (i.e., either the sum of both contraction constants or the sum of their limits if they are time-varying sequences is less than unity) is improved as follows.

Corollary 2.12. *Assume that X is a Banach space which has an associate complete metric space (X, d) with the metric $d : X \times X \to \mathbf{R}_{0+}$ being translation-invariant and homogeneous. Assume that A_1 and A_2 are nonempty, disjoint, convex, and closed subsets of X. Assume also that $T_i : A_1 \cup A_2 \to A_1 \cup A_2$ $(i = 1, 2)$ are both 2-cyclic either contractive under Corollary 2.4(i) with (or asymptotically contractive under Corollary 2.4(ii)) self-mappings and, furthermore, (see* (2.29)*). Then, any iterates $T^j(x, y)$, respectively, $T^{j+1}(x, y)$ converge to a best proximity point of either A_1, respectively, A_2, or conversely, as $j \to \infty$ for any given $x \in A_1$, $y \in A_2$.*

Proof. Take some real constant $k \in [k_1 + k_2, 1)$ and note from (1.1) that (2.8) is modified as follows:

$$\begin{aligned}
d(T(x, y), T(x', y')) &\le (k_1 + k_2) \max(d(x, x'), d(y, y')) + (2 - k_1 - k_2)\, \mathrm{dist}(A_1, A_2) \\
&\le k \max(d(x, x'), d(y, y')) + (1 - k)\, \mathrm{dist}(A_1, A_2), \\
&\qquad \forall x, y \in A_1,\ \forall x', y' \in A_2,
\end{aligned} \tag{2.31}$$

which holds since

$$\begin{aligned}
&(k - k_1 - k_2) \max(d(x, x'), d(y, y')) - (k - k_1 - k_2)\, \mathrm{dist}(A_1, A_2) \ge 0 \\
&\quad \Longleftrightarrow \max(d(x, x'), d(y, y')) \ge \mathrm{dist}(A_1, A_2), \quad \forall x, y \in A_1,\ \forall x', y' \in A_2.
\end{aligned} \tag{2.32}$$

Thus,

$$\begin{aligned} d\left(T^j(x,y),T^j(x',y')\right) &\le (k_1+k_2)\max(d(x,x'),d(y,y')) + (2-k_1-k_2)\operatorname{dist}(A_1,A_2) \\ &\le k^j \max(d(x,x'),d(y,y')) + (1-k)\operatorname{dist}(A_1,A_2)\left(\sum_{\ell=0}^{j} k^{\ell}\right) \\ &= k^j \max(d(x,x'),d(y,y')) + \left(1-k^j\right)\operatorname{dist}(A_1,A_2), \\ &\qquad \forall x,y\in A_1,\ \forall x',y'\in A_2, \end{aligned} \tag{2.33}$$

so that the limit $\lim_{j\to\infty} d(T^j(x,y),T^j(x',y')) = \operatorname{dist}(A_1,A_2)$ exists modifying (2.18) in Corollary 2.4(i). Then, Theorem 2.11 holds with $\lambda = 1$ and $\widehat{A}_i$ is a subset of the boundary of A_i containing the best proximity points of A_i to the set A_{i+1} $(i = 1,2)$ with $A_3 = A_1$. Thus, since $\{T^j(x,y)\}_{j\in\mathbf{Z}_+}$ is a Cauchy sequence since (X,d) is complete, then $T^j(x,y)$ converges to a best proximity point of A_2 (resp., A_1) if $j \to \infty$ and is odd (resp., even). A close proof leading to a similar modified limit follows by modifying (2.19) in Corollary 2.4(ii) by using the limits $\overline{k}_i$ of the time-varying asymptotically contractive sequences for $i = 1,2$, instead of the constants k_i, satisfying $\overline{k}_i \in [0,1/2)$ implying $\overline{k}_1 + \overline{k}_2 < 1$ for $\overline{k} \in [\overline{k}_1 + \overline{k}_2, 1)$. □

Note that Corollary 2.4 and its referred to part of Corollary 2.12 also hold in a more general version for limits $\overline{k}_i \in [0,1)$ with $\overline{k}_1 + \overline{k}_2 < 1$ and $\overline{k} \in [\overline{k}_1 + \overline{k}_2, 1)$.

Some illustrative examples follow.

Example 2.13. Consider the scalar differential equations

$$\dot{x}_i(t) = a_i x_i(t) + r_i, \quad x_i(0) = x_{i0}, \quad i = 1,2, \tag{2.34}$$

with $a_i < 0$ for $i = 1,2$. The solution trajectories are defined by contractive self-mappings $T_i : \mathbf{R} \to \mathbf{R}$ for $i = 1,2$ as follows:

$$x_i(t,x_{i0}) = (T_i x_{i0})(t) = e^{a_i t}\left(x_{i0} + r_i \int_0^t e^{-a_i\tau} d\tau\right), \quad i = 1,2;\ \forall x_{i0} \in \mathbf{R},\ \forall t \in \mathbf{R}_{0+}. \tag{2.35}$$

Both solutions converge to respective unique fixed points $z_i = r_i/|a_i|$ as $t \to \infty$ which are also stable equilibrium points, for $i = 1,2$ since by using the Euclidean metric, it follows trivially for any two initial conditions $x_{i0}, \overline{x}_{i0}$ $(i = 1,2)$

$$|x_i(t,x_{i0}) - x_i(t,\overline{x}_{i0})| \le e^{-|a_i|t}|x_{i0} - \overline{x}_{i0}| \to 0 \quad \text{as } t \to \infty \text{ for } i = 1,2, \tag{2.36}$$

and $x_i(t,x_{i0}) \to z_i$ with $x_{i0} \in \mathbf{R}$ as $t \to \infty$ for $i = 1,2$. Note that the same property holds by redefining the self-mappings $\overline{T}_{ih} : \mathbf{R} \to \mathbf{R}$ for $i = 1,2$ so as to pick up by successive iterates starting from initial conditions the sequence of points of the solutions $\{x_i(kh, x_{i0})\}_{k\in\mathbf{Z}_{0+}}$ for some $h \in \mathbf{R}_+$ as

$$x_i((k+1)h, x_{i0}) = \overline{T}_{ih} x_i(kh) = e^{a_i h}\left(x_i(kh) + r_i \int_0^h e^{-a_i\tau} d\tau\right), \quad i = 1,2;\ \forall x_{i0} \in \mathbf{R},\ \forall t \in \mathbf{R}_{0+}, \tag{2.37}$$

which are both contractive since $|x_i((k+1)h, x_{i0}) - x_i((k+1)h, \overline{x}_{i0})| \le e^{-|a_i|h}\ |x_i(kh) - \overline{x}_i(kh)| \to 0$ as $\mathbf{Z}_{0+} \ni k \to \infty$ for $i = 1,2$ and $x_i(kh, x_{i0}) \to z_i$ with $x_{i0} \in \mathbf{R}$ as $\mathbf{Z}_{0+} \ni k \to \infty$ for $i = 1,2$. Define the self-mapping $T : \mathbf{R} \to \mathbf{R}$ by $T(x,y)(t) = (T_1 x)(t) + (T_2 y)(t)$ for any $k \in \mathbf{Z}_+$; for all $x, y \in \mathbf{R}$. Note that $T(x,y)(t) \to T(z,z)$ and $\overline{T}_h^k(x,y) \to T(z,z)\ \mathbf{Z}_{0+} \ni k \to \infty$ for any $x, y \in \mathbf{R}$ with $z = z_1 + z_2 = r_1/|a_1| + r_2/|a_2|$. Now, assume that the initial conditions are restricted to fulfil the constraint $|x_{i0}| \le M_i$ for $i = 1,2$. Consider a convex closed real interval $A = B \equiv [-(M_1 + M_2 + r_1/|a_1| + r_2/|a_2|), M_1 + M_2 + r_1/|a_1| + r_2/|a_2|]$. Then, $T(x,y)(t)$ and $\overline{T}_h^k(x,y)$ are in such an interval and converge to the fixed point z for any initial conditions $x, y \in A$ satisfying the constraint. Note that $T_i(A), T_i(B) \subseteq A \equiv B$ and $\overline{T}_{ih}(A), \overline{T}_{ih}(B) \subseteq A \equiv B$ are trivially 2-cyclic contractive self-mappings from A to A (Theorem 2.5).

Example 2.14. Now, assume the replacement $r_1 \to r_1(t)$ in Example 2.13 where it exists the integral $\int_0^t e^{|a_1|\tau} r_1(\tau) d\tau = O(e^{|a_1|t})$; for all $t \in \mathbf{R}_{0+}$. Then, the above results still hold with the individual and combined resulting mappings still being contractive leading to respective unique fixed points $z_1 = \overline{r}_1$, $z_2 = r_2/|a_2|$, and $z = \overline{r}_1 + (r_2/|a_2|)$ with $\overline{r}_1 := \int_0^t e^{-|a_1|(t-\tau)} r_1(\tau) d\tau$, subject to $\widehat{r}_1 := \sup_{t\in\mathbf{R}_+}(\int_0^t e^{-|a_1|(t-\tau)} r_1(\tau) d\tau) < \infty$, and $A \equiv B = [-(M_1 + M_2 + \widehat{r}_1 + (r_2/|a_2|)), M_1 + M_2 + \widehat{r}_1 + r_2/|a_2|]$.

Example 2.15. Consider self-mappings $T_i : \overline{A} \cup B \to \overline{A} \cup B$ $(i = 1,2)$ defined for any given $h, \varepsilon \in \mathbf{R}_+$, $a_i \in \mathbf{R}_-$, and $r_i \in \mathbf{R}$ $(i = 1,2)$ by

$$\begin{aligned}
&T_i x_i(kh) = x_i((k+1)h, x_{i0}),\\
&x_i((k+1)h, x_{i0})\\
&\quad = \begin{cases} \overline{x}_i((k+1)h, x_{i0}) = e^{-|a_i|h} x_i(kh) + \dfrac{r_i(kh)}{|a_i|}\left(e^{|a_i|h} - 1\right), & \text{if } |\overline{x}_i((k+1)h, x_{i0})| \ge \varepsilon,\\ \varepsilon\ \operatorname{sign} \overline{x}_i((k+1)h, x_{i0}), & \text{otherwise}, \end{cases}\\
&r_i(kh) = \frac{|a_i|}{e^{|a_i|h} - 1}\left((-1)^{k+i} k_i - e^{-|a_i|h}\right) x_i(kh),\\
&\overline{x}_i((k+1)h, x_{i0}) = e^{-|a_i|h}\left(x_i(kh) + r_i \int_0^h e^{-a_i\tau} d\tau\right); \quad i = 1,2;\ \forall x_{10} \in A;\ \forall x_{20} \in B,\ \forall t \in \mathbf{R}_{0+},
\end{aligned} \tag{2.38}$$

for all $k \in \mathbf{Z}_{0+}$, $k_i \in [0,1)$ being given contraction constants for $i = 1,2$ for disjoint sets $A \equiv \mathbf{R}_{2\varepsilon-} \subset \overline{A} \equiv \mathbf{R}_{\varepsilon-}$, $B \equiv \mathbf{R}_{\varepsilon+} \equiv -A \equiv -\mathbf{R}_{\varepsilon-} := \{z \in \mathbf{R}_+ : z \ge \varepsilon\}$ for some given $\varepsilon \in \mathbf{R}_+$. The above equations are interpreted as the solution of the discretized versions of the differential equations (2.34) discussed in Examples 2.13 and 2.14 subject to saturated values on the near boundaries of the real subintervals $\overline{A}$ and B. Note that $T_i(A) \subseteq B$ and $T_i(B) \subseteq A$ for $i = 1, 2$ so that $T_i : A \cup B \to A \cup B$ $(i = 1, 2)$ are 2-cyclic self-mappings for which any succession of iterates converges to best proximity points $-2\varepsilon \in \overline{A}$ and $\varepsilon \in B$ for any bounded initial conditions which restricts the iterates to be real successions on bounded convex disjoint subsets of $\mathbf{R}$. The composed self-mapping $T : (A \cup B) \times (A \cup B) \to A \cup B$ restricted to $T : (\overline{A} \cup B) \times (\overline{A} \cup B) \to A \cup B$ by a domain restriction converges to best proximity points $-\varepsilon \in A$ and $\varepsilon \in B$ (Corollary 2.12). If now $a_1 = \varepsilon = x_{10} = 0$ then $A \cap B = \{0\}$ and $z = 0$ is a fixed point of $T : (A \cup B) \times (A \cup B) \to A \cup B$. Note that the mapping T_1 is continuous although it is noncontractive while some Krasnoselskii-type theorems still apply.

The above examples are easily extended for nonscalar cases by using the same tools.

3. Results for Mappings Defined by $P(\geq 2)$-Cyclic Self-Mappings

Define $\overline{n} := \{1, 2, \ldots, n\}$ for any $n \in \mathbf{Z}_+$. Then, the above results are easily extended to the case of $\ell(\geq 2)$ $p(\geq 2)$ p-cyclic $T_\ell : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$; for all $\ell \in \overline{n}$ self-mappings which define a mapping $T : ((\bigcup_{i \in \overline{p}} A_i) \times (\bigcup_{i \in \overline{p}} A_i)) \to X$ defined by $T(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} T_i x_i$, for any $x_i \in A_j$ and any given $j \in \overline{p}$ $T^j : ((\bigcup_{i \in \overline{p}} A_i) \times (\bigcup_{(i \in \overline{p})} A_i)) \to \bigcup_{(i \in \overline{p})} A_i$. To facilitate the exposition, assume that

(1) $T_\ell : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ are given p-cyclic noncontractive self-mappings for $\ell \in \overline{n}_1 \subset \overline{n}$ where $\operatorname{card} \overline{n} - \operatorname{card} \overline{n}_2 \geq \operatorname{card} \overline{n}_1 \geq 0$,

(2) $T_\ell : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ are p-cyclic contractive self-mappings for $\ell \in \overline{n}_2 := \overline{n} \backslash \overline{n}_1 \subseteq \overline{n}$, where $\operatorname{card} \overline{n}_2 \geq 1$, with contraction constants $k_\ell \in [0, 1)$; for all $\ell \in \overline{n}_2$.

That is, we allocate, with no loss of generality, the n-cyclic self-mappings which are not contractive, while they are typically nonexpansive or continuous, all in the first strictly ordered set of integers and those being contractive in the second strictly ordered one. The identities of sets $A_{i+p} \equiv A_i$; for all $i \in \overline{p}$ are assumed in all the necessary notations as associated with p-cyclic self-mappings. It is also used that any p-cyclic nonexpansive self-mappings have identical distances $\overline{d} := \operatorname{dist}(A_i, A_{i+1}) > 0$; for all $i \in \overline{p}$ between disjoint adjacent sets [6]. The following three results can be proven in a similar way as those counterparts of Section 2 for $p = q = 2$.

Theorem 3.1. *Assume that $T_i : (\bigcup_{\ell \in \overline{p}} A_i) \to (\bigcup_{\ell \in \overline{p}} A_i)$ for $i \in \overline{n}$ are p-cyclic nonexpansive for Part (i), with at least one of them is being contractive $(p - 1 \geq p_2 \geq 1)$ for Part (ii). Then, Theorem 2.1 holds with the subsequent replacements:*

$$
\begin{gathered}
2 \longrightarrow n = n_2 \geq 2; \qquad (\operatorname{diam} A_1 + \operatorname{diam} A_2) \longrightarrow \left(\sum_{i=1}^{p} \operatorname{diam} A_i \right), \qquad (k_1 + k_2) \longrightarrow \left(\sum_{i=1}^{p} k_i \right), \\
(\max(\min)(k_1, k_2)) \longrightarrow \left(\max(\min)\left(k_i : i \in \overline{p}\right) \right), \qquad \operatorname{dist}(A_1, A_2) \longrightarrow \left(\operatorname{dist}(A_i, A_{i+1}),\ \forall i \in \overline{p} \right), \\
(\forall x, y \in A_1,\ \forall x', y' \in A_2) \longrightarrow (\forall x, y \in A_i,\ \forall x', y' \in A_{i+1};\ \forall i \in \overline{p}).
\end{gathered} \tag{3.1}
$$

Corollary 3.2. *Assume that $T_i : (\bigcup_{\ell\in\overline{p}} A_i) \rightarrow (\bigcup_{(\ell\in\overline{p})} A_i)$ for $i \in \overline{n}$ are p-cyclic asymptotically contractive self-mappings. Then,* Corollary 2.2 *holds with the replacements:*

$$\left(k_i \underset{\infty\leftarrow j}{\longleftarrow} k_{ij} \in [0,1],\ i = 1,2,\ \forall j \in \mathbf{Z}_+\right) \longrightarrow \left(k_i \underset{\infty\leftarrow j}{\longleftarrow} k_{ij} \in [0,1],\ i \in \overline{p},\ \forall j \in \mathbf{Z}_+\right) \tag{3.2}$$

and all further necessary replacements borrowed from Theorem 3.1.

Corollary 3.3. *Assume that $T_i : (\bigcup_{\ell\in\overline{p}} A_i) \rightarrow (\bigcup_{\ell\in\overline{p}} A_i)$ for $i \in \overline{n}$ are all p-cyclic contractive for Part (i) and p-cyclic nonexpansive and asymptotically contractive for Part (ii). Thus,* Corollary 2.4 *holds by replacing:*

$$\begin{gathered}
\{k_{1j} + k_{2j}\}_{j\in\mathbf{Z}_+} \longrightarrow \left\{\sum_{i=1}^{n} k_{ij}\right\}_{j\in\mathbf{Z}_+}, \qquad \left(k \leq \max_{j\in\hat{\mathbf{Z}}_+}(k_{1j} + k_{2j})\right) \longrightarrow \left(k \geq \max_{j\in\hat{\mathbf{Z}}_+}\left\{\sum_{i=1}^{n} k_{ij}\right\}_{j\in\mathbf{Z}_+}\right), \\
\left(k' := \min_{j\in\mathbf{Z}_+}(k_{1j} + k_{2j}) \geq 0\right) \longrightarrow \left(k' := \min_{j\in\hat{\mathbf{Z}}_+}\left\{\sum_{i=1}^{n} k_{ij}\right\}_{j\in\mathbf{Z}_+}\right), \\
\left(\overline{k} := \max\left(\overline{k}_1, \overline{k}_2\right) < \frac{1}{2}\right) \longrightarrow \left(\overline{k} := \max\left(\overline{k}_i : i \in \overline{n}\right) < \frac{1}{n}\right), \\
\left(\overline{k'} := \min\left(\overline{k}_1, \overline{k}_2\right) \geq 0\right) \longrightarrow \left(\overline{k'} := \min\left(\overline{k}_i : i \in \overline{n}\right) \geq 0\right),
\end{gathered} \tag{3.3}$$

and all further necessary replacements borrowed from Theorem 3.1 *or* Corollary 2.2.

The following results based on Krasnoselskii theorems follow for fixed and best proximity points in a Banach space endowed with a translation-invariant and homogeneous metric. The results are a direct extension from the results for mappings $T : (A_1 \cup A_2) \times (A_1 \cup A_2) \rightarrow A_1 \cup A_2$ built with mixed continuous and contractive 2-cyclic self-mappings of Section 2 to mappings $T : ((\bigcup_{i\in\overline{p}} A_i) \times (\bigcup_{i\in\overline{p}} A_i)) \rightarrow \bigcup_{i\in\overline{p}} A_i$ built with $p(\geq 2)$-cyclic self-mappings, some of them being continuous while the remaining ones are contractive. The underlying ideas are (a) a mapping which is a sum of continuous mappings is continuous, (b) a set of contractive p-cyclic self-mappings for the same set of convex and closed subsets of X has each a unique fixed point if the p subsets involved in the p-cyclic contractive self-mappings intersect. The sum of all those contractive self-mappings has a unique fixed point which is the sum of the whole set of contractive self-mappings, (c) the combinations of both results yields the existence of a fixed point from Krasnoselskii fixed point theorems of $T : ((\bigcup_{i\in\overline{p}} A_i) \times (\bigcup_{i\in\overline{p}} A_i)) \rightarrow \bigcup_{i\in\overline{p}} A_i$ if all the subsets of the p-cyclic structure intersect. If those sets do not intersect, there is a limiting best proximity point at any of the p subsets of X for any p-cyclic iterates for initial points in between adjacent subsets.

Theorem 3.4. *Assume that X is a Banach space which has an associate complete metric space (X, d) with the metric $d : X \times X \rightarrow \mathbf{R}_{0+}$ being translation-invariant and homogeneous. Assume also that A_i; for all $i \in \overline{p}$ are nonempty, convex, and closed subsets of X with $\bigcap_{i\in\overline{p}} A_i \neq \emptyset$. Assume also that $T_\ell : \bigcup_{i\in\overline{p}} A_i \rightarrow \bigcup_{i\in\overline{p}} A_i$; for all $\ell \in \overline{n}$ are p-cyclic contractive self-mappings fulfilling $T((\bigcap_{i\in\overline{p}} A_i) \times (\bigcap_{i\in\overline{p}} A_i)) \subseteq \bigcap_{i\in\overline{p}} A_i$. Then, there is a unique fixed point $z = \sum_{i=1}^{n} z_i$ of $T : \bigcup_{i\in\overline{p}} A_i \rightarrow X$ in $\bigcap_{i\in\overline{p}} A_i$*

which satisfies $\sum_{i=1}^{n} T_i z = z$, *where* $z_\ell \in \mathrm{Fix}(T_\ell) \subseteq \bigcap_{i\in\overline{p}} A_i$; *for all* $\ell \in \overline{n}$ *are also the respective unique fixed points of* $T_\ell : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$; *for all* $\ell \in \overline{n}$.

Theorem 3.4 above and Corollary 3.5 below extend, respectively, Theorem 2.5 and Corollary 2.6 to the existence of fixed points for a mapping built from a set of n contractive, respectively, asymptotically contractive, $p(\geq 2)$-cyclic self-mappings.

Corollary 3.5. *Theorem 3.4 also holds if* $T_\ell : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$; *for all* $\ell \in \overline{n}$ *are* p*-cyclic asymptotically contractive self-mappings with respective time-varying contraction sequences being in* $[0,1]$ *with limits in* $[0,1)$.

Theorem 2.7 extends directly as follows

Theorem 3.6. *Assume that* X *is a Banach space which has an associate complete metric space* (X, d) *with the metric* $d : X\times X \to \mathbf{R}_{0+}$ *being translation-invariant and homogeneous. Assume that* A_1 *and* A_2 *are nonempty, convex, and closed subsets of* X*, which intersect and have convex union. Assume that* $T_\ell : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$; *for all* $\ell \in \overline{n}$ *are* p*-cyclic self-mappings, of which* n_1 *are continuous and* n_2 *are contractive subject to* $1 \leq n_1 = n - n_2$ *and* $n_2 \geq 1$*. Assume, furthermore, that the mapping* $T : \bigcup_{i\in\overline{p}} A_i \to X$ *satisfies*

$$T\left(\left(\bigcup_{i\in\overline{p}} A_i\right)\times\left(\bigcup_{i\in\overline{p}} A_i\right)\right) \subseteq \bigcup_{i\in\overline{p}} A_i, \qquad T\left(\left(\bigcap_{i\in\overline{p}} A_i\right)\times\left(\bigcap_{i\in\overline{p}} A_i\right)\right) \subseteq \bigcup_{i\in\overline{p}} A_i. \tag{3.4}$$

Then, there exists a (in general, nonunique) fixed point $z \in \bigcup_{i\in\overline{p}} A_i$ *of* $T : ((\bigcup_{i\in\overline{p}} A_i)\times(\bigcup_{i\in\overline{p}} A_i)) \to X$ *which has the properties below:*

$$\sum_{\ell=1}^{n} T_\ell^i z = z, \quad \forall i \in \mathbf{Z}_+, \qquad \exists \lim_{j\to\infty} T_i^j z = z - y = \sum_{\ell=1}^{n} T_\ell^i z - y, \quad \forall i \in \overline{n}_1, \tag{3.5}$$

where $y(= \sum_{\ell=n_1+1}^{n} T_\ell^i y_\ell = \sum_{\ell=n_1+1}^{n} y_\ell)$, $y_\ell \in \mathrm{Fix}(T_\ell) \equiv \{y_\ell\} \subseteq \bigcap_{i\in\overline{p}} A_i$; *for all* $\ell \in \overline{n}_2$, *for all* $i \in \mathbf{Z}_+$, *is the unique fixed point of the* p*-cyclic self-mapping* $T_\ell : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$; *for all* $\ell \in \overline{n}$.

Theorem 2.7 extends directly as follows

Theorem 3.7. *Theorem 3.6 holds if* $T_\ell : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$; *for all* $\ell \in \overline{n}_2$ *is a* p*-cyclic large contraction instead of being a contraction, for all* $\ell \in \overline{n}$.

Theorem 3.8. *Theorem 3.6 holds if the hypothesis* $T((\bigcup_{i\in\overline{p}} A_i)\times(\bigcup_{i\in\overline{p}} A_i)) \subseteq \bigcup_{i\in\overline{p}} A_i$ *is replaced by* $(x = \sum_{\ell=1}^{n_1} T_\ell^i x + \sum_{\ell=n_1+1}^{n} T_\ell^i y;\ \forall y \in \bigcup_{i\in\overline{p}} A_i) \Rightarrow x \in \bigcup_{i\in\overline{p}} A_i$.

One gets by extending Corollary 2.12 based on the ad hoc extensions of Theorem 2.20 and Corollary 2.4 for the composition of p-cyclic self-mappings.

Theorem 3.9. *Assume that* X *is a uniformly convex Banach space which has an associate complete metric space* (X, d) *with the metric* $d : X \times X \to \mathbf{R}_{0+}$ *being translation-invariant and homogeneous. Assume also that* A_i *for* $i \in \overline{p}$ *are nonempty, disjoint, convex and closed subsets of* X*. Furthermore, assume that* $T : ((\bigcup_{i\in\overline{p}} A_i)\times(\bigcup_{i\in\overline{p}} A_i)) \to \bigcup_{i\in\overline{p}} A_i$*. Then, the following properties hold.*

(i) *Assume that* $T_i : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ *are p-cyclic contractive self-mappings with time-varying real sequences* $\{k_{ij}\}_{j\in\mathbf{Z}}$*; for all* $i \in \overline{p}$ *such that the sum sequence* $\{\sum_{i=1}^{n} k_{ij}\}_{j\in\mathbf{Z}_+}$ *is in* $[0,1]$ *with* $k \le \max_{j\in\widehat{\mathbf{Z}}_+}(\sum_{i=1}^{n} k_{ij}) \in [0,1)$ *and* $k' := \min_{j\in\mathbf{Z}_+}(\sum_{i=1}^{n} k_{ij}) \ge 0$ *where* $\widehat{\mathbf{Z}}_+$ *is some infinite subset of* $\mathbf{Z}_+$*. Then, any iterates* $T^{j+i}(x,y)$ *converges as* $j \to \infty$ *to a best proximity point of* $A_{i+\ell} \equiv A_{i+\ell-p}$ *for some integer p subject to* $2p-i \ge \ell \ge p-i-1$ *for any given* $x \in A_\mu$*,* $y \in A_{\mu+1}$*; for all* $\mu \in \overline{p}$*.*

(ii) *Assume that* $T_i : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ *are p-cyclic asymptotically contractive self-mappings, and, furthermore, with the sequences* $\{\sum_{i=1}^{n} k_{ij}\}_{j\in\mathbf{Z}_+}$ *in* $[0,1]$ *and having limits* $\overline{k}_i$ *for* $i \in \overline{n}$ *satisfying* $\overline{k} := \max(\overline{k}_i : i \in \overline{n}) < 1/2$*,* $\overline{k'} := \min(\overline{k}_i : i \in \overline{n}) \ge 0$*. Then, any iterates* $T^{j+i}(x,y)$ *converges as* $j \to \infty$ *to a best proximity point of* $A_{i+\ell} \equiv A_{i+\ell-p}$ *for some integer p subject to* $2p-i \ge \ell \ge p-i-1$ *for any given* $x \in A_\mu$*,* $y \in A_{\mu+1}$*; for all* $\mu \in \overline{p}$*.*

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