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Observability and control of parabolic equations on networks with loops

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Abstract. Network theory can be useful for studying complex systems such as those that arise, for example, in physical sciences, engineering, economics and sociology. In this paper, we prove the observability of parabolic equations on networks with loops. By using a novel Carleman inequality, we find that the observability of the entire network can be achieved under certain hypothesis about the position of the observation domain. The main difficulty we tackle, due to the existence of loops, is to avoid entering into a circular fallacy, notably in the construction of the auxiliary function for the Carleman inequality. The difficulty is overcome with a careful treatment of the boundary terms on the junctions. Finally, we use the observability to prove the null controllability of the network and to obtain the Lipschitz stability for an inverse problem consisting on retrieving a stationary potential in the parabolic equation from measurements on the observation domain.

1. Introduction

1.1. Presentation of the problem and state of the art

Network theory can be useful for studying complex systems such as those that arise, for example, in physical sciences, engineering, economics and sociology. These systems can be modeled as networks, also known as metric graphs, and their elements and interactions or links are identified respectively by vertices and edges. During the last decades, the use of networks has been helpful and effective, among others, in the study of pipes, neural systems (the brain can be thought of as a network of neurons), the flow of traffic on roads, the global economy and the human circulatory system (see, for example, [9, Chapter 9], [11,22,43,55]).

In this work, we consider the propagation of heat on a network with loops. We seek to control these networks by acting in its interior with a source term, and to estimate the solutions with an observation domain located in the interior of the network. Indeed, the main purpose of this research is to extend the results of [38] to networks with

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loops. This is relevant considering that loops arise naturally in pipe systems, transport systems, etc.

Recent important works involving the control of parabolic equations on networks are the followings: [56], where the controllability of the discretized heat equation is studied, [12], where bilinear controls are analyzed on networks, [52], where the optimal control is studied in time-fractional diffusion equations and, [5], where the controllability is analyzed with vanishing viscosity. Note that the literature related to the controllability of hyperbolic equations on networks is more extensive, on which we may highlight the book [15] and the paper [37]. Particularly, [15] mainly analyzes the problem of propagation, observation and control of waves on planar one-dimensional networks, using groundbreaking developments related to non-harmonic Fourier series, Diophantine approximation, graph theory and wave propagation techniques (d'Alembert formula, for example).

Let us now outline the main breakthroughs in the literature regarding the background and the trajectory of the study of controllability for the heat equation:

- The first results on the study of the null controllability of the heat equation date back to the 1970s, when the controllability is derived from that of the wave equation. In [57], boundary controllability is proved for parabolic PDEs using some observability estimates of hyperbolic equations. Then, in [53,54] a null controllability result is given for the heat equation using the transmutation method and proving more accurate bounds of the controllability cost. The transmutation method relates the null controllability of the heat equation to the exact controllability of the wave equation in a direct way, different from the indirect complex and harmonic analysis method used in [57]. Similar methods allow to obtain more precise results, as: [23], where the estimates on the cost of controllability are improved assuming the geometric control condition, [16], where a lower bound of the reachable set of the one-dimensional heat equation is obtained and, [17], where better bounds were obtained for the 1D heat equation. There are other results such as [60], where the work of [53] is improved by giving new estimates on the norm of the operator associating the minimal norm control to any initial state driving the system to zero.
- In [47] in 1995, the control to zero of the heat equation is constructed thanks to some spectral inequalities proved there by using Carleman inequalities and duality arguments. Based on this method, several studies have been carried out:
  - The null controllability is studied when the control domain is a measurable set with positive measure in [2,62,65].
  - New observability inequalities are presented in [3], where bounded Lipschitz locally star-shaped domains are studied, and with these inequalities the controllability is obtained.
  - Appropriate observability inequalities are proved and null controllability of parabolic equations with fourth or higher order derivatives is obtained in [24, 25].

- A new method is provided in 1996 in [35], where an observability inequality is proved using global Carleman parabolic inequalities and energy methods. It extends the result of null controllability to parabolic equations whose second order coefficient depends on space and time variables. This outcome also allowed other researchers to later obtain controllability results for more general parabolic linear and non-linear equations as in [31,32,35]. Additional examples are given when the diffusion coefficient is not continuous in [7,20,46]. Moreover, it is well-known that this method can be generalized to cubes and to any Cartesian product of C^2 domains (see, for instance, [36]). More recently, the heat equation in pseudo-cylindrical domains has been studied in [4].
- Finally, the flatness approach introduced in [34] provides the possibility to prove the null controllability of the heat equation in a bounded cylinder in [51], giving an explicit and very regular control. The flatness approach parametrizes the solution of the equation and the control by the derivatives of a flat output. This technique is first introduced for parabolic one-dimensional equations in [44] for motion planning.

The paper is organized as follows. In Sect. 1.2 we recall basic definitions from network theory and its functional framework. In Sect. 1.3, we introduce the control problem based on a parabolic system that models the dynamics of the flux in a network with loops, we next set up the required hypotheses for the observation domain in order to control those networks, and finally state the main results regarding the null controllability and the existence of a feedback control. In Sect. 1.4, we complete the introduction by presenting a result that gives us a Lipschitz stability to obtain the solution of an inverse problem related to our parabolic system. Then, in Sect. 2 we prove an observability inequality for our system. In particular, in Sect. 2.1 we construct explicitly the necessary auxiliary function for the Carleman inequality, proved later in Sect. 2.2. Lastly, in Sect. 3 we present applications of the previous Carleman inequality to the null and feedback controllability (see Sect. 3.1) and to an inverse problem (see Sect. 3.2), and we conclude with some open problems related to Control and Network research areas (see Sect. 3.3).

1.2. Basic definitions

We first define some concepts related to graph theory that we use in this work. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph.

- An edge $e \in \mathcal{E}$ that is incident to the vertices v and $\tilde{v} \in \mathcal{V}$ is expressed as $e = v\tilde{v}$, where v and $\tilde{v}$ are the *ends* of e. The set of ends of e is denoted by $\mathcal{V}(e)$. Similarly, for every vertex $v \in \mathcal{V}$, $\mathcal{E}(v)$ denotes the *set of edges incident to* v. The *degree of a vertex* $v \in \mathcal{V}$, denoted by $d(v)$, is $|\mathcal{E}(v)|$.
- $\mathcal{V}_0 = \{v : |\mathcal{E}(v)| \geq 2\}$ denotes the *set of inner vertices*, and $\mathcal{V}_\partial = \mathcal{V} \backslash \mathcal{V}_0$ denotes the *set of boundary vertices* of the graph.

- A sequence of vertices $v_0 v_1 \dots v_{n-1} v_n$ such that $v_i \in \mathcal{V}$ for all $i = 0, \dots, n$ and such that $v_{i-1} v_i \in \mathcal{E}$ for all $i = 1, \dots, n$
  is called a *walk*. If all the vertices are distinct, it is called a *path*, and if all the vertices are distinct except $v_0 = v_n$ it is called a *cycle*.
- A graph is *connected* if there is a walk joining each pair of vertices.
- A graph isomorphic to $(\{v_0, \dots, v_n\}, \{v_0 v_1, \dots, v_{n-1} v_n\})$ for some $n \in \mathbb{N}$
  is called *path graph*.

We define a *network* as a tuple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I})$, where $\mathcal{V}$ is a finite *set of vertices*, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the *set of edges*, and $\mathcal{I}$ is the *identification* of each edge $e = v\tilde{v}$ and its ends v and $\tilde{v}$ with a closed interval $[0, L^e]$ and the ends 0 and L^e respectively. Formally, the identification can be viewed as a function from $\mathcal{E}$ to $X \times \mathcal{V} \times \mathcal{V}$, where X is the set of compact intervals. Notably, $I(e) = ([0, L^e], v, \tilde{v})$, where v is the end of e identified with 0, and $\tilde{v}$ is the other end of e, which we identify with L^e. This definition is also referred in the literature as *metric graph*. With the identification $\mathcal{I}$, for every edge $e \in \mathcal{E}$, we define the following numbers for the vertex v identified with 0 and for the vertex $\tilde{v}$ identified with L^e:

$$n^e(v) = -1 \text{ and } n^e(\tilde{v}) = 1.$$

This allows to define the operator $\partial_{n^e(v)} y = n^e(v) \partial_x y^e(v)$, which can be shorten to $\partial_{n^e} y$ when the vertex is clear. Usually, we make a small abuse of notation and do not write the identification $\mathcal{I}$ explicitly when we denote the network $\mathcal{G}$.

In this paper we work in the functional spaces $W^{k,p}_{pw}(\mathcal{E})$, which denotes the set of functions that belong to $W^{k,p}(e)$ for all $e \in \mathcal{E}$, $k \in \mathbb{N}$ and $1 \le p \le \infty$, and $W^{1,p}(\mathcal{E}) := W^{k,p}_{pw}(\mathcal{E}) \cap C^0(\overline{\mathcal{E}})$. Here "pw" stands for piecewise. Similar definitions apply to $H^k_{pw}(\mathcal{E})$ and $H^k(\mathcal{E})$. In this context, given a function $f \in W^{k,p}_{pw}(\mathcal{E})$, we define by $\partial_x f$ the derivative in each of the edges. Clearly, if $f \in W^{k,p}_{pw}(\mathcal{E})$, then $\partial_x f \in W^{k-1,p}_{pw}(\mathcal{E})$.

1.3. The controllability result

The problem that we study here is the dynamics of the flux and the control, which can be modelled by the following parabolic system:

$$\begin{cases} a\partial_t y - \mu \partial^2_{xx} y + b\partial_x y + cy = f 1_\omega, & \text{in } (0,T) \times \mathcal{E}, \\ y = 0, & \text{on } (0,T) \times \mathcal{V}_\partial, \\ y^{e_i} = y^{e_j}, & \text{on } (0,T) \times \mathcal{V}_0, \ \forall e_i, e_j \in \mathcal{E}(v), \\ \displaystyle\sum_{e \in \mathcal{E}(v)} \mu^e \partial_{n^e} y^e = \gamma y, & \text{on } (0,T) \times \mathcal{V}_0, \\ y(0,\cdot) = y_0, & \text{in } \mathcal{E}. \end{cases} \tag{1.1}$$

In this model y denotes the flux of the heat on the entire network. Throughout this paper, we denote the restriction of a function to an edge e by adding the superscript

e. In addition, a and μ are positive coefficients and b and c are coefficients which characterize the properties of the pipes of the network (roughness or properties of the heat flux, for example). Moreover, γ is a real coefficient measuring the flux of the heat on junctions, and $\omega \subset \mathcal{E}$ is the control domain. Here, when writing $\omega \subset \mathcal{E}$ we make a small abuse of notation to mean $\omega \subset \cup_{e\in\mathcal{E}} e$. In addition, by coefficients we mean functions which model the properties of the systems like the heat diffusivity and, unless stated otherwise, depend on the time and spatial variables.

It is trivial to prove in system (1.1) the usual energy estimations in $L^2(0, T; H^1(\mathcal{E}))\cap C^0([0, T]; L^2(\mathcal{E}))$ and regularity result in $L^2(0, T; H^2_{pw}(\mathcal{E})) \cap H^1(0, T; L^2(\mathcal{E}))$ for parabolic equations. This can be done by multiplying the first equation of the system by y and y_t and integrating it in $(0, T) \times \mathcal{E}$ (see [21] for a particular case).

In order to solve the controllability and inverse problems with respect to the parabolic system (1.1), we study the observability properties of the adjoint system, which is given by:

$$\begin{cases} -a\partial_t\varphi - \mu\partial^2_{xx}\varphi - \partial_x b\varphi - b\partial_x\varphi + c\varphi = 0, & \text{in } (0, T) \times \mathcal{E}, \\ \varphi = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ \varphi^{e_i} = \varphi^{e_j}, & \text{on } (0, T) \times \mathcal{V}_0,\ \forall e_i, e_j \in \mathcal{E}(v), \\ \displaystyle\sum_{e\in\mathcal{E}(v)} \mu^e\partial_{n^e}\varphi^e = \left(-\sum_{e\in\mathcal{E}(v)} n^e b^e + \gamma \right)\varphi, & \text{on } (0, T) \times \mathcal{V}_0, \\ \varphi(T, \cdot) = \varphi_T, & \text{in } \mathcal{E}. \end{cases} \tag{1.2}$$

System (1.2) might not be observable unless the control domain intersects a sufficient number of edges. In particular, in order to avoid some of those non-observable cases, we assume that the control domain intersects a sufficient number of edges:

Hypothesis 1. (*Existence of an indexing function*) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network and $\omega \subset \mathcal{E}$ an open subdomain. We suppose that:

1. ω intersects all the cycles of $\mathcal{G}$. That is, if $v_1, \dots, v_n \in \mathcal{V}$ such that $e_1 := v_1v_2, \dots, e_{n-1} := v_{n-1}v_n, e_n := v_nv_1$ satisfy $e_i \in \mathcal{E}$ for all $i = 1, \dots, n$, then there is $k \in \{1, \dots, n\}$ such that $e_k \cap \omega \neq \emptyset$.

Moreover, we suppose that there exists a function $u : \{e \in \mathcal{E} : e \cap \omega = \emptyset\} \mapsto \mathcal{V}_0$ such that:

2. u is injective.
3. e is incident to $u(e)$.

Roughly speaking, the state of the equation in the edge e is controlled by ω if $e \cap \omega \neq \emptyset$, and by $u(e)$ otherwise, which is controlled by the rest of the adjacent edges. Identifying the right hypothesis, in the sense that allows us to prove the results without being too restrictive, is not trivial and is one of the contributions of our paper. Indeed, the main breakthrough with respect to the previous work, and notably [38],

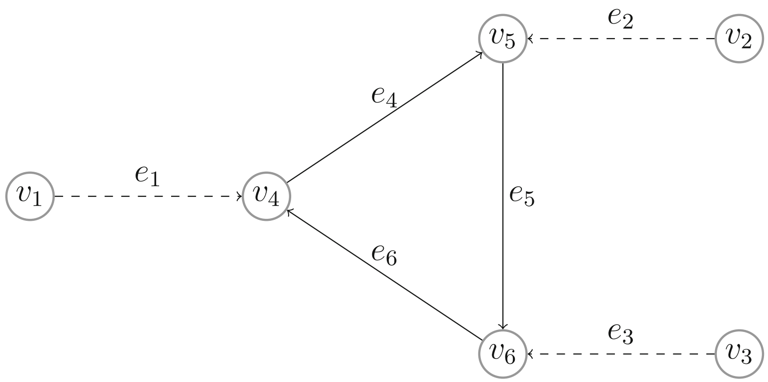

Figure 1. An example of a graph on which the heat equation is not always controllable. All the edges are identified with the interval [0, 1]. Moreover, the vertex on the tail of the arrow is identified with 0, and that of the head with 1. The control domain ω is represented by the dashed edges

is to make sure that we do not enter a circular reasoning fallacy. This is done with Hypothesis 1, as the proof of the controllability follows in a fluid way.

Remark 1.1. (On the necessity of having controls in cycles) Item 1 in Hypothesis 1 is necessary to ensure controllability. Let us consider, for example, a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{v_1, \dots, v_6\}$, and $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ for $e_1 = v_1v_4 \simeq [0, 1]$, $e_2 = v_2v_5 \simeq [0, 1]$, $e_3 = v_3v_6 \simeq [0, 1]$, $e_4 = v_4v_5 \simeq [0, 1]$, $e_5 = v_5v_6 \simeq [0, 1]$ and $e_6 = v_6v_4 \simeq [0, 1]$. In addition, we consider $\omega = e_1 \cup e_2 \cup e_3$ (see Fig. 1). It is clear that ω does not intersects the central cycle, but there exists a function that satisfies Items 2 and 3 of Hypothesis 1, for instance $u(e_4) = v_5$, $u(e_5) = v_6$ and $u(e_6) = v_4$. The parabolic system (1.1) may not be approximately controllable on such a network. For instance, if $a = \mu = 1$, $\gamma = 0$ and $b = c = 0$, then the network is not approximately controllable. In fact, for those parameters the system (1.2) does not satisfy the unique continuation principle, given that there is an eigenfunction of the Laplacian null on e_1, e_2 and e_3; for example:

$$\psi(x) = \begin{cases} 0, & x \in e_1 \cup e_2 \cup e_3, \\ \sin(2\pi x), & x \in e_4 \cup e_5 \cup e_6. \end{cases}$$

Remark 1.2. (On the necessity of injectivity in Hypothesis 1) Item 2 in Hypothesis 1 is necessary to ensure controllability. We can find graphs where we cannot define an injective function u as in Hipothesis 1, even if Items 1 and 3 can be satisfied. Let us consider, for example, a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{v_1, \dots, v_9\}$, and $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ for $e_1 = v_1v_7 \simeq [0, 1]$, $e_2 = v_2v_7 \simeq [0, 1]$, $e_3 = v_3v_8 \simeq [0, 1]$, $e_4 = v_4v_8 \simeq [0, 1]$, $e_5 = v_5v_9 \simeq [0, 1]$, $e_6 = v_6v_9 \simeq [0, 1]$, $e_7 = v_7v_8 \simeq [0, 1]$, $e_8 = v_8v_9 \simeq [0, 1]$ and $e_9 = v_9v_7 \simeq [0, 1]$. Also, we consider

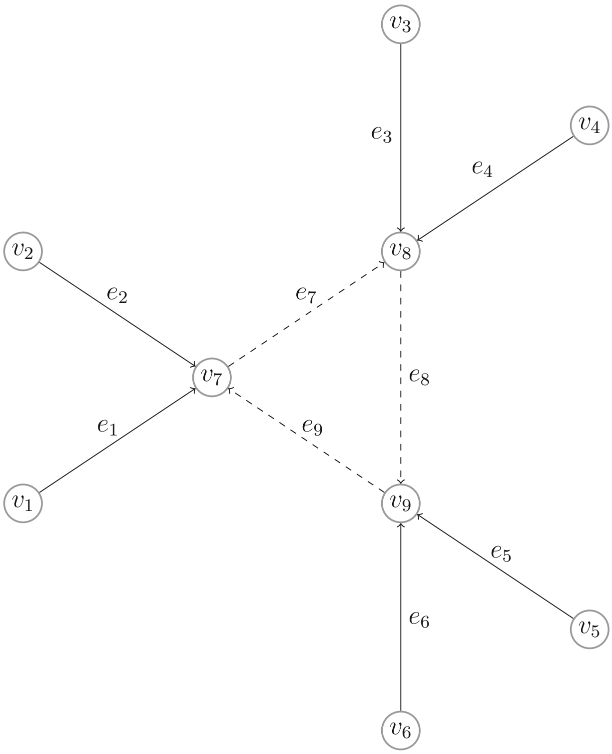

Figure 2. Another example of a graph on which the heat equation is not always controllable. All the edges are identified with the interval [0, 1]. Moreover, the vertex on the tail of the arrow is identified with 0, and that of the head with 1. The control domain ω is represented by the dashed edges

$\omega = e_7 \cup e_8 \cup e_9$ (see Fig. 2). In that network, we cannot find any injective function u because $|\mathcal{V}_0| = |\{v_7, v_8, v_9\}| = 3$ and because $|\{e_1, e_2, e_3, e_4, e_5, e_6\}| = 6$. The system (1.1) may not be approximately controllable on such a network. For instance, if $a = \mu = 1$, $\gamma = 0$ and $b = c = 0$, then the network is not approximately controllable. In fact, for those parameters the system (1.2) does not satisfy the unique continuation principle, given that there is an eigenfunction of the Laplacian null on e_7, e_8 and e_9; for example:

$$\psi(x) = \begin{cases} \sin(2\pi x), & x \in e_1 \cup e_2 \cup e_3 \cup e_4 \cup e_5 \cup e_6, \\ 0, & x \in e_7 \cup e_8 \cup e_9. \end{cases}$$

Remark 1.3. (Identification of edges) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network and let ω be a control domain such that Hypothesis 1 is satisfied with an indexing function u that we fix. In order to identify an edge e such that $\omega \cap e = \emptyset$ with an interval $[0, L^e]$, we establish that the end identified with L^e is the vertex $u(e)$. This assignment simplifies some computations in the proof of Proposition 2.1.

The main observability result in this paper is a Carleman inequality (see Proposition 2.5 in Sect. 2.2). With that inequality, we prove the following null controllability result regarding open-loop control:

Theorem 1.4. (Controllability of the heat equation on networks with loops) *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network satisfying Hypothesis 1, $a, \mu \in W^{1,\infty}((0,T); L^\infty(\mathcal{E})) \cap L^\infty((0,T); W^{1,\infty}_{pw}(\mathcal{E}))$ such that $\inf a, \inf \mu > 0$, $b \in L^\infty((0,T); W^{1,\infty}_{pw}(\mathcal{E}))$, $c \in L^\infty((0,T) \times \mathcal{E})$ and $\gamma \in L^\infty((0,T) \times \mathcal{V}_0)$. Then, there exists $C > 0$ such that for all $y_0 \in L^2(\mathcal{E})$ there is $f \in L^2((0,T) \times \omega)$ such that:*

$$\|f\|_{L^2((0,T)\times\omega)} \leq C\|y_0\|_{L^2(\mathcal{E})},$$

and the solution of (1.1) satisfies $y(T, \cdot) = 0$.

Theorem 1.4 is proved in Sect. 3.1 by duality.

Next, we focus on feedback control, also known as closed-loop control, in order to prove the stabilization properties for the simplified system of (1.1) when $b = c = 0$.

Many applications of feedback type controls can be found in industry and engineering: water level controller, air conditioner, adaptive measurement in quantum systems or servo voltage stabilizer (see [1,18,61,66]). The study and construction of feedback controls is a well-established research topic (see for instance, [13,48,58] and the survey [14]) and, in this work, we obtain a feedback control from Theorem 1.4.

In the works [58,59] we see the relation between the mild solution of Riccati equation and the null controllability of a system. The purpose of introducing Riccati equations into the study of control theory (see [42] and [63, Chapter 2]) is to design feedback controls for linear quadratic control problems. We apply Theorem 1.4 to obtain the existence of feedback controls for a simplified version of system (1.1), based mainly on the theory of Riccati equations and it is formulated in Theorem 1.5. In order to state our result we need first to define the following concepts:

- A and B are bounded operators defined as:

$$A : H_0^1(\mathcal{E}) \cap H^2(\mathcal{E}) \longrightarrow L^2(\mathcal{E}) \text{ and } B : L^2(\mathcal{E}) \longrightarrow L^2(\mathcal{E}).$$

- $\sum^+(L^2(\mathcal{E}))$ is the Banach space of all symmetric and positive operators acting in $L^2(\mathcal{E})$.
- $C_S\left([0,T); \sum^+(L^2(\mathcal{E}))\right)$ denotes the set of all mappings $S : [0,T) \longrightarrow \sum^+(L^2(\mathcal{E}))$ such that $S(\cdot)z_0$ is continuous in $[0,T)$ for each $z_0 \in L^2(\mathcal{E})$.
- For each $T > 0$ and $g \in L^2((0,T) \times \mathcal{E})$, we denote $(g(t))(x) := g(t,x)$ for $(t,x) \in (0,T) \times \mathcal{E}$.
- A^* and B^* denote the adjoint operators of the above mentioned corresponding operators A and B.

- A function P is called a *mild solution* to the Riccati equation:

$$P'(t) + A^*P(t) + P(t)A - P(t)BB^*P(t) = 0 \text{ on } [0, T), \tag{1.3}$$

if for each $\delta \in (0, T)$ the function P satisfies:

$$P(T-\delta-t)z_0 = e^{tA^*}P(T-\delta)e^{tA}z_0 \\ - \int_0^t e^{(t-s)A^*}P(T-\delta-s)BB^*P(T-\delta-s)e^{(t-s)A}z_0 ds,$$

for each $t \in [0, T-\delta]$ and $z_0 \in L^2(\mathcal{E})$, and the equality $\lim_{(s,z)\to(T,z_0)} \langle P(s)z, z\rangle = +\infty$ holds for each $z_0 \in L^2(\mathcal{E})$ and $z_0 \neq 0$.

Theorem 1.5. (Existence of feedback control) *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network satisfying the hypotheses of Theorem 1.4. Then, for each $z_0 \in H_0^1(\mathcal{E})$ and $T > 0$, there exists the following feedback control:*

$$f(t, x) := -a^{-1}1_\omega P(t)y(t),$$

for the solution y to (1.1) with $b = c = 0$ and $a, \mu \in \mathbb{R}$ constant coefficients, where $P \in C_S([0, T); \sum^+(L^2(\mathcal{E})))$ is the unique mild solution of the Riccati system (1.3) for $A = a^{-1}\mu\partial_{xx}^2$ and $B = a^{-1}1_\omega$.

Theorem 1.5 is proved in Sect. 3.1 by using auxiliary results from [58].

1.4. Application to the resolution of inverse problems

Carleman estimates can also be used to obtain results in the field of inverse problems, which is an additional objective of our paper. In fact, the link between Carleman inequalities and their applications is well known. Some important references regarding this topic include [39,40], and detailed surveys are included in [6,64].

In this paper, we seek to generalize the results of [38] to systems with loops. With that purpose, let us consider the system:

$$\begin{cases} \partial_t y - \mu\partial_{xx}^2 y + py = 0, & \text{in } (0, T) \times \mathcal{E}, \\ y = 0, & \text{on } (0, T) \times \mathcal{V}_\partial, \\ y^{e_i} = y^{e_j}, & \text{on } (0, T) \times \mathcal{V}_0,\ \forall e_i, e_j \in \mathcal{E}(v), \\ \displaystyle\sum_{e\in\mathcal{E}(v)} \mu^e \partial_{n^e} y^e = \gamma y, & \text{on } (0, T) \times \mathcal{V}_0, \\ y(0, \cdot) = y_0, & \text{in } \mathcal{E}, \end{cases} \tag{1.4}$$

for μ a piecewise constant function, γ a real parameter, y_0 the initial state and p the static potential. Moreover, we denote by $y[p, y_0]$ the solution of (1.4).

Our objective is to recover the potential p by making measurements on the flux of the heat at a time $t_0 > 0$ and also on the observation domain ω but throughout the whole time interval $(0, T)$. In particular, we prove the following result:

Theorem 1.6. (Resolution of an inverse problem) *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network satisfying Hypothesis 1, $p \in L^\infty(\mathcal{E})$, $r > 0$ and $y_0 \in L^2(\mathcal{E})$ such that $y[p, y_0] \in H^1(0, T; H^2_{pw}(\mathcal{E})) \cap H^2(0, T; L^2_{pw}(\mathcal{E}))$ and such that for some $t_0 \in (0, T)$ the following estimate holds:*

$$|y[p, y_0](t_0, \cdot)| \geq r \quad \text{in } \mathcal{E}. \tag{1.5}$$

Then, for any $m > 0$, there is a constant $C(m, r, T, \|\partial_t y[p, y_0]\|_{L^\infty((0,T)\times\mathcal{E})})$ such that for any $q \in L^\infty(\mathcal{E})$ satisfying:

$$\|q\|_{L^\infty(\mathcal{E})} \leq m,$$

we have:

$$\begin{aligned}\|q - p\|_{L^2(\mathcal{E})} \leq C \big(&\|y[p, y_0](t_0, \cdot) - y[q, y_0](t_0, \cdot)\|_{H^2(\mathcal{E})} \\ &+\|y[p, y_0] - y[q, y_0]\|_{H^1(0,T;L^2(\omega))}\big).\end{aligned} \tag{1.6}$$

The proof of Theorem 1.6 can be found in Sect. 3.2. In fact, this result is an easy consequence of the Carleman inequality proved in Sect. 2.2.

2. The observability problem

In this section we prove the observability inequality for system (1.2). With that purpose, in Sect. 2.1 we construct an auxiliary function of Fursikov–Imanuvilov type, and in Sect. 2.2, using appropriate weights, we obtain the observability of system (1.2) with a Carleman inequality.

2.1. Construction of the auxiliary function

In this section we construct an auxiliary function that is required to define the Fursikov–Imanuvilov weights in Sect. 2.2. Throughout this section we consider an open subdomain $\tilde{\omega} \subset \omega$ such that $\overline{\tilde{\omega}} \subset \omega$ and such that, for all $e \in \mathcal{E}$, $\tilde{\omega} \cap e \neq \emptyset$ if and only if $\omega \cap e \neq \emptyset$.

The construction of the auxiliary function is one of the main contributions of the paper. We need to make sure that for all edge e, if $e \cap \tilde{\omega} \neq \emptyset$, the maximum of η^e is achieved in $\tilde{\omega}$ and if $e \cap \tilde{\omega} = \emptyset$, the maximum of η^e is achieved on $u(e)$, being its derivative small near $u(e)$. As the "smallness" depends on the coefficients of the system, we get a family of auxiliary functions whose derivatives near $u(e)$ are as small as needed, and such that they are uniformly bounded in $W^{2,\infty}_{pw}(\mathcal{E})$.

Proposition 2.1. (Construction of the auxiliary function) *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network, and $\tilde{\omega}$ be a domain satisfying Hypothesis 1 with the indexing function u. We identify the edges of $\mathcal{G}$ as in Remark 1.3. Then, there is $C > 0$ such that for all $\delta \in [0, 1]$ there exists a function η satisfying:*

1. *The function $\eta \in C^0(\overline{\mathcal{E}}) \cap C^2_{pw}(\mathcal{E})$ and $\|\eta\|_{W^{2,\infty}_{pw}} \leq C$.*

2. *For all edges e such that $e \cap \tilde{\omega} = \emptyset$, then:*
   - $\partial_x \eta^e \geq \delta$ *on e,*
   - $\partial_{n^e} \eta^e(0) = -\partial_x \eta^e(0) = -1$,
   - $\partial_{n^e} \eta^e(L^e) = \partial_x \eta^e(L^e) = \delta$.
3. *If an edge e that we identify with $[0, L^e]$ satisfies $e \cap \tilde{\omega} \neq \emptyset$, then:*
   - $|\partial_x \eta| = 1$ *on $e \backslash \tilde{\omega}$,*
   - $\partial_{n^e} \eta^e(0) = -\partial_x \eta^e(0) = -1$,
   - $\partial_{n^e} \eta^e(L^e) = \partial_x \eta^e(L^e) = -1$.

The proof of the existence of such function is based on an induction on the number of edges of $\mathcal{G}$, and is one of the contributions of the paper. In order to prove Proposition 2.1, we first need to study the case of an edge assuming we have some restrictions on the boundary. This is done with Lemmas 2.2 and 2.3, whose proofs are standard (see [35]), but which we prove for the sake of completeness. We first study the construction of the auxiliary function for edges that have no intersection with $\tilde{\omega}$.

Lemma 2.2. (Extension of the auxiliary function with one constraint) *Let $\tilde{\omega} \subset \mathcal{E}$ be an open subdomain, $e \simeq [0, L^e]$ be an edge such that $\tilde{\omega} \cap e = \emptyset$, $R \in \mathbb{R}$, $p \in \{0, L^e\}$ and $\delta \in [0, 1]$. Then, there is a function $\eta^e \in C^2(\overline{e})$ such that:*

- $\partial_x \eta^e \geq \delta$ *on* $[0, L^e]$,
- $\|\eta^e\|_{L^\infty(0,L^e)} \leq |R| + L^e$, $\|\partial_x \eta^e\|_{L^\infty(0,L^e)} \leq 1$, $\|\partial_{xx} \eta^e\|_{L^\infty(0,L^e)} \leq \frac{1}{L^e}$,
- $\partial_{n^e} \eta^e(0) = -\partial_x \eta^e(0) = -1$,
- $\partial_{n^e} \eta^e(L^e) = \partial_x \eta^e(L^e) = \delta$,
- $\eta^e(p) = R$.

Proof. For the case $p = 0$ it suffices to consider:

$$\eta^e(x) = R + x - \frac{1-\delta}{2L^e} x^2, \quad \forall x \in [0, L^e].$$

Indeed,

- $\partial_x \eta^e(x) = 1 - \frac{1-\delta}{L^e} x$, so, on $[0, L^e]$ we have that:

$$\partial_x \eta^e(x) \geq \partial_x \eta^e(L^e) = 1 - (1 - \delta) = \delta.$$

- As η^e is increasing on $[0, L^e]$ and as $\delta \leq 1$:

$$\begin{aligned} \|\eta^e\|_{C^0([0,L^e])} &\leq \max\{|\eta^e(0)|, |\eta^e(L^e)|\} \\ &= \max\left\{|R|, \left|R + L^e \frac{1+\delta}{2}\right|\right\} \leq |R| + L^e. \end{aligned}$$

- $\partial_x \eta^e(x) = 1 - \frac{1-\delta}{L^e} x$, so $\|\partial_x \eta^e\|_{L^\infty(0,L^e)} = \partial_x \eta^e(0) = 1$. Indeed, $\partial_x \eta^e$ is decreasing and positive on $[0, L^e]$.
- $\partial_{xx} \eta^e = \frac{\delta - 1}{L^e}$, so $\|\partial_{xx} \eta^e\|_{C^0([0,L^e])} \leq \frac{1}{L^e}$.
- Clearly, $-\partial_x \eta^e(0) = -1$, $\partial_x \eta^e(L^e) = \delta$ and $\eta^e(0) = R$.

Similarly, for $p = L^e$ we have the auxiliary function:

$$\eta^e(x) = R + \delta(x - L^e) + \frac{\delta - 1}{2L^e}(x - L^e)^2, \ \forall x \in [0, L^e].$$

Indeed,

- $\partial_x \eta^e(x) = \delta + \frac{1-\delta}{L^e}(L^e - x)$, so, on $[0, L^e]$ we have that:

$$\partial_x \eta^e(x) \geq \partial_x \eta^e(L^e) = \delta.$$

- As η^e is increasing on $[0, L^e]$ and as $\delta \leq 1$:

$$\begin{aligned}\|\eta^e\|_{C^0([0,L^e])} &\leq \max\{|\eta^e(0)|, |\eta^e(L^e)|\} \\ &= \max\left\{\left|R - L^e\frac{1+\delta}{2}\right|, |R|\right\} \leq |R| + L^e.\end{aligned}$$

- $\partial_x \eta^e(x) = \delta - \frac{1-\delta}{L^e}(x - L^e)$, so $\|\partial_x \eta^e\|_{L^\infty(0,L^e)} = \partial_x \eta^e(0) = 1$. Indeed, $\partial_x \eta^e$ is decreasing and positive on $[0, L^e]$.
- $\partial_{xx} \eta^e = \frac{\delta-1}{L^e}$, so $\|\partial_{xx} \eta^e\|_{L^\infty(0,L^e)} \leq \frac{1}{L^e}$.
- Clearly, $-\partial_x \eta^e(0) = -1$, $\partial_x \eta^e(L^e) = \delta$ and $\eta^e(L^e) = R$.

□

Next, in the following Lemma, we study the construction of the auxiliary function for edges which intersect $\tilde{\omega}$.

Lemma 2.3. (Extension of the auxiliary function with two constraints) *Let $\tilde{\omega} \subset \mathcal{E}$ be an open subdomain, $e \simeq [0, L^e]$ be an edge such that $\tilde{\omega} \cap e \neq \emptyset$ and $R_1, R_2 \in \mathbb{R}$. Then, there is a function $\eta^e \in C^2(\overline{e})$ such that:*

- *$\|\eta^e\|_{W^{2,\infty}_{pw}(0,L^e)} \leq C(|R_1|, |R_2|, L^e, \tilde{\omega})$, for C increasing with $|R_1|$ and $|R_2|$ for a fixed L^e and $\tilde{\omega}$,*
- *$|\partial_x \eta^e| = 1$ on $[0, L^e]\backslash\tilde{\omega}$,*
- *$\partial_{n^e} \eta^e(0) = -\partial_x \eta^e(0) = -1$,*
- *$\partial_{n^e} \eta^e(L^e) = \partial_x \eta^e(L^e) = -1$,*
- *$\eta^e(0) = R_1$, $\eta^e(L^e) = R_2$.*

Proof. Let us fix an interval $(p_1, p_2) \subset \tilde{\omega} \cap e$ and let us consider χ a C^∞ function such that $\chi(x) = 0$ for all $x \leq 0$ and $\chi(x) = 1$ for all $x \geq 1$. Then, it suffices to consider:

$$\eta^e(x) = (R_1 + x)\chi\left(1 - \frac{x - p_1}{p_2 - p_1}\right) + (R_2 + L^e - x)\chi\left(\frac{x - p_1}{p_2 - p_1}\right).$$

It is easy to prove that η^e satisfies the required properties. In particular, the second one is satisfied because $(p_1, p_2) \subset \tilde{\omega}$ implies $[0, L^e]\backslash\tilde{\omega} \subset [0, p_1] \cup [p_2, L^e]$. □

Remark 2.4. Of course, Lemma 2.3 may be applied in a context of fewer constraints. In that case, we proceed as follows: if we are given the value that η^e takes in L^e, that is, R_2, we assume that $R_1 = 0$, and if we are given the value that η^e takes in 0, that is, R_1, we assume that $R_2 = 0$. This is relevant for proving Proposition 2.1.

Proof of Proposition 2.1. We prove the existence by induction on the number of edges on graphs that satisfy Hypothesis 1 and are connected. If they are not connected, it suffices to apply the result to each of the connected components.

The base case, a connected graph with one edge, is trivial, as a metric graph with one edge is just a segment, and the intersection with $\tilde{\omega}$ is non-trivial (since $\mathcal{V}_0 = \emptyset$, $\tilde{\omega}$ intersects the segment). Consequently, the existence on the base case follows from Lemma 2.3 applied with $R_1 = R_2 = 0$.

Next, we consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with an indexing function u and suppose that the result is proved for any graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ such that $|\mathcal{E}'| \le |\mathcal{E}| - 1$ and which satisfies Hypothesis 1. We recall that each edge is identified with a segment $[0, L^e]$ and that the identification of the edges is done in accordance with Remark 1.3; that is, when $\tilde{\omega} \cap e = \emptyset$, the value L^e is identified with the vertex $u(e)$. Also, in the following, if we have an indexing function u defined in all $e \in \mathcal{E}$ such that $\tilde{\omega} \cap e = \emptyset$ and $\mathcal{E}' \subset \mathcal{E}$, $u_{|\mathcal{E}'}$ denotes the indexing function u restricted to all $e \in \mathcal{E}'$ which satisfies that $\tilde{\omega} \cap e = \emptyset$. This is a small abuse of notation but which makes the proof more readable.

In order to prove the inductive case, we consider a division of cases based on the structure of $\mathcal{G}$, whose union covers all possible graphs:

1. $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a path graph. For the sake of simplicity we denote $\mathcal{V} = \{v_0, \dots, v_n\}$ and $\mathcal{E} = \{e_1, \dots, e_n\}$, where $e_i = v_{i-1}v_i \ \forall\, i = 1, \dots, n$. Let $k \in \{1, \dots, n\}$ be such that $e_k \cap \tilde{\omega} \neq \emptyset$ (as $|\mathcal{V}_0| = n - 1$ and $|\mathcal{E}| = n$, such edge exists). Then, we have the following three possibilities:
   (a) $e_1 \cap \tilde{\omega} = \emptyset$. Then, necessarily $u(e_1) = v_1$, as it is the only end of e_1 that is in $\mathcal{V}_0$. Thus, $\mathcal{G}' = (\{v_1, \dots, v_n\}, \{e_2, \dots, e_n\})$ satisfies Hypothesis 1 with $u = u_{|\{e_2,\dots,e_n\}}$ (if $e_2 \cap \tilde{\omega} = \emptyset$, then $u(e_2) = v_2$ because of the injectivity of u). Thus, there is a function η which satisfies the conclusion of Proposition 2.1 in $\mathcal{G}'$. Finally, we can extend it to e_1 by using Lemma 2.2 with $R = \eta^{e_2}(v_1)$ and $p = L^{e_1}$, as by Remark 1.3, v_1 is the end identified with L^{e_1}.
   (b) $e_n \cap \tilde{\omega} = \emptyset$. By symmetry this is analogous to the previous case.
   (c) $e_1 \cap \tilde{\omega} \neq \emptyset$ and $e_n \cap \tilde{\omega} \neq \emptyset$ (see Fig. 3 for an example). Then, there is $k \in \{1, \dots, n-1\}$ such that $v_k \notin \mathrm{range}(u)$. Indeed, the number of edges that do not intersect $\tilde{\omega}$ is at most $n-2$. Thus, the graph $\mathcal{G}' = \{\{v_0, \dots, v_k\}, \{e_1, \dots, e_k\}\}$ satisfies the inductive hypothesis with $u = u_{|\{e_1,\dots,e_k\}}$, so we may define a function η_1 in $\mathcal{G}'$ satisfying the conclusions of Proposition 2.1. Similarly, $\mathcal{G}'' = \{\{v_k, \dots, v_n\}, \{e_{k+1}, \dots, e_n\}\}$ satisfies the inductive hypothesis with $u = u_{|\{e_{k+1},\dots,e_n\}}$, so we may define function η_2 in $\mathcal{G}''$ satisfying the conclusions of Proposition 2.1. Consequently, we can construct the function η as follows:

$$\eta(x) = \begin{cases} \eta_1(x), & x \in \cup_{i=1}^{k} e_i, \\ \eta_2(x) - \eta_2(v_k) + \eta_1(v_k), & x \in \cup_{i=k+1}^{n} e_i. \end{cases}$$

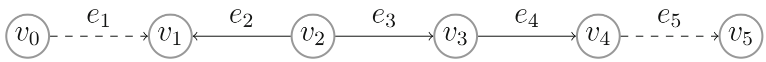

Figure 3. An example for Case 1c, where the domain $\tilde{\omega}$ is represented by the dashed edges. The tail and head of each arrow denotes the vertices identified with 0 and L^e respectively. Here $u(e_2) = v_1$, $u(e_3) = v_3$ and $u(e_4) = v_4$, and consequently $v_2 \notin \operatorname{range}(u)$. For this example, we define η_1 in $\mathcal{G}' = \{\{v_0, v_1, v_2\}, \{e_1, e_2\}\}$ and η_2 in $\mathcal{G}'' = \{\{v_2, v_3, v_4, v_5\}, \{e_3, e_4, e_5\}\}$ with the inductive hypothesis. With that, we define η as follows:

$$\eta(x) = \begin{cases} \eta_1(x), & x \in e_1 \cup e_2, \\ \eta_2(x) - \eta_2(v_2) + \eta_1(v_2), & x \in e_3 \cup e_4 \cup e_5. \end{cases}$$

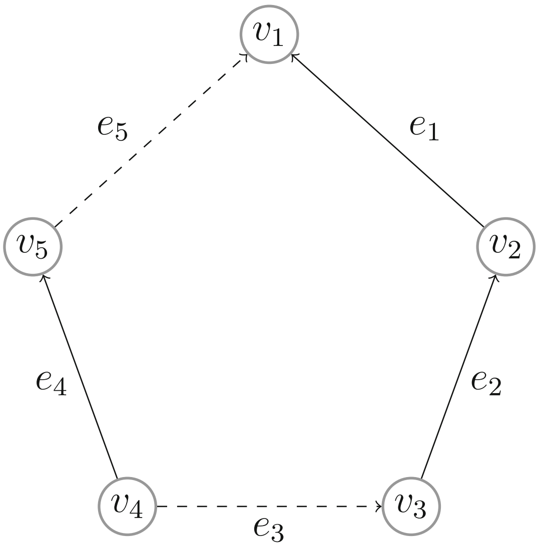

Figure 4. An example for Case 2, where the domain $\tilde{\omega}$ is represented by the dashed edges. The tail and head of each arrow denotes the vertices identified with 0 and L^e respectively. Here $u(e_1) = v_1$, $u(e_2) = v_2$, $u(e_4) = v_5$. We first construct η_1 with Lemma 2.2 taking $p = 0$ and $R = 0$, then η_2 with Lemma 2.2 taking $p = L^e$ and $R = \eta_1(v_2)$, then η_3 with Lemma 2.3 taking $R_1 = 0$ and $R_2 = \eta_2(v_3)$, then η_4 with Lemma 2.2 taking $p = 0$ and $R = \eta_3(v_4)$, and finally η_5 with Lemma 2.3 taking $R_1 = \eta_4(v_5)$ and $R_2 = \eta_1(v_1)$

2. The network is a cycle made up of the sets $\mathcal{V} = \{v_1, \ldots, v_n\}$ and $\mathcal{E} = \{e_1, \ldots, e_n\}$, with $e_i = v_i v_{i+1}$ for $i = 1, \ldots, n-1$, and $e_n = v_n v_1$ (see Fig. 4 for an example). We may assume, by changing the indexes, that $e_n \cap \tilde{\omega} \neq \emptyset$. We define the function η_1 in e_1 as follows:

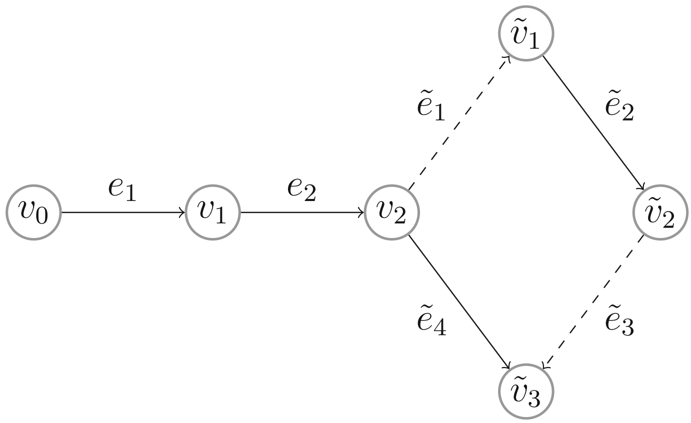

Figure 5. An example for Case 3, where the domain $\tilde{\omega}$ is represented by the dashed edges. The tail and head of each arrow denotes the vertices identified with 0 and L^e respectively. Here $u(e_1) = v_1$, $u(e_2) = v_2$, $u(\tilde{e}_2) = \tilde{v}_2$ and $u(\tilde{e}_4) = \tilde{v}_3$. For this example, we define η in $\mathcal{G}' = \{\{v_2, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}, \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}\}$ by the inductive hypothesis. Next, we extend η to e_2 with Lemma 2.2 and taking $p = L^e$ and $R = \eta(v_2)$. Finally, we extend η to e_1 with Lemma 2.2 and taking $p = L^e$ and $R = \eta(v_1)$

(a) If $e_1 \cap \omega = \emptyset$, we use Lemma 2.2 with $R = 0$ and $p = 0$.
(b) If $e_1 \cap \omega \neq \emptyset$, we use Lemma 2.3 with $R_1 = R_2 = 0$.

In addition, for $i = 2, \dots, n-1$, we define the functions η_i in e_i recursively as follows:

(c) If $e_i \cap \tilde{\omega} = \emptyset$ and v_i is identified with 0, then we use Lemma 2.2 with $p = 0$ and $R = \eta_{i-1}(v_i)$.
(d) If $e_i \cap \tilde{\omega} = \emptyset$ and v_i is identified with L^{e_i}, then we use Lemma 2.2 with $p = L^{e_i}$ and $R = \eta_{i-1}(v_i)$.
(e) If $e_i \cap \tilde{\omega} \neq \emptyset$ and v_i is identified with 0, then we use Lemma 2.3 with $R_1 = \eta_{i-1}(v_i)$ and $R_2 = 0$.
(f) If $e_i \cap \tilde{\omega} \neq \emptyset$ and v_i is identified with L^{e_i}, then we use Lemma 2.3 with $R_1 = 0$ and $R_2 = \eta_{i-1}(v_i)$.

Finally, we define η_n in e_n by using Lemma 2.3 with $R_1 = \eta_{n-1}(v_n)$ and $R_2 = \eta_1(v_1)$ if v_n is identified with 0, and with $R_1 = \eta_1(v_1)$ and $R_2 = \eta_{n-1}(v_n)$ if v_n is identified with L^{e_n}. Thus, the function that satisfies the conclusion of Proposition 2.1 is the following:

$$\eta(x) = \begin{cases} \eta_i(x), & \text{if } x \in e_i, \ i = 1, \dots, n. \end{cases}$$

3. There is one vertex v_0 such that $d(v_0) = 1$ and a path $v_0 v_1 \dots v_n$ with $d(v_i) = 2$ for $i = 1, \dots, n-1$ and $d(v_n) \geq 3$ (see Fig. 5). We denote $e_i = v_{i-1} v_i$. It

is trivial to check that the network $\mathcal{G}' = (\mathcal{V}\backslash\{v_0, \ldots, v_{n-1}\}, \mathcal{E}\backslash\{e_1, \ldots, e_n\})$ satisfies Hypothesis 1. Indeed, $v_n \in \mathcal{V}_0(\mathcal{G}')$ as its degree in $\mathcal{G}$ is at least 3 (see Fig. 5 for an example). Thus, there is a function η defined in $\mathcal{G}'$ which satisfies the conclusions of Proposition 2.1. We can extend this function to $\{e_1, \ldots, e_n\}$ by constructing an auxiliary function in each edge with one boundary constraint, which can be done thanks to Lemmas 2.2 and 2.3. This must be done in a recursive way: first we extend it to e_n with Lemma 2.2 if $e_n \cap \tilde{\omega} = \emptyset$ and with Lemma 2.3 otherwise, then we extend it to e_{n-1} with Lemma 2.2 if $e_{n-1} \cap \tilde{\omega} = \emptyset$ and with Lemma 2.3 otherwise, etc. In this extension we must proceed as follows for $i = n, n-1, \ldots, 1$:

- If $e_i \cap \tilde{\omega} = \emptyset$ and v_i is identified with 0, then we use Lemma 2.2 with $p = 0$ and $R = \eta(v_i)$.
- If $e_i \cap \tilde{\omega} = \emptyset$ and v_i is identified with L^e, then we use Lemma 2.2 with $p = L^e$ and $R = \eta(v_i)$.
- If $e_i \cap \tilde{\omega} \neq \emptyset$ and v_i is identified with 0, then we use Lemma 2.3 with $R_1 = \eta(v_i)$ and $R_2 = 0$.
- If $e_i \cap \tilde{\omega} \neq \emptyset$ and v_i is identified with L^e, then we use Lemma 2.3 with $R_1 = 0$ and $R_2 = \eta(v_i)$.

4. The degree of all the vertices are at least 2 and there is one vertex with degree at least 3.

   Then, we have two possibilities, not necessarily disjoint:

   (a) There is a cycle $v_1 v_2 \ldots v_n$ such that $d(v_1) \geq 3$ and $d(v_i) = 2$ for all $i = 2, \ldots, n$. Let us now distinguish two subcases:

   i. If $d(v_1) \geq 4$, let us define $e_i = v_i v_{i+1}$ for $i = 1, \ldots, n-1$, and $e_n = v_n v_1$ (see Fig. 6 for an example). We consider the subgraphs $\mathcal{G}' = (\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_n\})$ and $\mathcal{G}'' = (\mathcal{V}\backslash\{v_2, \ldots, v_n\}, \mathcal{E}\backslash\{e_1, \ldots, e_n\})$. As the degree of v_1 is at least 2, we have $v_1 \in \mathcal{V}_0(\mathcal{G}_2)$. Thus, both $\mathcal{G}'$ and $\mathcal{G}''$ satisfy the inductive hypothesis with the restriction of u, so we can define an auxiliary function η_1 and η_2 in each graph, respectively. With these functions we can define a function η in $\mathcal{G}$ which is globally continuous:

$$\eta(x) = \begin{cases} \eta_1(x), & x \in \{e_1, \ldots, e_n\}, \\ \eta_2(x) - \eta_2(v_1) + \eta_1(v_1), & x \in \mathcal{E}\backslash\{e_1, \ldots, e_n\}. \end{cases}$$

   ii. If $d(v_1) = 3$ (see Fig. 7 for an example), then we consider the following two graphs:

$$\mathcal{G}' = (\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_n\}),$$

   and

$$\mathcal{G}'' = \left(\mathcal{V}\backslash\{v_2, \ldots, v_n\} \cup \{v^*\}, (\mathcal{E} \cup \{v_1 v^*\})\backslash\{e_1, \ldots, e_n\}\right),$$

   for v^* and additional vertex that we define, which we joint to v_1 by the new edge $v_1 v^*$. Clearly, $\mathcal{G}'$ satisfies Hypothesis 1 with $u_{|\{e_1,\ldots,e_n\}}$, so we

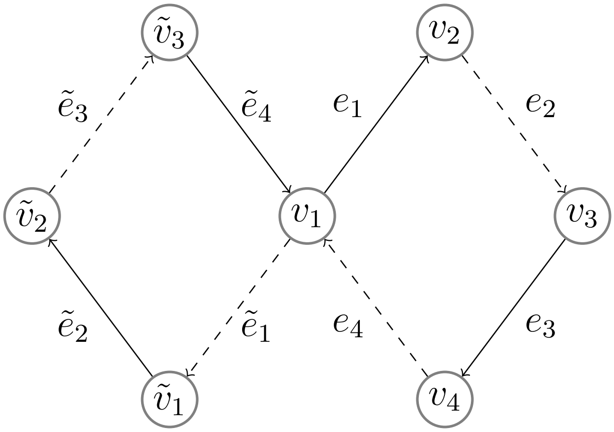

Figure 6. An example for Case 4(a)i, where the domain $\tilde{\omega}$ is represented by the dashed edges. The tail and head of each arrow denotes the vertices identified with 0 and L^e respectively. For this example, we define η_1 in $\mathcal{G}' = \{\{v_1, v_2, v_3, v_4\}, \{e_1, e_2, e_3, e_4\}\}$ and η_2 in $\mathcal{G}'' = \{\{v_1, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}, \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}\}$ with the inductive hypothesis. With that, we define η as follows:

$$\eta(x) = \begin{cases} \eta_1(x), & x \in e_1 \cup e_2 \cup e_3 \cup e_4, \\ \eta_2(x) - \eta_2(v_1) + \eta_1(v_1), & x \in \tilde{e}_1 \cup \tilde{e}_2 \cup \tilde{e}_3 \cup \tilde{e}_4. \end{cases}$$

can define there a function η_1. In addition, $\mathcal{G}''$ satisfies Hypothesis 1 with $\tilde{\omega}$ replaced by $\tilde{\omega} \cup \{v_1 v^*\}$ and with indexing function $u_{|(\mathcal{E}\cup\{v_1 v^*\})\setminus\{e_1,\ldots,e_n\}}$. Thus, as $\mathcal{G}''$ has less edges than $\mathcal{G}'$, we can define a function η_2 in $\mathcal{G}''$. With these two functions, we can define the auxiliary function for $\mathcal{G}$:

$$\eta(x) = \begin{cases} \eta_1(x), & x \in \{e_1, \ldots, e_n\}, \\ \eta_2(x) - \eta_2(v_1) + \eta_1(v_1), & x \in \mathcal{E}\setminus\{e_1, \ldots, e_n\}. \end{cases}$$

(b) There is a path $v_{-m}v_{-m+1}\ldots v_0 v_1 \ldots v_n$ for some $m \geq 0$ and $n \geq 1$ such that $d(v_{-m}) \geq 3$, $d(v_n) \geq 3$, $v_{-m} \neq v_n$ and $d(v_i) = 2$ for $-m < i < n$ and such that the edge joining v_0 and v_1 verifies $v_0 v_1 \cap \tilde{\omega} \neq \emptyset$. Again, we define $e_i = v_{i-1}v_i$ (see Fig. 8 for an example). Then, the network $\mathcal{G}' = (\mathcal{V}\setminus\{v_{-m+1}, \ldots, v_{n-1}\}, \mathcal{E}\setminus\{e_{-m+1}, \ldots e_n\})$ satisfies Hypothesis 1, and we can define an auxiliary function η there. Thus, we just have to prolong η to $\{e_{-m+1}, \ldots, e_n\}$. The process consists of prolonging η to the edges $e_{-m+1}, \ldots, e_0$ and $e_1, \ldots, e_n$ with one boundary constraint, which can be done with Lemmas 2.2 and 2.3 as in Case 3, and to e_1 with two boundary constraints, which can be done with Lemma 2.3 as $e_1 \cap \tilde{\omega} \neq \emptyset$.

Cases 4a and 4b include all the instances of Case 4. Indeed, we have to consider an edge e such that $e \cap \tilde{\omega} \neq \emptyset$. We call v_0 and v_1 the ends of e. If $d(v_0) \geq 3$ and

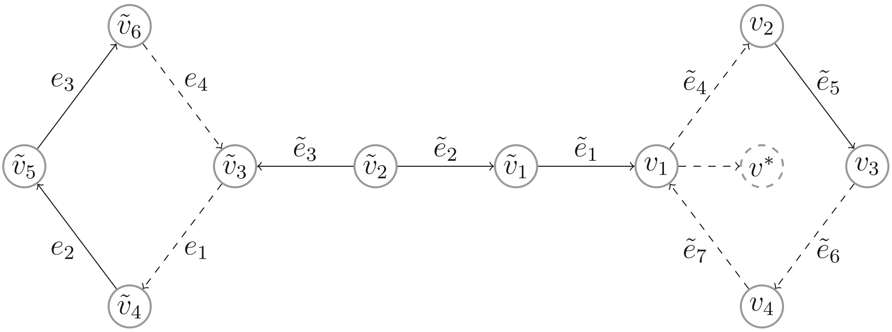

Figure 7. An example for Case 4(a)ii, where the domain $\tilde{\omega}$ is represented by the dashed edges. The tail and head of each arrow denotes the vertex identified with 0 and L^e respectively. Here $\mathcal{G}$ is all the vertices except v^* and all the edges except $v_1 v^*$. Using the inductive hypothesis, we define a function η_1 in $\mathcal{G}' = \{\{v_1, v_2, v_3, v_4\}, \{e_1, e_2, e_3, e_4\}\}$ and η_2 in $\mathcal{G}'' = \{\{v^*, v_1, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_5, \tilde{v}_6\}, \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6, \tilde{e}_7\}\}$ with the inductive hypothesis. With that, we define η as follows:
$$\eta(x) = \begin{cases} \eta_1(x), & x \in e_1 \cup e_2 \cup e_3 \cup e_4, \\ \eta_2(x) - \eta_2(v_1) + \eta_1(v_1), & x \in \cup_{i=1}^{7} \tilde{e}_i. \end{cases}$$

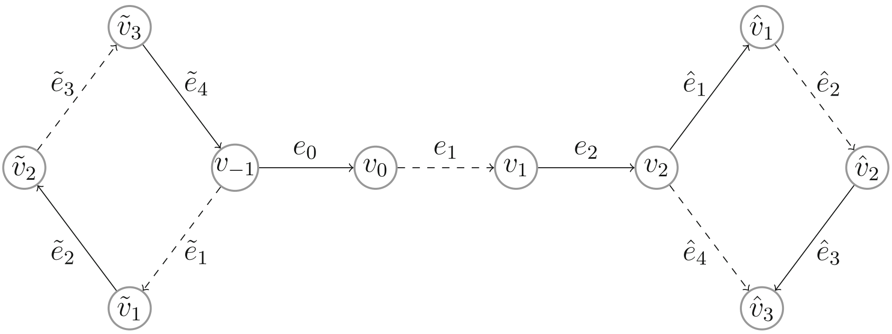

Figure 8. An example for Case 4b, where the domain $\tilde{\omega}$ is represented by the dashed edges. The tail and head of each arrow denotes the vertices identified with 0 and L^e respectively. For this example, we define η in $\mathcal{G}' = \{\{v_{-1}, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, v_2, \hat{v}_1, \hat{v}_2, \hat{v}_3\}, \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}\}$ thanks to the inductive hypothesis. Then, with Lemma 2.2 we prolong it to e_0 with $p = 0$ and $R = \eta(v_{-1})$ and to e_1 considering $p = L^e$ and $R = \eta(v_2)$. Finally, we prolong it to e_1 using Lemma 2.3 with $R_1 = \eta(v_0)$ and $R_2 = \eta(v_1)$

$d(v_1) \geq 3$, we are in Case 4b. If $d(v_1) = 2$, then, there is a sequence $v_1 v_2 \cdots v_n$ such that $d(v_i) = 2$ for $i = 2, \ldots, n-1$, and $d(v_n) \geq 3$. Similarly, if $d(v_0) = 2$, there is a sequence $v_0 v_{-1} v_{-2} \cdots v_{-m}$ such that $d(v_{-i}) = 2$ for $i = 1, \ldots, m-1$, and $d(v_{-m}) \geq 3$. So, if $v_{-m} = v_n$ we are in Case 4a (here $d(v_n) > 2$ as $\mathcal{G}$ is connected and have a vertex of degree at least 3), and otherwise in Case 4b.

□

2.2. A new Carleman inequality

The auxiliary function constructed in the previous section allows us to define the usual Fursikov–Imanuvilov weights:

$$\alpha(t,x) := \frac{e^{6\lambda\|\eta\|_\infty} - e^{\lambda(4\|\eta\|_\infty + \eta(x))}}{t(T-t)}, \quad \forall (t,x) \in (0,T) \times \mathcal{E}$$
$$\xi(t,x) := \frac{e^{\lambda(4\|\eta\|_\infty + \eta(x))}}{t(T-t)}, \quad \forall (t,x) \in (0,T) \times \mathcal{E}, \tag{2.1}$$

where η is defined in Proposition 2.1 for $\tilde{\omega}$ an open domain compactly included in ω such that $e \cap \tilde{\omega} = \emptyset$ if and only if $e \cap \omega = \emptyset$ for all $e \in \mathcal{E}$, and for $\delta > 0$ a sufficiently small parameter to be defined later on in the proof of Proposition 2.5 for absorbing boundary terms. Bearing this in mind, we state and prove the next Carleman inequality:

Proposition 2.5. (A new Carleman inequality) *Let $a, \mu \in W^{1,\infty}((0,T); L^\infty(\mathcal{E})) \cap L^\infty((0,T); W^{1,\infty}_{pw}(\mathcal{E}))$ such that $\inf a, \inf \mu > 0$, $h \in L^\infty((0,T) \times \mathcal{V}_0)$ and $g \in L^2(Q)$. Then, there is $C > 0$ depending on $\mathcal{G}$, ω, a, and μ such that for all $\varphi_T \in L^2(\mathcal{E})$, $\lambda \geq C$ and $s \geq C(T + T^2)$ the following inequality is satisfied:*

$$s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C\left(s^3\lambda^4 \iint_{Q_\omega} e^{-2s\alpha}\xi^3|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-2s\alpha}|g|^2 \,\mathrm{d}x\,\mathrm{d}t\right), \tag{2.2}$$

for α and ξ the weights defined in (2.1), and φ the solution of:

$$\begin{cases} -a\partial_t\varphi - \mu\partial^2_{xx}\varphi = g, & \text{in } (0,T) \times \mathcal{E}, \\ \varphi = 0, & \text{on } (0,T) \times \mathcal{V}_\partial, \\ \varphi^{e_i} = \varphi^{e_j}, & \text{on } (0,T) \times \mathcal{V}_0,\ \forall e_i, e_j \in \mathcal{E}(v), \\ \displaystyle\sum_{e \in \mathcal{E}(v)} \mu^e \partial_{n^e}\varphi^e = h\varphi, & \text{on } (0,T) \times \mathcal{V}_0, \\ \varphi(T,\cdot) = \varphi_T, & \text{in } \mathcal{E}. \end{cases} \tag{2.3}$$

In the proof we denote by $o(J(s,\lambda))$ a function depending on s, λ such that for all $\epsilon > 0$ there is $C > 0$ depending on $\mathcal{G}, a, \mu$ and ω such that if $\lambda \geq C$ and $s \geq C(T+T^2)$, then $|o(J(s,\lambda))| < \epsilon J(s,\lambda)$. Throughout the proof, we denote $Q := (0,T) \times \mathcal{E}$ and

$Q_\omega := (0,T)\times\omega$, and the constants $c, C > 0$ depend on $\mathcal{G}$, a, μ and ω and their value may be different from line to line.

Proof. As in [35], we consider the change of variables $\psi = e^{-s\alpha}\varphi$. From (2.3), we obtain that ψ satisfies the equation:

$$L_1\psi + L_2\psi = L_3\psi, \tag{2.4}$$

where:

$$\begin{cases} L_1\psi := -2s\lambda\mu\xi\partial_x\eta\partial_x\psi + a\partial_t\psi, \\ L_2\psi := s^2\lambda^2\mu|\partial_x\eta|^2\xi^2\psi + \mu\partial_{xx}^2\psi + sa\partial_t\alpha\psi, \\ L_3\psi := s\lambda\mu\partial_{xx}^2\eta\xi\psi + s\lambda^2\mu|\partial_x\eta|^2\xi\psi - e^{-s\alpha}g. \end{cases} \tag{2.5}$$

Indeed, from the equality:

$$e^{-s\alpha}(a\partial_t + \mu\partial_{xx}^2)(e^{s\alpha}\psi) = -e^{-s\alpha}g,$$

we get that:

$$a\partial_t\psi + as\partial_t\alpha\psi + \mu\partial_{xx}^2\psi + 2s\mu\partial_x\alpha\partial_x\psi + s\mu\partial_{xx}\alpha\psi + s^2\mu(\partial_x\alpha)^2\psi = -e^{-s\alpha}g. \tag{2.6}$$

Now, combining (2.6) with $\partial_x\alpha = -\lambda\partial_x\eta\xi$ and $\partial_{xx}^2\alpha = -\lambda\partial_{xx}^2\eta\xi - (\lambda\partial_x\eta)^2\xi$, we obtain that ψ satisfies (2.4).

We now proceed as in [5]. The main differences are on how to deal with the boundary terms at junctions and that the observation domain is in the interior. As usual, we denote by $(L_i\psi)_j$ the j-th term in the expression of $L_i\psi$ given above, for $i \in \{1,2,3\}$ and $j \in \{1,2,3\}$. From (2.4), we get:

$$\begin{aligned} &\|a^{-1/2}L_1\psi\|_{L^2(Q)}^2 + \|a^{-1/2}L_2\psi\|_{L^2(Q)}^2 + 2\sum_{\substack{i=1,2\\ j=1,2,3}} (a^{-1/2}(L_1\psi)_i, a^{-1/2}(L_2\psi)_j)_{L^2(Q)} \\ &= \|a^{-1/2}(L_1\psi + L_2\psi)\|_{L^2(Q)}^2 = \|a^{-1/2}L_3\psi\|_{L^2(Q)}^2. \end{aligned} \tag{2.7}$$

As usual, we estimate the scalar product to obtain the Carleman inequality, which is done in steps 1 and 2, and we then conclude in step 3 by using (2.7).

Step 1: *Estimates in the interior.* In this step we perform integrations by parts in the spirit of [5,35], but keeping track of the boundary terms appearing at the junctions. This part of the proof is standard and is presented here for the sake of completeness.

To begin with, we compute:

$$\begin{aligned}
&(a^{-1/2}(L_1\psi)_1, a^{-1/2}(L_2\psi)_1)_{L^2(Q)}\\
&\quad= -2s^3\lambda^3 \iint_Q a^{-1}\mu^2|\partial_x\eta|^2\partial_x\eta\xi^3\psi\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= 3s^3\lambda^4 \iint_Q a^{-1}\mu^2|\partial_x\eta|^4\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad \underbrace{-s^3\lambda^3\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)}\int_0^T (a^e)^{-1}(\mu^e)^2|\partial_x\eta^e|^2\partial_{n^e(v)}\eta^e(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t}_{=:I_1}\\
&\qquad + o\left(s^3\lambda^4\iint_Q \xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned} \tag{2.8}$$

Secondly, integration by parts (with respect to the time variable) yields, for $\lambda \geq C$ and $s \geq C(T+T^2)$,

$$\begin{aligned}
&(a^{-1/2}(L_1\psi)_2, a^{-1/2}(L_2\psi)_1)_{L^2(Q)}\\
&\quad= s^2\lambda^2\iint_Q \mu|\partial_x\eta|^2\xi^2\psi\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t = o\left(s^3\lambda^4\iint_Q\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned} \tag{2.9}$$

In addition,

$$\begin{aligned}
&(a^{-1/2}(L_1\psi)_1, a^{-1/2}(L_2\psi)_2)_{L^2(Q)} = -2s\lambda\iint_Q a^{-1}\mu^2\partial_x\eta\xi\partial_{xx}^2\psi\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= s\lambda^2\iint_Q a^{-1}\mu^2|\partial_x\eta|^2\xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad \underbrace{-s\lambda\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)}\int_0^T (a^e)^{-1}(\mu^e)^2\partial_{n^e(v)}\eta^e\xi^e|\partial_x\psi^e|^2(t,v)\,\mathrm{d}t}_{=:I_2}\\
&\qquad + o\left(s\lambda^2\iint_Q\xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned} \tag{2.10}$$

Moreover, we can prove that:

$$\begin{aligned}
&(a^{-1/2}(L_1\psi)_2, a^{-1/2}(L_2\psi)_2)_{L^2(Q)}\\
&\quad= \iint_Q \mu\partial_t\psi\partial_{xx}^2\psi\,\mathrm{d}x\,\mathrm{d}t = \underbrace{\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)}\int_0^T \mu^e\partial_{n^e(v)}\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t}_{=:I_3}\\
&\qquad + o\left(s^{-1}\iint_Q\xi^{-1}|\psi_t|^2dxdt + s\lambda^2\iint_Q\xi|\partial_x\psi|^2dxdt\right).
\end{aligned}$$

(2.11)

Finally,

$$(a^{-1/2}L_1\psi, a^{-1/2}(L_2\psi)_3)_{L^2(Q)} = \underbrace{-s^2\lambda\sum_{v\in\mathcal{V}}\sum_{e\in\mathcal{E}(v)}\int_0^T \mu^e\partial_{n^e(v)}\eta^e\partial_t\alpha^e\xi^e|\psi^e|^2(t,v)\,\mathrm{d}t}_{=:I_4} + o\left(s^3\lambda^4\iint_Q \xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{2.12}$$

Step 2: *Estimation of the boundary terms.* In this part of the proof we estimate the boundary terms I_1, I_2, I_3 and I_4. The exterior vertices, if any, can be treated as in [35], since those terms appear when studying the heat equation in a segment with Dirichlet boundary conditions. However, the boundary terms at junctions require new more precise computations:

Step 2.1: *Estimation of* I_1 *and* I_4. To begin with, let us deal with the boundary term I_1 and I_4, in (2.8) and (2.12) respectively. By using the Dirichlet boundary conditions of ψ on $\mathcal{V}_\partial$ we have that:

$$\begin{aligned}&-s^3\lambda^3\sum_{e\in\mathcal{E}(v)}\int_0^T (a^e)^{-1}(\mu^e)^2|\partial_x\eta^e|^2\partial_{n^e(v)}\eta^e(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\\ &-s^2\lambda\sum_{e\in\mathcal{E}(v)}\int_0^T \mu^e\partial_{n^e(v)}\eta^e\partial_t\alpha^e\xi^e|\psi^e|^2(t,v)\,\mathrm{d}t = 0 \quad \forall v\in\mathcal{V}_\partial.\end{aligned} \tag{2.13}$$

Let us now, estimate the terms in $\mathcal{V}_0$. For each interior node $v\in\mathcal{V}_0$ as the function ξ and ψ are continuous at junctions we get that:

$$\begin{aligned}&-s^3\lambda^3\sum_{e\in\mathcal{E}(v)}\int_0^T (a^e)^{-1}(\mu^e)^2|\partial_x\eta^e|^2\partial_{n^e(v)}\eta^e(\xi^e)^3|\psi^e|^2(t,v)\,\mathrm{d}t\\ &\geq cs^3\lambda^3\int_0^T \xi^3|\psi|^2(t,v)\,\mathrm{d}t \quad \forall v\in\mathcal{V}_0.\end{aligned} \tag{2.14}$$

Indeed, if $v\notin \mathrm{Range}(u)$, $\partial_{n^e(v)}\eta^e = -1$ for all $e\in\mathcal{E}(v)$ and (2.14) is straightforward. Otherwise, we have to use that $\partial_{n^{\tilde{e}}(v)}\eta^{\tilde{e}}(v) = \delta$ for the edge $\tilde{e} = u^{-1}(v)$ and $\partial_{n^e(v)}\eta^e(v) = -1$ for all $e\in\mathcal{E}(v)\backslash\tilde{e}$, so we obtain (2.14) by choosing δ small enough just depending on a^e, μ^e and on the number of edges adjacent to each junction (see Item 2 of Hypothesis 1). Moreover, since

$$\begin{aligned}&-s^2\lambda\sum_{e\in\mathcal{E}(v)}\int_0^T \mu^e\partial_{n^e(v)}\eta^e\partial_t\alpha^e\xi^e|\psi^e|^2(t,v)\,\mathrm{d}t\\ &= o\left(s^3\lambda^3\int_0^T \xi^3|\psi|^2(t,v)dt\right) \quad \forall v\in\mathcal{V}_0,\end{aligned} \tag{2.15}$$

combining (2.13)–(2.15), we obtain that:

$$I_1 + I_4 \geq cs^3\lambda^3 \sum_{v\in\mathcal{V}_0} \int_0^T \xi^3|\psi|^2(t,v)\,\mathrm{d}t. \tag{2.16}$$

Step 2.2: *Estimation of* I_2. To continue with, let us study the boundary term I_2 given in (2.10) for each $v \in \mathcal{V}$, i.e.

$$-s\lambda \sum_{e\in\mathcal{E}(v)} \int_0^T (a^e)^{-1}(\mu^e)^2\partial_{n^e(v)}\eta^e\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t.$$

If $v \notin \mathrm{Range}(u)$, then $\partial_{n^e(v)}\eta^e = -1$ for all $e \in \mathcal{E}(v)$ by Proposition 2.1. Consequently, we have the estimate:

$$\begin{aligned} &-s\lambda \sum_{e\in\mathcal{E}(v)} \int_0^T (a^e)^{-1}(\mu^e)^2\partial_{n^e(v)}\eta^e\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \\ &\geq cs\lambda \sum_{e\in\mathcal{E}(v)} \int_0^T \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)dt \quad \forall v \notin \mathrm{Range}(u). \end{aligned} \tag{2.17}$$

Note that this includes all $v \in \mathcal{V}_\partial$, as $\mathrm{Range}(u) \subset \mathcal{V}_0$. Otherwise, let us denote $\tilde{e} = u^{-1}(v)$. From Proposition 2.1 we have that $\partial_{n_{\tilde{e}}}\eta^{\tilde{e}}(v) = \delta$ and $\partial_{n^e}\eta^e(v) = -1$ for all $e \in \mathcal{E}(v)\backslash\{\tilde{e}\}$. Thus, we have that:

$$\begin{aligned} &-s\lambda \sum_{e\in\mathcal{E}(v)} \int_0^T (a^e)^{-1}(\mu^e)^2\partial_{n^e(v)}\eta^e\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \\ &= s\lambda \sum_{e\in\mathcal{E}(v)\backslash\{\tilde{e}\}} \int_0^T (a^e)^{-1}(\mu^e)^2\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \\ &\quad -\delta s\lambda \int_0^T (a^{\tilde{e}})^{-1}(\mu^{\tilde{e}})^2\xi^{\tilde{e}}|\partial_{n^{\tilde{e}}(v)}\psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t \\ &\geq cs\lambda \sum_{e\in\mathcal{E}(v)\backslash\{\tilde{e}\}} \int_0^T \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \\ &\quad -\delta s\lambda \int_0^T (a^{\tilde{e}})^{-1}(\mu^{\tilde{e}})^2\xi^{\tilde{e}}|\partial_{n^{\tilde{e}}(v)}\psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t. \end{aligned} \tag{2.18}$$

So, we have to absorb the boundary term of the edge $\tilde{e}$. For that, we differentiate $\psi = \varphi e^{-s\alpha}$ and use Cauchy-Schwarz inequality to get:

$$\begin{aligned} &\delta s\lambda \int_0^T \left(a^{\tilde{e}}\right)^{-1}(\mu^{\tilde{e}})^2\xi^{\tilde{e}}|\partial_{n^{\tilde{e}}}\psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t \\ &\leq 2\delta s^3\lambda^3 \int_0^T \left(a^{\tilde{e}}\right)^{-1}(\mu^{\tilde{e}})^2|\partial_{n^{\tilde{e}}}\eta^{\tilde{e}}|^2(\xi^{\tilde{e}})^3|\psi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t \\ &\quad + 2\delta s\lambda \int_0^T \left(a^{\tilde{e}}\right)^{-1}\xi^{\tilde{e}}e^{-2s\alpha^{\tilde{e}}}|\mu^{\tilde{e}}\partial_{n^{\tilde{e}}}\varphi^{\tilde{e}}|^2(t,v)\,\mathrm{d}t \\ &= I_5 + I_6. \end{aligned} \tag{2.19}$$

We can absorb I_5 by (2.14) if δ is sufficiently small. As for the second one, using the continuity of ξ, α and ψ on the junctions, the condition on the junctions $(2.3)_4$, the facts that $\partial_x \alpha = -\lambda \partial_x \eta \xi$, $e^{-s\alpha}\partial_{n^e}\varphi^e = \partial_{n^e}\psi^e + s\partial_{n^e}\alpha\psi^e$, and that $(x_1+\cdots+x_n)^2 \leq n(x_1^2+\cdots+x_n^2)$ for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$ (we shall use this for $n = |\mathcal{E}(v)|$):

$$
\begin{aligned}
I_6 &= 2\delta s\lambda \int_0^T (a^{\tilde{e}})^{-1}\xi^{\tilde{e}} e^{-2s\alpha^{\tilde{e}}} \left| \sum_{e\in\mathcal{E}(v)\backslash\tilde{e}} \mu^e \partial_{n^e}\varphi^e - h\varphi^e \right|^2 (t,v)\,\mathrm{d}t \\
&\leq 2|\mathcal{E}(v)|\delta s\lambda \int_0^T (a^{\tilde{e}})^{-1} \sum_{e\in\mathcal{E}(v)\backslash\tilde{e}} \xi^e(\mu^e)^2 \left| e^{-s\alpha^e}\partial_{n^e}\varphi^e \right|^2 (t,v)\,\mathrm{d}t \\
&\quad + 2|\mathcal{E}(v)| \|h\|^2_{L^\infty((0,T)\times\mathcal{V}_0)} \delta s\lambda \int_0^T (a^{\tilde{e}})^{-1}\xi|\psi|^2(t,v)\,\mathrm{d}t \\
&\leq 4|\mathcal{E}(v)|\delta s\lambda \sum_{e\in\mathcal{E}(v)\backslash\tilde{e}} \int_0^T (a^{\tilde{e}})^{-1}\xi^e(\mu^e)^2 \left|\partial_{n^e}\psi^e\right|^2 (t,v)\,\mathrm{d}t \\
&\quad + 4|\mathcal{E}(v)|\delta s^3\lambda^3 \sum_{e\in\mathcal{E}(v)\backslash\tilde{e}} \int_0^T (a^{\tilde{e}})^{-1}|\partial_{n^e}\eta^e|^2(\xi^e)^3(\mu^e)^2 \left|\psi^e\right|^2 (t,v)\,\mathrm{d}t \\
&\quad + 2|\mathcal{E}(v)| \|h\|^2_{L^\infty((0,T)\times\mathcal{V}_0)} \delta s\lambda \int_0^T (a^{\tilde{e}})^{-1}\xi|\psi|^2(t,v)\,\mathrm{d}t. \qquad (2.20)
\end{aligned}
$$

Taking δ small enough and using the continuity of ξ and ψ on junctions, we can absorb the second and third term on the right-hand side of (2.20) by (2.14). As for the first term in the right-hand side of (2.20), by taking δ small enough it can be absorbed by the first term in the right hand side of (2.18).

Summing up, from (2.16)–(2.20) we obtain that:

$$
\begin{aligned}
I_1 + I_2 + I_4 \geq{}& cs^3\lambda^3 \sum_{v\in\mathcal{V}_0} \int_0^T \xi^3|\psi|^2(t,v)\,\mathrm{d}t \\
&+ cs\lambda \sum_{v\notin\mathrm{Range}(u)} \sum_{e\in\mathcal{E}(v)} \int_0^T \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)dt \\
&+ cs\lambda \sum_{v\in\mathrm{Range}(u)} \sum_{e\in\mathcal{E}(v)\backslash u^{-1}(v)} \int_0^T \xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t. \qquad (2.21)
\end{aligned}
$$

Step 2.3: *Estimation of* I_3. Let us study the boundary term I_3 in (2.11). If $v \in \mathcal{V}_\partial$, then $\partial_t\psi = 0$ because of the Dirichlet boundary conditions; that is:

$$
\sum_{e\in\mathcal{E}(v)} \int_0^T \mu^e\partial_{n^e}\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t = 0 \;\; \forall v \in \mathcal{V}_\partial.
$$

Otherwise, if $v \in \mathcal{V}_0$, from $\partial_{n^e}\psi^e = -s\partial_{n^e}\alpha^e\psi + e^{-s\alpha^e}\partial_{n^e}\varphi^e$, (2.3)$_4$, from that $\partial_t\psi^e$ is continuous on junctions, and from $\partial_t\xi \le s\xi^2$ we obtain:

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}(v)}\int_0^T \mu^e\partial_{n^e}\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t\\
&= s\lambda\sum_{e\in\mathcal{E}(v)}\int_0^T \mu^e\partial_{n^e}\eta^e\xi^e\psi^e\partial_t\psi^e(t,v)\,\mathrm{d}t + \int_0^T h\psi\partial_t\psi(t,v)\,\mathrm{d}t\\
&= -\frac{s\lambda}{2}\sum_{e\in\mathcal{E}(v)}\int_0^T \partial_t(\mu^e\partial_{n^e}\eta^e\xi^e)|\psi^e|^2\,\mathrm{d}t - \frac{1}{2}\int_0^T\partial_t h|\psi|^2\,\mathrm{d}t\\
&= o\left(s^3\lambda^3\int_0^T\xi^3|\psi|^2(t,v)\,\mathrm{d}t\right)\quad \forall v\in\mathcal{V}_0.
\end{aligned}\tag{2.22}
$$

We can absorb both terms by (2.14) integrating by parts the time variable, as μ^e and h belongs to $W^{1,\infty}((0,T)\times\mathcal{V}_0)$. Consequently, from (2.21) and (2.22) we have proved that for δ small enough, $s \ge C(T+T^2)$ and $\lambda \ge C$ the following estimate holds:

$$
\begin{aligned}
&cs^3\lambda^3\sum_{v\in\mathcal{V}_0}\int_0^T\xi^3|\psi|^2(t,v)\,\mathrm{d}t + cs\lambda\sum_{v\notin\mathrm{Range}(u)}\ \sum_{e\in\mathcal{E}(v)}\int_0^T\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)dt\\
&+ cs\lambda\sum_{v\in\mathrm{Range}(u)}\ \sum_{e\in\mathcal{E}(v)\setminus u^{-1}(v)}\int_0^T\xi^e|\partial_{n^e(v)}\psi^e|^2(t,v)\,\mathrm{d}t \le \sum_{i=1}^4 I_i,
\end{aligned}
$$

and, in particular,

$$
\sum_{i=1}^4 I_i \ge 0. \tag{2.23}
$$

Step 3: *Conclusion of the proof.* Combining (2.8)–(2.23) and taking into account that $\inf\limits_{\mathcal{E}\setminus\tilde{\omega}}|\partial_x\eta| > 0$, we obtain the following estimate:

$$
\begin{aligned}
&s\lambda^2\iint_Q\xi|\partial_x\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4\iint_Q\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le (L_1\psi, L_2\psi)_{L^2(Q)} + s^3\lambda^4\iint_{Q_{\tilde{\omega}}}\xi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s\lambda^2\iint_{Q_{\tilde{\omega}}}\xi|\partial_x\psi|^2.
\end{aligned}\tag{2.24}
$$

From (2.24), it is classical to obtain (2.2) as in [35]: we add $\frac{1}{2}(\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)})$ at both sides of (2.24); we consider that $\|L_1\psi + L_2\psi\|^2_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}$; we absorb $(L_3\psi)_1$ and $(L_3\psi)_2$; we estimate the terms on $\partial_t\psi$ and $\partial^2_{xx}\psi$ by considering (2.5)$_1$ and (2.5)$_2$ respectively; we then estimate the local term of the gradient and, finally, we revert the transformation by using that $\varphi = e^{s\alpha}\psi$. $\square$

As an easy consequence of Proposition 2.5, we have the following result:

Corollary 2.6. (Observability of the heat equation on networks with loops) *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a network satisfying Hypothesis 1, $a, \mu \in W^{1,\infty}((0,T); L^\infty(\mathcal{E})) \cap L^\infty((0,T); W^{1,\infty}_{pw}(\mathcal{E}))$ such that $\inf a, \inf \mu > 0$, $b \in L^\infty((0,T); W^{1,\infty}_{pw}(\mathcal{E}))$, $c \in L^\infty((0,T) \times \mathcal{E})$ and $\gamma \in L^\infty((0,T) \times \mathcal{V})$. Then, there exists $C > 0$ such that for all $\varphi_T \in L^2(\mathcal{E})$ we have the inequality:*

$$\|\varphi(0,\cdot)\|_{L^2(\mathcal{E})} \le C\|\varphi\|_{L^2(Q_\omega)},$$

for φ the solution of (1.2).

3. Applications of the Carleman inequality and open problems

In this section we show some applications of the Carleman inequality proved in Proposition 2.5. Notably, in Sect. 3.1 we show the controllability of (1.1) and prove Theorems 1.4 and 1.5, in Sect. 3.2 we estimate the potential and prove Theorem 1.6 and, finally, in Sect. 3.3 we present some problems that remain open.

3.1. The controllability problem

We start this section giving the proof of Theorem 1.4.

Proof of Theorem 1.4. Theorem 1.4 follows from Corollary 2.6 and the Hilbert Uniqueness Method (see [35,49,50]), which assures that the null controllability is equivalent to prove an observability inequality for the adjoint equation (in our case (1.2)). □

We now prove the result that provides a feedback control for the simplified case of system (1.1).

Proof of Theorem 1.5. Let $y_0 \in H^1_0(\mathcal{E})$ and the system:

$$\begin{cases} a\partial_t y - \mu\partial^2_{xx} y = 1_\omega v, & \text{in } (0,T) \times \mathcal{E}, \\ y = 0, & \text{on } (0,T) \times \mathcal{V}_\partial, \\ y^{e_i} = y^{e_j}, & \text{on } (0,T) \times \mathcal{V}_0, \ \forall e_i, e_j \in \mathcal{E}(v), \\ \displaystyle\sum_{e \in \mathcal{E}(v)} \mu^e \partial_{n^e} y^e = \gamma y, & \text{on } (0,T) \times \mathcal{V}_0, \\ y(0,x) = y_0(x), & \text{in } \mathcal{E}, \end{cases} \tag{3.1}$$

where $v \in L^2(0,T; L^2(\mathcal{E}))$. Due to the fact y_0 is in $H^1_0(\mathcal{E})$, we know that for each $v \in L^2(0,T; L^2(\mathcal{E}))$, there exists a unique solution $y \in C((0,T]; H^1_0(\mathcal{E}))$ to (3.1). Now, we can rewrite (3.1) in this way:

$$\begin{cases} y'(t) = a^{-1}(\mu y_{xx}(t) + 1_\omega v(t)), & t \in (0,T), \\ y(0) = y_0. \end{cases}$$

We establish the next optimal control problem:

$$\min\left\{\int_0^T ||v(t)||^2_{L^2(\mathcal{E})}dt;\ \ y' = a^{-1}(\mu y_{xx} + 1_\omega v),\ \ y(0) = y_0,\ \ y(T) = 0\right\}, \tag{3.2}$$

and the Riccati equation (1.3). Then, using [58, Theorem 2.1, (ii)], its proof and the null controllability result for the parabolic system (1.1) we obtained in Theorem 1.4 with $b = c = 0$, we deduce that there exists a unique mild solution $P \in C_S([0, T); \sum^+(L^2(\mathcal{E})))$ to problem (1.3) for $A = a^{-1}\mu\partial^2_{xx}$ and $B = a^{-1}1_\omega$. In addition, due to [58, Lemma 2.1], we can conclude that $v(t) = -a^{-1}1_\omega P(t)y(t)$ is the optimal feedback control to problem (3.2). □

3.2. The inverse problem

Let us now prove Theorem 1.6. For that, we follow the steps in [38], which uses the technique of obtaining the inverse problem result from a Carleman inequality dating back to paper [39] in 1998:

Proof of Theorem 1.6. By a change of variables, it suffices to prove the result for $t_0 = T/2$. Let us define:

$$z[q, y_0] = y[p, y_0] - y[q, y_0],$$

which we shorten to z. Then, z is a solution of:

$$\begin{cases} \partial_t z - \mu\partial^2_{xx} z + qz = (q-p)y[p, y_0], & \text{in } (0,T)\times\mathcal{E},\\ z = 0, & \text{on } (0,T)\times\mathcal{V}_\partial,\\ z^{e_i} = z^{e_j}, & \text{on } (0,T)\times\mathcal{V}_0,\ \forall e_i, e_j \in \mathcal{E}(v),\\ \displaystyle\sum_{e\in\mathcal{E}(v)} \mu^e \partial_{n^e} z^e = \gamma z, & \text{on } (0,T)\times\mathcal{V}_0,\\ z(0,\cdot) = 0, & \text{in } \mathcal{E}. \end{cases}$$

This implies that $\partial_t z$ is a solution of:

$$\begin{cases} \partial_t(\partial_t z) - \mu\partial^2_{xx}(\partial_t z) + q(\partial_t z) = (q-p)\partial_t(y[p, y_0]), & \text{in } (0,T)\times\mathcal{E},\\ \partial_t z = 0, & \text{on } (0,T)\times\mathcal{V}_\partial,\\ \partial_t z^{e_i} = \partial_t z^{e_j}, & \text{on } (0,T)\times\mathcal{V}_0,\ \forall e_i, e_j \in \mathcal{E}(v),\\ \displaystyle\sum_{e\in\mathcal{E}(v)} \mu^e \partial_{n^e}(\partial_t z^e) = \gamma(\partial_t z), & \text{on } (0,T)\times\mathcal{V}_0,\\ \partial_t z(0,\cdot) = (q-p)y_0, & \text{in } \mathcal{E}. \end{cases} \tag{3.3}$$

To continue, by applying Proposition 2.5 to the solution of (3.3) we obtain that:

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}\left((s\xi)^{-1}|\partial_{tt}z|^2 + s\lambda^2\xi|\partial_{tx}z|^2 + s^3\lambda^4\xi^3|\partial_t z|^2\right) dx\,dt \\
&\leq C\left(\iint_Q e^{-2s\alpha}(|\partial_t z|^2 + |(q-p)\partial_t(y[p,y_0])|^2)\,dx\,dt + s^3\lambda^4\right. \\
&\qquad \left.\iint_{Q_\omega} e^{-2s\alpha}\xi^3|\partial_t z|^2\,dx\,dt\right),
\end{aligned}$$

which implies that for s large enough:

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}\left((s\xi)^{-1}|\partial_{tt}z|^2 + s\lambda^2\xi|\partial_{tx}z|^2 + s^3\lambda^4\xi^3|\partial_t z|^2\right) dx\,dt \\
&\leq C\left(\iint_Q e^{-2s\alpha}|(q-p)\partial_t(y[p,y_0])|^2\,dx\,dt + s^3\lambda^4\iint_{Q_\omega} e^{-2s\alpha}\xi^3|\partial_t z|^2\,dx\,dt\right).
\end{aligned} \tag{3.4}$$

Since $\lim_{t\to 0} e^{-2s\alpha} = 0$, we obtain from (3.4) that:

$$\begin{aligned}
&\int_{\mathcal{E}} e^{-2s\alpha(T/2,x)}\left|\partial_t z\left(\frac{T}{2},x\right)\right|^2 dx \\
&\quad = \int_0^{T/2}\frac{\partial}{\partial t}\left(\int_{\mathcal{E}} e^{-2s\alpha(t,x)}\left|\partial_t z\,(t,x)\right|^2 dx\right) dt \\
&\quad \leq C\iint_Q e^{-2s\alpha}(s^{-1}\lambda^{-2}\xi^{-1}|\partial_{tt}z|^2 + s^2\lambda^2\xi^2|\partial_t z|^2)\,dx\,dt \\
&\quad \leq \frac{C}{\lambda^2}\iint_Q e^{-2s\alpha}|(q-p)\partial_t(y[p,y_0])|^2\,dx\,dt \\
&\qquad + Cs^3\lambda^2\iint_{Q_\omega} e^{-2s\alpha}\xi^3|\partial_t z|^2\,dx\,dt.
\end{aligned}$$

Next, we consider that, because of (1.5),

$$\begin{aligned}
&\int_{\mathcal{E}} e^{-2s\alpha(T/2,x)}(q(x)-p(x))^2\,dx \\
&\leq C\int_{\mathcal{E}} e^{-2s\alpha(T/2,x)}|(q(x)-p(x))y[p,y_0](T/2,x)|^2\,dx \\
&\leq C\int_{\mathcal{E}} e^{-2s\alpha(T/2,x)}\left|\partial_t z\left(\frac{T}{2},x\right)\right|^2 dx + C\left\|z\left(\frac{T}{2},\cdot\right)\right\|^2_{H^2_{pw}(\mathcal{E})} \\
&\leq Cs^3\lambda^2\iint_{Q_\omega} e^{-2s\alpha}\xi^3|\partial_t z|^2\,dx\,dt + C\left\|z\left(\frac{T}{2},\cdot\right)\right\|^2_{H^2_{pw}(\mathcal{E})} \\
&\quad + \frac{C}{\lambda^2}\iint_Q e^{-2s\alpha}|(q-p)\partial_t(y[p,y_0])|^2\,dx\,dt.
\end{aligned} \tag{3.5}$$

Thus, by considering $s \geq C(T + T^2)$ and $\lambda \geq C$ large enough and by estimating the weight α we obtain (1.6) from (3.5). □

3.3. Open problems

We highlight the following problems that remain open and may be considered for future work:

- **Bang-bang controls and semi-linear heat equation.** One of the main applications of the Carleman parabolic inequalities are bang-bang controls and, with that, the controllability of the semi-linear heat equation. Indeed, in parabolic equations whose domains are smooth manifolds it is known that if the non-linearity grows smoothly, then the system is controllable to trajectories (see, for example, [19,26,28,32,33] and, more recently, [27,45]). An open problem is whether or not our results may be applied to obtain the controllability of the following equation:

$$\begin{cases} a\partial_t y - \mu\partial_{xx}^2 y + G(y, \nabla y) = f 1_\omega, & \text{in } (0,T)\times\mathcal{E}, \\ y = 0, & \text{on } (0,T)\times\mathcal{V}_\partial, \\ y^{e_i} = y^{e_j}, & \text{on } (0,T)\times\mathcal{V}_0,\ \forall e_i, e_j \in \mathcal{E}(v), \\ \displaystyle\sum_{e\in\mathcal{E}(v)} \mu^e \partial_{n^e} y^e = \gamma y, & \text{on } (0,T)\times\mathcal{V}_0, \\ y(0,\cdot) = y_0, & \text{in } \mathcal{E}, \end{cases}$$

  for $G \in C^1$ satisfying the same growth hypothesis as in [19]. Indeed, we should be careful as regularity results on networks are not as powerful as in segments, as regular solutions belong to $L^2(0,T; H^2_{pw}(\mathcal{E}))$ instead of $L^2(0,T; H^2(\mathcal{E}))$.
- **Minimum number of edges on which the control acts.** Another open problem is the characterization of the minimum number of edges where the control domain has to be positioned so that the system (1.1) is controllable. The construction of algorithms for that purpose also remains an open problem.
- **Non-controllable cases.** The counterexamples in Remarks 1.1 and 1.2 show that system (1.1) is not controllable for arbitrary coefficients, but it suggests that such problems only arise for some critical coefficients when the quantity of controls is small. Since Carleman inequalities do not depend on the coefficients, it is likely that it is not the right tool for approaching this problem. A possible approach is to use spectral-oriented results, like those in [8,10]. Additionally, a possible solution is to use the characterization of controllability obtained in [41].
- **Less regular coefficients.** To obtain controllability results with less regular coefficients, for instance, when they all belong to L^∞. This may be done, for instance, using the techniques presented in [29,30].

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REFERENCES

[1] C. Altafini and F. Ticozzi. Modeling and control of quantum systems: An introduction. IEEE T. Automat. Contr., 57(8):1898–1917, 2012.

[2] J. Apraiz and L. Escauriaza. Null-control and measurable sets. ESAIM: COCV, 19(1):239–254, 2013.

[3] J. Apraiz and L. Escauriaza. Observability inequalities and measurable sets. J. Eur. Math. Soc., 16(11):2433–2475, 2014.

[4] J. A. Bárcena-Petisco. Null controllability of the heat equation in pseudo-cylinders by an internal control. ESAIM: COCV, 26(122):1–34, 2020.

[5] J. A. Bárcena-Petisco, M. Cavalcante, G. M. Coclite, N. de Nitti, and E. Zuazua. Control of hyperbolic and parabolic equations on networks and singular limits. hal-03233211, 2021.

[6] M. Bellassoued and M. Yamamoto. Carleman estimates and applications to inverse problems for hyperbolic systems. Springer, 2017.

[7] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau. Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem. J. Math. Anal. Appl., 336(2):865–887, 2007.

[8] K. Bhandari, F. Boyer, and V. Hernández-Santamaría. Boundary null-controllability of 1-D coupled parabolic systems with Kirchhoff-type conditions. Math. Control Signal, pages 1–59, 2021.

[9] V. D. Blondel, E. D. Sontag, M. Vidyasagar, and J. C. Willems. Open Problems in Mathematical Systems and Control Theory. Communication and Control Engineering Series. Springer, London, 1999.

[10] F. Boyer and G. Olive. Boundary null-controllability of some multi-dimensional linear parabolic systems by the moment method. hal-03175706, 2021.

[11] J. Brouwer, I. Gasser, and M. Herty. Gas pipeline models revisited: Model hierarchies, non-isothermal models and simulations on networks. Multiscale Model. Simul., 9:601–623, 2011.

[12] P. Cannarsa, A. Duca, and C. Urbani. Exact controllability to eigensolutions of the bilinear heat equation on compact networks. Discret. Contin. Dyn. S. - S, 15(6): 1377–1401, 2022. arXiv:2111.02250

[13] S. Chen and I. Lasiecka. Feedback exact null controllability for unbounded control problems in Hilbert space. J Optim. Theory App., 74(2):191–219, 1992.

[14] J.-M. Coron. Control and nonlinearity. Number 136. American Mathematical Society, 2007.

[15] R. Dager and E. Zuazua. Wave propagation, observation and control in 1-d flexible multi-structures, volume 50 of Mathematics & Applications. Springer Verlag, Berlin, 2006.

[16] J. Dardé and S. Ervedoza. On the reachable set for the one-dimensional heat equation. SIAM J. Control and Optim., 56(3):1692–1715, 2018.

[17] J. Dardé and S. Ervedoza. On the cost of observability in small times for the one-dimensional heat equation. Anal. and PDE, 12(6):1455–1488, 2019.

[18] A. C. Doherty, S. Habib, K. Jacobs, H. Mabuchi, and S. M. Tan. Quantum feedback control and classical control theory. Phys. Rev. A, 62:012105, 2000.

[19] A. Doubova, E. Fernández-Cara, M. González-Burgos, and E. Zuazua. On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. SIAM J. Control. Optim., 41(3):798–819, 2002.

[20] A. Doubova, A. Osses, and J.-P. Puel. Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients. ESAIM: COCV, 8:621–661, 2002.

[21] H. Egger and N. Philippi. On the transport limit of singularly perturbed convection-diffusion problems on networks. Math. Methods Appl. Sci., 44, 2021.

[22] K. Egger and T. Kugler. Damped wave systems on networks: exponential stability and uniform approximations. Numer. Math., 138(4):839–867, 2018.

[23] S. Ervedoza and E. Zuazua. Observability of heat processes by transmutation without geometric restrictions. Math. Control Related F., 1(2):177–187, 2011.

[24] L. Escauriaza, S. Montaner, and C. Zhang. Observation from measurable sets for parabolic analytic evolutions and applications. J. Math. Pure. Appl., 104(5):837–867, 2015.

[25] L. Escauriaza, S. Montaner, and C. Zhang. Analyticity of solutions to parabolic evolutions and applications. SIAM J. Control and Optim., 49(5):4064–4092, 2017.

[26] C. Fabre, J.-P. Puel, and E. Zuazua. Approximate controllability of the semilinear heat equation. Proc. R. Soc. E. A.-Ma., 125(1):31–61, 1995.

[27] L. A. Fernández. Controllability properties for some semilinear parabolic PDE with a quadratic gradient term. Appl. Math. Lett., 25(12):2184–2187, 2012.

[28] L. A. Fernández and E. Zuazua. Approximate controllability for the semilinear heat equation involving gradient terms. J. Optim. Theory Appl., 101(2):307–328, 1999.

[29] E. Fernández-Cara, M. González-Burgos, S. Guerrero, and J.-P. Puel. Null controllability of the heat equation with boundary Fourier conditions: the linear case. ESAIM:COCV, 12(3):442–465, 2006.

[30] E. Fernández-Cara and S. Guerrero. Global Carleman estimates for solutions of parabolic systems defined by transposition and some applications to controllability. Appl. Math. Research Express, 2006:75090, 2006.

[31] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations: the linear case. Adv. Differential Equ., 5(4-6):465–514, 2000.

[32] E. Fernández-Cara and E. Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. Ann. I. H. Poincare-An., 17(5):583–616, 2000.

[33] E. Fernández-Cara. Null controllability of the semilinear heat equation. ESAIM: COCV, 2:87–103, 1997.

[34] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of non-linear systems: Introductory theory and examples. Int. J. Control, 61(6):1327–1361, 1995.

[35] A. V. Fursikov and O. Y. Imanuvilov. Controllability of evolution equations, volume 34 of Lecture Notes Series. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[36] M. González-Burgos and L. de Teresa. Some results on controllability for linear and nonlinear heat equations in unbounded domains. Adv. Differential Equ., 12(11):1201–1240, 2007.

[37] F. M. Hante, G. Leugering, A. Martin, L. Schewe, and M. Schmidt. Challenges in optimal control problems for gas and fluid flow in networks of pipes and canals: From modeling to industrial applications. In Industrial mathematics and complex systems, pages 77–122. Springer, 2017.

[38] L. Ignat, A. F. Pazoto, and L. Rosier. Inverse problem for the heat equation and the Schrödinger equation on a tree. Inverse Prob., 28(1):015011, 2011.

[39] O. Y. Imanuvilov and M. Yamamoto. Lipschitz stability in inverse parabolic problems by the Carleman estimate. Inverse Prob., 14:1229–1245, 1998.

[40] O. Y. Imanuvilov and M. Yamamoto. Global Lipschitz stability in an inverse hyperbolic problem by interior observations. Inverse Prob., 17(4):717, 2001.
[41] S. Iwasaki. Observability for the heat equation in equilateral metric graphs. In 2021 60th Annual Conference of the Society of Instrument and Control Engineers of Japan (SICE), pages 1270–1275. IEEE, 2021.
[42] R. E. Kalman. Contributions to the theory of optimal control. Bol. Soc. Mat. Mexicana, 5:102–119, 1960.
[43] L. E. Lagnese, G. Leugering, and E. J. P. G. Schmidt. Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures, volume 19 of Systems Control: Foundations Applications. Springer Science+Business Media, New York, 1994.
[44] B. Laroche, P. Martin, and P. Rouchon. Motion planning for the heat equation. Int. J. Robust Nonlinear Control, 10(8):629–643, 2000.
[45] K. Le Balc'h. Global null-controllability and nonnegative-controllability of slightly superlinear heat equations. J. Math. Pure. Appl., 135:103–139, 2020.
[46] J. Le Rousseau and L. Robbiano. Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces. Invent. Math., 183(2):245–336, 2011.
[47] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. Commun. Part. Diff. Eq., 20(1):335–356, 1995.
[48] P. Lin. Global blowup controllability of heat equation with feedback control. Commun. Contemp. Math., 20(5):1750062–1–11, 2018.
[49] J. L. Lions. Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, Contrôlabilité exacte. With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch, volume 8 of Recherches en Mathématiques Appliqués. Masson, Paris, 1988.
[50] J. L. Lions. Exact controllability, stabilization and perturbations for distributed systems. SIAM Review, 30(1):1–68, 1988.
[51] P. Martin, L. Rosier, and P. Rouchon. Null controllability of the heat equation using flatness. Automatica, 50(12):3067–3076, 2014.
[52] V. Mehandiratta, M. Mehra, and G. Leugering. Optimal control problems driven by time-fractional diffusion equations on metric graphs: optimality system and finite difference approximation. SIAM J. Control and Optim., 59(6):4216–4242, 2021.
[53] L. Miller. Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time. J. Differ. Equation, 204(1):202–226, 2004.
[54] L. Miller. The control transmutation method and the cost of fast controls. SIAM J. Control and Optim., 45(2):762–772, 2006.
[55] M. Newman, A. L. Barabási, and D. J. Watts. The Structure and Dynamics of Networks, volume 19 of Princeton Studies in Complexity. Princeton University Press, 2011.
[56] G. Notarstefano and G. Parlangeli. Controllability and observability of grid graphs via reduction and symmetries. IEEE T. Automat. Contr., 58(7):1719–1731, 2013.
[57] D. L. Russell. A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. Stud. Appl. Math., 52(3):189–211, 1973.
[58] M. Sîrbu. A Riccati equation approach to the null controllability of linear systems. Comm. Appl. Anal., 164–177(2), 2002.
[59] M. Sîrbu and G. Tessitore. Null controllability of an infinite dimensional sde with state and control-dependent noise. Syst. Control Lett., 385–394(44), 2001.
[60] G. Tenenbaum and M. Tucsnak. New blow-up rates for fast controls of Schrödinger and heat equations. J. Differ. Equations, 243(1):70–100, 2007.
[61] A. Thosar, A. Patra, and S. Bhattacharyya. Feedback linearization based control of a variable air volume air conditioning system for cooling applications. ISA Transactions, 47:339–349, 2008.
[62] G. Wang. L^∞-null controllability for the heat equation and its consequences for the time optimal control problem. SIAM J. Control and Optim., 47(4):1701–1720, 2008.
[63] W. M. Wonham. Linear Multivariable Control, a Geometric Approach, volume 10 of Applications of Mathematics. Springer-Verlag, 1985.
[64] M. Yamamoto. Carleman estimates for parabolic equations and applications. Inverse Prob., 25(12):123013, 2009.

[65] C. Zhang. An observability estimate for the heat equation from a product of two measurable sets. J. Math. Anal. Appl., 396(1):7–12, 2012.

[66] J. Zhao and X. Zhang. Inverse tangent functional nonlinear feedback control and its application to water tank level control. Processes, 8(3: 347), 2020.

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