THE DUAL INTEGRATED FORCE METHOD APPLIED TO UNIDIRECTIONAL COMPOSITES

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ABSTRACT

This paper shows the use of the Dual Integrated Force Method in Finite Elements Method and its application to composite materials. This method was developed by S.N. Patnaik in isotropic materials, considering not only the equilibrium equations but also the compatibility conditions. In the Dual Integrated Force Method, the principal unknowns are the displacements and the structure of governing equation is similar to the stiffness method. It is shown that the governing equation of the Dual Integrated Force Method is the same than in the case of the hybrid method of Pian. The method is applied to two examples: A cantilever beam of orthotropic material loaded at the end and one off-axis tensile test in a unidirectional composite specimen. The results of this method have been compared with the ones obtained from the application of the Stiffness Method and with analytical results.

Keywords: Dual Integrated Force Method; Hybrid element; Unidirectional composite; Stiffness method

1. INTRODUCTION

The Stiffness Method (SM) or displacement method applied to the Finite Element Method (FEM) is one usual calculation procedure in structural mechanics and mechanics of solids [1]. This method is based on the equilibrium equations of each element, relating the forces acting on the nodes with displacements at the same nodes. The equilibrium equations are derived from the principle of virtual work.

The Integrated Force Method (IFM) was developed by Patnaik, who published an article in 1973 about this Method on discrete structures [2]. In that work it appears the idea of using not only the equilibrium equations, but also the compatibility conditions. A comparison between IFM and the Force Method (FM)[3] based on Airy's and Beltrami-Michell's formulations was carried out. In 1986, he developed a variational functional for the IFM [4] based on the principles of virtual work and complementary virtual work. In Elasticity, Patnaik et al. applied these principles in order to complete the Beltrami-Michell's formulation and finally, he obtained the equations of IFM in continuous systems [5,6].

Patnaik et al. developed a systematic way to get the compatibility equations applied to both discrete and continuous systems [7,8] and then he applied the IFM to FEM [9]. They generated the IFM governing matrix of various elements in two dimensions [10] and three dimensions [11], assuming isotropic materials.

The IFMD was also developed by Patnaik [12] from the same formulation than the IFM, being the structure of the governing equation similar to SM, i.e., their primary unknowns being displacements. The IFM and IFMD have similar governing equations, equivalent formulations and provide identical solutions for stresses and displacements.

Fraeijs de Veubeke [13] showed that the SM and FM are lower and upper bounds of the actual solution. SM is based on the assumption of a compatible displacement field. The application of the Principle of Virtual Work (PVW) ensures the fulfilment of equilibrium equations as a function of the nodal displacements. Since all the admissible displacements of the system are not included, the system is more constrained than in reality is and the solution is over-stiff, being its energy a lower bound [14]. On the other hand, FM is based on the assumption of an equilibrated stress field. The application of the Principle of Complementary Virtual Work (PCVW) ensures that equations of compatibility are satisfied as a function of the assumed stress parameters. Since all the admissible stresses of the system are not present, the system has fewer constraints than in the actual case, it is more flexible and the energy is an upper bound. As a consequence, Fraeijs de Veubeke proposed a dual analysis based on the alternate use of displacement and equilibrium modes for obtaining a quantitative estimate of the convergence to the true solution, by comparison of the upper and lower bounds obtained.

Otherwise, in 1964, the hybrid stress element was formulated by Pian based on the principle of stationary complementary energy [15]. Later he formulated that element by the application of the Hellinger- Reissner principle [16], assuming independent stress and displacement fields. The term hybrid element is defined as the one which is formulated by multifield variational functional, yet the resulting matrix equations consist of only the nodal values of displacements as unknown. [17].

A disadvantage of the hybrid method is the additional computational cost of the inversion of the flexibility matrix required to construct the stiffness matrix. This disadvantage has been solved by Zhang et al. [18,19] by means of the diagonalization of the flexibility matrix.

In the present work the formulation concerning SM, based on the PVW and IFMD, based on both PVW and on the PCVW, are summarized. Then, it is shown that the hybrid formulation of Pian and IFMD are equivalent. Finally, IFMD is applied to two examples of unidirectional composites in plane stress state and results are compared with SM.

2. THE STIFFNESS METHOD

The SM is achieved by applying the PVW. In this principle, the virtual work done by the external forces is equal to the virtual work of the stresses, that is:

$$\int_v \{\delta\varepsilon\}^T \{\sigma\}\, dv = \{\delta a^i\}^T \{P^i\} \tag{1}$$

Where V is the volume of the element; $\{\delta\varepsilon\}$ and $\{\delta a^i\}$ are virtual variations of strain and nodal displacement vectors, respectively; $\{\sigma\}$ is the stress vector; and $\{P^i\}$ is the equivalent nodal forces vector.

The displacements at any point of the element are obtained from the nodal displacements by means of interpolation functions. In matrix form it results:

$$\{u\} = [N]\{a^i\} \tag{2}$$

Where $\{u\}$ is the vector of displacements at any point of the element and $[N]$ is the matrix of displacement interpolation functions. The displacements in the element $\{u\}$ are related to the strains $\{\varepsilon\}$ through the matrix of the differential operators $[L]$:

$$\{\varepsilon\} = [L]\{u\} \tag{3}$$

Combining equation (2) and (3) the strains at any point of the element are obtained by means of the nodal displacements, being:

$$\{\varepsilon\} = [L][N]\{a^i\} = [B]\{a^i\} \tag{4}$$

Where $[B]$ is the nodal displacement-strain matrix. Stresses $\{\sigma\}$ and strains $\{\varepsilon\}$ are related by the constitutive matrix of the material $[D]$:

$$[\sigma] = [D]\{\varepsilon\} \tag{5}$$

Replacing equations (4) and (5) in equation (1) it results:

$$\{\delta a^i\}^T \left(\int_v [B]^T [D][B]\, dv \right) \{a^i\} = \{\delta a^i\}^T \{P^i\} \tag{6}$$

As $\{\delta a^i\}$ are arbitrary, the governing equation of SM is:

$$[K]\{a^i\} = \{P^i\} \tag{7}$$

Where $[K]$ is the stiffness matrix given by:

$$[K] = \int_v [B]^T [D][B]\, dv \tag{8}$$

The stiffness matrix of the structure is obtained by assembling the stiffness matrices of the elements.

According to this formulation, equation (7) represents the fulfilment of equilibrium equations by the application of the PVW. Stresses are obtained from equations (5), (3) and (2) as a function of nodal displacements and they satisfy equilibrium equations in the element.

3. DUAL INTEGRATED FORCE METHOD

3.1 EQUILIBRIUM MATRIX

Stresses at any point of the element are expressed as:

$$\{\sigma\} = [Y]\{F^j\} \tag{9}$$

Where $\{\sigma\}$ is the vector of stress components; [Y] is the matrix of force interpolation function, that satisfy equilibrium equations [20]; and $\{F^j\}$ are the vector of stress parameters. Index j is used instead of i, due to the number of independent forces is lesser that the number of independent displacements. Then, replacing equation (9) in equation (1) it results:

$$\{\delta a^i\}^T \left(\int_v [B]^T [Y] dv \right) \{F^j\} = \{\delta a^i\}^T \{P^i\} \tag{10}$$

Then, the principle of virtual work can be written in the following alternative form [21]:

$$[E]\{F^j\} = \{P^i\} \tag{11}$$

Where $[E] = \int_v [B]^T [Y] dv$ is defined as the equilibrium matrix.

3.2 DEFORMATION DISPLACEMENT RELATION

This relation is obtained by the equality between the internal energy (*IE*) stored in the structure and the work (*W*) done by external loads [7,22]. In lineal elastic behaviour, the internal energy stored in the structure is derived from the internal forces and the deformations caused by these forces, being

$$IE = \frac{1}{2}\{F^j\}^T \{\beta^j\} \tag{12}$$

Where $\{\beta^j\}$ is the vector of deformations linked to internal forces. For instance, in the case of a beam, the deformation for the axial force is the extension and for the bending moment it is the curvature. The work done by external loads (*W*) is derived from the external loads and from displacements caused by these loads. For liner elastic behaviour:

$$W = \frac{1}{2}\{P^i\}^T \{a^i\} \tag{13}$$

Equating equations (12) and (13) it results:

$$\{F^j\}^T \{\beta^j\} = \{P^i\}^T \{a^i\} \tag{14}$$

Replacing equation (11) in equation (14):

$$\{F^j\}^T\{\beta^j\} = \{F^j\}^T [E]^T \{a^i\} \tag{15}$$

Equation (15) can be written as:

$$\{F^j\}^T\left([E]^T\{a^i\} - \{\beta^j\}\right) = 0 \tag{16}$$

As $\{F^j\}^T$ are independent, the deformation-displacements relation (DDR) is given by:

$$\{\beta^j\} = [E]^T\{a^i\} \tag{17}$$

3.3 FORCE- DEFORMATION RELATION

By applying the PCVW, internal forces and deformations are related by means of the flexibility matrix [21,23]. This principle states that:

$$\int_v \{\delta\sigma\}^T\{\varepsilon\}\,dv = \{\delta P^i\}^T\{a^i\} \tag{18}$$

Replacing equation (9) into (18) gives

$$\int_v \{\delta F^j\}^T [Y]\{\varepsilon\}\,dv = \{\delta P^i\}^T\{a^i\} \tag{19}$$

The stresses and strains are related by the compliance matrix [*S*], being

$$\{\varepsilon\} = [S]\{\sigma\} \tag{20}$$

Replacing equations (9), (11) and (20) into (19) gives

$$\{\delta F^j\}^T\left(\int_v [Y]^T[S][Y]\,dv\right)\{F^j\} = \{\delta F^j\}^T[E]^T\{a^i\} \tag{21}$$

And from equation (17)

$$\{\delta F^j\}^T\left(\int_v [Y]^T[S][Y]\,dv\right)\{F^j\} = \{\delta F^j\}^T\{\beta^j\} \tag{22}$$

As $\{\delta F^j\}$ are arbitrary, equation (21) gives:

$$\int_v\left([Y]^T[S][Y]\,dv\right)\{F^j\} = \{\beta^j\} \tag{23}$$

Thus, Force Deformation Relation (FDR) is

$$\{\beta^j\} = [G]\{F^j\} \tag{24}$$

Being $[G]$ the flexibility matrix given by

$$[G] = \int_v [Y]^T [S][Y] dv \tag{25}$$

3.4 GOVERNING EQUATION OF THE DUAL INTEGRATED FORCE METHOD

The IFMD formulation is derived eliminating $\{\beta^j\}$ from FDR in equation (24) and DDR in equation (17), resulting in:

$$[G]\{F^j\} = [E]^T \{a^i\} \tag{26}$$

Extracting $\{F^j\}$ from equation (26), internal forces and displacements are related by:

$$\{F^j\} = [G]^{-1} [E]^T \{a^i\} \tag{27}$$

Replacing equation (27) into (11), the governing equation of IFMD is:

$$[E][G]^{-1}[E]^T \{a^i\} = \{P^i\} \tag{28}$$

Equation (28) can be written as:

$$[K]\{a^i\} = \{P^i\} \tag{29}$$

Where $[K]$ is the Stiffness Matrix of IFMD:

$$[K] = [E][G]^{-1}[E]^T \tag{30}$$

The Stiffness Matrix of the structure is obtained by following standard assembly procedure, in the same manner than in SM.

Nodal displacements $\{a^i\}$ are obtained from equation (29) and replacing them in equation (27) the vector of stress parameters $\{F^j\}$ is obtained. It is worth noting that, according to equation (30), the equilibrium matrix $[E]$ obtained from the PVW and the flexibility matrix $[G]$ obtained from the PCVW are included in the stiffness matrix [K]. Since displacements parameters that stiffen the system and stress parameters that make the system more flexible are included in the formulation, it is expected that the solution obtained will be between the upper and lower bounds that correspond to SM and FM, respectively.

4. EQUIVALENCE OF IFMD AND THE HYBRID STRESS ELEMENT OF PIAN.

In 1964, Pian developed a method to obtain the element stiffness matrix by an assumed stress distribution [15]. Initially, the variational method for the formulation of the hybrid stress finite element was based on the principle of stationary complementary energy, but it was realized that is more convenient to construct the element stiffness matrices by using the Hellinger-Reissner functional [16].

In a solid with surface tractions prescribed on S_T and in absence of prescribed displacements, the stress distribution over the element can be determined by the Hellinger - Reissner variational principle [17, 24]. The functional is:

$$\pi_{HR} = \int_v \left[-\tfrac{1}{2}\{\sigma\}^T [S]\{\sigma\} + \{\sigma\}^T \left([L]\{u\}\right) - \{u\}^T \{F_v\} \right] dv - \int_{S_T} \{u\}^T \{F_s\} ds \tag{31}$$

Where $\{u\}$ is the displacement vector, $[L]$ is the matrix of differential operators, $[S]$ is the compliance matrix, $\{F_v\}$ is the vector of body forces and $\{F_s\}$ is the vector of surface forces.

Replacing equations (2) and (9) into (31),

$$\begin{aligned}\pi_{HR} = -\tfrac{1}{2}\{F^j\}^T \left(\int_v [Y]^T [S][Y] dv \right) \{F^j\} + \{F^j\}^T \left(\int_v [Y]^T [L][N] dv \right) \{a^i\} - \\ -\{a^i\}^T \left(\int_v [N]^T \{F_v\} dv + \int_{S_T} [N]^T \{F_s\} ds \right)\end{aligned} \tag{32}$$

And then,

$$\pi_{HR} = -\tfrac{1}{2}\{F^j\}^T [G]\{F^j\} + \{F^j\}^T [T]\{a^i\} - \{a^i\}^T \{P^i\} \tag{33}$$

Where $[G]$ is the flexibility matrix of equation (25), $[T]$ is the leverage matrix and $\{P^i\}$ is the vector of equivalent nodal forces. In this functional, the independent variables subjected to variation are $\{F^j\}$ and $\{a^i\}$. From the partial stationary condition respect to $\{F^j\}$, the relation between assumed stress coefficients $\{F^j\}$ and nodal displacements $\{a^i\}$ is,

$$[G]\{F^j\} = [T]\{a^i\} \tag{34}$$

and extracting $\{F^j\}$,

$$\{F^j\} = [G]^{-1} [T]\{a^i\} \tag{35}$$

Now, substituting equations (35) into (33),

$$\pi_{HR} = \tfrac{1}{2}\{a^i\}^T [T]^T [G]^{-1} [T]\{a^i\} - \{a^i\}^T \{P^i\} \tag{36}$$

By the partial stationary condition with respect to $\{a^i\}$, the governing equation of the element is obtained.

$$[T][G]^{-1}[T]^T\{a^i\}-\{P^i\}=0 \tag{37}$$

And then,

$$[K]\{a^i\}=\{P^i\} \tag{29}$$

In equations (26) and (34) it can be seen that the transpose of the leverage matrix of Pian [16,25] is the equilibrium matrix of IFMD, that is, $[T]=[E]^T$ .Therefore, both methods have the same analytical expression and the equilibrium equation (11) can be written as:

$$[T]^T\{F^j\}=\{P^i\} \tag{38}$$

Further analysis verifying the validity of the method for general quadrilaterals has been carried out by Pian and Sumihara [26].

5. APLICATION OF IFMD TO RECTANGULAR MEMBRANE ELEMENT

5.1 MATRIX OF ELEMENT EQUILIBRIUM [E]

The four nodes rectangular membrane element [27] of figure 1, is used as example for implementing IFMD.

The same lineal Lagrange polynomials than in SM are used as displacement interpolation functions for this element [1], as follows:

$$u(x,y)=\frac{1}{4}\{(1-\xi)(1-\eta)X_1+(1+\xi)(1-\eta)X_3+(1+\xi)(1+\eta)X_5+(1-\xi)(1+\eta)X_7\} \tag{39}$$

$$v(x,y)=\frac{1}{4}\{(1-\xi)(1-\eta)X_2+(1+\xi)(1-\eta)X_4+(1+\xi)(1+\eta)X_6+(1-\xi)(1+\eta)X_8\} \tag{40}$$

Where $\xi= x/a$ and $\eta=y/b$; and X_1, X_2... X_8 are the eight displacement degrees of freedom of the element.

The selected force interpolation functions [3,20] have been derived from the Airy stress functions, using a complete polynomial of third order, being:

$$[Y]=\begin{bmatrix}1 & \eta & 0 & 0 & 0\\ 0 & 0 & 1 & \xi & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix} \tag{41}$$

Using equation (41), equation (9) can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{1}{t} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{1}{t} \begin{bmatrix} 1 & \eta & 0 & 0 & 0 \\ 0 & 0 & 1 & \xi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} \tag{42}$$

Where N_x, N_y, N_{xy} are forces per unit length; $F_1, F_2 ... F_5$ are independent internal forces per unit length. Then, by differentiation of the displacement interpolation functions, the displacement-strain matrix $[B]$ of equation (4) is given by:

$$[B] = \frac{1}{4} \begin{bmatrix} \frac{-1}{a}(1-\eta) & 0 & \frac{1}{a}(1-\eta) & 0 & \frac{1}{a}(1+\eta) & 0 & \frac{-1}{a}(1+\eta) & 0 \\ 0 & \frac{-1}{b}(1-\xi) & 0 & \frac{-1}{b}(1+\xi) & 0 & \frac{1}{b}(1+\xi) & 0 & \frac{-1}{b}(1-\xi) \\ \frac{-1}{b}(1-\xi) & \frac{-1}{a}(1-\eta) & \frac{-1}{b}(1+\xi) & \frac{1}{a}(1-\eta) & \frac{1}{b}(1+\xi) & \frac{1}{a}(1-\eta) & \frac{1}{b}(1-\xi) & \frac{-1}{a}(1+\eta) \end{bmatrix} \tag{43}$$

In this case, as thickness is constant along the element, equilibrium matrix is extended to the surface of the element, being

$$[E] = \int_s [B]^T [Y] ds \tag{44}$$

Applying equation (44), $[E]$ of a four nodes rectangular membrane element is

$$[E] = \begin{bmatrix} -b & b/3 & 0 & 0 & -a \\ 0 & 0 & -a & a/3 & -b \\ b & -b/3 & 0 & 0 & -a \\ 0 & 0 & -a & -a/3 & b \\ b & b/3 & 0 & 0 & a \\ 0 & 0 & a & a/3 & b \\ -b & -b/3 & 0 & 0 & a \\ 0 & 0 & a & -a/3 & -b \end{bmatrix} \tag{45}$$

4.2 FLEXIBILITY MATRIX [G]

The compliance matrix of a unidirectional off-axis composite is [28,29]:

$$[S] = \begin{bmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ S_{sx} & S_{sy} & S_{ss} \end{bmatrix} \tag{46}$$

Where the compliance coefficients are:

$$
\begin{aligned}
S_{xx} &= m^4 S_{11} + 2m^2n^2 S_{12} + n^4 S_{22} + m^2n^2 S_{66} \\
S_{xy} &= m^2n^2 S_{11} + m^2n^2 S_{22} + \left(m^2 + n^2\right) S_{12} - m^2n^2 S_{66} \\
S_{yy} &= n^4 S_{11} + 2m^2n^2 S_{12} + m^4 S_{22} + m^2n^2 S_{66} \\
S_{xs} &= 2m^3n S_{11} + 2\left(mn^3 - nm^3\right) S_{12} - 2n^3m S_{22} + \left(mn^3 - nm^3\right) S_{66} \\
S_{ys} &= 2n^3m S_{11} + 2\left(nm^3 - mn^3\right) S_{12} - 2m^3n S_{22} + \left(nm^3 - mn^3\right) S_{66} \\
S_{ss} &= 4m^2n^2 S_{11} - 8m^2n^2 S_{12} + 4m^2n^2 S_{22} + \left(m^2 - n^2\right)^2 S_{66}
\end{aligned}
\qquad (47)
$$

Where $m = \cos\alpha$ and $n = \sin\alpha$ being α the fibre orientation angle. The compliance coefficients in the principal directions of orthotropy 1, 2 are:

$$S_{11} = 1/E_1 \,, S_{12} = -\upsilon_{12}/E_1 \,, S_{22} = 1/E_2 \quad S_{66} = 1/G_{12}$$

Where E_1 is the Young modulus in the fiber direction, E_2 is the modulus in the transverse direction; G_{12} is the in-plane shear modulus; and ν_{12} is the major Poisson ratio.

As in the case of the equilibrium matrix [*E*], the flexibility matrix of equation (24) is extended to the surface of the element, being:

$$[G] = \int_S [Y]^T [S][Y] ds \qquad (48)$$

Then, from equations (41), (46) and (48), flexibility matrix expression is obtained:

$$[G] = \frac{4ab}{t} \begin{bmatrix} S_{xx} & 0 & S_{xy} & 0 & S_{xs} \\ 0 & \frac{1}{3}S_{xx} & 0 & 0 & 0 \\ S_{xy} & 0 & S_{yy} & 0 & S_{ys} \\ 0 & 0 & 0 & \frac{1}{3}S_{yy} & 0 \\ S_{xs} & 0 & S_{ys} & 0 & S_{ss} \end{bmatrix} \qquad (49)$$

Replacing Equilibrium matrix (45) and Flexibility matrix (49) in equation (30) the stiffness matrix of the element is obtained. The stiffness matrix of the structure is obtained by assembling the stiffness matrices of the elements, in the same manner than in SM.

After knowing the nodal displacements, replacing the Flexibility matrix (49) and Equilibrium matrix (45) in the equation (27), the stress parameters of each element are obtained. Finally, by means of equation (42) the stresses in any point of the element are determined.

5. EXAMPLES OF APLICATION

5.1 CANTILEVER BEAM. ORTHOTROPIC MATERIAL

An IFMD analysis of the cantilever beam shown in Fig 2 has been done. The fibers are in longitudinal direction. The unidirectional material analyzed is a carbon/epoxy composite being the elastic properties: E_1=142GPa; E_2=8.9GPa; ν_{12}=0.28; G_{12}=4.8GPa.

Displacements and stresses obtained by IFMD have been compared with those obtained by SM and the analytical approach related to Timoshenko beam theory, which includes transverse shear effects. The vertical displacement at any point of the middle line is given by:

$$v = \frac{6Px^2}{E_1\, t\, h^3}\left(-L + \frac{x}{3}\right) - \frac{3}{5}\frac{Px}{G_{12}ht} \tag{50}$$

Otherwise, maximum stress is:

$$\sigma_x = \frac{6P}{th^2}\left(-L + x\right) \tag{51}$$

The dimensions and the load are: L = 100mm; h = 4mm; t = 2mm and P = 10N. Thus, maximum displacement and stress are:

$$\begin{aligned} v_{\max} &= 2.232\, mm \\ \sigma_x &= 187.5\, MPa \end{aligned} \tag{52}$$

Figure 3 shows the displacements obtained by IFMD and SM in comparison with those obtained with the analytical approach from equation (50). In IFMD analysis, the results obtained with four and eight elements are very similar to the ones obtained from equation (50). However, in SM analysis, the displacements are much lower than the analytical values.

Figure 4 shows the stress distribution in IFMD and SM analysis in comparison with the theoretical distribution from equation (51). The error in the maximum stress by means of IFMD with eight elements is around 6%. The stresses obtained by IFMD and SM are uniform along the length of the element. In the case of IFMD, the average values of the nodes agree with the values of Equation (51) except at the ends. In the case of SM, the maximum difference occurs at the clamped end.

The results obtained in IFMD are better than in SM because in IFMD the PVW and the PCVW have been applied to obtain stresses as displacements. However, in SM the displacements are obtained by applying the PVW and then, stresses are obtained by differentiation of the displacements.

Figure 5 shows the theoretical values of maximum displacement and maximum stress and the results obtained by means of IFMD and SM varying the number of elements. In IFMD, the maximum displacement value converges faster to the analytic solution than that obtained by SM.

5.2 OFF-AXIS TENSILE TEST. ANISOTROPIC MATERIAL

The example of Figure 6 shows a beam of the same unidirectional composite material than in the previous case subjected to an off-axis tensile test. The orientation of the fiber with respect to the longitudinal direction of the beam is 10º. The results corresponding to IFMD and SM are compared with the analytical approach proposed by Mujika [30].

Horizontal and vertical displacements *u* and *v*, respectively, in the lowest line of the specimen are:

$$u = \frac{Px}{A}\left(S_{xx} + S_{xs}k_1\right)$$
$$v = k_1 P\left[\frac{S_{xx}x^2}{I}\left(-2x+3L\right)+\tfrac{6}{5}\frac{S_{ss}x}{A}\right]+\frac{S_{xs}Px}{A} \quad (53)$$

Where $A = ht$ and $I = \frac{2}{3}th^3$. The stresses σ_x and τ_s in the lowest line of figure 6 are given by

$$\left(\sigma_x\right)_{\max} = \sigma_0\left(1+3k_1\,c\right)$$
$$\tau_s = 0 \quad (54)$$

Where $\sigma_0 = P/th$ is the applied nominal stress. The parameter k_1 depends on the compliance coefficients and on the length-to-width ratio *c*, being:

$$k_1 = \frac{-S_{xs}}{S_{xx}c^2 + 1{,}2\,S_{ss}} \quad (55)$$

Figure 7 shows the horizontal and vertical displacements of the lowest line obtained from equations (53) and the ones obtained by SM and IFMD with 8 elements. The agreement between SM and IFMD is much better than in the previous example. This is attributed to the fact that external constrains of the tensile case reduce the admissible displacements of the system. Therefore, displacement results obtained by SM are better when the system is more constrained.

Figure 8 shows the values of σ_x in the lowest line obtained by equation (54) in comparison with the average values obtained by IFMD and SM with 4 and 8 elements. The agreement with analytic results is better in the case of IFMD. In the case of SM, the difference with respect to analytic results is similar in the case of 4 and 8 elements, being maximum at the ends of the specimen. This fact is attributed to the differences between analytic and numeric slopes of displacements of Figure 7 that are maxima at the ends of the specimen in both cases. As strains in SM are obtained by differentiation, the errors of strains and stresses are maxima at the ends. Nevertheless, stresses in IFMD are obtained directly without differentiation. Therefore, the results of SM can be attributed to the process of differentiation of displacements.

6. SUMMARY AND CONCLUSIONS

The Finite Element Method is implemented in four node membrane elements using two different formulations: The SM based on the PVW and the IFMD based on the PVW and on the PCVW. It is shown that the resulting method is equivalent to the hybrid method of Pian, based on the Hellinger-Reissner stationary principle.

Two examples concerning unidirectional composites have been solved and compared with analytical approaches: An orthotropic cantilever beam and an off-axis tensile test. The results obtained show that displacements and stresses obtained by IFMD agree better with analytical results than the ones obtained by SM. This is due to both factors:

- Taking into account that the PVW is equivalent to equilibrium equations and that the PCVW is equivalent to compatibility conditions, IFMD satisfy the conditions of the problem better than the SM based only on the PVW.
- While stresses in IFMD are obtained directly from displacements, in SM it is necessary the differentiation of displacements.

The cantilever beam has less deformation restrictions than the tensile specimen that is clamped at both ends. The displacements of both examples show that the results obtained by SM are better when the constraints of the system increase. Otherwise, differences in stresses obtained by SM and IFMD are significant in the second example at the ends of the specimen, in spite of the differences between displacements are not. This is due to the differentiation of displacements needed in SM for obtaining stresses.

NOMENCLATURE

$[K]$ = Stiffness Matrix
$\{a^i\}$ = Nodal displacements vector
$\{P^i\}$ = External forces vector
$[K]_{Ifmd}$ = IFMD stiffness matrix
$\{F^j\}$ = Stress parameters vector
IE = Internal energy
W = Work of the external forces
$\{\beta^j\}$ = Deformations vector
$[G]$ = Flexibility matrix
$[E]$ = Equilibrium matrix
FDR = Flexibility relation (Forces- displacements relation)
DDR = Deformations - displacements relation
$[Y]$ = Matrix of forces interpolation functions
$[N]$ = Matrix of displacements interpolation functions
$[L]$ = Matrix of differential operators
$[B]$ = Displacement- strain matrix
$\{u\}$ = Vector of displacements at any point in the element
U_e = Strain energy
$\{N_x, N_y, N_{xy}\}$ = Vector of forces per unit length at any point in the element
$[S]$ = Compliance matrix of the material
$[D]$ = Stiffness matrix of the material
$\{\varepsilon\}$ = Strain field vector

$\{\sigma\}$	= Stress field vector
$\{\delta a^i\}$	= Virtual displacement
$\{\delta F^j\}$	= Virtual internal force
$[T]$	= Leverage matrix
u	= Horizontal displacement
v	= Vertical displacement
α	= Fiber orientation angle with respect to the longitudinal direction of the beam
m	= $\cos \alpha$
n	= $\sin \alpha$

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FIGURE CAPTIONS

Figure 1. Four nodes rectangular element.
Figure 2. Cantilever beam with two tip forces at the end.
Figure 3. Vertical displacements of the cantilever beam: a/ 4 elements. b/ 8 elements.
Figure 4. Stress distribution of the cantilever beam: a/ 4 elements. b/ 8 elements.
Figure 5. Maximum values depending on discretization: a/ Displacement; b/ Stress.
Figure 6. Off-axis tensile test.
Figure 7. Displacements in the off-axis tensile specimen: a/ Horizontal displacements, *u*; b/ Vertical displacements, *v*.
Figure8. Stress distribution in the lowest line of the off-axis tensile test: a/ 4 elements; b/ 8 elements.

Figure 1. Four nodes rectangular element

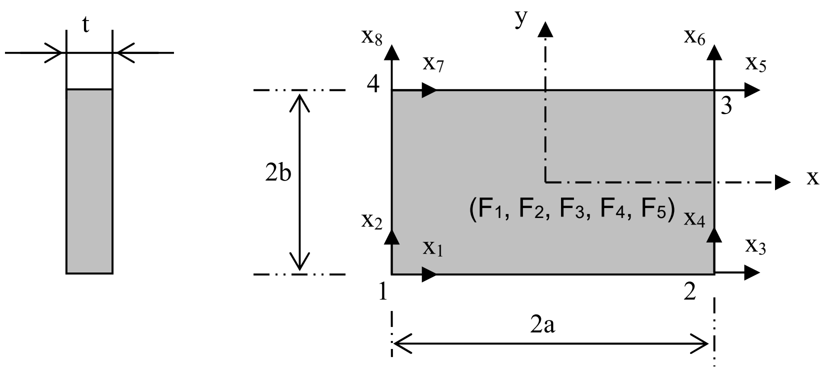


Figure 2. Cantilever beam with two tip forces at the end.

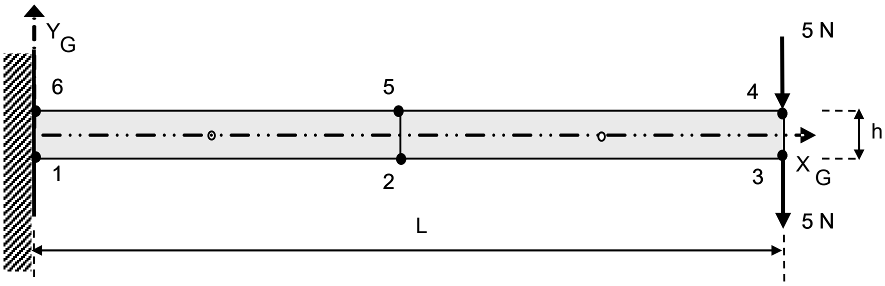
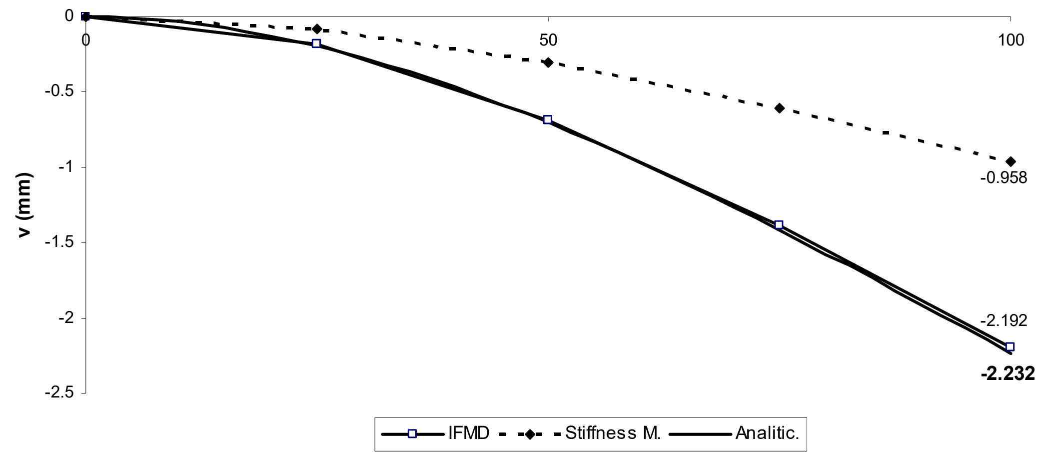
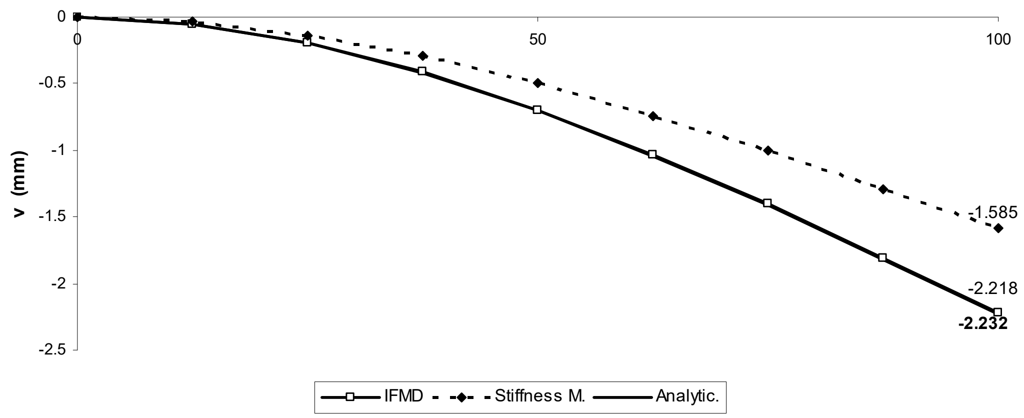


Figure 3. Vertical displacements of the cantilever beam:
a/ 4 elements; b/ 8 elements.



a/



b/

Figure 4. Stress distribution of the cantilever beam:
a/ 4 elements; b/ 8 elements.

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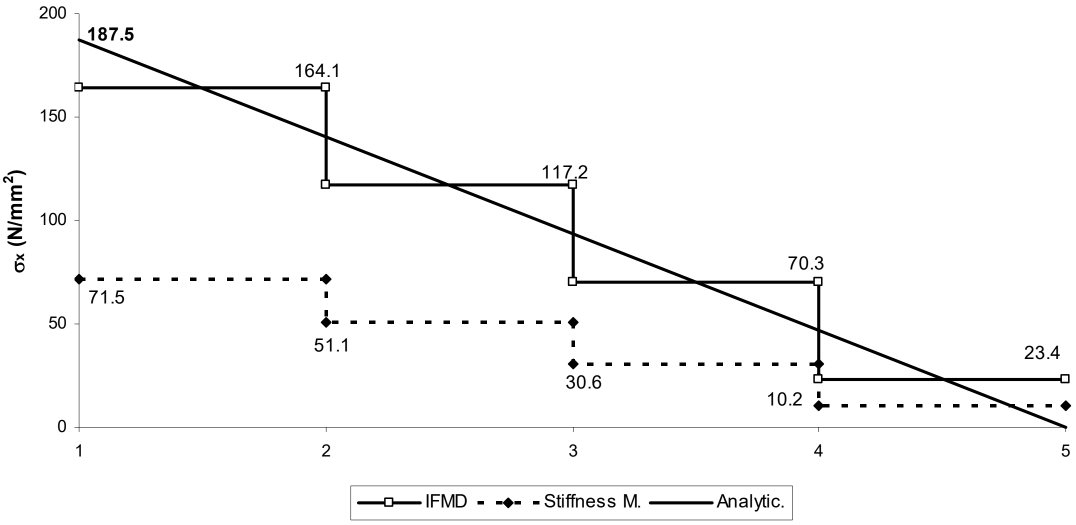
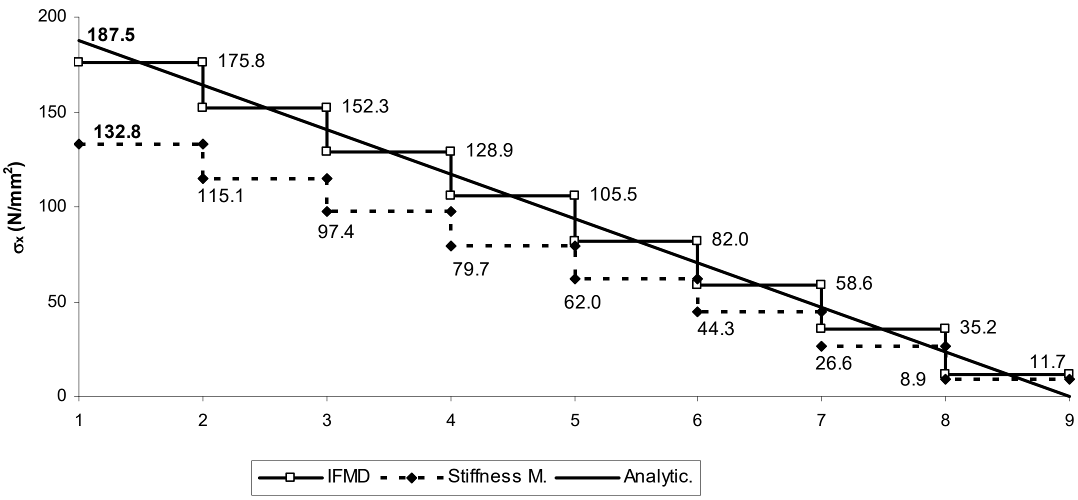
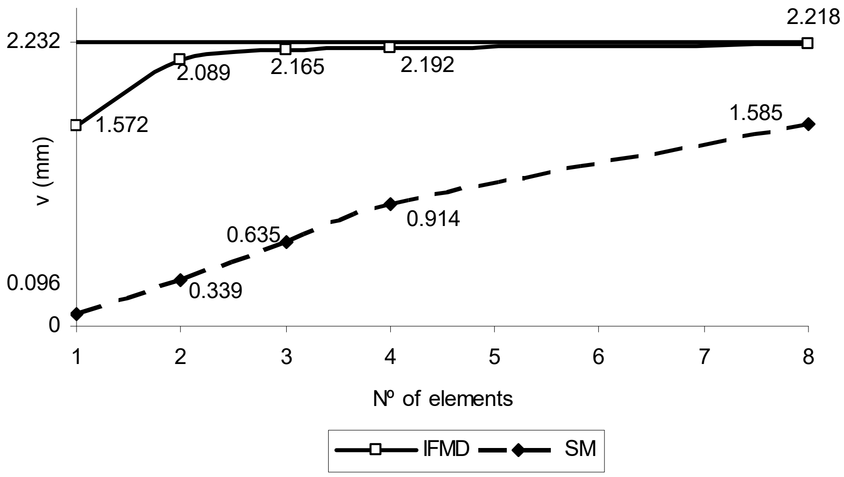


a/

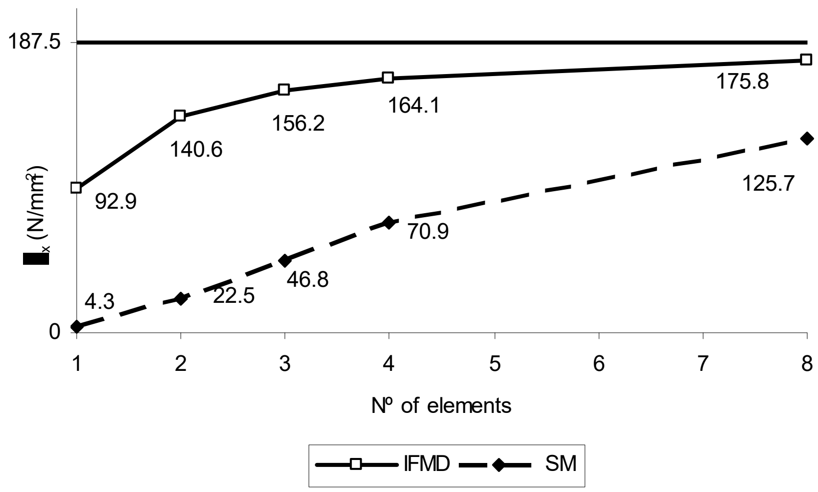


b/

Figure 5. Maximum values depending on discretization:
a/ Displacement; b/ Stress.



a/



b/

Figure 6. Off-axis tensile test.

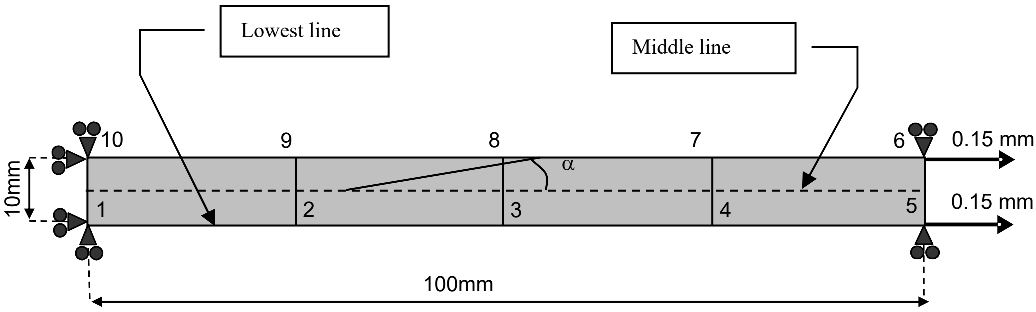
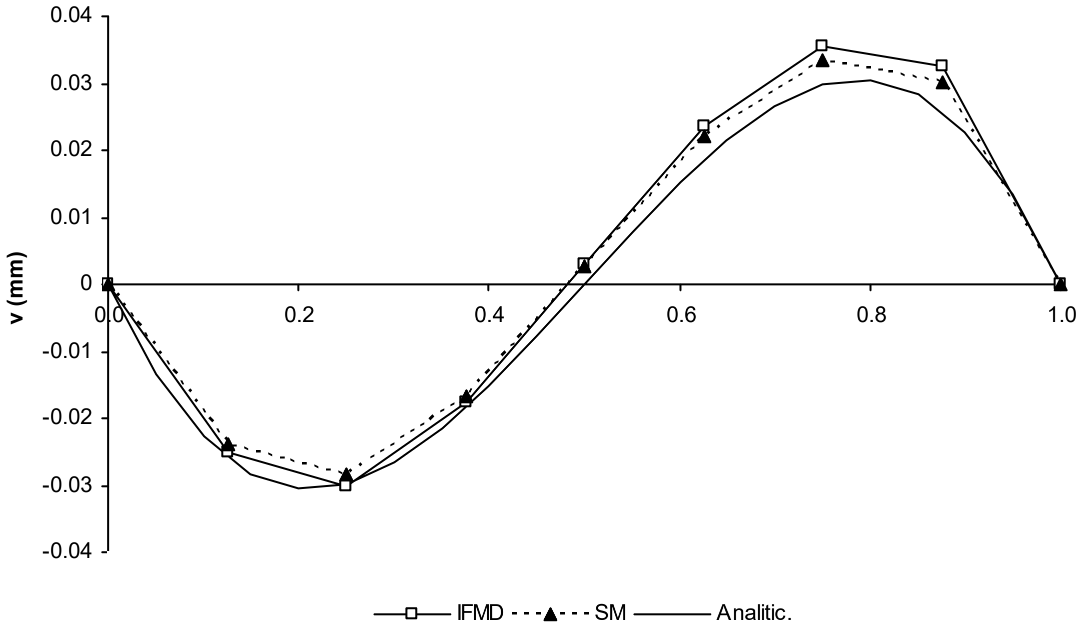
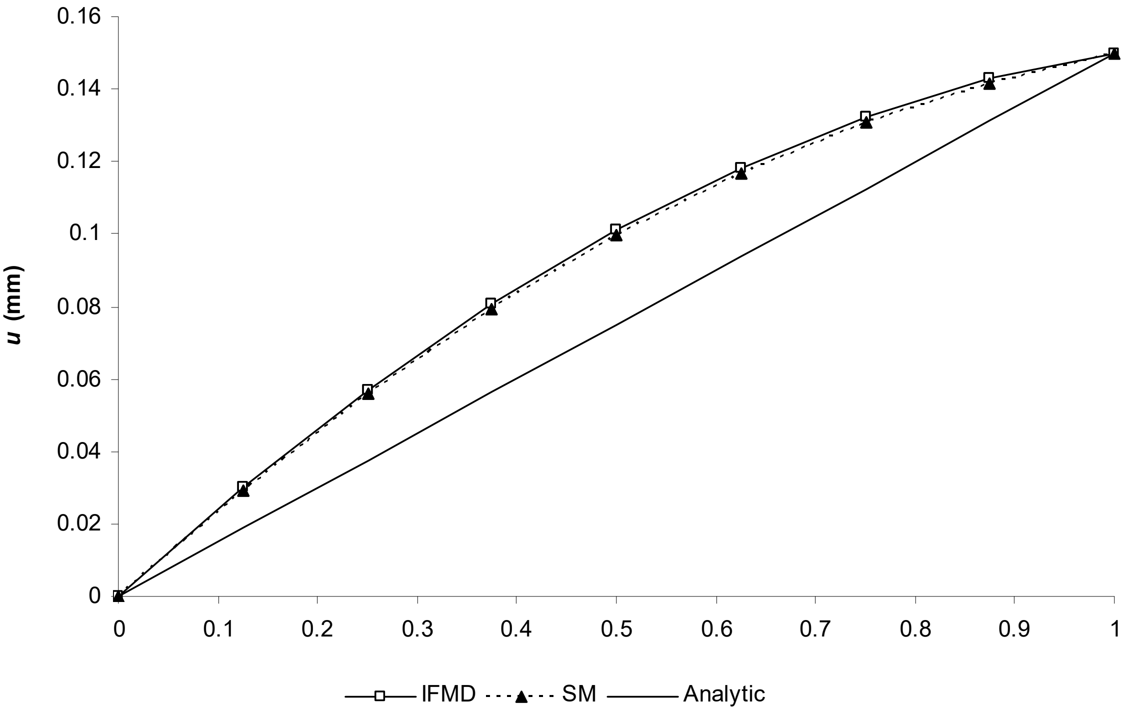


Figure 7. Displacements in the off-axis tensile specimen:
a/ Horizontal displacements, *u*; b/ Vertical displacements, *v*.

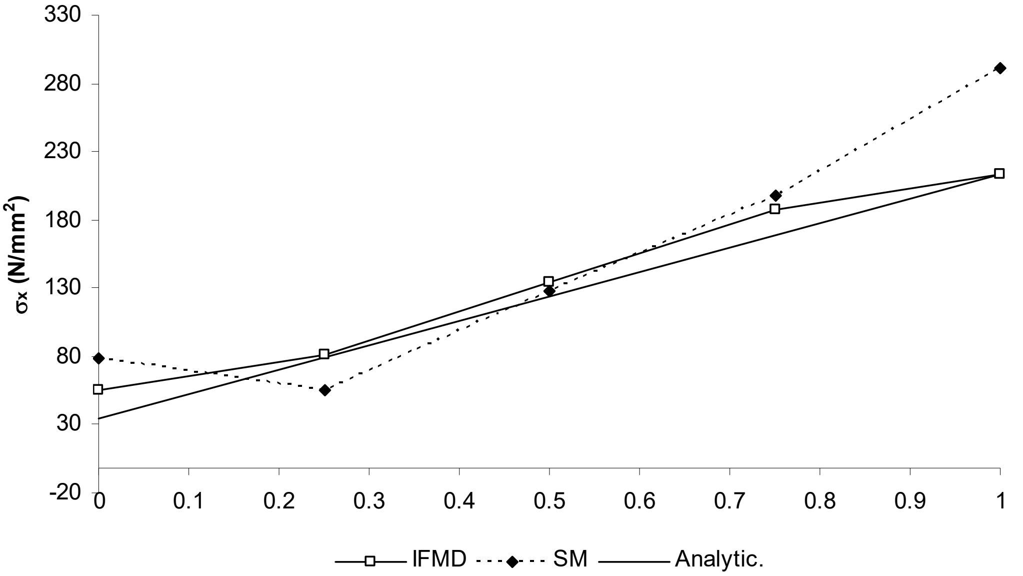


a/

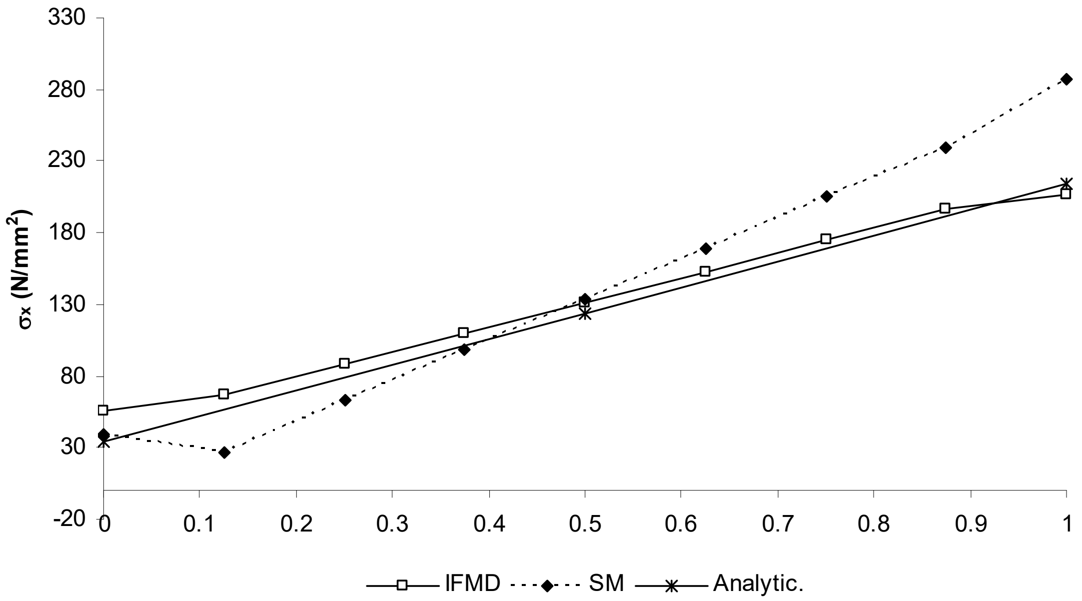


b/

Figure8. Stress distribution in the lowest line of the off-axis tensile test:
a/ 4 elements; b/ 8 elements.



a/



b/