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**ON MONOTONIC CORE ALLOCATIONS
FOR COALITIONAL GAMES WITH VETO
PLAYERS**

by

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On monotonic core allocations for coalitional games with veto players*

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Abstract

We characterize a monotonic core concept defined on the class of veto balanced games. We also discuss what restricted versions of monotonicity are possible when selecting core allocations. We introduce a family of monotonic core concepts for veto balanced games and we show that, in general, the nucleolus per capita is not monotonic.

Keywords: Monotonicity, Core, TU games, nucleolus per capita

1 Introduction

Young (1985) formulates an impossibility result for the problem of finding core concepts satisfying monotonicity¹ on the domain of balanced TU games. A positive result holds on the domain of convex games where the Shapley value (Shapley, 1953) is a core concept satisfying several monotonicity requirements. On the domain of veto balanced games, Arin and Feltkamp (2005) introduce a monotonic core concept that does not entirely accomplish the goals we would expect from a *fair* solution concept. The solution is very extreme and divides the worth of the grand coalition equally among the veto players (we call this the *all for veto players solution*). We also argue that the per capita nucleolus is not monotonic on the class of veto balanced games even if it does satisfy *strong N-monotonicity*. Therefore, the existence of monotonic core concepts in the class of veto balanced games is still an open question.

This paper deals with this problem. We discuss the possibilities of combining monotonicity and core selection. We introduce restricted monotonicity

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¹See Section 2 and 3 for formal definitions of core concept and monotonicity.

requirements and given these restricted versions of monotonicity properties, we characterizes a core concept on the class of veto balanced games.

The core concept characterized proves to be a member of a family of core concepts that also includes the *all for veto players solution*.

2 Preliminaries

2.1 TU Games

A *cooperative n-person game in characteristic function form* is a pair (N, v) , where N is a finite set of n elements and $v : 2^N \rightarrow \mathbb{R}$ is a real-valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of N is called a *coalition*. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \setminus S$ the set of players of N that are not in S . We only consider games where the worth of all coalitions is non negative. A distribution of $v(N)$ among the players is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i . A distribution satisfying $x_i \geq v(i)$ for all $i \in N$ is called an *imputation* and the set of imputations is denoted by $I(v)$. We denote $\sum_{i \in S} x_i$ by $x(S)$. The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$C(v) = \{x \in I(v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

A game with a non-empty core is called a balanced game. *Player i is a veto player* if $v(S) = 0$ for all S where player i is not present. A balanced game with at least one veto player is called a veto balanced game. We denote by Γ_B the class of balanced games and by Γ_{BV} the class of veto balanced games.

A solution ϕ on a class of games Γ_0 is a correspondence that associates with every game (N, v) in Γ_0 a set $\phi(N, v)$ in \mathbb{R}^N such that $x(N) \leq v(N)$ for all $x \in \phi(N, v)$. This solution is *efficient* if this inequality holds with equality. The solution is *single-valued* if the set contains a unique element for every game in the class.

Given $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game v is defined as $e(S, x) := v(S) - x(S)$. Let $\theta(x)$ be the vector of all excesses at x arranged in non-increasing order. The lexicographic order \prec_L between two vectors x and y is defined by $x \prec_L y$ if there exists an index k such that $x_l = y_l$ for all $l < k$ and $x_k < y_k$ and the weak lexicographic order \preceq_L by $x \preceq_L y$ if $x \prec_L y$ or $x = y$.

Schmeidler (1969) introduced the *nucleolus* of a game v , denoted by $\eta(v)$, as the unique imputation that lexicographically minimizes the vector of non increasingly ordered excesses over the set of imputations. In formula:

$$\{\eta(N, v)\} = \{x \in I(N, v) | \theta(x) \preceq_L \theta(y) \text{ for all } y \in I(N, v)\}.$$

For any game v with a non-empty imputation set, the nucleolus is a single-valued solution, is contained in the kernel and lies in the core provided that the core is non-empty.

The per capita nucleolus is defined analogously by using the concept of per capita excess instead of excess. Given S and x the per capita excess of S at x is

$$e^{pc}(S, x) := \frac{v(S) - x(S)}{|S|}$$

In this paper, we study solution concepts that select precisely one core allocation for each balanced game. We call such concepts *core concepts*.

Let ϕ be a single-valued solution concept defined on a class of games Γ_0 . We say that ϕ satisfies **equal treatment property** (ETP) if for each $(N, v) \in \Gamma_0$ interchangeable players i, j are treated equally, i.e., $\phi_i(N, v) = \phi_j(N, v)$. Players i and j are interchangeable if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

2.2 On monotonicity properties

We present some monotonicity properties for single-values solution concepts.

Let ϕ be a single-valued solution on a class of games Γ_0 . We say that solution ϕ satisfies:

Monotonicity: if for all $v, w \in \Gamma_0$, such that for all T containing player i , $v(T) \leq w(T)$, and for all $S \subseteq N \setminus \{i\}$ $v(S) = w(S)$, then, $\phi_i(w) \geq \phi_i(v)$.

coalitional monotonicity: if for all $v, w \in \Gamma_0$, if for all $S \neq T$, $v(S) = w(S)$ and $v(T) < w(T)$, then for all $i \in T$, $\phi_i(v) \leq \phi_i(w)$.

Strong coalitional monotonicity: if for all $v, w \in \Gamma_0$, if for all $S \neq T$, $v(S) = w(S)$ and $v(T) < w(T)$, then for all $i, j \in T$, $\phi_i(w) - \phi_i(v) = \phi_j(w) - \phi_j(v) \geq 0$.

N -monotonicity (Meggido, 1974): if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i \in N$, $\phi_i(v) \leq \phi_i(w)$.

strong N -monotonicity: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\phi_i(w) - \phi_i(v) = \phi_j(w) - \phi_j(v) \geq 0$.

The strong versions of the properties capture the idea of equal treatment of players equally affected (by the monotonic change).

Monotonicity and *coalitional monotonicity* are equivalent whenever each coalitional change generates games belonging to the domain in which the solution has been defined². *Coalitional monotonicity* implies N -monotonicity. It

²Let ϕ be a single-valued solution defined on the class of convex games satisfying monotonicity and coalitional monotonicity. Let $N = \{1, 2, 3\}$ a set of players and consider the following 3-person convex games:

$$v(S) = \begin{cases} 2 & \text{if } S = \{3\} \\ 8 & \text{if } S \in \{\{1, 3\}, \{2, 3\}\} \\ 6 & \text{if } S = \{1, 23\} \\ 14 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \quad \text{and } w(S) = \begin{cases} 6 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

is also immediately apparent that strong *coalitional monotonicity* implies strong *N-monotonicity*

In the class of veto balanced games there exist core concepts satisfying monotonicity. We call the following core concept the *all for veto players solution*.

Let (N, v) be a veto balanced game with T as the set of veto players and let σ be the solution for veto balanced games defined as follows:

$$\sigma_i = \begin{cases} \frac{v(N)}{|T|} & \text{for all } i \in T \\ 0 & \text{for all } i \in N \setminus T. \end{cases}$$

This solution does not satisfy the strong versions of the monotonicity properties.

The next section discusses the impossibility of finding a core concept (on the class of veto balanced games) that satisfies the strong monotonicity requirements. But still is possible to find core concepts *more monotonic* than the *all for veto players solution*.

3 On monotonic core selections

3.1 Examples and Properties

We use three examples to illustrate what kind of restricted monotonicity properties are allowed for core concepts.

Example 1 Let $N = \{1, 2, 3\}$ a set of players and consider the following 3-person veto balanced games:

$$v(S) = \begin{cases} 6 & \text{if } S \in \{\{1, 2\}, \{1, 3\}\} \\ 6 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \quad \text{and } w(S) = \begin{cases} 6 & \text{if } S = \{1, 2\} \\ 6 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

A core concept satisfying ETP should choose $(6, 0, 0)$ and $(3, 3, 0)$ in games (N, v) and (N, w) .

Player 3 (unlike player 1) does not receive any benefit from the fact that he is a member of the only coalition changing its worth by increasing it. Therefore, core concepts do not satisfy strong *coalitional monotonicity*. However there are core concepts that satisfy *strong N-monotonicity*.

Example 2 Let $N = \{1, 2, 3, 4\}$ a set of players and consider the following

From game (N, w) to game (N, v) we can apply monotonicity of ϕ . But from game (N, w) to game (N, v) we cannot apply coalitional monotonicity since not all the resulting games are convex (for any order in the coalitions that have changed their worth).

4-person veto balanced games:

$$v(S) = \begin{cases} 6 & \text{if } 1 \in S \text{ and } |S| = 3 \\ 6 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \quad \text{and } w(S) = \begin{cases} 6 & \text{if } 1 \in S \text{ and } |S| = 3 \\ & \text{or } S = \{1, 2\} \\ 6 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

A core selection should choose $(6, 0, 0, 0)$ in the games (N, v) and (N, w) .

Players 1 and 2 do not receive any benefit from the fact that they are the members of the only coalition changing its worth by increasing it.

Therefore, core selections impose some restrictions on the monotonicity properties we can ask for. The fact that, given a game, only one coalition increases its worth does not imply necessarily that all members of the coalition should benefit from this increase.

Example 3 Let $N = \{1, 2, 3, 4\}$ a set of players and consider the following 4-person veto balanced games:

$$v(S) = \begin{cases} 6 & \text{if } 1 \in S \text{ and } |S| = 3 \\ 10 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases}, \quad w(S) = \begin{cases} 6 & \text{if } S = \{1, 2\} \\ v(S) & \text{otherwise.} \end{cases}$$

$$\text{and } q(S) = \begin{cases} 6 & \text{if } S = N \\ w(S) & \text{otherwise.} \end{cases}$$

Assume ϕ is a core concept on Γ_{BV} that satisfies *strong N-monotonicity*. Then $\phi(N, v) = (7, 1, 1, 1)$. It is also necessary that $\phi(N, w) = (7, 1, 1, 1)$. Otherwise *strong N-monotonicity* will be violated since $\phi(N, q) = (6, 0, 0, 0)$

Even if core restriction allow for a strictly positive increasing of the payoffs of players 1 and 2 while *moving* from game (N, v) to game (N, w) such a change is not possible if *strong N-monotonicity* is required.

The aim of a monotonic core concept, given these restrictions, should be to be *as strongly monotonic as possible*. This notion is formalized below. The first problem is to detect whether or not increasing the worth of a coalition is strictly profitable for some (maybe all) members of the coalition or no. The answer is suggested by Examples 1, 2 and 3.

In the following, we set a bound for each player. The bound limits which coalitional changes are beneficial for the player and which are not. Let (N, v) be a game with veto players and let player 1 be a veto player. Define for each player i a value d_i as follows:

$$d_i = \max_{S \subseteq N \setminus \{i\}} v(S).$$

Example 2 suggest that if the worth of a coalition changes, and the d -values do not change then the members of the coalition would not expect to benefit from this change.

Examples 1 and 3 suggest that if the worth of a coalition changes, the members of the coalition with d -values higher than the worth of that coalition would not expect to make any profit from the change.

The following properties are an attempt at combining monotonicity and core requirements.

Let ϕ be a solution defined on Γ_0

- **Property I (Restricted strong coalitional monotonicity):** Let $v, w \in \Gamma_0$, such that $v(T) < w(T)$, and for all $S \neq T$ $v(S) = w(S)$ and let $P = \{i \in T; d_i(N, v) = d_i(N, w) \leq v(T)\}$. Then ϕ satisfies Property I if for all $i, j \in P$, $\phi_i(w) - \phi_i(v) = \phi_j(w) - \phi_j(v) \geq 0$.
- **Property II (Restricted strong monotonicity):** Let $v, w \in \Gamma_0$ and let \mathbf{Q} be a set of coalitions of N , such that $v(T) < w(T)$, for all $T \in \mathbf{Q}$ and $v(S) = w(S)$ for all $S \notin \mathbf{Q}$. Let $T_1 = \arg \min_{T \in \mathbf{Q}} v(T)$ and let $P = \left\{ i \in \bigcap_{T \in \mathbf{Q}} T; d_i(N, v) = d_i(N, w) \leq v(T_1) \right\}$. Then ϕ satisfies Property II if for all $i, j \in P$, $\phi_i(w) - \phi_i(v) = \phi_j(w) - \phi_j(v) \geq 0$.

Property I and Property II are equivalent on the class of veto balanced games. Note that Property I (with efficiency) implies *strong N -monotonicity*.

We will use the term **monotonic core concept** to refer to any core concept satisfying these two properties. A **monotonic core allocation** is any allocation selected by a monotonic core concept.

3.2 A procedure for selecting a monotonic core allocation.

The way in which a monotonic core allocation is selected given any veto balanced game is described below. The procedure has $n-1$ steps and is defined recursively. Each step has 2 substeps.

Let $(N, v) \in \Gamma_{BV}$ and let $(d_1, d_2, \dots, d_n, d_{n+1})$ be its vector of d -values where $d_{n+1} = v(N)$ and assume that players are renamed according to the nondecreasing order of these values. That is player 1 is a veto player (and therefore $d_1 = 0$), player 2 is the player with second lowest d -value and so on. Proceed as follows:

- Step 1

a.- Let (N, v^1) be defined as follows:

$$v^{1a}(S) = \min(v(S), d_2) \text{ for all } S \subseteq N.$$

The only core allocation of this game is

$$(d_2, 0, \dots, 0).$$

b.- Let (N, v^{1b}) be defined as follows:

$$v^{1b}(S) = \begin{cases} d_3 & \text{if } S = N \\ v^{1a}(S) & \text{otherwise} \end{cases}$$

By applying Property I to the pair $\{(N, v^{1a}), (N, v^{1b})\}$ a monotonic core concept must select

$$(d_2 + \frac{d_3 - d_2}{n}, \frac{d_3 - d_2}{n}, \dots, \frac{d_3 - d_2}{n}).$$

• Step 2

a.- Let (N, v^{2a}) be defined as follows:

$$v^{2a}(S) = \min(v(S), d_3) \text{ for all } S \subseteq N.$$

By applying Property II to the pair $\{(N, v^{1b}), (N, v^{2a})\}$ a monotonic core concept must select

$$(d_2 + \frac{d_3 - d_2}{2}, \frac{d_3 - d_2}{2}, 0, \dots, 0).$$

b.- Let (N, v^{2b}) be defined as follows:

$$v^{2b}(S) = \begin{cases} d_4 & \text{if } S = N \\ v^{2a}(S) & \text{otherwise} \end{cases}$$

By applying Property I to the pair $\{(N, v^{2a}), (N, v^{2b})\}$ a monotonic core concept must select the allocation x^2 where

$$x_i^2 = \begin{cases} \sum_{i=l}^4 \frac{d_{i+1} - d_i}{i} & \text{for all } i \in \{1, 2\} \\ \frac{d_4 - d_3}{n} & \text{for all } i \in N \setminus \{1, 2\} \end{cases}$$

• Step k ($k < n$)

a.- Let (N, v^{ka}) be defined as follows:

$$v^{ka}(S) = \min(v(S), d_{k+2}) \text{ for all } S \subseteq N.$$

By applying Property II to the pair $\{(N, v^{k-1,b}), (N, v^{ka})\}$ a monotonic core selector must select y^k where

$$y_i^k = \begin{cases} \sum_{i=l}^{k+1} \frac{d_{i+1} - d_i}{i} & \text{for all } i \in \{1, 2, \dots, k\} \\ 0 & \text{for all } i \in N \setminus \{1, 2, \dots, k\} \end{cases}$$

b.- Let (N, v^4) be defined as follows:

$$v^{kb}(S) = \begin{cases} d_{k+2} & \text{if } S = N \\ v^{ka}(S) & \text{otherwise} \end{cases}$$

By applying Property I to the pair $\{(N, v^{ka}), (N, v^{kb})\}$ a monotonic core selector must select x^k where

$$x_i^k = \begin{cases} \sum_{i=l}^{k+1} \frac{d_{i+1} - d_i}{i} & \text{for all } i \in \{1, 2, \dots, k+1\} \\ \frac{d_{k+2} - d_{k+1}}{n} & \text{for all } i \in N \setminus \{1, 2, \dots, k+1\} \end{cases}$$

- The procedure ends after $n - 1$ steps. In substep b of step n the game is the one with which the procedure started. In this last step the final outcome is x^n where

$$x_l^n = \sum_{i=l}^n \frac{d_{i+1} - d_i}{i} \text{ for all } l \in \{1, \dots, n\}.$$

Note that in the described procedure all games are veto balanced games.

We illustrate the procedure by means of the following example.

Example 4 Let $N = \{1, 2, 3, 4\}$ a set of players and consider the following 4-person veto balanced game (N, v) where $v(N) = 12$, $v(\{1, 2, 3\}) = 8$, $v(\{1, 2, 4\}) = 5$, $v(\{1, 3, 4\}) = 3$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 3$, $v(\{1, 4\}) = 2$, and $v(S) = 0$ otherwise.

In this game

$$(d_1, d_2, d_3, d_4, d_5) = (0, 3, 5, 8, 12).$$

Applying the procedure we get;

- Step 1

a.- The game (N, v^{1a}) results:

$$v^{1a}(N) = 3, v^{1a}(\{1, 2, 3\}) = 3, v^{1a}(\{1, 2, 4\}) = 3, v^{1a}(\{1, 3, 4\}) = 3, v^{1a}(\{1, 2\}) = 3, v^{1a}(\{1, 3\}) = 3, v^{1a}(\{1, 4\}) = 2, \text{ and } v^{1a}(S) = 0 \text{ otherwise.}$$

The only core allocation is

$$(3, 0, 0, 0).$$

b.- The game (N, v^{1b}) results:

$$v^{1b}(S) = \begin{cases} 5 & \text{if } S = N \\ v^{1a}(S) & \text{otherwise.} \end{cases}$$

By applying Property I to the pair $\{(N, v^{1a}), (N, v^{1b})\}$ a monotonic core concept must select

$$(3 + \frac{2}{4}, \frac{2}{4}, \frac{2}{4}, \frac{2}{4}).$$

• Step 2

a.- The game (N, v^{2a}) results:

$v^{2a}(N) = 5$, $v^{2a}(\{1, 2, 3\}) = 5$, $v^{2a}(\{1, 2, 4\}) = 5$, $v^{2a}(\{1, 3, 4\}) = 3$, $v^{2a}(\{1, 2\}) = 4$, $v^{2a}(\{1, 3\}) = 3$, $v^{2a}(\{1, 4\}) = 2$, and $v^{2a}(S) = 0$ otherwise.

By applying Property II players 1 and 2 should increase their payoffs by the same amount. By core restrictions player 3 and 4 should receive 0. Therefore, for this game the monotonic core allocation results:

$$(4, 1, 0, 0).$$

b.- The game (N, v^{2b}) results:

$$v^{1b}(S) = \begin{cases} 8 & \text{if } S = N \\ v^{1a}(S) & \text{otherwise.} \end{cases}$$

By applying Property I to the pair $\{(N, v^{2a}), (N, v^{2b})\}$ a monotonic core concept must select

$$(4 + \frac{3}{4}, 1 + \frac{3}{4}, \frac{3}{4}, \frac{3}{4}).$$

• Step 3

a.- The game (N, v^{3a}) results:

$v^{3a}(N) = 8$, $v^{3a}(\{1, 2, 3\}) = 8$, $v^{3a}(\{1, 2, 4\}) = 5$, $v^{3a}(\{1, 3, 4\}) = 3$, $v^{3a}(\{1, 2\}) = 4$, $v^{3a}(\{1, 3\}) = 3$, $v^{3a}(\{1, 4\}) = 2$, and $v^{3a}(S) = 0$ otherwise.

By applying Property II players 1,2 and 3 should increase their payoffs by the same amount. By core restrictions player 4 should receive 0. Therefore, for this game the monotonic core allocation results:

$$(4 + \frac{3}{4} + \frac{3}{12}, 1 + \frac{3}{4} + \frac{3}{12}, \frac{3}{4} + \frac{3}{12}, 0) = (5, 2, 1, 0).$$

b.- The game (N, v^{3b}) proves to be the initial game since

$$v^{1b}(S) = \begin{cases} 12 & \text{if } S = N \\ v^{3a}(S) & \text{otherwise.} \end{cases}$$

By applying Property I to the pair $\{(N, v^{3a}), (N, v^{3b})\}$ a monotonic core concept must select

$$(5 + \frac{4}{4}, 2 + \frac{4}{4}, 1 + \frac{4}{4}, \frac{4}{4}) = (6, 3, 2, 1).$$

The procedure can also be used as the proof of the following Theorem.

Theorem 5 Let ϕ be a monotonic core concept on Γ_{BV} and let $(N, v) \in \Gamma_{BV}$. Then

$$\phi_l = \sum_{i=l}^n \frac{d_{i+1} - d_i}{i} \text{ for all } l \in N.$$

The following example illustrates how this monotonic core concept behaves.

Example 6 Let $N = \{1, 2, 3, 4\}$ a set of players and consider the following 4-person veto balanced game (N, v) where

$$v(S) = \begin{cases} 8 & \text{if } S \in \{\{1, 2, 3\}, \{1, 2, 4\}\} \\ 6 & \text{if } S = \{1, 3, 4\} \\ 12 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Computing the vector of d -values we get:

$$(d_1, d_2, d_3, d_4, d_5) = (0, 6, 8, 8, 12).$$

Applying the formula,

$$\begin{aligned} \phi_1 &= \frac{d_2 - d_1}{1} + \frac{d_3 - d_2}{2} + \frac{d_4 - d_3}{3} + \frac{d_5 - d_4}{4} = 8 \\ \phi_2 &= \frac{d_3 - d_2}{2} + \frac{d_4 - d_3}{3} + \frac{d_5 - d_4}{4} = 2 \\ \phi_3 &= \frac{d_4 - d_3}{3} + \frac{d_5 - d_4}{4} = 1 \\ \phi_4 &= \frac{d_5 - d_4}{4} = 1. \end{aligned}$$

The formula suggests that the solution follows a serial rule principle. Each player i has a right, a veto power, over the amount $v(N) - d_i$ and the amount is divided equally among the players with that right, veto power, over it.

This serial rule satisfies monotonicity.

3.3 Per capita nucleolus and strong N -monotonic core concepts

The per capita nucleolus is a core concept that in the class of all balanced games satisfies *strong N -monotonicity*. This is why some authors consider this core concept a good candidate when trying to select monotonic core allocations. The following example shows that the per capita nucleolus violates the monotonicity principle and therefore, even if it does satisfy *strong N -monotonicity*, it can hardly be seen as a **monotonic** core concept (at least in the class of all balanced games and in the subclass of veto balanced games).

Example 7 Let $N = \{1, 2, \dots, 6\}$ a set of players and consider the following 6-person veto balanced games:

$$v(S) = \begin{cases} 6 & \text{if } 1 \in S \text{ and } |S| = 5 \\ 6 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases} \quad \text{and } w(S) = \begin{cases} 8 & \text{if } S \in \left\{ \begin{array}{l} N \setminus \{6\}, \\ N \setminus \{5\}, N \end{array} \right\} \\ v(S) & \text{otherwise.} \end{cases}$$

A core concept must choose the allocation $(6, 0, 0, 0, 0, 0)$ in the first game. The per capita nucleolus of the second game is $(5.75, 0.75, 0.75, 0.75, 0, 0)$. Therefore, the per capita nucleolus is not monotonic since player 1 receives a lower payoff in the second game.

In the class of veto balanced games there are core concepts (other than the serial rule introduced in the previous subsection) that satisfy monotonicity and *strong N-monotonicity* such as the following modified version of the *all for veto players solution*.

Let (N, v) be a veto balanced game with T as the set of veto players and let σ^n be a solution for veto balanced games defined as follows:

$$\sigma_i^n = \begin{cases} \frac{d_n}{|T|} + \frac{d_{n+1}-d_n}{|N|} & \text{for all } i \in T \\ \frac{d_{n+1}-d_n}{|N|} & \text{for all } i \in N \setminus T. \end{cases}$$

Similarly, *more monotonic* core concepts can be defined all of them violating Property I. Let (N, v) be a veto balanced game with T as the set of veto players and let σ^k be a solution for veto balanced games defined as follows:

$$\sigma_l^k = \begin{cases} \frac{d_k}{|T|} + \sum_{i=k}^n \frac{d_{i+1}-d_i}{i} & \text{for all } l \in T \\ \sum_{i=\max\{l,k\}}^n \frac{d_{i+1}-d_i}{i} & \text{for all } l \in N \setminus T. \end{cases}$$

The *all for veto players solution*, σ^{n+1} , is the least monotonic solution of the family and the serial rule, σ^2 , is the most monotonic solution of the family (the only one satisfying Property I).

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