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AN AXIOMATIZATION OF SUCCESS

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Abstract. In this paper we give an axiomatic characterization of three families of measures of success defined by Laruelle and Valenciano (2005) for voting rules.

Key words: collective decision-making, voting rules, axiomatization

1. Introduction

The aim of this paper is to provide an axiomatization of the measures of success in voting rules. We look for a set of axioms, that is, assumptions that, whatever their plausibility, have a clear meaning and make sense one by one, independently of the others. What we obtain in this paper are the three families of measures of success for voting rules defined by Laruelle and Valenciano (2005). These measures are associated with probability distributions p over the set of all possible vote configurations. Measure Ω^p, which is formalized in the following section, gives the probability for a voter of having the result he voted for. Measure Ω^{p+} gives the probability for a voter of having the result he voted for conditioned on voting yes. And measure Ω^{p-} gives the probability for a voter of having the result he voted for conditioned on voting no.

In this paper we give three axiomatic characterizations. One for the family of measures $\{\Omega^p\}_{p\in P}$, where P denotes the set of all the possible probability distributions. Other for the family of measures $\{\Omega^{p+}\}_{p\in P}$. And the last one for $\{\Omega^{p-}\}_{p\in P}$. The axioms we employ are some common ones together with others which are specific for each family.

In the following section we present the measures of success defined by Laruelle and Valenciano (2005), and in Section 3, 4 and 5 we give the axiomatic characterizations of the three families.

2. Background

We consider voting rules to make dichotomous choices (acceptance and rejection) by a voting body. Let $N = \{1, 2, .., n\}$ denote the set of seats. If any vote different from 'yes' is assimilated into 'no', there are 2^n possible *vote configurations*. Each vote configuration can be represented by the set $S \subseteq N$ of 'yes' voters. An N-*voting rule* is fully specified by the set W_N of *winning vote configurations*, that is, those

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which lead to the acceptance of a proposal (the others would lead to the rejection of the proposal):

$$W_N = \{S : S \text{ leads to a final 'yes'}\}.$$

When N is obvious from the context, we omit the subscript'N' and write W instead of W_N. In order to exclude unconsistent voting rules, we assume that the set W satisfies the following conditions: (i): The unanimous 'yes' leads to the acceptance of the proposal: $N \in W$; (ii): The unanimous 'no' leads to the rejection of the proposal: $\emptyset \notin W$; (iii): If a vote configuration is winning, then any other configuration containing it is also winning: If $S \in W$, then $T \in W$ for any T containing S; (iv): If one vote configuration leads to the acceptance of a proposal, the opposite configuration will not: If $S \in W$, then $N \backslash S \notin W$. Let VR_N denote the set of voting rules with set of seats N. A voting rule can also be described by its set of minimal winning configurations. A configuration S is minimal winning if $S \in W$ and for any $i \in S$, $S \setminus i \notin W$. The set of minimal winning configurations of rule W is denoted $M(W)$. A seat i is said to be a *dictator seat* if for all S we have $S \in W$ if and only if $i \in S$. The T-*unanimity* rule, denoted W^T, is the voting rule

$$W^T = \{S \subseteq N : S \supseteq T\}$$

The extreme cases are when $T = N$ (unanimity) and $T = \{i\}$ (seat i is a dictator seat). For any voting rule $W \in VR_N$ such that $W \neq U^N$, and any $T \in M(W)$, the *modified voting rule* W_T^* is the voting rule such that $W_T^* = W \setminus \{T\}$.

Let G_N denote the set of transferable utility games with player set N. That is, G_N is formed by the mappings w from 2^N into $\mathbb{R}^N$ such that $w(\emptyset) = 0$. And SG_N denote the subset of G_N formed by simple superadditive games such that the worth of N is 1. That is, by the mappings $w \in G_N$ such that $w(S) \in \{0, 1\}$ for any $S \subseteq N$, $w(N) = 1$ and $w(S \cup T) \geq w(S) + w(T)$ whenever $S \cap T = \emptyset$. Notice that superadditivity implies monotonicity, that is, $w(T) \geq w(S)$ whenever $S \subseteq T$. Then we can obviously identify VR_N with SG_N, by associating $W \in VR_N$ with the game $w \in SG_N$ that satisfies $w(S) = 1$ if and only if $S \in W$. We distinguish the game and the procedure by using the small letter in the first case and the capital letter in the second case.

Laruelle and Valenciano (2005) define some measures of success. They consider a probability distribution over the set of all possible vote configurations, which can be interpreted as a "common prior" about the voters voting behavior. Let p denote a probability distribution over the set of vote configurations, and let $p(S)$ denote, for each $S \subseteq N$, the probability of S being the vote configuration. Let P denote the set of all probability distributions. For a given p let

$$\gamma_i(p) := Prob \ \ (i \text{ votes 'yes'}) = \sum_{S: i \in S} p(S).$$

In the following we will assume that $0 < \gamma_i(p) < 1$.

A voter's probability of being successful (having the result one voted for) for a voter i is given by

$$\Omega_i^p(W) = Prob \ \ (i \text{ is successful}) = \sum_{S: i \in S \in W} p(S) + \sum_{S: i \notin S \notin W} p(S).$$

We will deal also with the following 'interim' evaluations (i.e., conditional expectations updated with the private information of each voter's own vote) for which we use the following notation:

$$\Omega_i^{p+}(W) = Prob\ (i \text{ is successful} \mid i \text{ votes 'yes'}) = \frac{\sum_{S:i\in S\in W} p(S)}{\gamma_i(p)}$$

and

$$\Omega_i^{p-}(W) := Prob\ (i \text{ is successful} \mid i \text{ votes 'no'}) = \frac{\sum_{S:i\notin S\notin W} p(S)}{1-\gamma_i(p)}.$$

We will consider Ω^p, Ω^{p+} and Ω^{p-} as mappings from VR_N into $\mathbb{R}^N$.

3. Characterization of $\{\Omega^p\}_{p\in P}$

The following axioms permit to characterize the family $\{\Omega^p\}_{p\in P}$. We represent by Φ a mapping from VR_N into $\mathbb{R}^N$ in this section and the following ones.

Transfer axiom states that the impact on a voter's index of deleting a minimal winning coalition from the list of winning ones is the same whatever the voting procedure in which the deleted coalition is minimal winning:

Transfer* (T*) : For all $V, W \in VR_N$, and all $S \in M(V) \cap M(W)$ $(S \neq N)$:

$$\Phi_i(V) - \Phi_i(V_S^*) = \Phi_i(W) - \Phi_i(W_S^*) \text{ for all } i \in N.$$

This axiom was introduced by Laruelle and Valenciano (2001) for simple superadditive games in order to characterize the Shapley-Shubik (1954) and Banzhaf (1965, 1966) indices. It was also employed by the same authors (2003) to characterize the semivalues.

Contrary Gain-Loss states that the effect of eliminating a minimal winning coalition is just the opposite for a voter inside the coalition and for a voter outside it.

Contrary Gain-Loss (ConGL): For all $W \in VR_N$, all $S \in M(W)$ $(S \neq N)$, and all $i \in S$, $j \notin S$,

$$\Phi_i(W) - \Phi_i(W_S^*) = \Phi_j(W_S^*) - \Phi_j(W).$$

The following axiom is equivalent to 'coalitional monotonicity' (Young, 1985) in the domain of simple games. It postulates something about the effects on the voters' index of a minimal modification of a voting procedure. Namely, when a minimal winning coalition is deleted from the list of winning ones that specifies it. The elimination of a minimal winning coalition diminishes the index of the voters within this coalition.

Coalitional Monotonicity* (CMon*): For all $W \in VR_N$, and all $S \in M(W)$ $(S \neq N)$:

$$\Phi_i(W) \geq \Phi_i(W_S^*) \text{ for all } i \in S.$$

The following axiom requires a dictator voter index to be equal to 1. Notice that if we consider Ω^p, the index of any voter is less or equal than 1.

Dictator Seat Axiom (DS): If i is a dictator seat in W then $\Phi_i(W) = 1$.

In the last axiom an upper bound is settled for the decrements associated with deletions of minimal winning coalitions in voting procedures.

Upper bound (UB): Let $i \in N$. For all $S \subsetneq N$, $S \neq \emptyset$, let $W^S \in VR_N$ such that $S \in M\left(W^S\right)$. Then,

$$\sum_{\substack{S \subsetneq N \\ i \in S}} \left(\Phi_i(W^S) - \Phi_i(\left(W^S\right)^*_S)\right) + \sum_{\substack{S \subsetneq N \\ S \neq \emptyset, i \notin S}} \left(\Phi_i(\left(W^S\right)^*_S) - \Phi_i(W^S)\right) \leq 1.$$

Observe that this axiom states an upper bound for the total amount of decrements associated with deletions of all likely minimal winning coalitions containing and not containing voter i.

These axioms characterize the family $\{\Omega^p\}_{p \in P}$. First we prove two lemmas. In the first VR_N is identified with SG_N.

Lemma 1. *If $\Phi : VR_N \to \mathbb{R}^N$ satisfies Transfer* then there exists a unique linear mapping $\overline{\Phi} : G_N \to \mathbb{R}^N$ such that $\overline{\Phi}(w) = \Phi(W)$ if $W \in VR_N$.*

Proof. The proof is similar to the beginning of the proof in Theorem 2.4 (Einy, 1987). □

Lemma 2. *$\Omega^p : VR_N \to \mathbb{R}^N$ satisfies T*, ConGL, CMon*, DS and UB.*

Proof. First of all notice that if $W \in VR_N$

$$\Omega^p_i(W) - \Omega^p_i(W^*_S) = \begin{cases} p(S) \text{ if } i \in S \\ -p(S) \text{ if } i \notin S. \end{cases}$$

This equality implies that Ω^p satisfies T*. It implies also ConGL since

$$\Omega^p_i(W) - \Omega^p_i(W^*_S) = p(S) = -\left(-p(S)\right) = \Omega^p_j(W^*_S) - \Omega^p_j(W)$$

when $i \in S$, $j \notin S$. CMon* is also satisfied because if $i \in S$

$$\Omega^p_i(W) - \Omega^p_i(W^*_S) = p(S) \geq 0,$$

where the inequality is true since p is a probability distribution.

On the other hand, if i is a dictator seat in W then

$$\Omega^p_i(W) = \sum_{S : i \in S} p(S) + \sum_{S : i \notin S} p(S) = 1,$$

where we have taken into account also that p is a probability distribution.

Finally, let $i \in N$ and for all $S \subsetneq N$, $S \neq \emptyset$, let $W^S \in VR_N$ such that $S \in M\left(W^S\right)$. Then

$$\sum_{\substack{S \subsetneq N \\ i \in S}} \left(\Phi_i(W^S) - \Phi_i(\left(W^S\right)^*_S)\right) + \sum_{\substack{S \subsetneq N \\ S \neq \emptyset, i \notin S}} \left(\Phi_i(\left(W^S\right)^*_S) - \Phi_i(W^S)\right) = \sum_{\substack{S \subsetneq N \\ S \neq \emptyset}} p(S),$$

and this expression is smaller or equal than 1 since p is a probability distribution. Therefore Ω^p satisfies UB. □

In the characterization theorem we employ the basis of G_N formed by the games u_S, $S \subseteq N$, $S \neq \emptyset$, defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. *A mapping $\Phi : VR_N \to \mathbb{R}^N$ satisfies T*, ConGL, CMon*, DS and UB if and only if there exists a probability distribution p on 2^N such that $\Phi = \Omega^p$.*

Proof. We have proved in Lemma 2 that Ω^p satisfies T*, ConGL, CMon*, DS and UB for all probablity distribution p.

Now let us prove the other implication. Let $\Phi : VR_N \to \mathbb{R}^N$ which satisfies the above axioms. By Lemma 1 there exists a linear mapping $\overline{\Phi} : G_N \to \mathbb{R}^N$ such that $\overline{\Phi}(w) = \Phi(W)$ when $W \in VR_N$.

Let $w \in G_N$. We have that

$$w = \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} w(S) \cdot u_S.$$

Since $\overline{\Phi}$ is linear then

$$\overline{\Phi}(w) = \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} w(S) \cdot \overline{\Phi}(u_S).$$

And taking into account Lemma 1, if $W \in VR_N$ then

$$\Phi(W) = \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} w(S) \cdot \overline{\Phi}(u_S). \tag{2}$$

Let $S \subsetneq N$, $S \neq \emptyset$. Consider $W \in VR_N$ such that $S \in M(W)$. If $i, j \in S$, ConGL and (2) imply

$$\overline{\Phi}_i(u_S) = \Phi_i(W) - \Phi_i(W_S^*) = \Phi_j(W) - \Phi_j(W_S^*) = \Phi_j(u_S).$$

So let

$$c_S = \overline{\Phi}_i(u_S), \tag{3}$$

where $i \in S$. And let us prove that if $k \notin S$ then

$$\overline{\Phi}_k(u_S) = -c_S. \tag{4}$$

Indeed, let $i \in S$. ConGL and (2) imply

$$\overline{\Phi}_k(u_S) = \Phi_k(W) - \Phi_k(W_S^*) = \Phi_i(W_S^*) - \Phi_i(W) = -c_S,$$

and (4) is obtained.

We have that $c_S \geq 0$. This inequality is implied by CMon*, (2) and (3) since given $i \in S$ and $W \in VR_N$ such that $S \in M(W)$,

$$c_S = \Phi_i(u_S) = \Phi_i(W) - \Phi_i(W_S^*) \geq 0.$$

Let us fix now $i \in N$. If we consider the voting rule W^d in which i is a dictator seat, then

$$1 = \Phi_i\left(W^d\right) = \sum_{\substack{S \subseteq N \\ i \in S}} \overline{\Phi}_i(u_S) = \sum_{\substack{S \subsetneq N \\ i \in S}} c_S + \overline{\Phi}_i(u_N), \tag{5}$$

where the three equalities are implied by DS, (2) and (3) in the same order.

Besides, UB implies

$$\sum_{\substack{S \subsetneq N \\ S \neq \emptyset}} c_S \leq 1.$$

This inequality and (5) imply

$$1 \leq 1 - \sum_{\substack{S \subsetneq N, \\ S \neq \emptyset, i \notin S}} c_S + \overline{\Phi}_i(u_N).$$

And therefore,

$$\overline{\Phi}_i(u_N) = c_0 + \sum_{\substack{S\subsetneq N,\\ S\neq\emptyset, i\notin S}} c_S, \tag{6}$$

with $c_0 \geq 0$.

Consider now $W \in VR_N$. The equalities (2), (3), (4) and (6) imply

$$\Phi_i(W) = c_0 + \sum_{\substack{S\subsetneq N\\ i\in S}} w(S)\cdot c_S + \sum_{\substack{S\subseteq N\\ S\neq\emptyset, i\notin S}} (1-w(S))\cdot c_S.$$

Let $a, b \geq 0$ such that $c_0 = a + b$ and define

$$p(S) = \begin{cases} c_S & \text{if } S \neq \emptyset, N \\ a & \text{if } S = N \\ b & \text{if } S = \emptyset. \end{cases}$$

Obviously, p is a probability distribution over the set of vote configurations. Then the above expression turns into

$$\Phi_i(W) = \sum_{S: i\in S\in W} p(S) + \sum_{S: i\notin S\notin W} p(S),$$

and the proof is complete. □

Remark 1. *UB does not depend on T*, ConGL, CMon* and DS. Indeed, let N be such that $|N| \geq 2$, $\varepsilon > 0$, and $\Psi_i^1(W) = \sum_{S: i\in S\in W} q(S) + \sum_{S: i\notin S\notin W} q(S)$ such that $q(N) = -\varepsilon, q(\emptyset) = 0, q(i) = (1+\varepsilon)/|N|$ for all $i \in N$ and $q(S) = 0$ otherwise. This mapping satisfies all these axioms except UB.*

Remark 2. *We can alternatively define an index as a mapping from VR_N into $[0,1]^N$. Then, DS would require the index for a dictator seat to be the maximum possible one, and UB would require the decrements to be bounded by this possible maximum index.*

4. Characterization of $\{\Omega^{p+}\}_{p\in P}$

Now we characterize the family $\{\Omega^{p+}\}_{p\in P}$.

We employ Transfer* (T*), Coalitional Monotonicity* (CMon*), Dictator Seat Axiom (DS) and the following axioms.

According to the first one, the elimination of a minimal winning coalition does not have any effect on the index of the voters outside this winning coalition.

No-Gain-No-Loss Out (NGNL-OUT): For all $W \in VR_N$, and all $S \in M(W)$ $(S \neq N)$:

$$\Phi_i(W) = \Phi_i(W_S^*) \text{ for all } i \in N\backslash S.$$

If the elimination of a minimal winning coalition does not affect on the index of a voter in this coalition, then the index of the other agents in the minimal winning coalition will not be affected either. Moreover, if the index of a voter is zero in the unanimity rule then the index of any voter is zero too. This is what the following axiom requires.

Symmetric Null Gain-Loss (SymNGL): Let $W \in VR_N$, $S \in M(W)$ $(S \neq N)$ and $i \in S$ such that $\Phi_i(W) - \Phi_i(W_S^*) = 0$. Then, $\Phi_j(W) - \Phi_j(W_S^*) = 0$ for all $j \in S$. Moreover, if $\Phi_i(U^N) = 0$ then $\Phi_j(U^N) = 0$ for all $j \in N$.

We also require a proportionality axiom.

Proportionality (Prop): Let $W \in VR_N$, $S_1, S_2, S_3 \in M(W)$ $(S_1, S_2, S_3 \neq N)$ and $i_1, i_2, i_3 \in N$ such that $i_1, i_3 \in S_1, i_1, i_2 \in S_2, i_2, i_3 \in S_3$. Then

$$\frac{\Phi_{i_1}(W) - \Phi_{i_1}(W_{S_1}^*)}{\Phi_{i_3}(W) - \Phi_{i_3}(W_{S_1}^*)} = \frac{\Phi_{i_1}(W) - \Phi_{i_1}(W_{S_2}^*)}{\Phi_{i_2}(W) - \Phi_{i_2}(W_{S_2}^*)} \frac{\Phi_{i_2}(W) - \Phi_{i_2}(W_{S_3}^*)}{\Phi_{i_3}(W) - \Phi_{i_3}(W_{S_3}^*)},$$

whenever $\Phi_{i_3}(W) - \Phi_{i_3}(W_{S_1}^*) \neq 0$, $\Phi_{i_2}(W) - \Phi_{i_2}(W_{S_2}^*) \neq 0$ and $\Phi_{i_3}(W) - \Phi_{i_3}(W_{S_3}^*) \neq 0$.

So the deletion of S_1 affects to i_1 and i_3 as the product of what affects the deletion of S_2 to i_1 and i_2, by what affects the deletion of S_3 to i_2 and i_3.

Notice that if $i_3 = i_2$ and $S_2 = S_3$ (or $S_1 = S_3$), then $i_1, i_2 \in S_1 \cap S_2$ and the equality in this axiom turns into

$$\frac{\Phi_{i_1}(W) - \Phi_{i_1}(W_{S_1}^*)}{\Phi_{i_2}(W) - \Phi_{i_2}(W_{S_1}^*)} = \frac{\Phi_{i_1}(W) - \Phi_{i_1}(W_{S_2}^*)}{\Phi_{i_2}(W) - \Phi_{i_2}(W_{S_2}^*)}.$$

That is, it requires the deletion of S_1 and S_2 to affect in the same proportion to i_1 and i_2.

And the last axiom requires the index to be non negative in the unanimity rule.

Non-negativity for the unanimity rule (NNU):

$$\Phi_i(U^N) \geq 0 \text{ for all } i \in N.$$

First let us prove that Ω^{p+} satisfies these axioms.

Lemma 4. *Ω^{p+} satisfies T*, CMon*, DS, NGNL-OUT, SymNGL, Prop and NNU.*

Proof. If $W \in VR_N$ then

$$\Omega_i^{p+}(W) - \Omega_i^{p+}(W_S^*) = \begin{cases} \frac{p(S)}{\gamma_i(p)} \text{ if } i \in S \\ 0 \text{ if } i \notin S. \end{cases}$$

This equality implies that Ω^{p+} satisfies T* and Sym NGL. And if $i \in S$ then

$$\Omega_i^{p+}(W) - \Omega_i^{p+}(W_S^*) = \frac{p(S)}{\gamma_i(p)} \geq 0,$$

since p is a probability distribution. Hence, CMon* is also satisfied.

If i is a dictator seat in W then

$$\Omega_i^{p+}(W) = \frac{\sum_{S:i\in S} p(S)}{\gamma_i(p)} = 1,$$

and therefore DS is satisfied.

NGNL-OUT is satisfied because for all $i \in N \backslash S$

$$\Omega_i^{p+}(W) = \frac{\sum_{T:i\in T\in W} p(T)}{\gamma_i(p)} = \Omega_i^{p+}(W_S^*),$$

Prop is satisfied because

$$\frac{\Omega_{i_1}^{p+}(W)-\Omega_{i_1}^{p+}(W_{S_1}^*)}{\Omega_{i_3}^{p+}(W)-\Omega_{i_3}^{p+}(W_{S_1}^*)}=\frac{\gamma_{i_3}(p)}{\gamma_{i_1}(p)}=\frac{\gamma_{i_2}(p)}{\gamma_{i_1}(p)}\frac{\gamma_{i_3}(p)}{\gamma_{i_2}(p)}$$

$$=\frac{\Omega_{i_1}^{p+}(W)-\Omega_{i_1}^{p+}(W_{S_2}^*)}{\Omega_{i_2}^{p+}(W)-\Omega_{i_2}^{p+}(W_{S_2}^*)}\frac{\Omega_{i_2}^{p+}(W)-\Omega_{i_2}^{p+}(W_{S_3}^*)}{\Omega_{i_3}^{p+}(W)-\Omega_{i_3}^{p+}(W_{S_3}^*)}.$$

And finally NNU is satisfied because if $i\in N$

$$\Omega_i^{p+}(U^N)=\frac{p(N)}{\gamma_i(p)}\geq 0.$$

□

Theorem 5. *A mapping $\Phi : VR_N\to\mathbb{R}^N$ satisfies T*, CMon*, DS, NGNL-OUT, SymNGL, Prop and NNU if and only if there exists a probability distribution p on 2^N such that $\Phi=\Omega^{p+}$.*

Proof. We have proved in Lemma 4 that Ω^{p+} satisfies T*, CMon*, DS, NGNL-OUT, SymNGL, Prop and NNU for all probablity distribution p.

To prove the other implication, let $\Phi : VR_N\to\mathbb{R}^N$ which satisfies the above axioms. Applying Lemma 1, there exists a linear mapping $\overline{\Phi} : G_N\to\mathbb{R}^N$ such that $\overline{\Phi}(w)=\Phi(W)$ when $W\in VR_N$. And given $w\in G_N$ we have that

$$\overline{\Phi}(w)=\sum_{\substack{S\subseteq N\\ S\neq\emptyset}} w(S)\cdot\overline{\Phi}(u_S).$$

Applying again Lemma 1,

$$\Phi(W)=\sum_{\substack{S\subseteq N\\ S\neq\emptyset}} w(S)\cdot\overline{\Phi}(u_S). \tag{8}$$

NGNL-OUT implies that $\overline{\Phi}_i(u_S)=0$ if $i\notin S$. Indeed, suppose that there exists $S\subseteq N$, $S\neq\emptyset$, such that $i\notin S$ and $\overline{\Phi}_i(u_S)\neq 0$. And let $W\in VR_N$ such that $S\in M(W)$. By expression (8) we have that

$$\Phi_i(W)-\Phi_i(W_S^*)=\overline{\Phi}_i(u_S),$$

and therefore this difference is not null. But by NGNL-OUT $\Phi_i(W)-\Phi_i(W_S^*)=0$, which is a contradiction. Hence,

$$\Phi_i(W)=\sum_{\substack{S\subseteq N\\ i\in S}} w(S)\cdot\overline{\Phi}_i(u_S). \tag{9}$$

Let us prove now that there exists $\{p(S)\in\mathbb{R} : S\subseteq N, S\neq\emptyset\}$ such that for all $i\in N$ and for all $S\ni i$

$$\overline{\Phi}_i(u_S)\cdot\sum_{T\ni i} p(T)=p(S).$$

Let us fix $i\in N$ and consider the folllowing system with unknowns $\left\{p(S)^i : S\ni i\right\}$,

$$\overline{\Phi}_i(u_S)\cdot\sum_{T\ni i} p(T)^i=p(S)^i \text{ where } S\ni i.$$

This is an homogeneous system with infinity solutions since DS and Lemma 1 imply

$$\sum_{S\ni i}\overline{\Phi}_i\left(u_S\right)=\overline{\Phi}_i\left(\sum_{S\ni i}u_S\right)=1.$$

This equality also implies that there exists $S_i\ni i$ such that $\overline{\Phi}_i\left(u_{S_i}\right)\neq 0$, and the solutions of the system are

$$p\left(S\right)^i=\frac{\overline{\Phi}_i\left(u_S\right)}{\overline{\Phi}_i\left(u_{S_i}\right)}p\left(S_i\right)^i\text{ where }p\left(S_i\right)^i\in\mathbb{R}.$$

Now let us prove that for all $i,j\in N$ we have that

$$p\left(S\right)^i=p\left(S\right)^j \tag{10}$$

for all $S\ni i,j$.

Let us fix $i,j\in N$. Given $S\ni i,j$, we have equality $p\left(S\right)^i=p\left(S\right)^j$ if

$$\frac{\overline{\Phi}_i\left(u_S\right)}{\overline{\Phi}_i\left(u_{S_i}\right)}p\left(S_i\right)^i=\frac{\overline{\Phi}_j\left(u_S\right)}{\overline{\Phi}_j\left(u_{S_j}\right)}p\left(S_j\right)^j. \tag{11}$$

If $\overline{\Phi}_i\left(u_S\right)=0$, by (9) we have that $\Phi_i\left(W\right)-\Phi_i(W_S^*)=0$ for some $W\in VR_N$ such that $S\in M\left(W\right)$. And by SymNGL we have that $\Phi_j\left(W\right)-\Phi_j(W_S^*)=0$, that is $\overline{\Phi}_j\left(u_S\right)=0$, if we take into account (9) again. Therefore, in this case we have $p\left(S\right)^i=0=p\left(S\right)^j$.

So suppose that $\overline{\Phi}_i\left(u_S\right)\neq 0$. In this case (11) can be rewritten as follows

$$p\left(S_i\right)^i=\frac{\overline{\Phi}_i\left(u_{S_i}\right)}{\overline{\Phi}_i\left(u_S\right)}\frac{\overline{\Phi}_j\left(u_S\right)}{\overline{\Phi}_j\left(u_{S_j}\right)}p\left(S_j\right)^j, \tag{12}$$

so we obtain this relation between $p\left(S_i\right)^i$ and $p\left(S_j\right)^j$.

Since (10) has to hold for any subset containing i and j, consider $S'\ni i,j$, with $S'\neq S$. Equality $p\left(S'\right)^i=p\left(S'\right)^j$ holds if

$$\frac{\overline{\Phi}_i\left(u_{S'}\right)}{\overline{\Phi}_i\left(u_{S_i}\right)}p\left(S_i\right)^i=\frac{\overline{\Phi}_j\left(u_{S'}\right)}{\overline{\Phi}_j\left(u_{S_j}\right)}p\left(S_j\right)^j,$$

which is true if $\overline{\Phi}_i\left(u_{S'}\right)=0$, and if $\overline{\Phi}_i\left(u_{S'}\right)\neq 0$ it will be true if

$$\frac{\overline{\Phi}_i\left(u_{S_i}\right)}{\overline{\Phi}_i\left(u_S\right)}\frac{\overline{\Phi}_j\left(u_S\right)}{\overline{\Phi}_j\left(u_{S_j}\right)}=\frac{\overline{\Phi}_i\left(u_{S_i}\right)}{\overline{\Phi}_i\left(u_{S'}\right)}\frac{\overline{\Phi}_j\left(u_{S'}\right)}{\overline{\Phi}_j\left(u_{S_j}\right)},$$

that is, if

$$\overline{\Phi}_j\left(u_S\right)\overline{\Phi}_i\left(u_{S'}\right)=\overline{\Phi}_j\left(u_{S'}\right)\overline{\Phi}_i\left(u_S\right),$$

and this equality holds by Prop. Just consider $W\in VR_N$ such that $S_1=S\in M(W),S_2=S_3=S'\in M(W)$, $i_1=i$, $i_2=j$ and $i_3=i_2$.

If there exists $k\in N$ such that there exists $S^*\ni j,k$ such that $\overline{\Phi}_k\left(u_{S^*}\right)\neq 0$, reasoning as above we take

$$p\left(S_k\right)^k=\frac{\overline{\Phi}_k\left(u_{S_k}\right)}{\overline{\Phi}_k\left(u_{S^*}\right)}\frac{\overline{\Phi}_j\left(u_{S^*}\right)}{\overline{\Phi}_j\left(u_{S_j}\right)}p\left(S_j\right)^j, \tag{13}$$

and we have to prove also that if there exists $\widetilde{S}\ni i,k$ such that $\overline{\Phi}_k\left(u_{\widetilde{S}}\right)\neq 0$, then $p\left(\widetilde{S}\right)^i=p\left(\widetilde{S}\right)^k$, that is

$$\frac{\overline{\Phi}_i\left(u_{\widetilde{S}}\right)}{\overline{\Phi}_i\left(u_{S_i}\right)}p\left(S_i\right)^i=\frac{\overline{\Phi}_k\left(u_{\widetilde{S}}\right)}{\overline{\Phi}_k\left(u_{S_k}\right)}p\left(S_k\right)^k.$$

Substituting expression (12) and (13) in this equality, it turns into

$$\frac{\overline{\Phi}_i\left(u_{\widetilde{S}}\right)}{\overline{\Phi}_i\left(u_{S_i}\right)}\frac{\overline{\Phi}_i\left(u_{S_i}\right)}{\overline{\Phi}_i\left(u_S\right)}\frac{\overline{\Phi}_j\left(u_S\right)}{\overline{\Phi}_j\left(u_{S_j}\right)}p\left(S_j\right)^j=\frac{\overline{\Phi}_k\left(u_{\widetilde{S}}\right)}{\overline{\Phi}_k\left(u_{S_k}\right)}\frac{\overline{\Phi}_k\left(u_{S_k}\right)}{\overline{\Phi}_k\left(u_{S^*}\right)}\frac{\overline{\Phi}_j\left(u_{S^*}\right)}{\overline{\Phi}_j\left(u_{S_j}\right)}p\left(S_j\right)^j.$$

Taking $p\left(S_j\right)^j\neq 0$ and simplifying,

$$\overline{\Phi}_i\left(u_{\widetilde{S}}\right)\overline{\Phi}_k\left(u_{S^*}\right)\overline{\Phi}_j\left(u_S\right)=\overline{\Phi}_i\left(u_S\right)\overline{\Phi}_k\left(u_{\widetilde{S}}\right)\overline{\Phi}_j\left(u_{S^*}\right),$$

which is true by Prop: consider $W\in VR_N$ such that $S_1=\widetilde{S}\in M(W), S_2=S^*\in M(W), S_3=S\in M(W)$, and $i_1=i$, $i_2=j$ and $i_3=k$.

Hence we have proved that there exists $\{p\left(S\right)\in\mathbb{R}:S\subseteq N,S\neq\emptyset\}$ such that for all $i\in N$ and for all $S\ni i$

$$\overline{\Phi}_i\left(u_S\right)\cdot\sum_{T\ni i}p\left(T\right)=p\left(S\right).$$

And moreover that there exist disjoint sets $N_1,...,N_k\subseteq N$ whose union is N such that for every $S\subseteq N_l, S\neq\emptyset$ there exists $c_S\in\mathbb{R}$ with which $p\left(S\right)=c_Sx\left(N_l\right)$. N_1 will be formed by players $i_1,j_1,k_1,...$ such that there exists $S_1'\ni i_1,j_1$ with $\overline{\Phi}_{i_1}\left(u_{S_1'}\right)\neq 0$, there exists $S_1^*\ni j_1,k_1$ such that $\overline{\Phi}_{j_1}\left(u_{S_1^*}\right)\neq 0$, and so on, being N_1 as greater as possible. N_2 will be formed by another players (if they exist) $i_2,j_2,k_2,...$ such that there exists $S_2'\ni i_2,j_2$ with $\overline{\Phi}_{i_2}\left(u_{S_2'}\right)\neq 0$, there exists $S_2^*\ni j_2,k_2$ such that $\overline{\Phi}_{j_2}\left(u_{S_2^*}\right)\neq 0$, and so on, being N_2 also as greater as possible. N_3 would be formed similarly and so on. Notice that furthermore, $x\left(N_l\right)$ can be any real number.

By CMon* and (9) we have that $\overline{\Phi}_i\left(u_S\right)\geq 0$ if $i\in S$ and $S\neq N$, and by NNU and (9) we have that $\overline{\Phi}_i\left(u_N\right)\geq 0$ if $i\in N$. Hence, $c_S\geq 0$ since in the expressions above $\overline{\Phi}_i\left(u_S\right)$ with $i\in S$ are the real numbers which can appear as coefficients of $x\left(N_l\right)$.

Finally let us define $p\left(\emptyset\right)$ by means of the equality

$$p\left(\emptyset\right)=1-\sum_{\substack{S\subseteq N\\ S\neq\emptyset}}c_Sx\left(N_l\right).$$

Since $x\left(N_l\right)$ can be any real number we take them so as $p\left(\emptyset\right)$ to be non negative. And the proof is complete. □

Remark 3. *Let N be such that $|N|\geq 2$. Mapping Ω^p satisfies T*, CMon*, DS, SymNGL, Prop and NNU, but not NGNL-OUT. Hence this axiom is not implied by the others.*

Remark 4. *SymNGL does not depend on T*, CMon*, DS, NGNL-OUT, Prop and NNU. Let N be such that $|N|\geq 2, i,j\in N, i\neq j$ and two probability distributions*

p and p^i over the set of vote configurations such that $p^i(ij) = 1$ and $p(S) = 0$ if $|S| \geq 2$. If we define

$$\Psi_k^2(W) = \begin{cases} \Omega_k^{p^k+}(W) & \text{if } k = i \\ \Omega_k^{p+}(W) & \text{otherwise,} \end{cases}$$

this mapping satisfies all the axioms but SymNGL.

Remark 5. *Prop does not depend on T*, CMon*, DS, NGNL-OUT, SymNGL and NNU. Let N be such that $|N| \geq 3$, $i_1, i_2, i_3 \in N$ such that $i_1 \neq i_2 \neq i_3$, $i_1 \neq i_3$ and $S_1, S_2, S_3 \subsetneq N$ such that $i_1, i_3 \in S_1, i_1, i_2 \in S_2, i_2, i_3 \in S_3$. For each $i \in N$ consider a probability distribution p^i over the set of vote configurations satisfying $p^i(S) \neq 0$ for all $S \subseteq N$, $p^{i_1}(S_1) = \varepsilon = p^{i_3}(S_3)$, $p^{i_1}(S_2) = 2\varepsilon = p^{i_3}(S_1)$ and $p^{i_2}(S_2) = p^{i_2}(S_3)$ for some $\varepsilon > 0$. Define*

$$\Psi_i^3(W) = \Omega_i^{p^i+}(W).$$

Ψ^3 satisfies all these axioms except Prop.

Remark 6. *NNU does not depend on the other axioms. To prove this, let N be such that $|N| \geq 2$, $\epsilon < \frac{1}{|N|-1}$ and consider mapping*

$$\Psi_i^4(W) = \frac{\sum\limits_{S: i \in S \in W} q(S)}{\gamma_i(q)},$$

where $q(N) = -\epsilon$, $q(i) = \frac{1+\epsilon}{|N|}$ and $q(S) = 0$ otherwise.

5. Characterization of $\{\Omega^{p-}\}_{p \in P}$

To obtain a characterization of the family $\{\Omega^{p-}\}_{p \in P}$ we employ Transfer* (T*), Upper Bound (UB) and the following axioms.

We consider a variation of CMon*. According to the new monotonicity axiom the elimination of a minimal winning coalition increases the index of the voters outside this coalition.

Coalitional Monotonicity$^{*\prime}$ (CMon$^{*\prime}$): For all $W \in VR_N$, and all $S \in M(W)$ $(S \neq N)$:

$$\Phi_i(W) \leq \Phi_i(W_S^*) \text{ for all } i \in N \backslash S.$$

The following axiom is also a variation of NGNL-OUT considering now inside voters instead of outside voters. Now, the elimination of a minimal winning coalition does not have any effect on the index of the voters inside this winning coalition.

No-Gain-No-Loss-In (NGNL-IN): For all $W \in VR_N$, and all $S \in M(W)$ $(S \neq N)$:

$$\Phi_i(W) = \Phi_i(W_S^*) \text{ for all } i \in S.$$

We have also a Symmetric Null Gain-Loss axiom considering outside voters instead of inside voters.

Symmetric Null Gain-Loss$'$ (SymNGL$'$): Let $W \in VR_N$, $S \in M(W)$ $(S \neq N)$ and $i \in N \backslash S$ such that $\Phi_i(W_S^*) - \Phi_i(W) = 0$. Then, $\Phi_j(W_S^*) - \Phi_j(W) = 0$ for all $j \in N \backslash S$.

There is also a proportionality axiom with outside voters.

Proportionality′ (Prop′): Let $W \in VR_N$, $S_1, S_2, S_3 \in M(W)$ ($S_1, S_2, S_3 \neq N$) and $i_1, i_2, i_3 \in N$ such that $i_1, i_3 \notin S_1, i_1, i_2 \notin S_2, i_2, i_3 \notin S_3$. Then

$$\frac{\Phi_{i_1}(W^*_{S_1}) - \Phi_{i_1}(W)}{\Phi_{i_3}(W^*_{S_1}) - \Phi_{i_3}(W)} = \frac{\Phi_{i_1}(W^*_{S_2}) - \Phi_{i_1}(W)}{\Phi_{i_2}(W^*_{S_2}) - \Phi_{i_2}(W)} \frac{\Phi_{i_2}(W^*_{S_3}) - \Phi_{i_2}(W)}{\Phi_{i_3}(W^*_{S_3}) - \Phi_{i_3}(W)},$$

whenever $\Phi_{i_3}(W^*_{S_1}) - \Phi_{i_3}(W) \neq 0$, $\Phi_{i_2}(W^*_{S_2}) - \Phi_{i_2}(W) \neq 0$ and $\Phi_{i_3}(W^*_{S_3}) - \Phi_{i_3}(W) \neq 0$.

We require any voter index to be equal to 1 in the voting procedure U^N. Notice that Ω^{p-} does not give greater value for any other procedure.

Unanimity rule axiom(URA):

$$\Phi_i(U^N) = 1 \text{ for all } i \in N.$$

The following axiom relates decrements associated with different coalitions not containing different voters.

Decrement equality (DE): For all $S \subsetneq N$, $S \neq \emptyset$, let $W^S \in VR_N$ such that $S \in M\left(W^S\right)$. Let $T \subsetneq N$, $T \neq \emptyset$ and $i, j \notin T$ such that $\Phi_i(\left(W^T\right)^*_T) - \Phi_i(W^T) \neq 0 \neq \Phi_j(\left(W^T\right)^*_T) - \Phi_j(W^T)$. Then,

$$\frac{1}{\Phi_i((W^T)^*_T) - \Phi_i(W^T)} \left(-\Phi_i(U^N) + \sum_{\substack{S \subsetneq N, \\ S \neq \emptyset, i \notin S}} \left(\Phi_i(\left(W^S\right)^*_S) - \Phi_i(W^S) \right) \right)$$

$$= \frac{1}{\Phi_j((W^T)^*_T) - \Phi_j(W^T)} \left(-\Phi_j(U^N) + \sum_{\substack{S \subsetneq N, \\ S \neq \emptyset, j \notin S}} \left(\Phi_j(\left(W^S\right)^*_S) - \Phi_j(W^S) \right) \right).$$

Observe that in this equality we can not withdraw N as a minimal winning coalition so we just write $\Phi_i(U^N)$ and $\Phi_j(U^N)$.

We prove that Ω^{p-} satisfies these axioms.

Lemma 6. *Ω^{p-} satisfies T*, CMon*′, NGNL-IN, SymNGL′, Prop′, URA, DE and UB.*

Proof. If $W \in VR_N$ then

$$\Omega_i^{p-}(W) - \Omega_i^{p-}(W^*_S) = \begin{cases} 0 \text{ if } i \in S \\ \frac{-p(S)}{1-\gamma_i(p)} \text{ if } i \notin S. \end{cases}$$

Hence, Ω^{p-} satisfies T* and Sym NGL′. And if $i \in N \backslash S$ then

$$\Omega_i^{p-}(W) - \Omega_i^{p-}(W^*_S) = \frac{-p(S)}{\gamma_i(p)} \leq 0,$$

since p is a probability distribution. Therefore, CMon*′ is also satisfied.

NGNL-IN is satisfied because for all $i \in S$

$$\Omega_i^{p-}(W) = \frac{\sum\limits_{T: i \notin T \notin W} p(T)}{1 - \gamma_i(p)} = \Omega_i^{p-}(W^*_S).$$

If $i \in N$ then

$$\Omega_i^{p+}(U^N) = \frac{\sum\limits_{S: i \notin S} p(S)}{1 - \gamma_i(p)} = 1,$$

and URA is implied. Prop$'$ is satisfied because

$$\frac{\Omega_{i_1}^{p-}(W_{S_1}^{*})-\Omega_{i_1}^{p-}(W)}{\Omega_{i_3}^{p-}(W_{S_1}^{*})-\Omega_{i_3}^{p-}(W)}=\frac{1-\gamma_{i_3}(p)}{1-\gamma_{i_1}(p)}=\frac{1-\gamma_{i_2}(p)}{1-\gamma_{i_1}(p)}\frac{1-\gamma_{i_3}(p)}{1-\gamma_{i_2}(p)}$$

$$=\frac{\Omega_{i_1}^{p-}(W_{S_2}^{*})-\Omega_{i_1}^{p-}(W)}{\Omega_{i_2}^{p-}(W_{S_2}^{*})-\Omega_{i_2}^{p-}(W)}\frac{\Omega_{i_2}^{p-}(W_{S_3}^{*})-\Omega_{i_2}^{p-}(W)}{\Omega_{i_3}^{p-}(W_{S_3}^{*})-\Omega_{i_3}^{p-}(W)}.$$

For DE, for all $S\subsetneq N$, $S\neq\emptyset$, let $W^S\in VR_N$ such that $S\in M\left(W^S\right)$, $T\subsetneq N$, $T\neq\emptyset$ and $i,j\notin T$ such that $\Phi_i(\left(W^T\right)_T^{*})-\Phi_i(W^T)\neq 0\neq\Phi_j(\left(W^T\right)_T^{*})-\Phi_j(W^T)$. Then,

$$\frac{1}{\Omega_i^{p-}((W^T)_T^{*})-\Omega_i^{p-}(W^T)}\left(-\Omega_i^{p-}(U^N)+\sum_{\substack{S\subsetneq N,\\ S\neq\emptyset,i\notin S}}\left(\Omega_i^{p-}(\left(W^S\right)_S^{*})-\Omega_i^{p-}(W^S)\right)\right)$$

$$=\frac{1-\gamma_i(p)}{p(T)}\left(-1+\sum_{\substack{S\subsetneq N,S\neq\emptyset\\ i\notin S}}\frac{p(S)}{1-\gamma_i(p)}\right)=\frac{1-\gamma_i(p)}{p(T)}\left(-1+\frac{1-\gamma_i(p)-p\left(\emptyset\right)}{1-\gamma_i(p)}\right)$$

$$=\frac{-p\left(\emptyset\right)}{p(T)}$$

$$=\frac{1}{\Omega_j^{p-}((W^T)_T^{*})-\Omega_j^{p-}(W^T)}\left(-\Omega_j^{p-}(U^N)+\sum_{\substack{S\subsetneq N,\\ S\neq\emptyset,j\notin S}}\left(\Omega_j^{p-}(\left(W^S\right)_S^{*})-\Omega_j^{p-}(W^S)\right)\right).$$

And finally UB is satisfied. Let $i\in N$. By NGNL-IN we have just to consider $S\subsetneq N$, $S\neq\emptyset$, and $W^S\in VR_N$ such that $S\in M\left(W^S\right)$ with $i\notin S$. And,

$$\sum_{\substack{S\subsetneq N,\\ S\neq\emptyset,i\notin S}}\left(\Omega_i^{p-}(\left(W^S\right)_S^{*})-\Omega_i^{p-}(W^S)\right)=\sum_{\substack{S\subsetneq N,\\ S\neq\emptyset,i\notin S}}\frac{p(S)}{1-\gamma_i(p)}$$

$$=\frac{1-\gamma_i(p)-p\left(\emptyset\right)}{1-\gamma_i(p)}\leq 1,$$

where in the inequality we take into account that p is a probability distribution. □

Theorem 7. *A mapping $\Phi:VR_N\to\mathbb{R}^N$ satisfies T*, CMon*$'$, NGNL-IN, SymNGL$'$, Prop$'$, URA, DE and UB if and only if there exists a probability distribution p on 2^N such that $\Phi=\Omega^{p-}$.*

Proof. In Lemma 6 it is proved that Ω^{p-} satisfies T*, CMon*$'$, NGNL-IN, SymNGL$'$, Prop$'$, URA, DE and UB for all probablity distribution p.

For the other implication, let $\Phi:VR_N\to\mathbb{R}^N$ which satisfies all the above axioms. Applying also Lemma 1, there exists a linear mapping $\overline{\Phi}:G_N\to\mathbb{R}^N$ such that $\overline{\Phi}\left(w\right)=\Phi\left(W\right)$ when $W\in VR_N$. And

$$\Phi\left(W\right)=\sum_{\substack{S\subseteq N\\ S\neq\emptyset}}w\left(S\right)\cdot\overline{\Phi}\left(u_S\right).$$

Let $S \neq N$. NGNL-IN implies that $\overline{\Phi}_i(u_S) = 0$ if $i \in S$. If not, there exists $S \subseteq N$ with $i \in S$ such that $\overline{\Phi}_i(u_S) \neq 0$. If we consider $W \in VR_N$ such that $S \in M(W)$, by the above expression we have that

$$\Phi_i(W) - \Phi_i(W_S^*) = \overline{\Phi}_i(u_S) \neq 0.$$

But by NGNL-IN $\Phi_i(W) - \Phi_i(W_S^*) = 0$, which is a contradiction. Hence, if $i \in N$,

$$\Phi_i(W) = \overline{\Phi}_i(u_N) + \sum_{\substack{S \subseteq N, \\ S \neq \emptyset, i \notin S}} w(S) \cdot \overline{\Phi}_i(u_S).$$

And applying URA,

$$\Phi_i(W) = 1 + \sum_{\substack{S \subseteq N, \\ S \neq \emptyset, i \notin S}} w(S) \cdot \overline{\Phi}_i(u_S).$$

Let us prove now that there exists $\{p(S) \in \mathbb{R} : S \subsetneq N\}$ such that for all $i \in N$ and for all $S \neq \emptyset$ such that $i \notin S$

$$\overline{\Phi}_i(u_S) \cdot \sum_{\substack{T \subseteq N \\ i \notin T}} p(T) = -p(S).$$

Let us fix $i \in N$ and consider the folllowing system with unknowns $\left\{p(S)^i : i \notin S\right\}$,

$$\overline{\Phi}_i(u_S) \cdot \sum_{\substack{T \subseteq N \\ i \notin T}} p(T)^i = -p(S)^i \text{ where } i \notin S.$$

This is an homogeneous system and therefore it has a solution.

We will see that this solution can be chosen as to be non null. If $\overline{\Phi}_i(u_S) = 0$ for all $S \neq \emptyset$ such that $i \notin S$, then $p(S)^i = 0$ for all $S \neq \emptyset$ such that $i \notin S$ and $p(\emptyset)^i$ can be any real number.

If there exists $S_i \neq \emptyset$ such that $i \notin S_i$ such that $\overline{\Phi}_i(u_{S_i}) \neq 0$, then the solutions $p(S)^i$ with $S \neq \emptyset$ of the system are

$$p(S)^i = \frac{\overline{\Phi}_i(u_S)}{\overline{\Phi}_i(u_{S_i})} p(S_i)^i \text{ where } p(S_i)^i \in \mathbb{R}.$$

Notice that this expression coincides with the one obtained in the proof of Theorem 5 for the solutions. Reasoning as in that proof, Prop′ guarantees different systems to give the same solutions.

Moreover, in this case $p(\emptyset)^i$ is a solution of the system and

$$p(\emptyset)^i = \frac{-p(S_i)^i}{\overline{\Phi}_i(u_{S_i})} \left(\left(\sum_{\substack{T \subseteq N, \\ T \neq \emptyset, i \notin T}} \overline{\Phi}_i(u_T) \right) + 1 \right). \tag{14}$$

We need $p(\emptyset)^i = p(\emptyset)^j$ if $i \neq j$. This equality holds taking into account DE and expression (12) if there exists $S \subseteq N, S \neq \emptyset$, such that $i, j \notin S$ and $\overline{\Phi}_i(u_S) \neq 0$. If there is not such a subset S, then $p(\emptyset)^i = p(\emptyset)^j$ will give $p(S_i)^i$ as a function of $p(S_j)^j$.

Similarly as in the proof of Theorem 5 we can take $p(S)$ non negative when $S \subsetneq N, S \neq \emptyset$ (observe that in this case the expressions $\overline{\Phi}_i(u_S)$ are non positive).

NGNL-IN, UB and (14) imply $p(\emptyset) \geq 0$ and finally $p(N)$ can be defined by means of the equality

$$p(N) = 1 - \sum_{S \subsetneq N} p(S),$$

in such a way that $p(N) \geq 0$. □

Remark 7. *Let N be such that $|N| \geq 2$. Mapping $\Psi_i^5(W) = \Omega_i^p(W) - \Omega_i^p(W^N) + 1$ satisfies T*, CMon*′, SymNGL′, Prop′, URA, DE and UB, but not NGNL-IN. Therefore this axiom is not implied by the others.*

Remark 8. *SymNGL′ does not depend on T*, CMon*′, NGNL-IN, Prop′, URA, DE and UB. Let N be such that $|N| \geq 3$, $i, j \in N$, $i \neq j$ and thwo probability distributions p and p^i over the set of vote configurations such that $p^i(k) = 1$ for some $k \in N \setminus \{i, j\}$, $p(S) = 0$ for all $S \subseteq N$, $S \neq \emptyset$, and $p(\emptyset) \neq 0$. If we define*

$$\Psi_k^6(W) = \begin{cases} \Omega_k^{p^k-}(W) & \text{if } k = i \\ \Omega_k^{p-}(W) & \text{otherwise,} \end{cases}$$

this mapping satisfies all the axioms but SymNGL′.

Remark 9. *Prop′ is not implied by T*, CMon*′, NGNL-IN, SymNGL′, URA, DE and UB. Let N be such that $|N| \geq 3$, $i_1, i_2, i_3 \in N$ and $S_1, S_2, S_3 \subseteq N$ such that $i_1, i_3 \notin S_1, i_1, i_2 \notin S_2, i_2, i_3 \notin S_3$. For each $i \in N$ consider a probability distribution p^i over the set of vote configurations satisfying $p^i(S) \neq 0$ for all $S \subseteq N$, $S \neq \emptyset$, $p^i(\emptyset) = 0$, $p^{i_1}(S_1) = \varepsilon = p^{i_3}(S_3)$, $p^{i_1}(S_2) = 2\varepsilon = p^{i_3}(S_1)$ and $p^{i_2}(S_2) = p^{i_2}(S_3)$ for some $\varepsilon > 0$. Define*

$$\Psi_k^7(W) = \Omega_i^{p^i-}(W).$$

Ψ^7 satisfies all the axioms except Prop′.

Remark 10. *DE does not depend on the other axioms either. Let N be such that $|N| \geq 3$ and let us fix $i \in N$. Consider two probability distributions p and p^i over the set of vote configurations such that $p(\emptyset) \neq 0$, $p^i(\emptyset) = 0$, $p^i(S) = p(S) \neq 0$ if $S \subsetneq N$, $S \neq \emptyset$. Mapping*

$$\Psi_k^8(W) = \begin{cases} \Omega_i^{p^i-}(W) & \text{if } k = i \\ \Omega_k^{p-}(W) & \text{otherwise,} \end{cases}$$

satisfies all the axioms but DE.

Remark 11. *UB is not implied by the other axioms. Let N be such that $|N| \geq 2$, and $q(S)$, $S \subseteq N$, such that $q(\emptyset) = -\epsilon$ and $q(S) > \epsilon$ for all $S \subseteq N$ for some $\epsilon \geq 0$. Then,*

$$\Psi_i^9(W) = \frac{\sum_{S: i \notin S \notin W} q(S)}{\sum_{S: i \notin S} q(S)}$$

satisfies all the axioms but UB.

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