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Shortcuts to adiabaticity in three-level systems using Lie transforms

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Sped-up protocols that drive a system quickly to the same populations than a slow adiabatic process may involve Hamiltonian terms which are difficult to realize. We use the dynamical symmetry of the Hamiltonian to find, by means of Lie transforms, alternative Hamiltonians that achieve the same goals without the problematic terms. We apply this technique to three-level systems (two interacting bosons in a double well, and beam splitters with two and three output channels) driven by Hamiltonians that belong to the four-dimensional algebra U3S3.

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I. INTRODUCTION

"Shortcuts to adiabaticity" are manipulation protocols that take the system quickly to the same populations, or even the same state, than a slow adiabatic process [1]. Adiabaticity is ubiquitous to prepare the system state in atomic, molecular and optical physics, so many applications of this concept have been worked out, both in theory and experiment [1]. Some of the engineered Hamiltonians that speed up the adiabatic process in principle, may involve terms which are difficult or impossible to realize in practice. In simple systems the dynamical symmetry of the Hamiltonian could be used to eliminate the problematic terms and provide instead feasible Hamiltonians. Examples are single particles transported or expanded by harmonic potentials [2, 3], or two-level systems [4–6]. In this paper we extend this program to three-level systems whose Hamiltonians belong to a four-dimensional dynamical algebra. This research was motivated by a recent observation by Opatrný and Mølmer [7]. Among other systems they considered two (ultra cold) interacting bosons in a double well within a three-state approximation. Specifically the aim was to speed up a transition from a "Mott-insulator" state with one particle in each well, to a delocalized "superfluid" state. The reference adiabatic process consisted on slowly turning off the inter particle interaction while increasing the tunneling rate. To speed up this process they applied a method to generate shortcuts based on adding a "counterdiabatic" (cd) term to the original time-dependent Hamiltonian [4, 8, 9], but the evolution with the cd-term turns out to be difficult to realize in practice [7]. In this paper we shall use the symmetry of the Hamiltonian (its dynamical algebra) to find an alternative shortcut by means of a Lie transform, namely, a unitary operator in the Lie group associated with the Lie algebra. Since other physical systems have the same Hamiltonian structure the results are applicable to them too. Specifically the analogy between the time-dependent Schrödinger equation and the stationary wave equation for a waveguide in the paraxial approximation [10–15] is used to design short-length optical beam splitters with two and three output channels.

In Sec. II we describe the theoretical model for two indistinguishable particles in two wells. In Sec. III we summarize the counterdiabatic or transitionless tracking approach and apply it to the bosonic system. Sec. IV sets the approach based on unitary Lie transforms to produce alternative shortcuts. In Sec. V we introduce the insulator-superfluid transition and apply the shortcut designed in the previous section. In Sec. VI we apply the technique to generate beam splitters with two and three output channels. Section VII discusses the results and open questions. Finally, in the Appendix A some features of the Lie algebra of the system are discussed.

II. THE MODEL

An interacting boson gas in a two-site potential is described within the Bose-Hubbard approximation [16, 17] by

$$H_0 = \frac{U}{2}\sum_{j=1}^{2} n_j(n_j-1) - J(a_1 a_2^\dagger + a_1^\dagger a_2), \qquad (1)$$

where a_j ($a_j^\dagger$) are the bosonic particle annihilation (creation) operators at the j-th site and n_j is the occupation number operator. The on-site interaction energy is quantified by the parameter U and the hopping energy by J. They are assumed to be controllable functions of time. For two particles the Hamiltonian in the occupation number basis $|2,0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|1,1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $|0,2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$, is given by [7]

$$H_0 = \begin{pmatrix} U & -\sqrt{2}J & 0 \\ -\sqrt{2}J & 0 & -\sqrt{2}J \\ 0 & -\sqrt{2}J & U \end{pmatrix} = UG_4 - 4JG_1, \quad (2)$$

where

$$G_1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (3)$$

This Hamiltonian belongs to the vector space (Lie algebra) spanned by G_1, G_4, and two more generators,

$$G_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, G_3 = \frac{1}{4} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad (4)$$

with nonzero commutation relations

$$[G_1, G_2] = iG_3, [G_2, G_3] = iG_1, [G_3, G_1] = iG_2,$$

$$[G_4, G_1] = iG_2, [G_2, G_4] = iG_1. \quad (5)$$

This 4-dimensional Lie algebra, U3S3 [18], is described in more detail in the Appendix A. To find the Hermitian basis we calculate $[G_1, G_4]$, and then all commutators of the result with previous elements. This operation is repeated for all operator pairs until no new, linearly independent operator appears.

To diagonalize the Hamiltonian (2) it is useful to parameterize U and J as [7]

$$U = E_0 \cos\varphi, \; J = \frac{E_0}{4} \sin\varphi, \quad (6)$$

where $E_0 = E_0(t)$ and $\varphi = \varphi(t)$, so that

$$H_0 = E_0 \begin{pmatrix} \cos\varphi & -\frac{1}{2\sqrt{2}}\sin\varphi & 0 \\ -\frac{1}{2\sqrt{2}}\sin\varphi & 0 & -\frac{1}{2\sqrt{2}}\sin\varphi \\ 0 & -\frac{1}{2\sqrt{2}}\sin\varphi & \cos\varphi \end{pmatrix}. \quad (7)$$

The instantaneous eigenvalues are

$$E_1 = \frac{E_0}{2}(\cos\varphi - 1), \quad (8)$$

$$E_2 = E_0 \cos\varphi, \quad (9)$$

$$E_3 = \frac{E_0}{2}(\cos\varphi + 1), \quad (10)$$

corresponding to the normalized eigenstates

$$|\phi_1\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1-\cos\varphi} \\ \frac{1}{\sqrt{2}}\sqrt{1+\cos\varphi} \\ \frac{1}{2}\sqrt{1-\cos\varphi} \end{pmatrix}, \quad (11)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad (12)$$

$$|\phi_3\rangle = \begin{pmatrix} \frac{1}{2}\sqrt{1+\cos\varphi} \\ -\frac{1}{\sqrt{2}}\sqrt{1-\cos\varphi} \\ \frac{1}{2}\sqrt{1+\cos\varphi} \end{pmatrix}. \quad (13)$$

III. COUNTERDIABATIC OR TRANSITIONLESS TRACKING APPROACH

For the transitionless driving or counterdiabatic approach formulated by Demirplak and Rice [8] or equivalently by Berry [9], the starting point is a time-dependent reference Hamiltonian

$$H_0(t) = \sum_n |n_0(t)\rangle E_n^{(0)}(t) \langle n_0(t)|. \quad (14)$$

The approximate time-dependent adiabatic solutions are

$$|\psi_n(t)\rangle = e^{i\xi_n(t)} |n_0(t)\rangle, \quad (15)$$

where the adiabatic phase reads

$$\xi_n(t) = -\frac{1}{\hbar}\int_0^t dt' E_n^{(0)}(t') + i\int_0^t dt' \langle n_0(t')|\partial_{t'} n_0(t')\rangle. \quad (16)$$

Defining now the unitary operator

$$A(t) = \sum_n e^{i\xi_n(t)} |n_0(t)\rangle\langle n_0(0)|, \quad (17)$$

a Hamiltonian $H(t) = i\hbar \dot{A} A^\dagger$ can be constructed to drive the system exactly along the adiabatic paths of $H_0(t)$ as

$$\begin{aligned} H(t) &= H_0(t) + H_{cd}(t), \\ H_{cd}(t) &= i\hbar \sum_n (|\dot{n}_0(t)\rangle\langle n_0(t)| \\ &- \langle n_0(t)|\dot{n}_0(t)\rangle |n_0(t)\rangle\langle n_0(t)|), \end{aligned} \quad (18)$$

where $H_{cd}(t)$ is purely non-diagonal in the $\{|n_0(t)\rangle\}$ basis and the dot represents time derivative.

For our system ($|n_0(t)\rangle \to |\phi_n\rangle$), the counterdiabatic term takes the form

$$H_{cd} = i\hbar(|\dot{\phi}_1\rangle\langle\phi_1| + |\dot{\phi}_3\rangle\langle\phi_3|). \quad (19)$$

Taking into account Eqs. (11), (12), (13) and their respective time derivatives we get

$$H_{cd} = -\hbar\dot{\varphi} G_2. \quad (20)$$

Implementing this interaction is quite challenging as discussed in detail in [7]. In particular, a rapid switching between G_1 and G_4, to implement G_2 through their commutator, is not a practical option [7]. Our goal in the following is to design an alternative Hamiltonan to perform the shortcut without G_2.

IV. ALTERNATIVE DRIVING PROTOCOLS VIA LIE TRANSFORMS

The main goal here is to define a new shortcut, different from the one described by $i\hbar\partial_t\psi(t) = H(t)\psi(t)$, where $H(t) = H_0(t) + H_{cd}(t)$. A wave function $\psi_I(t)$, that represents the alternative dynamics, is related to $\psi(t)$ by a unitary operator $B(t)$,

$$\psi_I(t) = B^\dagger(t)\psi(t), \quad (21)$$

and obeys $i\hbar\partial_t\psi_I(t) = H_I(t)\psi_I(t)$, where

$$H_I(t) = B^\dagger(t)(H(t) - K(t))B(t), \quad (22)$$

$$K(t) = i\hbar\dot{B}(t)B^\dagger(t). \quad (23)$$

These are formally the same expressions that define an interaction picture. However, in this application the "interaction picture" portrays a different physical setting

from the original one [6]. In other words, H_I is not a mathematical aid to facilitate a calculation in some transformed space, but rather a physically realizable Hamiltonian different from H. Similarly ψ_I represents in general different dynamics from ψ. The transformation provides indeed an alternative shortcut if $B(0) = B(t_f) = 1$, so that $\psi_I(t_f) = \psi(t_f)$ for a given initial state $\psi_I(0) = \psi(0)$. Moreover, if $\dot{B}(0) = \dot{B}(t_f) = 0$ also the Hamiltonians coincide at initial and final times, $H(0) = H_I(0)$ and $H(t_f) = H_I(t_f)$. These boundary conditions may be relaxed in some cases as we shall see.

We carry out the transformation by exponentiating a member G of the dynamical Lie algebra of the Hamiltonian,

$$B(t) = e^{-i\alpha G}, \tag{24}$$

where $\alpha = \alpha(t)$ is a time dependent real function to be determined. This type of unitary operator $B(t)$ constitutes a "Lie transform". Lie transforms have been used for example to develop efficient perturbative approaches that try to set the perturbation term of a Hamiltonian in a convenient form both in classical and quantum systems [19, 20].

Note that K in Eq. (23) becomes $-\hbar\dot{\alpha}G$ and commutes with G. Then, H_I, given now by

$$\begin{aligned} B^\dagger(H-K)B &= e^{i\alpha G}(H-K)e^{-i\alpha G} \\ &= H - \hbar\dot{\alpha}G + i\alpha[G,H] - \frac{\alpha^2}{2!}[G,[G,H]] \\ &- i\frac{\alpha^3}{3!}[G,[G,[G,H]]] + \cdots \end{aligned} \tag{25}$$

depends only on G, H, and its repeated commutators with G, so it stays in the algebra. If we can choose G and α so that the undesired generator components in H cancel out and the boundary conditions for B are satisfied, the method provides a feasible, alternative shortcut. In the existing applications of the method [1, 6], and in this paper we proceed by trial an error, testing different generators. In the present application we want the Hamiltonian H_I to keep the structure of the original one, with non-vanishing components proportional to G_1 and G_4. We may quickly discard by inspection G_1, G_2, and G_3 as candidates for G. Choosing $G \to G_4$ in Eq. (24), and substituting into Eqs. (22) and (25), the series of repeated commutators may be summed up. H_I becomes

$$\begin{aligned} H_I &= (E_0\cos\varphi - \hbar\dot{\alpha})\,G_4 \\ &- (E_0\sin\varphi\cos\alpha + \hbar\dot{\varphi}\sin\alpha)\,G_1 \\ &- (E_0\sin\varphi\sin\alpha - \hbar\dot{\varphi}\cos\alpha)\,G_2. \end{aligned} \tag{26}$$

To cancel the G_2 term, we choose

$$\alpha(t) = \operatorname{arccot}\left[\frac{E_0(t)}{\hbar\dot{\varphi}(t)}\sin(\varphi(t))\right]. \tag{27}$$

Substituting Eq. (27) into Eq. (26) we have finally

$$H_I = \left[\frac{\cos\varphi E_0^3\sin^2\varphi + \hbar^2\sin\varphi\dot{E}_0\dot{\varphi} + \hbar^2 E_0\left(2\cos\varphi\dot{\varphi}^2 - \sin\varphi\ddot{\varphi}\right)}{E_0^2\sin^2\varphi + \hbar^2\dot{\varphi}^2}\right]G_4 - \left[E_0\sin\varphi\sqrt{1+\frac{\hbar^2\csc^2\varphi\dot{\varphi}^2}{E_0^2}}\right]G_1, \tag{28}$$

which has the same structure (generators) as the reference Hamiltonian but different time-dependent coefficients.

V. INSULATOR-SUPERFLUID TRANSITION

Changing the U/J ratio, the system may go from a "Mott insulator" (the two particles isolated in separate wells) to a "superfluid" state (in which each particle is distributed with equal probability in both wells). From Eq. (11), the Mott-insulator ground state is $|\phi_1\rangle = |1,1\rangle$ and in the superfluid regime the ground state becomes $|\phi_1\rangle = \frac{1}{2}|2,0\rangle + \frac{1}{\sqrt{2}}|1,1\rangle + \frac{1}{2}|0,2\rangle$. To design a reference process (one that performs the transition when driven slowly enough) we consider polynomial functions for $E_0(t)$ and $\varphi(t)$. Since we want to drive the system from $|1,1\rangle$ to $\frac{1}{2}|2,0\rangle + \frac{1}{\sqrt{2}}|1,1\rangle + \frac{1}{2}|0,2\rangle$, we impose in Eq. (11)

$$\varphi(0) = 0, \varphi(t_f) = \pi/2. \tag{29}$$

To have the wells isolated at $t = 0$ but connected (allowing the particles to pass from one to the other) at $t = t_f$ we also set

$$E_0(0) = 0, E_0(t_f) \neq 0, \tag{30}$$

so that $J(0) = U(0) = 0$ and $J(t_f) \neq 0$. Moreover, for a smooth connection with the asymptotic regimes ($t < 0$, $t > t_f$) we set

$$\dot{\varphi}(0) = 0, \dot{\varphi}(t_f) = 0. \tag{31}$$

This implies that $H_{cd}(0) = H_{cd}(t_f) = 0$, see Eq. (20). The condition

$$\ddot{\varphi}(t_f) = 0 \tag{32}$$

is also needed to implement alternative shortcuts, in particular, to satisfy $\dot{B}(t_f) = 0$. At intermediate times, we interpolate the functions as $E_0(t) = \sum_{j=0}^{1} a_j t^j$ and $\varphi(t) = \sum_{j=0}^{4} b_j t^j$, where the coefficients are found by solving the equations for Eqs. (29), (30), (31) and (32). These functions are shown in Fig. 1. In this and other figures $\tau = E_0^{max} t/\hbar$, where E_0^{max} is the maximum value of $E_0(t)$.

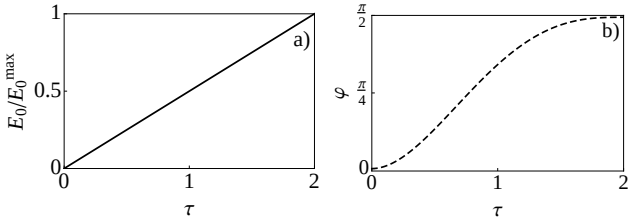

FIG. 1: Functions in $H_I(t)$: (a) $E_0(t)$ and (b) $\varphi(t)$. Parameters: $\tau = E_0^{max} t/\hbar$ where E_0^{max} is the maximum value of $E_0(t)$ and $\tau_f = 2$.

The actual time evolution of the state

$$|\Psi(t)\rangle = c_1(t)|2,0\rangle + c_2(t)|1,1\rangle + c_3(t)|0,2\rangle, \qquad (33)$$

is given by solving Schrödinger's equation with the different Hamiltonians. For this particular transition, $|\Psi(0)\rangle = |\phi_1(0)\rangle$ and the ideal target state is (up to a global phase factor) $|\Psi(t_f)\rangle = |\phi_1(t_f)\rangle$. The dynamics

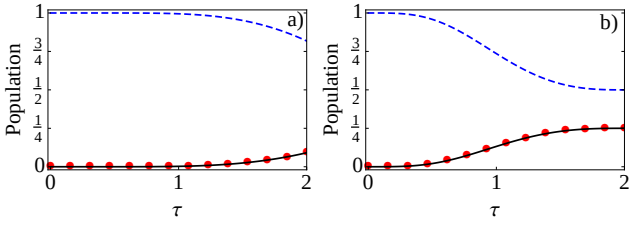

FIG. 2: (Color online). Bare-state populations for (a) $H_0(t)$, (b) $H(t)$ and $H_I(t)$. $|c_1(t)|^2$ (red circles), $|c_2(t)|^2$ (short-dashed blue line) and $|c_3(t)|^2$ (solid black line). Parameters: $\tau = E_0^{max} t/\hbar$ with E_0^{max} the maximum value of $E_0(t)$, $\tau_f = 2$.

versus time τ is shown in Fig. 2 for $\tau_f = 2$. For this short time $H_0(t)$ fails to drive the populations to 1/2 and 1/4, whereas when $H_{cd}(t)$ is added the intended transition occurs successfully. As for the alternative Hamiltonian in Eq. (28), with $B = e^{-i\alpha G_4}$, and α in Eq. (27), we find

$$B(t_f) = 1, \dot{B}(0) = \dot{B}(t_f) = 0, \qquad (34)$$

(Eq. (32) is necessary to have $\dot{\alpha}(t_f) = 0$ and consequently $\dot{B}(t_f) = 0$), whereas

$$B(0) = \begin{pmatrix} e^{-i\pi/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\pi/2} \end{pmatrix} \neq 1. \qquad (35)$$

However $B^\dagger(0)|1,1\rangle = |1,1\rangle$ so $\psi^I(0) = \psi(0)$ and H_I provides the desired shortcut.

Solving numerically the dynamics for $H_I(t)$ we obtain a perfect insulator-superfluid transition (see Fig. 2 (b)). Notice that, as G_4 is diagonal in the bare basis, the bare-populations are the same for the dynamics driven by H and H_I, see Fig. 2 (b).

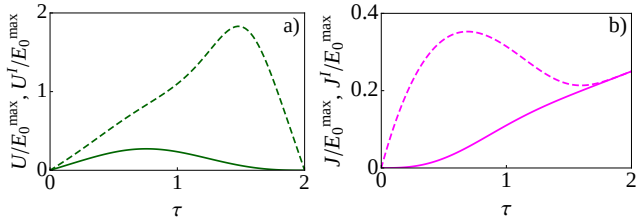

FIG. 3: (Color online). (a) Interaction energy for the reference Hamiltonian H_0 (solid green line) and for H_I (short-dashed green line). (b) Hopping energy for H_0 (solid magenta line) and H_I (short-dashed magenta line). Same parameters as Fig. 1.

In order to compare our approach with other protocols we reformulate H_I as

$$H_I = \begin{pmatrix} U^I & -\sqrt{2}J^I & 0 \\ -\sqrt{2}J^I & 0 & -\sqrt{2}J^I \\ 0 & -\sqrt{2}J^I & U^I \end{pmatrix} = U^I G_4 - 4J^I G_1. \qquad (36)$$

Comparing Eqs. (36) and (28) we find that

$$\begin{aligned} U^I &= \frac{1}{(E_0)^2 \sin^2\varphi + \hbar^2(\dot{\varphi})^2}\Big[\cos\varphi (E_0)^3 \sin^2\varphi \\ &+ \hbar^2 \sin\varphi \dot{E}_0 \dot{\varphi} + \hbar^2 E_0 \left(2\cos\varphi(\dot{\varphi})^2 - \sin\varphi\ddot{\varphi}\right)\Big], \\ J^I &= \frac{1}{4} E_0 \sin\varphi \sqrt{1 + \frac{\hbar^2 \csc^2\varphi (\dot{\varphi})^2}{(E_0)^2}}. \end{aligned} \qquad (37)$$

Figure 3 shows the functions U_I and J_I. We have set $H_I(t_b) = H_0(t_b)$, for $t_b = 0, t_f$, since $H_{cd}(t_b) = 0$ and $\dot{B}(t_b) = 0$. In the same way as Eq. (6) we can rewrite the above energies as

$$U^I = E_0^I \cos\varphi^I, \; J^I = \frac{E_0^I}{4}\sin\varphi^I, \qquad (38)$$

where $E_0^I = E_0^I(t)$ and $\varphi^I = \varphi^I(t)$. The inverse transformation is

$$\varphi^I = \arctan\left(4\frac{J^I}{U^I}\right), \; E_0^I = \frac{U^I}{\cos\varphi'}. \qquad (39)$$

Consider a simple protocol with $E_0(t) = E_0^M(t) = const.$ and a linear $\varphi^M(t)$ from 0 and $\pi/2$ [7]. Setting the value of E_0^M so that $\int E_0^M dt = \int E_0^I dt$, it is found that the simple protocol needs $\tau_f = 18.8$ to perform the transition with a 0.9999 fidelity. In other words, the protocol based on H_I is 9.4 times faster according to this criterion.

VI. BEAM SPLITTERS

The three-level Hamiltonian (2) describes other physical systems apart from two bosons in two wells. For example it represents in the paraxial approximation and substituting time by a longitudinal coordinate three coupled waveguides [10–15], where J is controlled by waveguide separation and U by the refractive index. In particular J and U may be manipulated to split an incoming wave in the central wave guide into two output channels (corresponding to the external waveguides) or three output chanels [14, 15]. The Hamiltonian also represents a single particle in a triple well [21], where U plays the role of the bias of the outer wells with respect to the central one and J the coupling coefficient between adjacent wells. The beam splitting may thus depict the evolution of the particle wave function from the central well either to the two outer wells or to three of them with equal probabilities.

For the three-well or three-waveguide systems[1] the minimal channel basis for left, center and right wave functions is $|L\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|C\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $|R\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

A. 1:2 beam splitter

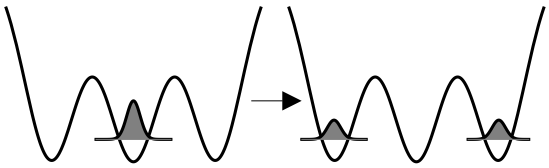

FIG. 4: Schematic representation of a $1:2$ beam splitter.

To implement a $1:2$ beam splitter, see Fig. 4, the goal is to drive the eigenstate from $|\phi_1(0)\rangle = |C\rangle$ to $|\phi_1(t_f) = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$. As in the previous section we use polynomial functions for $E_0(t)$ and $\varphi(t)$ to set a reference process. We impose

$$\varphi(0) = 0, \varphi(t_f) = \pi \tag{40}$$

in Eq. (11). The wells (waveguides) should be isolated at initial and final times. If morever all wells are at equal heights at those times we set

$$E_0(0) = E_0(t_f) = 0, E(t_f/2) \neq 0, \tag{41}$$

to satisfy $H_0(0) = H_0(t_f) = 0$. We also impose

$$\dot{\varphi}(0) = 0, \dot{\varphi}(t_f) = \pi \tag{42}$$

to smooth the functions at the time boundaries and make $H_{cd}(t_b) = 0$. In addition

$$\ddot{\varphi}(t_f) = 0 \tag{43}$$

is imposed to satisfy $\dot{B}(t_f) = 0$. At intermediate times $E_0(t) = \sum_{j=0}^{2} a_j t^j$ and $\varphi(t) = \sum_{j=0}^{4} b_j t^j$, with the coefficients deduced from Eqs. (40), (41), (42) and (43). These functions are shown in Fig. 5.

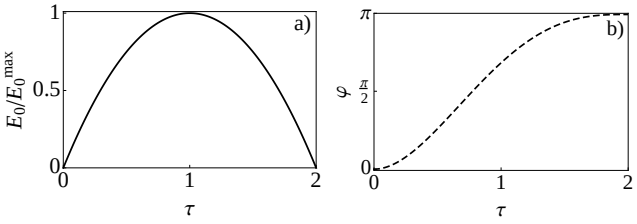

FIG. 5: (a) $E_0(t)$ and (b) $\varphi(t)$. $\tau = E_0^{max} t/\hbar$ where E_0^{max} is the maximum value of $E_0(t)$. $\tau_f = 2$.

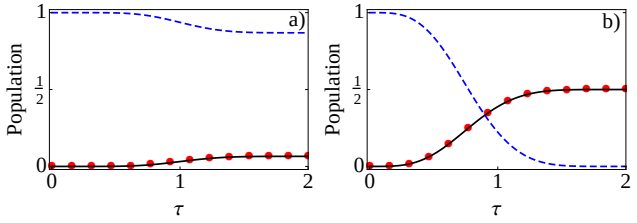

FIG. 6: (Color online). Bare-state populations for (a) $H_0(t)$, (b) $H(t)$ and $H_I(t)$. $|c_1(t)|^2$ (red circles), $|c_2(t)|^2$ (short-dashed blue line) and $|c_3(t)|^2$ (solid black line). Parameters: $\tau = E_0^{max} t/\hbar$ with E_0^{max} the maximum value of $E_0(t)$, $\tau_f = 2$.

Figure 6 shows the dynamics for $\tau_f = 2$. This time (corresponding to the splitter length in the optical system) is too short for the reference Hamiltonian $H_0(t)$ to drive the bare-basis populations to 0 and 1/2. Adding $H_{cd}(t)$ the transition occurs as desired. As in Sec. IV, we construct an alternative shortcut $H_I(t)$ without G_2 using the transformation $B = e^{-i\alpha G_4}$. With α in Eq. (27), $\dot{B}(0) = \dot{B}(t_f) = 0$, whereas

$$B(0) = B(t_f) = \begin{pmatrix} e^{-i\pi/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\pi/2} \end{pmatrix}. \tag{44}$$

This is enough for our objective as $B^\dagger(0)|C\rangle = |C\rangle$, and $B^\dagger(t_f)|\psi(t_f)\rangle = -i|\psi(t_f)\rangle$. Solving numerically the dynamics for $H_I(t)$ we obtain a perfect $1:2$ beam splitting (see Figs. 7 and 6 (c)).

To compare the new shortcut and the simple approach with $E_0^M = const.$ and $\varphi^M(t) = \frac{t}{t_f}\pi$ we set $\int E_0^M dt = \int E_0^I dt$. The constant-$E_0$ protocol needs $\tau_f \geq 18.6$ to achieve 0.9999 fidelity, so the protocol driven by H_I is 9.3 times faster.

[1] The Hamiltonian (2) also describes a three-level atom under appropriate laser interactions, see [13].

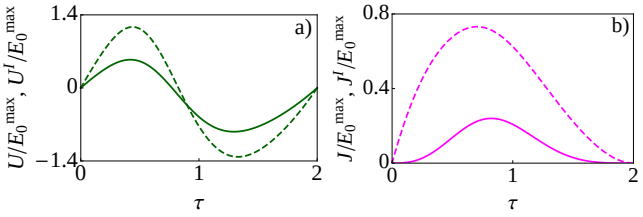

FIG. 7: (Color online). (a) Interaction energy for H_0 (solid green line) and H_I (short-dashed green line). (b) Hopping energy for H_0 (solid magenta line) and H_I (short-dashed magenta line). Same parameters as Fig. 5.

B. 1:3 beam splitter

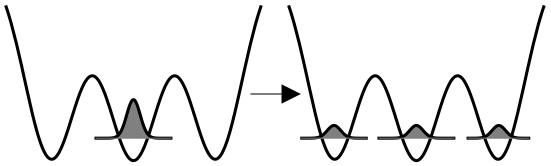

FIG. 8: Schematic representation of the beam splitter 1 : 3.

We also describe briefly a 1 : 3 beam splitter, see Fig. 8. The aim is to drive the system from $|\phi_1(0)\rangle = |C\rangle$ to equal populations in $|L\rangle$, $|C\rangle$, and $|R\rangle$. To design a reference protocol we use polynomial interpolation for $E_0(t)$ and $\varphi(t)$, see Fig. 9, with the same boundary conditions of the 1 : 2 splitter but with $\varphi(t_f) = 0.60817\pi = \arccos(-1/3)$ and the additional condition $\dot{E}_0(t_f) = 0$ (to satisfy $U^I(t_f) = U(t_f)$ so that $H_I(t_f) = H_0(t_f)$). The Lie transform may be applied as before on the protocol with the counterdiabatic correction, see Fig. 10 (b). A

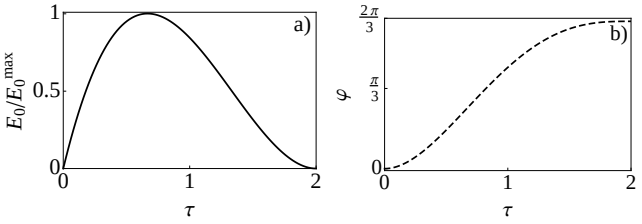

FIG. 9: (a) $E_0(t)$ and (b) $\varphi(t)$. $\tau = E_0^{max}t/\hbar$, where E_0^{max} is the maximum value of $E_0(t)$. $\tau_f = 2$.

simple protocol with E_0^M and $\varphi(t) = \frac{t}{t_f}0.60817\pi$ needs $\tau_f = 22$, if $\int E_0^M dt = \int E_0^I dt$, for a 0.9999 fidelity, so the protocol based on H_I is 11 times faster.

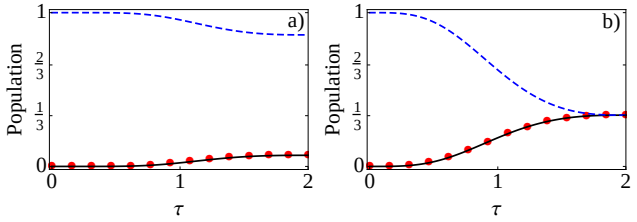

FIG. 10: (Color online). Bare-state populations for (a) $H_0(t)$, (b) $H(t)$ and $H_I(t)$. $|c_1(t)|^2$ (red circles), $|c_2(t)|^2$ (short-dashed blue line) and $|c_3(t)|^2$ (solid black line). Parameters: $\tau = E_0^{max}t/\hbar$ with E_0^{max} the maximum value of $E_0(t)$, $\tau_f = 2$.

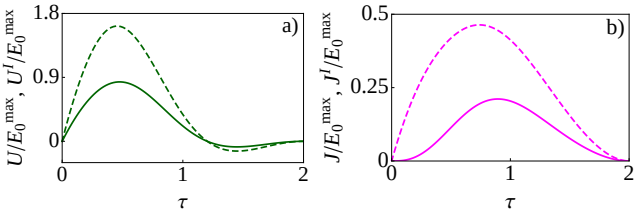

FIG. 11: (Color online). (a) Interaction energy for H_0 (solid green line) and H_I (short-dashed green line). (b) Hopping energy for H_0 (solid magenta line) and H_I (short-dashed magenta line). Same parameters as in Fig. 9

VII. DISCUSSION

We have started with shortcuts to adiabaticity for three-level systems with U3S3 symmetry (a four-dimensional Lie algebra) that include Hamiltonian terms which are difficult to implement in the laboratory. Alternative shortcuts without them have then been found by means of Lie transforms. These transformations are formally equivalent to interaction picture (IP) transformations. However the resulting IP-Hamiltonian and state represent a different physical process from the original (Schrödinger) Hamiltonian and dynamics. We have set shortcuts for different physical systems. For two particles in two wells we have implemented a fast insulator-superfluid transition. For coupled waveguides or a particle in a triple well we have implemented fast beam splitting with one input channel and two or three output channels. In all cases the IP Hamiltonian involves only two realizable terms (generators).

In a companion paper a related method has been worked out [22]. Both approaches rely on Lie algebraic methods and aim at constructing shortcuts to adiabaticity. However we do not use dynamical invariants explicitly in the current approach, whereas the bottom-up approach in [22] engineers the Hamiltonian making explicit use of its relation to dynamical invariants. In contrast, we start here from an existing, known shortcut –for example the one generated by a counter-diabatic method–; then, a Lie transform is applied to generate alternative,

feasible or more convenient shortcuts, as in [6]. A connection between the transformation method and dynamical invariants is sketched briefly in the Appendix but it deserves a separate study. We note that the dynamics of all our examples takes place in a degenerate eigenspace of an algebraic invariant which is not proportional to the unit matrix and commutes with all members of the algebra. The degeneracy is required to produce non-trivial dynamics, so identifying degenerate subspaces of nontrivial invariants, as well as the conditions allowing to cancel certain generators will be instrumental in finding further applications in systems described by other Lie algebras.

Optimal control theory (OCT) offers an alternative way to generate fast dynamics [23, 24]. In this paper no optimization has been attempted, but the combination of shortcut-to-adiabaticity techniques offering multiple exact protocols with perfect fidelity, such as the one based on Lie transforms, and OCT, has been shown to be fruitful. OCT may select among the protocols generated the ones that optimize a physically significant variable [25–27].

Within the scope of the algebra U3S3, other physical systems that could be treated are in quantum optics (three level atoms) [28, 29], nanostructures (triple wells or dots) [30], optics (mode converters) [31, 32], or Bose-Einstein condensates in an accelerated optical lattice [33].

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Appendix A: Lie algebra

The algebra of this three-level system is a four-dimensional Lie Algebra U3S3 according to the classification of 4-dimensional Lie algebras in [18]. (For comparison with that work it is useful to rewrite the generators in the skew-Hermitian base $\tilde{G}_k = -iG_k$, $k = 1, 2, 3, 4$.) $U3S3$ is a direct sum of the one dimensional algebra spanned by the invariant $G_4 - G_3$, that commutes with all members of the algebra, and a three-dimensional SU(2) algebra spanned by $\{G_1, G_2, G_3\}$. Notice that this realization of the 3D algebra is not spanned by the matrices

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{(A1)}$$

that correspond, in the subspace $|2,0\rangle, |1,1\rangle, |0,2\rangle$, to the operators

$$J_x = \frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \quad \text{(A2)}$$

$$J_y = \frac{1}{2i}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right), \quad \text{(A3)}$$

$$J_z = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right). \quad \text{(A4)}$$

In particular we cannot get the matrices for J_y or J_z by any linear combination of our G_k matrices (see Eqs. (3-4)). A second-quantized form for the G_k consistent with the matrices includes quartic terms in annihilation/creation operators:

$$\begin{aligned} G_1 &= \frac{1}{4}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \\ G_2 &= \frac{1}{4i}\left[a_1^\dagger a_2^\dagger \left(a_1 a_1 + a_2 a_2\right)\right. \\ &- \left.\left(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger\right) a_1 a_2\right], \\ G_3 &= \frac{1}{8}\left[\left(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger\right) a_1 a_1 - 4 a_1^\dagger a_2^\dagger a_1 a_2\right. \\ &+ \left.\left(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger\right) a_2 a_2\right], \\ G_4 &= \frac{1}{4}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)^2, \end{aligned} \quad \text{(A5)}$$

These second-quantized operators do not form a closed algebra under commutation but their matrix elements for two particles do.

An invariant (defined in a Lie-algebraic sense) commutes with any member of the algebra. There are generically two independent invariants for $U3S3$ [34]. For the matrix representation in Eqs. (3) and (4) they are

$$\begin{aligned} I_1 &= G_1^2 = G_2^2 = G_3^2 = \frac{1}{8}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \\ I_2 &= G_4 - G_3 = \frac{1}{4}\begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}. \end{aligned} \quad \text{(A6)}$$

I_1, which is not in the algebra, has eigenvalues

$$\lambda_1^{(2)} = 1, \ \lambda_1^{(1,3)} = \frac{1}{2}, \quad \text{(A7)}$$

and I_2, a member of the algebra, has eigenvalues

$$\lambda_2^{(2)} = 0, \ \lambda_2^{(1,3)} = \frac{1}{4}. \quad \text{(A8)}$$

The two invariants have the same eigenvectors,

$$\begin{aligned} |u^{(1)}\rangle &= \frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle), \\ |u^{(2)}\rangle &= \frac{1}{\sqrt{2}}(|2,0\rangle - |0,2\rangle), \\ |u^{(3)}\rangle &= |1,1\rangle. \end{aligned} \quad \text{(A9)}$$

with $|u^{(1)}\rangle$ and $|u^{(3)}\rangle$ spanning a degenerate subspace.

Lie-algebraic invariants constructed with time-independent coefficients satisfy as well the equation

$$i\hbar\frac{\partial I_{1,2}}{\partial t} + [H(t), I_{1,2}] = 0 \qquad \text{(A10)}$$

so they are also dynamical invariants [35] (i.e., operators that satisfy Eq. (A10) whose expectation values remain constant). The degenerate subspace of eigenvectors allows the existence of time-dependent eigenstates of time-independent invariants. In particular, in all the examples in the main text, the dynamics takes place within the degenerate subspace: the initial state is $|u^{(3)}\rangle$ at $t = 0$ and ends up in some combination of $|u^{(1)}\rangle$ and $|u^{(3)}\rangle$ at t_f. The specific state as a function of time is known explicitly, $|\psi_I(t)\rangle = e^{i\alpha(t)G_4}e^{-i\int_0^t E_1 dt'}|\phi_1(t)\rangle$, see Eq. (21). Note that $|\phi_1\rangle$ and $|\phi_3\rangle$ in Eqs. (11) and (13) are two orthogonal combinations of $|u^{(1)}\rangle$ and $|u^{(3)}\rangle$. Also $|u^{(2)}\rangle = |\phi_2\rangle$, see Eq. (12). In the non degenerate subspace spanned by $|u^{(2)}\rangle$ "nothing evolves", other than a phase factor, but the initial states in the examples do not overlap with it.

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