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Unions, Monetary Shocks and the Labour Market Cycle

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Abstract. This paper provides a new growth model by considering strategic behaviour in the supply of labour. Workers form a labour union with the aim of manipulating wages in their own benefit. We analyse the implications on labour market dynamics at business cycle frequencies of getting away from the price-taking assumption. A calibrated monetary version of the union model does quite a reasonable job in replicating the dynamic features of labour market variables observed in post-war U.S. data.

JEL Classification: E24, E32

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1 Introduction

Starting with Kydland and Prescott (1982) and Long and Plosser (1983), literature has analysed the role of productivity shocks in explaining aggregate economic fluctuations. Various models have been proposed to generate business cycles on real aggregate variables that are fairly consistent with those observed in actual data. Here, we add to this literature an object already explored in a static framework by Fernández-de-Córdoba and Moreno-García (2006). Specifically, we build a variant of the standard neoclassical growth model where the suppliers of labour engage in strategic behaviour through an institution that we call a "union".

The purpose of introducing unions is twofold. First, the paper studies the ability of the union model to match the dynamic features of labour market variables observed in U.S. data at business cycle frequencies. Second, the paper analyses how the union model is capable of accounting for the high persistence of the employment level observed in actual data. Standard real business cycle (RBC) models have trouble in accounting for these labour market features. We depart from the lotteries model proposed by Hansen (1985) and the home-produced consumption good model introduced by Benhabib, Rogerson and Wright (1991) and incorporate unions into the aggregate economy. The behaviour of unions is first described in a simple setup of the model in order to gain intuition. We then proceed to analyse a more complicated setup that is more suitable for analysing the implications of unions on labour market dynamics. The two alternative setups share a common characteristic: unions manipulate the supply of labour in order to maximise the wage share of the economy.

The model contains two types of agent: capitalists and workers. The first type owns capital whereas the second owns labour. We assume that there is a large number of identical (non-unionised) capitalists and that all workers belong to a union capable of manipulating wages by controlling labour supply.[1] We first consider a simple model where the union is myopic and does not value leisure. These two features allow us to characterise the implications of unions on labour market dynamics based on analytical results. In particular, we show in this simple framework that a union with monopoly power chooses labour supply such

[1] We do not enter into the labour union formation problem or issues regarding union stability. In particular, we assume that cyclical fluctuations do not affect union stability.

that labour-capital ratio is constant. Then, labour (a flow variable) behaves as a stock variable (capital) by responding sluggishly to technology shocks. This feature is qualitatively preserved when moving from this simple setup to a *generalized* union model that (i) relaxes the assumption that the union is myopic; (ii) considers leisure in the unions' objective function; and (iii) introduces the use of money through a cash-in-advance constraint. Since workers are not allowed to hold capital, they have no access to capital markets and thus they respond sharply to monetary shocks.

In order to assess relative performance the business cycle properties of the generalised union model are compared with those exhibited by U.S. data and those obtained from a standard RBC model. Quantitative evaluation based on second moment statistics shows that the union model features depend crucially on the relative size and persistence of technology versus monetary shocks. On the one hand, the union monopoly power reduces the effects of technology shocks on aggregate volatility due to the small, slow reaction to these shocks. On the other hand, the fact that workers have no access to capital markets implies that monetary shocks, by affecting inflationary expectations, have large effects on consumption-leisure substitution decisions, which results in large movements in labour supply. In particular, we show that when monetary shocks are larger than technology shocks the union model proposed in this paper provides a better characterisation of labour market dynamics than a standard RBC model.

The rest of the paper is organised as follows. Section 2 introduces the basic union model. Section 3 sets up a monetary business cycle model with unions and discusses the implications on labour market dynamics. Section 4 concludes.

2 The Basic Model

Before analysing the optimization problem faced by the union's planner, it is useful to highlight several features that distinguish this model from the standard neoclassical model. The model assumes two types of agent: workers and capitalists. They are assumed to be different in three important dimensions. First, workers only own labour input whereas capitalists only own capital factor. Second, workers unionise to manipulate labour supply whereas capitalists are assumed to behave competitively.[2] Finally, since workers are not allowed to accumulate capital (i.e. they do not have access to capital markets) they face a strong financial constraint which makes it more difficult to smooth their consumption intertemporally.

The basic model of a unionised economy is one where the labour force is controlled by a single union. The union maximises workers' income in each period without considering the effects of today's manipulation of labour supply on future capital accumulation. Before analysing the union's problem, let us first study the problem faced by the stand-in owner of capital.

2.1 The stand-in owner of capital problem

The stand-in owner of capital faces the following optimisation program

$$\max_{\{k_{t+1},c_{kt}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_{kt}^{1-\eta}}{1-\eta}$$

$$\begin{aligned} s.t.\ c_{kt} + k_{t+1} - k_t &= (r_t - \delta)k_t, \\ k_0 &= \overline{k}, \end{aligned}$$

where β is the discount factor, η is the (constant) risk aversion parameter, δ is the rate of depreciation, E_0 denotes the conditional expectation operator, c_{kt} denotes capitalist consumption, k_t is capital stock and r_t is the real return on capital. The Lagrangian associated with the stand-in owner of capital is given by

$$L(c_{kt}, k_{t+1}, \lambda_t) = E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_{kt}^{1-\eta}}{1-\eta} + \lambda_t \left((r_t - \delta)k_t - c_{kt} - k_{t+1} + k_t \right) \right].$$

The first-order conditions (F.O.C.) of the stand-in owner of capital are:

$$\frac{\partial L(c_{kt}, k_t, \lambda_t)}{\partial c_{kt}} = c_{kt}^{-\eta} - \lambda_t = 0, \tag{1}$$

[2] In this paper, we explore the implications of labour unions on equilibrium dynamics. We leave for future research the implications derived from the existence of capitalist clubs. Moreover, we refrain from tackling the issues associated with the endogenous formation of the two groups of agents.

$$\frac{\partial L(c_{kt}, k_t, \lambda_t)}{\partial k_{t+1}} = E_t \left[\beta^{t+1} \lambda_{t+1} \left(r_{t+1} - \delta + 1 \right) - \beta^t \lambda_t \right] = 0, \tag{2}$$

$$\frac{\partial L(c_{kt}, k_t, \lambda_t)}{\partial \lambda_t} = (r_t - \delta) k_t - c_{kt} - k_{t+1} + k_t = 0. \tag{3}$$

Capital owners produce the consumption commodity using the following constant returns to scale technology:

$$y_t = A_t \left(\varepsilon k_t^{\rho} + (1 - \varepsilon) n_t^{\rho} \right)^{\frac{1}{\rho}}, \quad \rho \in (-\infty, 0), \tag{4}$$

where A_t and n_t denote the productivity shock and the supply of labour, respectively. Capital owners are price takers and their factor demands must thus satisfy the condition that marginal productivity of each factor should equal its rental rate:

$$w_t = A_t \left(1 - \varepsilon \right) n_t^{\rho - 1} \left(\varepsilon k_t^{\rho} + (1 - \varepsilon) n_t^{\rho} \right)^{\frac{1}{\rho} - 1}, \tag{5}$$

$$r_t = A_t \varepsilon k_t^{\rho - 1} \left(\varepsilon k_t^{\rho} + (1 - \varepsilon) n_t^{\rho} \right)^{\frac{1}{\rho} - 1}, \tag{6}$$

where w_t is the marginal productivity of labour (real wage).

2.2 The union problem

Assume the existence of a benevolent union planner maximising the (expected) intertemporal consumption stream of its workers:

$$\max_{\{c_{ut}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_{ut}^{1-\eta}}{1 - \eta},$$

$$\begin{aligned} s.t.\ c_{ut} &= w_t(k_t) n_t, \\ n_t &\leq L, \end{aligned}$$

where c_{ut} denotes the union's total consumption and L is the endowment of time.

The union's planner takes into account and manipulates the demand for labour generated by the constant returns to scale technology. Using wage equation (5), total income for workers is given by

$$w_t n_t = A_t \left(1 - \varepsilon \right) n_t^{\rho} \left(\varepsilon k_t^{\rho} + (1 - \varepsilon) n_t^{\rho} \right)^{\frac{1}{\rho} - 1}. \tag{7}$$

By substituting the budget constraint into the objective function the utility maximisation program faced by the union's planner can be rewritten as an optimisation problem where a discounted stream of labour returns is maximised:

$$\max_{\{n_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{(w_t n_t)^{1-\eta}}{1-\eta}.$$

More explicitly, the optimisation problem can be written as follows

$$\max_{\{n_t\}} E_0 \left[... + \beta^t \frac{[w_t(k_t(n_{t-1}))n_t]^{1-\eta}}{1-\eta} + \beta^{t+1} \frac{[w_{t+1}(k_{t+1}(n_t))n_{t+1}]^{1-\eta}}{1-\eta} + ... \right].$$

Since a forward-looking union does not take prices (wages and rental rate of capital) as given, it takes into account that the choice of n_t has an impact on the amount of next period capital and thus on the next period real wage. That is, the ratio $\frac{\partial k_{t+1}}{\partial n_t}$ is different from zero unless the union is myopic. Formally, the F.O.C. of the union's intertemporal optimisation problem is given by

$$E_t \left\{ (w_t n_t)^{-\eta} \left[\frac{\partial w_t}{\partial n_t} n_t + w_t \right] + \beta \left(w_{t+1} n_{t+1} \right)^{-\eta} \left[\frac{\partial w_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial r_t} \frac{\partial r_t}{\partial n_t} n_{t+1} \right] \right\} = 0. \quad (8)$$

Notice that the intertemporal plan characterised by (8) is not time consistent since in each period, when deciding today's employment and real wage, the union has an incentive to deviate from this optimal plan by ignoring the effect that the manipulation of labour supply, n_t, has on the stock of capital one period ahead, k_{t+1}. Later we assume in Section 3 the existence of a commitment technology that allows F.O.C. (8) to be implemented, but let us consider first the case where the union maximises its current labour income in each period. In this case, the optimality condition is given by

$$\frac{\partial w_t}{\partial n_t} n_t + w_t = 0. \quad (9)$$

Notice that the optimality condition (9) can also be obtained directly from (8) by assuming that the union is myopic (i.e. the discount factor of the union is zero). Taking into account the real wage expression (5), the optimality condition (9) can be written as

$$\begin{aligned} 0 &= \left[\frac{\partial w_t}{\partial n_t} n_t + w_t \right] = A_t \left(1-\varepsilon\right) n_t^{\rho-1} \left(\varepsilon k_t^{\rho} + (1-\varepsilon) n_t^{\rho} \right)^{\frac{1}{\rho}-1} \\ & \left[\rho + (1-\rho)\left(1-\varepsilon\right) n_t^{\rho} \left(\varepsilon k_t^{\rho} + (1-\varepsilon) n_t^{\rho} \right)^{-1} \right]. \end{aligned} \quad (10)$$

Thus, the term in brackets must be zero, which implies that

$$n_t = \left(\frac{-\rho\varepsilon}{1-\varepsilon}\right)^{1/\rho} k_t, \qquad \text{if } n_t < L. \tag{11}$$

Equation (11) establishes that the capital-labour ratio is constant whenever it is optimal to supply a lower amount of labour than the total endowment of time, L. Notice that equation (10) (or (11)) can be written in terms of (the inverse of) the elasticity of labour demand as follows

$$\left[\frac{\partial w_t}{\partial n_t} n_t + w_t\right] = \left[\zeta_{w_t,n_t} + 1\right] w_t = 0,$$

or alternatively $\zeta_{w_t,n_t} = -1$. The intuition is that a union with monopoly power would choose a supply of labour n_t such that the demand for labour is at the point of unitary elasticity. At this point any further restriction on the labour supply is offset by an increase in wages that is smaller than the restriction on labour, making workers' income $w_t n_t$ lower.

It is worth noting an important aspect of the discretionary (no-commitment) equilibrium that we are describing. The steady state of this economy coincides with the steady state of the neoclassical model with no unions, because equation (11) establishes that if the capital stock k_t is low, then the union will produce a shortage of labour below L. But if the economy is on the balanced growth path, the stock of capital will eventually reach a level k^* such that $n^* = \left(\frac{-\rho\varepsilon}{1-\varepsilon}\right)^{1/\rho} k^* = L$. Figure 1 illustrates this result. It shows two economies with the same initial stock of capital. Unions restrict the supply of labour below the endowment level, L, in order to maximise wage income. Then, output and the real return on capital are lower than in the standard neoclassical economy. As the capital stock grows, the union increases labour supply up to a point marked as t^* in the figure. At that point, the restriction imposed by equation (11) is no longer binding and the supply of labour coincides with the total endowment of time L. At that point, the economy with a union behaves as a standard neoclassical economy with an endowment of capital at $t = 0$, $k_0 = k^*$, and an endowment of labour L. The consequence is that the steady state is the same for both economies. After t^*, the restrictions imposed by the union are no longer binding. This implies that positive shocks on A_t above its steady-state level do not produce reactions by the union, however, negative shocks make equation (11) binding.

As emphasised above, a crucial feature of the union economy is that the capital-labour ratio is constant along the transition path. This result implies

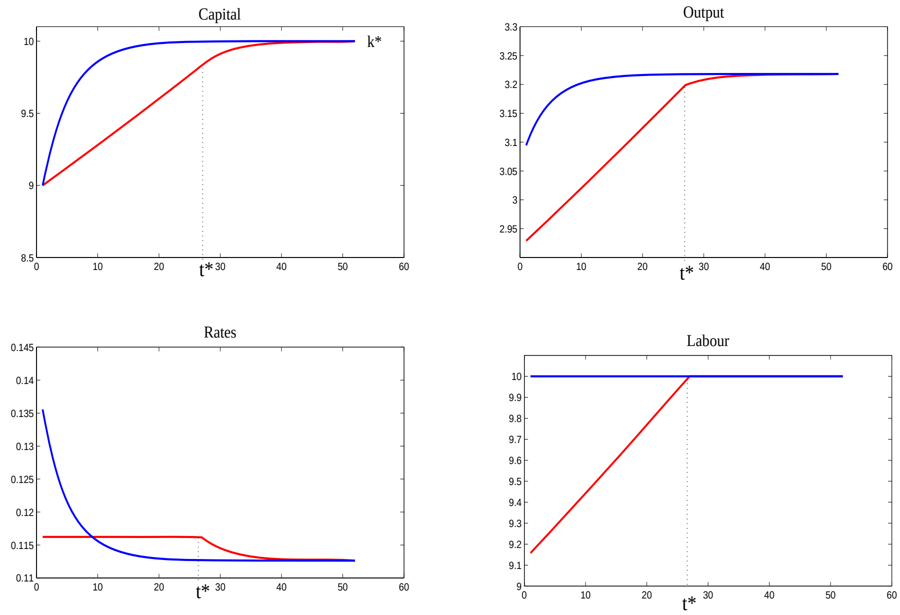

Figure 1: Transition to the steady-state

that labour (a flow variable) behaves as capital (a stock variable), which implies that the responses of both capital and labour to technology shocks display a great deal of persistence that is consistent with the highly persistent employment time series observed in actual data. We show in the next section that this important qualitative feature of labour is preserved when considering a sounder model for the analysis of business cycle dynamics than the simple one studied in this section.

The simple union model described above serves to fix the intuition of the main force that drives the unionised economy. However, it has two major caveats for the analysis of business cycles. First, the distribution of labour is truncated and only deviations below the total endowment of time can be observed. Second, workers have no means of saving.

The first caveat is illustrated in Figure 2 by considering two alternative scenarios. Figure 2 depicts a labour supply with two segments. The inelastic segment

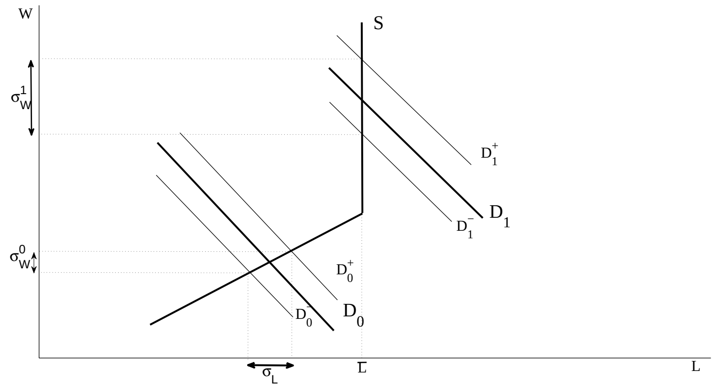

Figure 2: No leisure in the workers' utility function

describes the scenario where capital is larger than k^* and the union supplies the whole endowment of labour L. This scenario is also characterised by the labour demand schedule D_1. In this scenario (moderate) productivity shocks result in highly volatile wages whereas the equilibrium level of labour remains constant. The second scenario is illustrated by the labour demand schedule D_0 intersecting with the elastic segment of labour supply where the union monopoly power mechanism works (i.e. equation (11) is binding), which results in lower volatility of wages and higher volatility of labour. The truncation of the distribution of labour can be overcome by introducing leisure into the utility function of workers. Notice that by misrepresenting labour supply the union will offer a lower amount of labour than under the competitive equilibrium for each level of the real wage.

The second caveat in the simple union model implies that workers face strong difficulties in smoothing their consumption, which results in a highly volatile aggregate consumption which is at odds with the evidence obtained from actual data. This limitation is overcome by allowing workers to hold bonds and introducing money through a cash-in-advance constraint.

The next section introduces a generalised union model where a commitment technology is assumed for the union: it values workers' leisure and workers are able to hold bonds in order to smooth their consumption and hold money to purchase consumption goods.

3 A Monetary Union Model

As in the previous section, workers are assumed to be represented by a union that behaves strategically in the supply of labour. In this section, however, we assume the existence of a commitment technology that allows the union to maximise the expected intertemporal utility stream of the representative worker forming the union.[3] Moreover, we build upon the simple model studied above by assuming that the union's planner values both workers consumption and leisure and money is essential for purchasing goods and services. This latter feature is introduced through a cash-in-advance constraint. Formally, the union faces the following problem:

$$\max_{\{c_{ut},n_t,m_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[c_{ut}^{\lambda}(1-n_t)^{1-\lambda}\right]^{1-\eta}}{1-\eta},$$

$$s.t.\ P_t c_{ut} \leq M_{t-1} + T_t,$$

$$P_t c_{ut} + M_t + B_t \leq P_t w_t(k_t) n_t + M_{t-1} + (1+i_{t-1})B_{t-1} + T_t.$$

The union's planner enters each period with nominal money balances M_{t-1} and bond holdings B_{t-1} and receives a nominal lump-sum transfer of T_t.[4] The first restriction represents the cash-in-advance (CIA) constraint. In real terms

$$c_{ut} \leq \frac{m_{t-1}}{\Pi_t} + \tau_t,$$

where $\Pi_t = (P_t/P_{t-1}) = 1 + \pi_t$ is one plus the rate of inflation between periods $t-1$ and t, $\tau_t = T_t/P_t$ and $m_t = M_t/P_t$. Following Cooley and Hansen (1989), we assume that the CIA constraint is binding. The second constraint is the flow budget constraint. In real terms,

$$c_{ut} + m_t + b_t \leq w_t(k_t) n_t + \frac{m_{t-1} + (1+i_{t-1})b_{t-1}}{\Pi_t} + \tau_t.$$

The union's planner takes into account and manipulates the demand for labour generated by the constant returns to scale technology described above. The

[3] Below, we also analyse the case where the commitmment technology is removed (that is, the union maximises labour income in each period) in order to compare the effects of commitment on model dynamics.

[4] In the aggregate, this transfer is related to the growth rate of the nominal supply of money. Letting the stochastic variable θ_t denote the rate of money growth ($M_t = (1+\theta_t)M_{t-1}$), the transfer will be $\theta_t M_{t-1}$. θ_t is known at the start of period t.

Lagrangian associated with the union's optimisation problem is given by

$$\begin{aligned} L(c_{ut}, n_t, , m_t, \kappa_t, \mu_t) &= E_0 \sum_{t=0}^{\infty} \beta^t \{ \frac{\left[c_{ut}^{\lambda}(1-n_t)^{1-\lambda}\right]^{1-\eta}}{1-\eta} + \kappa_t \left(\frac{m_{t-1}}{\Pi_t} + \tau_t - c_{ut} \right) + \\ & \mu_t \left[w_t(k_t(n_{t-1}))n_t + \frac{m_{t-1} + (1+i_{t-1})b_{t-1}}{\Pi_t} + \tau_t - c_{ut} - m_t - b_t \right] \}. \end{aligned}$$

The F.O.C. of the union's intertemporal optimisation problem are given by

$$\left[c_{ut}^{\lambda}(1-n_t)^{1-\lambda}\right]^{-\eta} \lambda c_{ut}^{\lambda-1}(1-n_t)^{1-\lambda} - \kappa_t - \mu_t = 0, \tag{12}$$

$$\begin{aligned} 0 &= -\left[c_{ut}^{\lambda}(1-n_t)^{1-\lambda}\right]^{-\eta}(1-\lambda)c_{ut}^{\lambda}(1-n_t)^{-\lambda} + \mu_t w_t + \mu_t \frac{\partial w_t}{\partial n_t} n_t + \\ & \beta E_t \mu_{t+1} \frac{\partial w_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial r_t} \frac{\partial r_t}{\partial n_t} n_{t+1}, \end{aligned} \tag{13}$$

$$\beta E_t \left(\frac{\kappa_{t+1}}{\Pi_{t+1}} + \frac{\mu_{t+1}}{\Pi_{t+1}} \right) - \mu_t = 0, \tag{14}$$

$$\beta E_t \left(\mu_{t+1} \frac{(1+i_t)}{\Pi_{t+1}} \right) - \mu_t = 0, \tag{15}$$

$$w_t n_t + \frac{m_{t-1} + (1+i_{t-1})b_{t-1}}{\Pi_t} + \tau_t - c_{ut} - m_t - b_t = 0, \tag{16}$$

$$\frac{m_{t-1}}{\Pi_t} + \tau_t - c_{ut} = 0. \tag{17}$$

The union's optimal plan described by equations (12)-(17) is not time consistent. As a benchmark case, we assume the existence of a commitment technology that forces the union to follow the optimal plan characterised by (12)-(17). The stand-in owner of capital's optimisation problem is the same as the one described in the previous section.

The competitive equilibrium of this economy is then described by (i) capitalist decisions on capitalist consumption, labour demand and capital investment, which are characterised by equations (1)-(6); (ii) unions' choices about workers' consumption, labour supply and money and bond demands, which are characterised by (12)-(17); (iii) money transfers to workers from the government; and (iv) the goods market equilibrium condition. The model is completed by assuming stochastic processes for the two shocks considered

$$\ln A_t = (1-\phi_A)\ln A_{ss} + \phi_A \ln A_{t-1} + v_{At}, \tag{18}$$

$$\theta_t - \theta_{ss} = \gamma(\theta_{t-1} - \theta_{ss}) + \upsilon_t. \tag{19}$$

Equation (18) assumes that (the steady-state log-deviations of) technology shocks follow a first-order autoregressive process where $0 < \phi_A < 1$ is a parameter measuring shock persistence and υ_{At} is a white noise innovation. Equation (19) establishes that the deviations of money growth from the steady state, θ_{ss}, follow a first-order autoregressive process with $0 < \gamma < 1$.

Deriving a log-linear approximation of the equilibrium around the steady state is quite straightforward and the solution of the resulting linear rational expectations system is obtained applying the methods of Sims (2002) and Lubik and Schorfheide (2003).[5] Next, we discuss the standard calibration of the parameter values carried out in order to solve the model numerically. We use U.S. data to calibrate the model. We believe that the type of unions formed at plant level (professional unions) makes manipulation of the type of complementarity between labour and capital studied in this paper more likely than the politically oriented (non-professional) unions that are more widespread in Europe than in the U.S.

3.1 Calibration

We follow standard procedures for calibrating this specific model economy using U.S. data (see, for instance, Cooley and Prescott, 1995). First, the law of motion for the capital stock in the steady state implies that the rate of depreciation, δ, is equal to the steady-state investment-capital ratio. The steady-state investment-capital ratio for the U.S. economy is 0.076, which implies that the quarterly depreciation rate is 0.019.

Second, from the steady-state characterisation we have, on the one hand, that

$$\frac{labour - share}{capital - share} = \frac{w_{ss}n_{ss}}{r_{ss}k_{ss}} = \frac{1}{\varepsilon}\left[\frac{r_{ss}}{\varepsilon}\right]^{\frac{\rho}{1-\rho}},$$

where $r_{ss} = \frac{1}{\beta} - 1 + \delta$. Since the steady-state labour-capital share ratio for the U.S. economy is about 1.5, we have that ε and ρ must satisfy

$$\frac{1-\varepsilon}{\varepsilon}\left[\frac{n_{ss}}{k_{ss}}\right]^{\rho} = 1.5. \tag{20}$$

[5] Appendix 1 describes the log-linear approximation and how the solution of the resulting linear rational expectations system is obtained.

On the other hand, we have that the steady-state annual capital-output ratio for the U.S. economy is 3.32, which implies that

$$\frac{h_{ss}}{k_{ss}^{\rho}} = \frac{1}{3.32}\frac{\varepsilon}{r_{ss}^{a}}, \tag{21}$$

where r_{ss}^{a} is the steady-state annualised real interest rate.

Using the steady-state definition for h_{ss} and taking into account (20) and (21) we have, after simplifying, that $r_{ss}^{a} = 0.12$. Using the definition of r_{ss} and $\delta = 0.019$, we then obtain that the quarterly calibrated values for β and r_{ss} are 0.989 and 0.03, respectively.

Finally, using (20) and taking into account the definitions of n_{ss} and k_{ss}, we have the following expression for calibrating ρ and ε

$$\varepsilon = \frac{r_{ss}^{\rho}}{2.5^{1-\rho}}. \tag{22}$$

Obviously, there are multiple values of ρ and ε satisfying (22). A large negative value of ρ, on the one hand, increases the degree of complementarity between labour and capital, enhancing the effects of union power. But on the other hand it implies a larger value of ε, which reduces the importance of labour in the production function, which in turn mitigates the effects of union power. As a benchmark value we chose $\rho = -0.3$, which implies $\varepsilon = 0.87$. Later we carry out a sensitivity analysis by choosing alternative values for these two parameters.

In regard to utility parameters, we assume a risk aversion parameter of $\sigma = 2$ whereas the value for λ ($= 0.343$) is determined by the time devoted to market activities on average, $n_{ss} = 0.2$.

Traditionally, technology shocks have been identified with standard Solow residuals.[6] As pointed out by King and Rebelo (2000), there are three major concerns about using standard Solow residuals as a measure of productivity shocks. First, there is evidence suggesting that the Solow residual can be forecast using

[6]Standard Solow residuals are obtained using data on aggregate output, capital and labour, and assuming a Cobb-Douglas production function. By using postwar U.S. data and fitting an AR (1) process to the standard Solow residual an estimate of the persistence parameter of around 0.95 is obtained, whereas the estimate of the standard deviation of innovation is around 0.007.

variables such as military spending (Hall, 1988) or lagged values of several monetary aggregates (Evans, 1992), which are hardly related to productivity shocks. Second, the large variance of Solow residuals leads to probabilities of technological regress that are implausibly large, as suggested by Burnside, Eichenbaum and Rebelo (1996). Finally, cyclical changes in labour effort and capital utilisation can result in overestimation of the variance of technology shocks when using the Solow residual as a measure of them. In particular, the union model considered in this paper displays two features that may bias the Solow residual as an estimate for productivity shocks. First, money is not neutral in this model. Therefore, changes in output are determined by both technology and monetary shocks. Second, the union model introduces variations in labour effort since the union manipulates labour in order to maximise the worker income stream. Therefore, the standard Solow residual gets contaminated by the presence of strategic labour behaviour.[7]

All these considerations lead us to take a lower value for the standard deviation of the productivity shock, σ_{vA}, than the one estimated from the standard Solow residual. Specifically, we consider $\sigma_{vA} = 0.002$. This value is in line with those assumed by King and Rebelo (2000), which imply small, plausible probabilities of technological regress. The value of the persistence parameter, ϕ_A, is assumed to be 0.95, in line with the values traditionally assumed in RBC literature.

Finally, we calibrate the money supply process. We choose a narrow measure of money supply which is consistent with the transaction motive of money assumed by the model through a CIA constraint. In particular, we consider the currency component of $M1$ as defined by the Federal Reserve. The money supply process parameters are calibrated by estimating an autoregression for the rate of growth of currency over the sample period 1954:1-2006:2. The estimation results are given by the following estimated equation:

$$\begin{aligned} \theta_t &= \underset{(0.0009)}{0.0048} + \underset{(0.0489)}{0.6962}\theta_{t-1}, \qquad \widehat{\sigma}_\nu = 0.0061. \end{aligned}$$

[7]Moreover, the use of Solow residuals as a measure of productivity shocks is more complicated in the present model because, as shown above, the CES production function assumed results in multiple combinations of parameters ε and ρ which are consistent with the long-run properties of actual data, and each of these combinations leads to a different estimate of the Solow residual.

The implied average growth rate of money is 1.57% per quarter. Table 1 summarises the benchmark parameterisation and Table 2 describes the associated steady state of the model.

Table 1. Benchmark parameterisation

$\rho = -0.30$	$\varepsilon = 0.87$	$\beta = 0.989$	$\eta = 2.0$
$\delta = 0.019$	$\lambda = 0.343$	$\phi_A = 0.95$	$\sigma_{vA} = 0.002$
$\gamma = 0.6962$	$\sigma_v = 0.0061$		

Table 2. Steady-state values

$n_{ss} = 0.2$	$y_{ss} = 32.67$	$k_{ss} = 434.26$	$c_{ss} = 24.42$
$c_{uss} = 19.59$	$c_{kss} = 4.83$	$r_{ss} = 0.03$	$\pi_{ss} = 0.0157$

3.2 Quantitative evaluation of the union model

We start this section by reporting some well known (see, for instance, King, and Rebelo, 2000) stylised facts of U.S. business cycles. We mainly focus on real business cycle features and, in particular, on aggregate labour market fluctuations. A summary of second-moment statistics for selected variables, taken from King and Rebelo (2000), is displayed in the second column of Table 3.[8] This column provides quantity measures of six well known stylised facts: (i) consumption is much less volatile than output; (ii) investment is much more volatile than output; (iii) total volatility of labour hours is similar to output volatility; (iv) wages are less volatile than output; (v) consumption, investment and labour hours are highly procyclical; (vi) wages are mildly procyclical; and (vii) the correlation between labour hours and wages is zero or slightly negative depending on the time series considered.

[8]Real business cycle statistics may change depending upon the sample period considered and the data sources used to compute them. For this reason, we also show in parentheses the minimum and maximum values reported in a number of prominent articles such as Kydland and Prescott (1982), Hansen (1985), Benhabib, Rogerson and Wright (1991), Hansen and Wright (1992), Gomme (1993), Cooley and Prescott (1995) and King and Rebelo (2000). Each interval should be understood as a rough measure of dispersion associated with each of the business cycle statistics considered.

In order to compare how the alternative features introduced by the generalised union model in the RBC model contribute to explaining the RBC stylised facts, we proceed in several steps. First, we analyse the properties exhibited by a standard RBC model that assumes a Cobb-Douglas production function (i.e. assuming that $\rho \to 0$). Column 3 in Table 3 shows the second moment statistics associated with this standard RBC model. This model is able to qualitatively reproduce some stylised facts, but it has trouble in reproducing (iii), (vi) and (vii). Moreover, it also has difficulties in reproducing stylised fact (i) from a quantitative perspective. That is, the standard RBC model implies that consumption and labour hour volatilities are too low; that the contemporaneous correlations between labour hours and output and between wages and output are too high; and that the correlation between labour hours and wages is high and positive. Unsurprisingly, the low value assumed for the volatility of productivity shocks (i.e. $\sigma_{vA} = 0.002$) implies that the share of output volatility explained by technology shocks is 0.38, much lower than the figure of 1.76 obtained using the standard Solow residual (i.e. $\sigma_{vA} = 0.007$). This small contribution of productivity shocks to output volatility is in line with the estimated forecast error variance decomposition obtained by Smets and Wouters (2007), among others, by estimating a DSGE model that allows for multiple exogenous shocks.

Second, we consider the standard RBC model and assume the existence of some degree of complementarity between the two production factors by using a CES production function with parameter values $\rho = -0.30$ and $\varepsilon = 0.87$. The fourth column in Table 3 displays the second moment statistics associated with this mild variation of the standard RBC model. Comparing columns 3 and 4 we observe that the presence of complementarity does not change much the RBC features displayed by the simple model in qualitative terms. From a quantitative perspective, there is, however, a 12.5% reduction in the volatility of labour hours. The intuition for this fall in labour hour volatility is simple, as explained above: since capital volatility is low (i.e. capital is a stock variable) the presence of complementarity between capital and labour leads to low volatility of hours worked.

Third, we consider the generalised union model without money. Apart from production factors being complements, the union model departs from standard RBC models in several important dimensions. There are two types of agent. Capitalists own capital and behave competitively whereas workers own labour and unionise to manipulate both labour supply and equilibrium wages. Moreover, since workers have no access to capital markets they find it hard to smooth their consumption intertemporally. The fifth column in Table 3 shows the RBC features associated with this model. Clearly, the union model leads to much higher volatility in consumption because workers cannot smooth consumption. Moreover, investment volatility is substantially reduced because capitalists try to smooth their consumption as well, since capitalist income depends entirely on the returns on investment. Furthermore, labour volatility tends to zero whereas wage volatility increases substantially. The intuition for these results works as follows. Workers' current consumption depends entirely on the current wage bill (i.e. wage times labour hours) and the only way for workers to smooth their consumption (since they have no access to capital markets) is to smooth labour hours because the wage is determined by the labour demand schedule that the union faces, which is shifted period by period by productivity shocks.

Table 3. Real business cycle features[9]

Statistic	US data economy	RBC model	RBC model with complementariness	Union model	Union model with money
σ_c/σ_y	0.74 (0.48, 0.76)	0.32	0.35	0.82	0.76
σ_I/σ_y	2.93 (2.82, 4.89)	2.97	2.91	1.55	1.75
σ_k/σ_y	0.36 (0.36, 0.38)	0.20	0.19	0.10	0.01
σ_n/σ_y	1.02 (0.82, 1.22)	0.56	0.49	0.03	1.03
σ_w/σ_y	0.38 (0.38, 0.70)	0.45	0.47	0.97	0.94
σ_n/σ_w	2.61 (1.37, 2.61)	1.24	1.04	0.03	1.09
$\rho_{c,y}$	0.88 (0.75, 0.90)	0.96	0.96	1.00	0.99
$\rho_{I,y}$	0.80 (0.80, 0.96)	1.00	1.00	1.00	0.99
$\rho_{n,y}$	0.88 (0.74, 0.88)	0.99	0.99	0.93	0.63
$\rho_{w,y}$	0.12 (0.12, 0.66)	0.99	0.98	1.00	0.29
$\rho_{n,w}$	(−0.35, 0.10)	0.96	0.95	0.93	−0.56

Finally, we consider the generalised union model with money. The presence of money improves the performance of the union model in many dimensions.

[9] The second moments associated with the alternative models are sample means of statistics computed from 500 simulations. Each simulation consists of 150 observations, which is of the same order of magnitude as the U.S. sample period considered in most RBC studies. The Hodrick-Prescott filter is used to isolate the business cycle component of the time series.

The volatility of consumption is closer to matching that observed in actual data. The same can be said with respect to volatility of hours worked, the correlation between labour hours and output and the correlation between wages and output. As explained below in the analysis of the impulse responses to a shock in the rate of growth of money, monetary shocks lead to changes in inflationary expectations that induce substitution effects between consumption and leisure which result in higher volatility of employment and output. Moreover, the presence of monetary shocks substantially reduces the high contemporaneous correlations between labour hours and output and between wages and output in contrast to the high correlations exhibited by RBC models induced by the presence of a single (technology) shock.

Figure 3 shows the impulse responses to a technology shock in the generalised union model. As expected, a positive technology shock induces positive responses in the rental rate of capital, wages, capital investment and output but induces negative responses in inflation and nominal interest rate. The impulse responses display a great deal of persistence, which is especially pronounced in employment (hours worked). By comparing the impulse responses of employment and capital we observe that the generalised union model preserves the feature emphasised in the analysis of the simple model studied above. Namely, employment (a flow variable) behaves in a persistent manner as the capital stock.

Figure 4 shows the impulse responses to a positive monetary shock in the nominal growth rate of money supply. As expected, an increase in the rate of money growth induces higher inflationary expectations due to the persistence of the money growth process. This results on the one hand in a higher inflationary tax that reduces the marginal utility of consumption and leads workers to substitute consumption for leisure, reducing employment and output. The fall in employment results in higher wages and lower real interest rate that decreases investment and consumption by capitalists. On the other hand, higher inflationary expectations result in higher inflation and nominal interest rates.

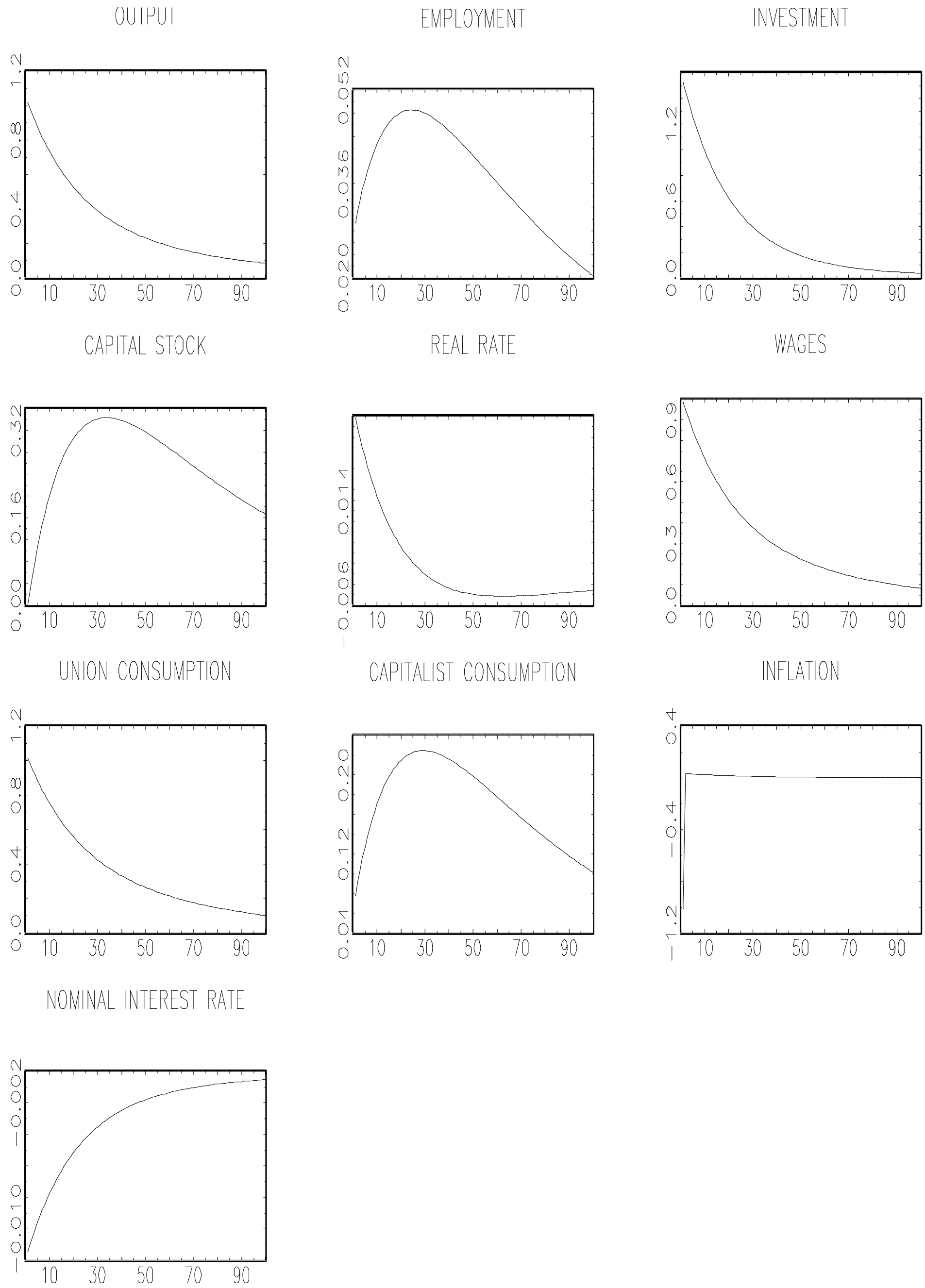

Figure 3: Impulse responses to a technology shock

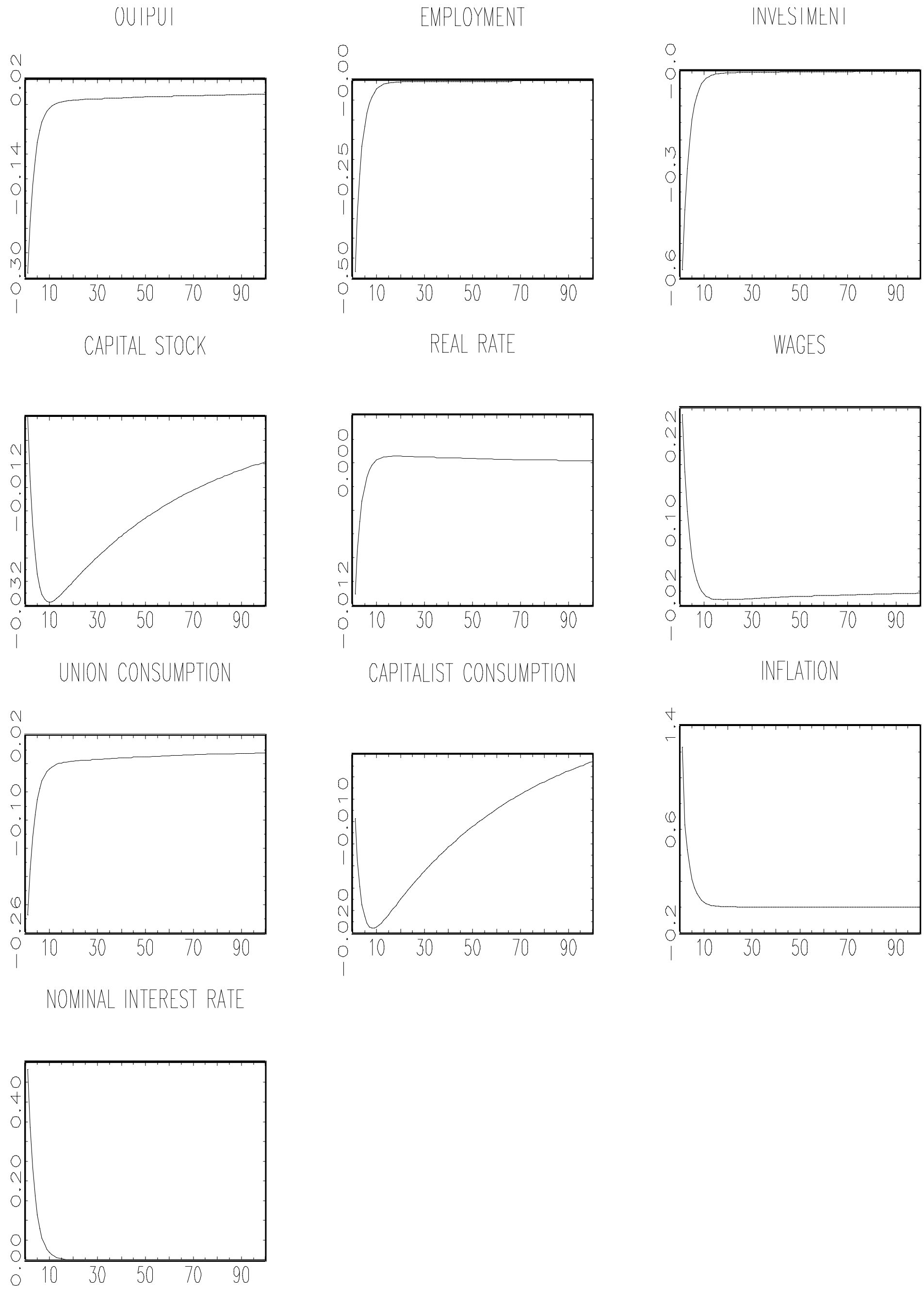

Figure 4: Impulse responses to a monetary shock

3.3 Sensitivity analysis

In this subsection, we carry out a sensitivity analysis on two fronts. First, we analyse the dynamic properties of the model when the union maximises the current income of workers in each period (i.e. the union follows a discretionary optimal plan). The second column in Table 4 shows the second moment statistics in this case. It is clear that the real business cycle features displayed by the model do not change substantially when the assumption of the existence of a commitment technology is removed.

Second, we consider two alternative parameterisations of ρ and ε in order to perform a sensitivity analysis. As mentioned above there are multiple combinations of values for ε and ρ which are consistent with the long-run properties of U.S. data. The selected moments associated with these two alternative parameterisations are also displayed in Table 4. The first alternative parameterisation assumes $\rho = -0.2$, which according to (22) implies $\varepsilon = 0.6127$. The second alternative parameterisation considers $\rho = -0.1$, which implies $\varepsilon = 0.4315$. The sensitivity analysis described in Table 4 shows that the generalised union model delivers quite robust features under the alternative parameterisations of the production function considered.

Table 4. Sensitivity analysis

Variable	No commitment	$\rho = -0.2$	$\rho = -0.1$
σ_y	0.33	0.34	0.35
σ_c/σ_y	0.75	0.79	0.83
σ_I/σ_y	1.78	1.70	1.64
σ_k/σ_y	0.10	0.10	0.09
σ_n/σ_y	1.09	1.06	1.06
σ_w/σ_y	0.85	0.88	0.82
σ_n/σ_w	1.16	1.20	1.29
$\rho_{c,y}$	0.99	1.00	1.00
$\rho_{I,y}$	0.99	0.99	1.00
$\rho_{n,y}$	0.66	0.67	0.70
$\rho_{w,y}$	0.21	0.28	0.29
$\rho_{n,w}$	-0.60	-0.52	-0.48

4 Conclusions

This paper introduces a model that departs from standard neoclassical business cycle models by assuming that the suppliers of labour engage in strategic behaviour through an institution referred to as a union. The paper shows that a monetary union model does a reasonable job in reproducing the labour market dynamics displayed by U.S. data at business cycle frequencies when monetary shocks are larger than technology shocks. The dynamic features exhibited by the monetary union model are the equilibrium outcomes of two features of the model that work in opposite directions. On the one hand, union monopoly power mitigates the effects of productivity shocks on aggregate volatility due to the small, slow reaction of employment to these shocks. On the other hand, the fact that workers have no access to capital markets implies that by affecting inflationary expectations monetary shocks have large effects on the marginal utility of consumption and then on consumption-leisure substitution choices, which results in large movements in labour supply.

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Appendix 1 (not intended for publication)

This appendix describes the log-linear approximation of the equilibrium around the steady state and how the solution of the resulting linear rational expectations system is obtained. In this appendix we consider that the production function is given by

$$y_t = \left(\varepsilon B_t^\rho k_t^\rho + (1-\varepsilon) A_t^\rho n_t^\rho\right)^{\frac{1}{\rho}},$$

in order to account for the possibility of technology bias. Notice that for $A_t = B_t$ this production function becomes (4).

We start by obtaining the expressions for the partial derivatives included in equation (13):

$$\begin{aligned} \frac{\partial w_t}{\partial n_t} &= A_t^\rho (1-\varepsilon)(\rho-1) n_t^{\rho-2} \left(\varepsilon B_t^\rho k_t^\rho + (1-\varepsilon) A_t^\rho n_t^\rho\right)^{\frac{1}{\rho}-1} \\ &\quad + A_t^{2\rho} (1-\varepsilon)^2 (1-\rho) n_t^{2(\rho-1)} \left(\varepsilon B_t^\rho k_t^\rho + (1-\varepsilon) A_t^\rho n_t^\rho\right)^{\frac{1}{\rho}-2}, \end{aligned} \tag{23}$$

$$\begin{aligned} \left[\frac{\partial w_t}{\partial n_t} n_t + w_t\right] &= A_t^\rho (1-\varepsilon) n_t^{\rho-1} \left(\varepsilon B_t^\rho k_t^\rho + (1-\varepsilon) A_t^\rho n_t^\rho\right)^{\frac{1}{\rho}-1} \\ &\quad \left[\rho + A_t^\rho (1-\rho)(1-\varepsilon) n_t^\rho \left(\varepsilon B_t^\rho k_t^\rho + (1-\varepsilon) A_t^\rho n_t^\rho\right)^{-1}\right], \end{aligned} \tag{24}$$

$$\frac{\partial w_{t+1}}{\partial k_{t+1}} = A_{t+1}^\rho \varepsilon (1-\varepsilon)(1-\rho) n_{t+1}^{\rho-1} k_{t+1}^{\rho-1} \left(\varepsilon k_{t+1}^\rho + (1-\varepsilon) A_{t+1}^\rho n_{t+1}^\rho\right)^{\frac{1}{\rho}-2}. \tag{25}$$

From the capitalist resource constraint, we have that

$$\frac{\partial k_{t+1}}{\partial r_t} = k_t, \tag{26}$$

$$\frac{\partial r_t}{\partial n_t} = \varepsilon (1-\rho)(1-\varepsilon) A_t^\rho B_t^\rho k_t^{\rho-1} n_t^{\rho-1} \left(\varepsilon B_t^\rho k_t^\rho + (1-\varepsilon) A_t^\rho n_t^\rho\right)^{\frac{1}{\rho}-2}. \tag{27}$$

In equilibrium, we have that $\tau_t = (M_t - M_{t-1})/P_t$. Substituting this expression in the cash-in-advance constraint, we obtain that $c_{ut} = m_t$. Using this expression together with equations (24), (25), (26), (27), and letting $h_t \equiv \varepsilon B_t^\rho k_t^\rho + (1-\varepsilon) A_t^\rho n_t^\rho$ (the introduction of some auxiliary variables simplifies the log-linear approximation of the equilibrium conditions carried out below) the union F.O.C. (13) becomes

$$\left[m_t^\lambda (1-n_t)^{1-\lambda}\right]^{-\eta} (1-\lambda) m_t^\lambda (1-n_t)^{-\lambda} = \mu_t (1-\varepsilon) A_t^\rho n_t^{\rho-1} h_t^{\frac{1}{\rho}-1} +$$

$$\mu_t (1-\varepsilon)(\rho-1) A_t^\rho n_t^{\rho-1} h_t^{\frac{1}{\rho}-1} + \mu_t (1-\varepsilon)^2 (1-\rho) A_t^{2\rho} n_t^\rho n_t^{\rho-1} h_t^{\frac{1}{\rho}-2} +$$

$$\beta E_t\left[\mu_{t+1}A_{t+1}^{\rho}\varepsilon^2(1-\varepsilon)^2(1-\rho)^2 n_{t+1}^{\rho-1}k_{t+1}^{\rho-1}h_{t+1}^{\frac{1}{\rho}-2}k_tA_t^{\rho}B_t^{\rho}k_t^{\rho-1}n_t^{\rho-1}h_t^{\frac{1}{\rho}-2}n_{t+1}\right]. \quad (28)$$

Since all workers are identical, nobody issues bonds in equilibrium and $b_t = 0$ for all t, which implies from F.O.C.'s (16) and (17) that $w_t n_t = m_t$ or $m_t = (1-\varepsilon)A_t^{\rho}n_t^{\rho}h_t^{\frac{1}{\rho}-1}$. Using this result, after performing a little algebra, we can write the union F.O.C. (28) in a more compact form

$$\left[m_t^{\lambda}(1-n_t)^{1-\lambda}\right]^{-\eta}(1-\lambda)m_t^{\lambda}(1-n_t)^{-\lambda} =$$

$$\mu_t(1-\varepsilon)A_t^{\rho}n_t^{\rho-1}h_t^{\frac{1}{\rho}-1}\left[\rho + A_t^{\rho}(1-\varepsilon)(1-\rho)n_t^{\rho}h_t^{-1}\right] +$$

$$\beta E_t\left[\mu_{t+1}A_{t+1}^{\rho}\varepsilon^2(1-\varepsilon)^2(1-\rho)^2 n_{t+1}^{\rho}k_{t+1}^{\rho-1}h_{t+1}^{\frac{1}{\rho}-2}A_t^{\rho}B_t^{\rho}k_t^{\rho}n_t^{\rho-1}h_t^{\frac{1}{\rho}-2}\right].$$

Using (12), F.O.C. (14) can be written

$$\mu_t = \beta\lambda E_t\left[m_{t+1}^{\lambda}(1-n_{t+1})^{1-\lambda}\right]^{-\eta}m_{t+1}^{\lambda-1}(1-n_{t+1})^{1-\lambda}\Pi_{t+1}^{-1}. \quad (29)$$

The conditions characterising the competitive equilibrium (where we include three additional auxiliary variables, x_t, s_t and z_t) are

$$h_t \equiv \varepsilon B_t^{\rho}k_t^{\rho} + (1-\varepsilon)A_t^{\rho}n_t^{\rho}, \quad (30)$$

$$y_t = h_t^{\frac{1}{\rho}}, \quad (31)$$

$$x_t \equiv \rho h_t + (1-\rho)(1-\varepsilon)A_t^{\rho}n_t^{\rho}, \quad (32)$$

$$s_t \equiv \mu_t m_t n_t^{-1}x_t, \quad (33)$$

$$z_t \equiv (1-\lambda)h_t m_t^{\lambda(1-\eta)}(1-n_t)^{-\lambda-\eta(1-\lambda)} \quad (34)$$

$$z_t = s_t + \beta E_t\left[\mu_{t+1}\varepsilon^2(1-\rho)^2 m_{t+1}k_{t+1}^{\rho-1}h_{t+1}^{-1}B_t^{\rho}m_t n_t^{-1}k_t^{\rho}\right], \quad (35)$$

$$\mu_t = \beta\lambda E_t\left[m_{t+1}^{\lambda-1-\lambda\eta}(1-n_{t+1})^{(1-\lambda)(1-\eta)}\Pi_{t+1}^{-1}\right], \quad (36)$$

$$\mu_t = \beta E_t\left(\mu_{t+1}\frac{(1+i_t)}{\Pi_{t+1}}\right), \quad (37)$$

$$E_t\left[\beta c_{kt+1}^{-\eta}\left(r_{t+1}-\delta+1\right) - c_{kt}^{-\eta}\right] = 0, \quad (38)$$

$$c_{kt} + k_{t+1} = r_t k_t + (1-\delta)k_t, \quad (39)$$

$$m_t = y_t - r_t k_t, \quad (40)$$

$$r_t = \varepsilon B_t^{\rho}k_t^{\rho-1}h_t^{\frac{1-\rho}{\rho}}, \quad (41)$$

From this set of equations it is easy to characterise the steady-state equilibrium:

$$h_{ss} = \varepsilon k_{ss}^{\rho} + (1-\varepsilon) n_{ss}^{\rho}, \tag{42}$$

$$y_{ss} = h_{ss}^{\frac{1}{\rho}}, \tag{43}$$

$$x_{ss} = \rho h_{ss} + (1-\rho)(1-\varepsilon) n_{ss}^{\rho}, \tag{44}$$

$$(1-\lambda) h_{ss} m_{ss}^{\lambda(1-\eta)} (1-n_{ss})^{-\lambda-\eta(1-\lambda)} = \mu_{ss} m_{ss} n_{ss}^{-1} x_{ss} + \beta\varepsilon^2 (1-\rho)^2 \mu_{ss} m_{ss}^2 k_{ss}^{2\rho-1} h_{ss}^{-1} n_{ss}^{-1}, \tag{45}$$

$$\mu_{ss} = \beta\lambda m_{ss}^{\lambda-1-\lambda\eta} (1-n_{ss})^{(1-\lambda)(1-\eta)} \Pi_{ss}^{-1}, \tag{46}$$

$$\beta(1+i_{ss}) = \Pi_{ss} = 1 + \pi_{ss}, \tag{47}$$

$$\beta\left(r_{ss} - \delta + 1\right) = 1, \tag{48}$$

$$c_{kss} = (r_{ss} - \delta) k_{ss}, \tag{49}$$

$$m_{ss} = y_{ss} - r_{ss} k_{ss}, \tag{50}$$

$$r_{ss} = \varepsilon k_{ss}^{\rho-1} h_{ss}^{\frac{1-\rho}{\rho}}, \tag{51}$$

where $A_{ss} = B_{ss} = 1$ is assumed.

From (48), we have that

$$r_{ss} = \frac{1}{\beta} - 1 + \delta.$$

Substituting this expression into (51),

$$\frac{1}{\beta} - 1 + \delta = \varepsilon k_{ss}^{\rho-1} h_{ss}^{\frac{1-\rho}{\rho}},$$

solving for h_{ss}, we obtain that

$$h_{ss} = a_1 k_{ss}^{\rho},$$

where

$$a_1 = \left[\frac{1}{\varepsilon}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\frac{\rho}{1-\rho}}.$$

Taking into account (42), we have

$$k_{ss} = \left(\frac{1-\varepsilon}{a_2}\right)^{\frac{1}{\rho}} n_{ss},$$

where $a_2 = a_1 - \varepsilon$.

Given a value for n_{ss} we can then solve for k_{ss}. Using (42) and (51), we solve for h_{ss} and r_{ss}. Using (43) and (44), we solve for y_{ss} and x_{ss}. Using (49) and (50), we solve for c_{kss} and m_{ss}. Assuming a value for the money growth rate in the steady state, θ_{ss}, (which is equal to the rate of inflation at the steady state, π_{ss}) and using (47), we solve for i_{ss}.

From (45) and (46), we obtain the expression of the utility function parameter λ that is consistent with n_{ss}:

$$\lambda = \frac{h_{ss}(1+\pi_{ss})n_{ss}}{\beta(1-n_{ss})\left[x_{ss}+\beta\varepsilon^2(1-\rho)^2 m_{ss}k_{ss}^{2\rho-1}h_{ss}^{-1}\right]+h_{ss}(1+\pi_{ss})n_{ss}}.$$

In order to derive the log-linear approximation of the set of equations characterising the equilibrium, variables are expressed as log-linear deviations around the steady state, except those already expressed in percentage terms. Log-linear deviations of a variable u around its steady-state value, u_{ss}, are denoted by $\widehat{u}$, where $\widehat{u}_t = \ln u_t - \ln u_{ss}$. That is,

$$u_t = u_{ss}e^{\widehat{u}_t} \approx u_{ss}(1+\widehat{u}_t).$$

Two basic rules are followed in deriving approximations (Uhlig, 1999). First for two variables u_t and z_t,

$$u_t z_t \approx u_{ss}(1+\widehat{u}_t)z_{ss}(1+\widehat{z}_t) \approx u_{ss}z_{ss}(1+\widehat{u}_t+\widehat{z}_t),$$

that is, we assume that product terms such as $\widehat{u}_t\widehat{z}_t$ are approximately zero. Second,

$$u_t^a \approx u_{ss}^a(1+\widehat{u}_t)^a \approx u_{ss}^a(1+a\widehat{u}_t).$$

- Equation (30)

$$h_t \equiv \varepsilon B_t^\rho k_t^\rho + (1-\varepsilon)A_t^\rho n_t^\rho,$$

$$h_{ss}(1+\widehat{h}_t) = \varepsilon k_{ss}^\rho(1+\rho\widehat{B}_t+\rho\widehat{k}_t)+(1-\varepsilon)n_{ss}^\rho(1+\rho\widehat{A}_t+\rho\widehat{n}_t),$$

using the expression for h_{ss} we have that

$$h_{ss}\widehat{h}_t = \varepsilon k_{ss}^\rho\rho(\widehat{B}_t+\widehat{k}_t)+(1-\varepsilon)n_{ss}^\rho\rho(\widehat{A}_t+\widehat{n}_t). \tag{52}$$

- Equation (31)

$$y_t = h_t^{\frac{1}{\rho}},$$

$$\ln y_t = \frac{1}{\rho} \ln h_t,$$

$$\ln y_{ss} + \widehat{y}_t = \frac{1}{\rho}\left(\ln h_{ss} + \widehat{h}_t\right),$$

using the expression for y_{ss} we have that

$$\widehat{y}_t = \frac{1}{\rho}\widehat{h}_t. \tag{53}$$

- Equation (32)

$$x_t \equiv \rho h_t + (1-\rho)(1-\varepsilon)A_t^{\rho} n_t^{\rho},$$

$$x_{ss}(1+\widehat{x}_t) = \rho h_{ss}(1+\widehat{h}_t) + (1-\rho)(1-\varepsilon)n_{ss}^{\rho}(1+\rho\widehat{A}_t + \rho\widehat{n}_t),$$

using the expression for x_{ss} we have that

$$x_{ss}\widehat{x}_t = \rho h_{ss}\widehat{h}_t + (1-\rho)(1-\varepsilon)n_{ss}^{\rho}\rho(\widehat{A}_t + \widehat{n}_t). \tag{54}$$

- Equation (33)

$$s_t = \mu_t m_t n_t^{-1} x_t,$$

$$\ln s_t = \ln \mu_t + \ln m_t - \ln n_t + \ln x_t,$$

$$\widehat{s}_t = \widehat{\mu}_t + \widehat{m}_t - \widehat{n}_t + \widehat{x}_t. \tag{55}$$

- Equation (34)

$$z_t \equiv (1-\lambda)h_t m_t^{\lambda(1-\eta)}(1-n_t)^{-\lambda-\eta(1-\lambda)},$$

$$\ln z_t = \ln(1-\lambda) + \ln h_t + \lambda(1-\eta)\ln m_t - [\lambda + \eta(1-\lambda)]\ln l_t,$$

where $l_t = 1 - n_t$ and $\widehat{l}_t = -(n_{ss}/(1-n_{ss}))\widehat{n}_t$. Therefore,

$$\widehat{z}_t = \widehat{h}_t + \lambda(1-\eta)\widehat{m}_t + [\lambda + \eta(1-\lambda)]\frac{n_{ss}}{1-n_{ss}}n_t. \tag{56}$$

- Equation (35)

$$z_t = s_t + \beta E_t\left[\mu_{t+1}\varepsilon^2(1-\rho)^2 m_{t+1}k_{t+1}^{\rho-1}h_{t+1}^{-1}B_t^{\rho}m_t n_t^{-1}k_t^{\rho}\right],$$

or

$$z_t = s_t + \beta E_t\left[q_{t+1}\right],$$

where

$$q_{t+1} = \mu_{t+1}\varepsilon^2(1-\rho)^2 m_{t+1}k_{t+1}^{\rho-1}h_{t+1}^{-1}B_t^{\rho}m_t n_t^{-1}k_t^{\rho},$$

$$z_{ss}(1+\widehat{z}_t) = s_{ss}(1+\widehat{s}_t) + \beta q_{ss}(1+E_t\widehat{q}_{t+1}),$$

$$z_{ss}\widehat{z}_t = s_{ss}\widehat{s}_t + \beta q_{ss}E_t\widehat{q}_{t+1}, \tag{57}$$

$$\widehat{q}_{t+1} = \widehat{\mu}_{t+1} + \widehat{m}_{t+1} + (\rho-1)\widehat{k}_{t+1} - \widehat{h}_{t+1} + \rho\widehat{B}_t + \widehat{m}_t - \widehat{n}_t + \rho\widehat{k}_t.$$

- Equation (36)

$$\mu_t = \beta\lambda E_t\left[m_{t+1}^{\lambda-1-\lambda\eta}(1-n_{t+1})^{(1-\lambda)(1-\eta)}\Pi_{t+1}^{-1}\right],$$

$$\widehat{\mu}_t = E_t\left[(\lambda-1-\lambda\eta)\widehat{m}_{t+1} - (1-\lambda)(1-\eta)\frac{n_{ss}}{1-n_{ss}}n_{t+1} - \widehat{\pi}_{t+1}\right]. \tag{58}$$

- Equation (37)

$$\mu_t = \beta E_t\left(\mu_{t+1}\frac{1+i_t}{1+\pi_{t+1}}\right),$$

$$\mu_{ss}(1+\widehat{\mu}_t) = \beta\mu_{ss}\frac{1+i_{ss}}{1+\pi_{ss}}E_t\left(1+\widehat{\mu}_{t+1}+\widehat{i}_t-\widehat{\pi}_{t+1}\right),$$

since $\frac{1+i_{ss}}{1+\pi_{ss}} = \beta^{-1}$, we have that

$$\widehat{\mu}_t = \widehat{i}_t + E_t\left(\widehat{\mu}_{t+1} - \widehat{\pi}_{t+1}\right). \tag{59}$$

- Equation (38)

$$E_t\left[\beta c_{kt+1}^{-\eta}\left(r_{t+1}-\delta+1\right)\right] = c_{kt}^{-\eta},$$

$$E_t\left[\ln\beta - \eta\ln c_{kt+1} + \ln R_{t+1}\right] = -\eta\ln c_{kt},$$

where $R_{t+1} = r_{t+1}-\delta+1$. Using the steady-state conditions

$$E_t\left[-\eta(\widehat{c}_{kt+1}-\widehat{c}_{kt}) + \widehat{R}_{t+1}\right] = 0, \tag{60}$$

where $\widehat{R}_{t+1} = \frac{r_{ss}}{R_{ss}}\widehat{r}_{t+1}$.

- Equation (39)

$$c_{kt} + k_{t+1} = r_t k_t + (1-\delta)k_t,$$

$$c_{kss}(1+\widehat{c}_{kt}) + k_{ss}(1+\widehat{k}_{t+1}) = r_{ss}(1+\widehat{r}_t)k_{ss}(1+\widehat{k}_t) + (1-\delta)k_{ss}(1+\widehat{k}_t),$$

since r_t is already measured as a percentage rate, we then consider $\widetilde{r}_t = r_t - r_{ss}(=\widehat{r}_t r_{ss})$ instead of $\widehat{r}_t = \frac{r_t - r_{ss}}{r_{ss}}$, (that is, the percentage deviation around the steady state). Using the steady-state condition and assuming that product term $\widehat{r}_t\widehat{k}_t$ is approximately zero, we obtain

$$c_{kss}\widehat{c}_{kt} + k_{ss}\widehat{k}_{t+1} = k_{ss}\widetilde{r}_t + r_{ss}k_{ss}\widehat{k}_t + (1-\delta)k_{ss}\widehat{k}_t. \tag{61}$$

- Equation (40)

$$m_t = y_t - r_t k_t,$$

$$m_{ss}(1+\widehat{m}_t) = y_{ss}(1+\widehat{y}_t) - r_{ss}(1+\widehat{r}_t)k_{ss}(1+\widehat{k}_t),$$

using the steady-state condition and assuming that product term $\widehat{r}_t\widehat{k}_t$ is approximately zero, we obtain

$$m_{ss}\widehat{c}_{ut} = y_{ss}\widehat{y}_t - k_{ss}\widetilde{r}_t - r_{ss}k_{ss}\widehat{k}_t. \qquad (62)$$

- Equation (41)

$$r_t = \varepsilon B_t^{\rho} k_t^{\rho-1} h_t^{\frac{1-\rho}{\rho}},$$

taking natural logs

$$\ln r_t = \ln \varepsilon + \rho \ln B_t + (\rho - 1)\ln k_t + \frac{1-\rho}{\rho}\ln h_t,$$

using the steady-state condition

$$\widetilde{r}_t = \rho r_{ss}\widehat{B}_t + (\rho-1)r_{ss}\widehat{k}_t + \frac{1-\rho}{\rho} r_{ss}\widehat{h}_t. \qquad (63)$$

- Equation (18)

$$\ln A_t = (1-\phi_A)\ln A_{ss} + \phi_A \ln A_{t-1} + v_{At},$$

and then

$$\widehat{A}_t = \phi \widehat{A}_{t-1} + v_{At}. \qquad (64)$$

- Equation (**??**)

$$\ln B_t = (1-\phi_B)\ln B_{ss} + \phi_B \ln B_{t-1} + v_{Bt},$$

$$\widehat{B}_t = \phi_B \widehat{B}_{t-1} + v_{Bt}. \qquad (65)$$

- Equation (19)

$$M_t = (1+\theta_t)M_{t-1},$$

writing this expression in real terms

$$(1+\pi_t)m_t = (1+\theta_t)m_{t-1},$$

$$m_{ss}(1+\widehat{m}_t)(1+\pi_{ss}+\widehat{\pi}_t) = (1+\theta_{ss}+\widehat{\theta}_t)m_{ss}(1+\widehat{m}_{t-1}),$$

where $\widehat{\pi}_t$ and $\widehat{\theta}_t$ denote the deviations of money growth rate and inflation from their respective steady state values (that is, $\widehat{\pi}_t \equiv \pi_t - \pi_{ss}$ and $\widehat{\theta}_t \equiv \theta_t - \theta_{ss}$). Since $\theta_{ss} = \pi_{ss}$ the latter expression can be written as

$$\widehat{m}_t = \widehat{m}_{t-1} + \widehat{\theta}_t - \widehat{\pi}_t. \tag{66}$$

Moreover, we assume that $\widehat{\theta}_t$ is determined by

$$\widehat{\theta}_t = \gamma\widehat{\theta}_{t-1} + \phi(\ln A_{t-1} - \ln A_{ss}) + v_t. \tag{67}$$

The system of equations characterising the log-linear approximation of the model (52)-(67) (together with seven extra identities involving forecast errors) can be written in matrix form as follows

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\overline{v}_t + \Pi\eta_t, \tag{68}$$

where

$$\begin{aligned} X_t \;\; = \;\; & (\widehat{k}_{t+1}, \widehat{n}_t, \widehat{h}_t, \widehat{x}_t, \widehat{s}_t, \widehat{z}_t, \widehat{y}_t, \widehat{c}_{kt}, \widehat{m}_t, \widehat{\mu}_t, \widetilde{r}_t, \widehat{i}_t, \widehat{\pi}_t, E_t\widehat{m}_{t+1}, \\ & E_t\widehat{c}_{kt+1}, E_t\widehat{n}_{t+1}, E_t\widehat{h}_{t+1}, E_t\widetilde{r}_{t+1}, E_t\widehat{\mu}_{t+1}, E_t\widehat{\pi}_{t+1}, \widehat{A}_t, \widehat{B}_t, \widehat{\theta}_t)', \end{aligned}$$

$$\overline{v}_t = (v_{At}, v_{Bt}, v_t)',$$

$$\begin{aligned} \eta_t \;\; = \;\; & (\widehat{m}_t - E_{t-1}\widehat{m}_t, \widehat{c}_{kt} - E_{t-1}\widehat{c}_{kt}, \widehat{n}_t - E_{t-1}\widehat{n}_t, \widehat{h}_t - E_{t-1}\widehat{h}_t, \\ & \widetilde{r}_t - E_{t-1}\widetilde{r}_t, \widehat{\mu}_t - E_{t-1}\widehat{\mu}_t, \widehat{\pi}_t - E_{t-1}\widehat{\pi}_t)' \end{aligned}$$

the seven extra identities are

$$\widehat{m}_t = E_{t-1}\widehat{m}_t + (\widehat{m}_t - E_{t-1}\widehat{m}_t),$$

$$\widehat{c}_{kt} = E_{t-1}\widehat{c}_{kt} + (\widehat{c}_{kt} - E_{t-1}\widehat{c}_{kt}),$$

$$\widehat{n}_t = E_{t-1}\widehat{n}_t + (\widehat{n}_t - E_{t-1}\widehat{n}_t),$$

$$\widehat{h}_t = E_{t-1}\widehat{h}_t + (\widehat{h}_t - E_{t-1}\widehat{h}_t),$$

$$\widetilde{r}_t = E_{t-1}\widetilde{r}_t + (\widetilde{r}_t - E_{t-1}\widetilde{r}_t),$$

$$\widehat{\mu}_t = E_{t-1}\widehat{\mu}_t + (\widehat{\mu}_t - E_{t-1}\widehat{\mu}_t),$$

$$\widehat{\pi}_t = E_{t-1}\widehat{\pi}_t + (\widehat{\pi}_t - E_{t-1}\widehat{\pi}_t),$$

and

$$
\Gamma_0 = \begin{pmatrix}
0 & -h_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_2 & -h_1 & 0 \\
0 & 0 & -1/\rho & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_2 & -x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -n_1 & -1 & 0 & 0 & 1 & 0 & 0 & -n_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Gamma_{6,1} & \overline{q} & 0 & 0 & -s_{ss} & z_{ss} & 0 & 0 & -\overline{q} & 0 & 0 & 0 & 0 & -\overline{q} & 0 & 0 & \overline{q} & 0 & -\overline{q} & 0 & 0 & -\rho\overline{q} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & -\eta & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \Gamma_{9,14} & 0 & \Gamma_{9,16} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
k_{ss} & 0 & 0 & 0 & 0 & 0 & 0 & c_{kss} & 0 & 0 & -k_{ss} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -y_{ss} & 0 & m_{ss} & 0 & k_{ss} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Gamma_{12,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r_{ss}\rho & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\Gamma_1 = \begin{pmatrix}
h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{q}\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{ss}\beta^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-r_{ss}k_{ss} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\rho-1)r_{ss} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_A & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_B & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi & \phi & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},
$$

where

$$\overline{q} = \beta q_{ss},$$

$$\Gamma_{6,1} = -\overline{q}(\rho - 1),$$

$$\Gamma_{9,14} = 1 - \lambda(1 - \eta),$$

$$\Gamma_{9,16} = (1 - \lambda)(1 - \eta)\frac{l_{ss}}{1 - l_{ss}},$$

$$\Gamma_{12,3} = -r_{ss}\frac{(1 - \rho)}{\rho},$$

$$h_1 = \frac{\varepsilon\rho k_{ss}^{\rho}}{h_{ss}},$$

$$h_2 = \frac{(1 - \varepsilon)\rho n_{ss}^{\rho}}{h_{ss}},$$

$$x_1 = \frac{\rho h_{ss}}{x_{ss}},$$

$$x_2 = \frac{\rho(1 - \rho)(1 - \varepsilon)n_{ss}^{\rho}}{x_{ss}},$$

$$n_1 = [\eta(1 - \lambda) + \lambda)]\frac{n_{ss}}{1 - n_{ss}},$$

$$n_2 = \lambda(1 - \eta),$$

$$\Psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Equation (68) is a linear rational expectations (LRE) system. Lubik and Schorfheide (2003) characterise the complete set of solutions of LRE models and provide a method for computing them that builds on Sims' (2001) approach.[10]

[10]The GAUSS code for computing the equilibria of LRE models was downloaded from Schorfheide's web-site.