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A Study of the Stability of Integro-Differential Volterra-Type Systems of Equations with Impulsive Effects and Point Delay Dynamics

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Abstract: This research relies on several kinds of Volterra-type integral differential systems and their associated stability concerns under the impulsive effects of the Volterra integral terms at certain time instants. The dynamics are defined as delay-free dynamics contriobution together with the contributions of a finite set of constant point delay dynamics, plus a Volterra integral term of either a finite length or an infinite one with intrinsic memory. The global asymptotic stability is characterized via Krasovskii–Lyapuvov functionals by incorporating the impulsive effects of the Volterra-type terms together with the effects of the point delay dynamics.

Keywords: Lyapunov's stability; Krasovskii–Lyapunov functionals; positive systems; Volterra integral equations; impulsive effects; Popov's inequality

MSC: 34A12; 34A34; 34D23; 45D05

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1. Introduction

Delayed effects appear often in the evolution of dynamic systems or in their control efforts. Well-known examples are the effects in electrical transmission lines, epidemic models, decentralized control operated by tandems of local controllers, control actions on ships (as in, for instance, Minorsky´s problem), economy-related evolution and decision processes, etc. Delays can happen either in the state dynamics, so-called internal delays, or in the controls and/or measurable outputs, which are commonly referred to as external delays. Sometimes, internal and external delays can be jointly present. On the other hand, the delays can be either point delays or distributed delays and can also be either constant or time-varying, eventually being time-differentiable with bounded time-derivative. The above delay types can appear in a mixed way, in the sense that point and distributed state delays or external delays can jointly influence the dynamics of certain dynamic systems. In general, the modeling issues of dynamic systems under the influences of delays in the state might be involved to deal with, since the resulting system is of infinite dimensionality. That assertion relies on the fact that there is no finite set of linearly independent functions, the combination of which is able to build the solution, since the dynamics of time-delay systems have infinitely many characteristic roots. This happens, although the dimension of its state vector is finite, and it does not increase under the presence of delays. In particular, the structure of the space of solutions and the periodicity and stability properties of both Volterra and ordinary differential equations are dealt with in detail in [1]. It can be pointed out that Volterra-type differential equations are a kind of functional differential equations involving distributed delayed dynamics. Specific links with fixed point theory are also discussed in [1]. In [2], several typical kinds of delays such as point delays, distributed delays, and neutral-type differential systems with delayed dynamics are described. The associated stability problems are also studied in a formal way via Lyapunov theory and the allocation of the characteristic roots in the stable region. In [3], a collection of works is included. The book analyzes complex time-delay systems including cases of multiple, distributed, and time-varying delays. Hybrid models of differential-difference type involving delays are also considered. A survey of the existing results concerning time-delay systems is given in [4]. In [5], three stability criteria are discussed by using mixed techniques of convex combination, Wirtinger-based integral inequalities, and Lyapunov–Krasovskii stability analysis. On the other hand, an adaptive control scheme for a class of systems with periodic dynamics, uncertainties, delays, and input saturation is proposed in [6]. In the case that impulsive feedback controls are combined with regular feedback controls, the stabilization has to be focused on by combining both effects together [7–9]. It can be pointed out that stability maintenance is a very relevant property to achieve in dynamic systems. In practice, the stabilization is achievable via feedback of the relevant measurable signals. On the other hand, the absolute stability refers to global asymptotic stability of dynamic systems subject to non-linear control devices whose characterization functions belong to a defined sector, namely the Lur´e or Popov sectors [10–12]. In this context, stabilization is achieved for the full class of a particular sector-type of nonlinear devices, rather than for a particular one. An extension of the absolute stability is the property of the asymptotic hyperstability, that is, the global asymptotic stability under any non-linear, and possibly time-varying, element belonging to a family of feedback controllers, all of them satisfying a given time integral inequality. For instance, the control and simulation of a model reference adaptive control is presented in [13] based on Popov hyperstability theory by using real-time testing to evaluate the action of disturbances. In [14], a neural network adaptive control of a class on nonlinear systems is discussed based on hyperstability theory. A case of interest in the context of hyperstability problems is that when the feed-forward systems are linear and time-invariant while being defined by a positive real or strictly positive real transfer function [15], while the feedback controller is any of the member of the family of controllers which satisfy for all time a Popov-type integral inequality [12,13,16]. In [17], the stability of composite systems in a combined parallel structure disposal having asymptotically hyperstable subsystems is discussed. Some additional stability conditions are needed for the remaining subsystems. Due to the fact that the transfer function of the open-loop system is positive real, another characteristic is that the input–output energy is bounded non-negative. This leads to the interesting property that the controlled system is stable under any hyperstable controller for any set of finite initial conditions. That property is of interest to control and stabilize systems under parametrical variations of the nominal controller parameterization due to either eventual ageing due to long use, or to dispersion of product related to the nominal standards along the fabrication process [12].

A specific type of time-delay system is the so-called Volterra system [1,2,18,19]. It can be pointed out that Volterra integro-differential equations exhibit dynamics of distributed delay type under the form of convolution or non-convolution integral contributions on each whole time interval previous to the time current instant where the solution is calculated. See, for instance [1,18,19] and the references therein.

On the other hand, impulsive controlling actions are of interest for the characterization, for instance, of switches between alternative operation modes of a dynamic system, since impulsive controls at certain time instants lead to finite jumps in the state or output. This fact leads to a useful formal framework to continue calculating the global solution evolution under successive switching from a parameterization or configuration to another potential one of such a system [20,21]. This analysis permits the description and stabilization of processes in which the parameterizations or the operation conditions change through time, as it happens in certain industrial diffusion processes, or in discrete processes when sampling is non-uniform or there are unmodeled dynamics [22–24].

It has to be pointed out that the impulses are neither directly stabilizing nor destabilizing since their influence on stability depends on the time instants where they are injected, and their amplitudes and signs combined with the effects of the regular dynamics. As a result, their positive or negative influences on the stability results are analyzed in this paper, together with the delays' effects and the parameterization through Krasovskii–Lyapunov functionals. However, some of the given results are derived based on norms of the contributions of the matrices of dynamics of delays and impulses considering the worst-case of the maximum sizes of the impulses to be tolerated being compatible with the stability. In this case, the impulsive effects are considered negative from the stability point of view. Those mentioned Volterra-type systems are succinctly described in the sequel. The point time delays act on the state trajectory solution affecting contributing terms in the dynamics which are not included in the integral terms of the differential system. The impulsive effects are typically characterized by Dirac distributions within the integrands of the integro-differential Volterra equations. Their contributions into finite jumps in the solutions of the integral terms of the integro-differential Volterra-type equations at certain time instants received relevant attention in the background literature. For instance, in [25], the discretization of Volterra integral equations of the first kind has been focused on. On the other hand, the admissibility problem of linear discrete Volterra operators is studied in [26]. Also, necessary and sufficient admissibility conditions in several spaces of sequences are proved. The obtained results are also used to study the existence of solutions and their boundedness and convergence properties. The dynamic properties and the asymptotic separation of solutions of convolution-type discrete Volterra equations are discussed in [27]. A polynomial lower-bound of the norm of the difference between a pair of distinct solutions is characterized as well, and the results are applied to fractional differential equations. In [28], nonlinear implicit Volterra discrete equations of convolution-type are studied and sufficient-type conditions are given for the solutions to converge to a finite limit. A linearized stability analysis of a kind of discrete Volterra equation is discussed in [29]. In [30], a difference scheme based on a quadrature formula is proposed in [30] for obtaining a second-order convergence in discrete maximum norm. Such a technique is applied for solving Volterra integro-differential equations involving delays and some kinds of neutral-type equations.

Some additional recent research on Volterra equations are as follows. In [31], some weakly compatible mappings are investigated in intuitionistic fuzzy metric spaces subject to an implicit constraint and the existence and a common solution uniqueness are investigated for a system of Volterra-type integral equations. A class of Volterra integral equations is investigated in [32] including S-function, Mittag-Leffler function with six parameters, and a generalized Mittag-Leffler function. Also, the existence and uniqueness as well as the growth rates of solutions of a class of implicit integro-differential Volterra equations on unbounded from above time scales is investigated in [33]. The investigation of Volterra equations in the formal fractional context is receiving certain attention in the literature. See, for instance [34,35] and some of the references therein.

The basic objective and novelty of this paper is the characterization of several Volterra-type integro-differential systems and their associated stability concerns when the dynamics have impulsive effects at certain time instants and, in parallel, there exist delayed dynamics caused by constant point delays. The main contributions of this paper are the study and characterization of the global asymptotic stability of integro-differential systems defined by a finite set of point delays together with a Volterra-type integral term which incorporates impulsive effects. Such a term plays the role of a distributed delay while its associated dynamics are subject, in general, to non-periodically distributed impulses of Dirac distribution type. The total size of the contribution of the distributed delay through the Volterra-type integral term can be either constant or time-varying but bounded. A particular considered system relies on the point-delay free case with a Volterra-type contribution which is also extended to the case of matrix-type impulsive contributions on the parameterization under the integral symbol of such a term. It can be pointed out that the impulsive actions within the Volterra-type integral generate finite jumps in the integral evaluation at the time instance where these impulses take place. As a result, the differential system has corresponding finite jumps in the time-derivative of its solution, whose value depends on the amplitude of the Dirac distribution associated impulses. The main stability results are derived through the "ad hoc" use of Krasovskii–Lyapunov stability analysis while some further results are derived based on the use of norms for the various dynamic contributions of delays and impulsive effects. A particular variation of the stability property is also analyzed under positivity conditions of the solution in the absence of point delays, namely, only being subject to a distributed delay either with or without impulsive effects. In the scalar case, the above results are extended to non-positivity constraints and the presence of impulsive terms in an arbitrary-order system.

The rest of the paper is organized as follows. Section 2 describes the various mentioned differential systems under consideration, as well as their associated stability results. The main tool involved is the use of Krasovskii–Lyapunov functionals for each of those systems while taking into account, in parallel, the impulsive effects. Section 3 gives some conclusions. The solutions of the relevant differential systems dealt with are given explicitly in Appendix A.

Notation

$$\boldsymbol{R}_{0+} = \boldsymbol{R}_{+} \cup \{0\} = \{x \in \boldsymbol{R} : x \geq 0\};\ \boldsymbol{R}_{+} = \{x \in \boldsymbol{R} : x > 0\}$$

$$\overline{p} = \{1, 2, \ldots, p\}$$

$\delta(.)$ is the Dirac distribution.

If a function f has a finite jump at t then, its left and right limits are denoted by $f(t^-)$ and $f(t) = f(t^+)$. To simplify the corresponding formulas, the notation $f(t)$ is used for $f(t^+)$ at discontinuity time instants t.

Let M be a square real matrix. Then, $M \succ 0$, $M \succeq 0$, $M \prec 0$, and $M \preceq 0$ denote, respectively, that M is positive definite, positive semidefinite, negative definite, negative semidefinite.

$M \rhd 0$ denotes that M is positive, that is, it is nonzero with at least one positive entry, $M \rhd= 0$ denotes that M is non-negative (its entries are positive or null including the case when $M = 0$), $M \rhd\rhd 0$ denotes that M is strictly positive, that is, all its entries are positive, $M \rhd M'$ denotes $(M - M') \rhd 0$, $M \rhd\rhd M'$ denotes $(M - M') \rhd\rhd 0$. A similar notation applies for real vectors. The reversed symbol $\lhd$ to $\rhd$ stands for the negativity property under the related "mutatis-mutandis" modified notations.

A square real matrix $M \in M_E$ denotes that M is Metzler, that is, the set of square matrices such that its off-diagonal entries are all non-negative.

A square real matrix M is called monomial, or generalized permutation matrix, if each column has a nonzero positive entry and all remaining entries are zero (this implies that M is nonsingular).

A^T denotes the transpose of the real matrix A.

I_n is the n-th identity matrix.

$\lambda_{max}(A)$ and $\lambda_{min}(A)$ are the maximum and minimum eigenvalues of a real square symmetric matrix A.

$\mu(A) = \lim\limits_{h \to 0} \frac{\|I_n + hA\| - h}{h}$ is the matrix measure (or logarithmic norm) of the square matrix A.

In particular, $\| \ \|_2$ denotes the $\updownarrow_2$ matrix norm and $\mu_2(A)$ is the -matrix measure, that is,

$$\mu_2(A) = \lim_{h \to 0} \frac{\|I_n + hA\|_2 - h}{h} = \frac{1}{2} \lambda_{max}\left(A + A^T\right).$$

2. Main Results

Consider the following differential system with r point constant delays h_i for $i \in \overline{r}$ and a distributed delay on an interval of length h with impulsive effects given by:

$$\dot{x}(t) = \sum_{i=0}^{r} A_i x(t - h_i) + \int_{-h}^{0} A(\tau) v_I(\tau) x(t + \tau) d\tau\ ;\ \forall t \in \boldsymbol{R}_{0+} \tag{1}$$

where:

(a) With no loss in generality, it is assumed that the non-necessarily commensurate point delays h_i (i.e., h_i can be distinct of iT for some real $T > 0$) satisfy the ordering constraints $h_i > h_{i-1}$; $\forall i \in \overline{r} \cup \{0\}$, with $h_0 = h_{-1} = 0$ and $h_r = \max\limits_{1 \le i \le r} h_i < +\infty$. The dynamic system (1) is subject to any given finite absolutely continuous function $\phi : \left[-\overline{h}\ ,\ 0\right] \to \boldsymbol{R}^n$, where $\overline{h} = max(h_r, h)$, $x : \left[-\overline{h},\ +\infty\right) \to \boldsymbol{R}^n$, with $x(t) = \phi(t)$ for $t \in \left[-\overline{h}\ ,\ 0\right]$, where $x_0 = x(0) = \phi(0)$, $A_i \in \boldsymbol{R}^{n \times n}$; $i \in \overline{r} \cup \{0\}$, $A : [-h\ ,\ 0] \to \boldsymbol{R}^{n \times n}$ is bounded piecewise-continuous, subject to $\tau_j < \tau_{j-1}$, with no loss in generality, and $\delta(.)$ is the Dirac distribution.

(b) The impulsive set of time instants for the state trajectory solution $x(t)$ within the time interval $[t - h\ ,\ t]$, $t(\ge h) \in \boldsymbol{R}_{0+}$ is $Imp(t) = \{t - \tau_i : i \in \overline{p}\}$ with $\tau_i \in [-h\ ,\ 0]$, $\forall i \in \overline{p}$.

(c) $v_I : [-h\ ,\ 0] \to \boldsymbol{R}^n$ is an impulsive function with p impulses in the integrand of (1) which is defined by:

$$v_I(t) = 1 + \sum_{i=1}^{p} \delta(t - \tau_i);\ t \in [-h\ ,\ 0],\ \tau_i \in [-h\ ,\ 0],\ \forall i \in \overline{p}. \quad (2)$$

Note from (1) and the above definition (2), that the testing matrix function of the impulsive integrand $A(\tau)v_I(\tau)x(t+\tau)$ is given by $A(\tau)x(t+\tau)$.

Note also from (1) and (2), that $A(\tau)x(t+\tau)$ is a well-posed test function for the Dirac distribution on $[-h\ ,\ 0]$; $\forall t \in \boldsymbol{R}_{0+}$. In particular, $\lim\limits_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} A(\tau)x(t+\tau)d\tau = A(0^-)x(t)$. Note also that (1) contains Dirac distributions under the integral symbol associated with the impulsive effects.

Assumption 1. *Assume that* $A : [-h\ ,\ 0] \to \boldsymbol{R}^{n \times n}$ *is bounded piecewise-continuous except at the discrete impulsive set* $Imp_A = \{-\tau_1, -\tau_2, \ldots, -\tau_p\}$ *where it is impulsive.*

Note that the impulsive set $Imp(t) = \{t - \tau_i : i \in \overline{p}\}.t \in \boldsymbol{R}_{0+}$ for the state-trajectory solution $x(t)$ is directly related uniquely to the corresponding impulsive set $Imp_A = \{-\tau_1, -\tau_2, \ldots, -\tau_p\}$ for the matrix function $A : [-h\ ,\ 0] \to \boldsymbol{R}^{n \times n}$. In general, we will refer to the impulsive contribution through the matrix function $A(t)$ to facilitate the use of a simplified notation since Imp_A does not depend on each current time instant. If $\tau_0(= -h) \in Imp_A$ then $\tau_1 = \tau_0$ but note that $\tau_0 = -h$ is not necessarily in Imp_A and that $\tau_{p+1}(= 0) \notin Imp_A$ by hypothesis. Note also that, since $\tau_0 = h$, $\tau_{p+1} = 0$ and $A_{-i} = A(-\tau_i)$; $\forall i \in \overline{p}$, and using Assumption 1, the integral term associated with the distributed delay can be integrated leading to the subsequent expression:

$$\begin{aligned}\int_{-h}^{0} A(\tau)v_I(\tau)x(t+\tau)d\tau &= \int_{-h}^{-\tau_1^-} A(\tau)x(t+\tau)d\tau + \sum_{i=1}^{p-1} \int_{-\tau_i}^{-\tau_{i+1}^-} A(\tau)x(t+\tau)d\tau + \int_{-\tau_p}^{0^-} A(\tau)x(t+\tau)d\tau \\ &\quad + \sum_{i=1}^{p} \int_{-\tau_i^-}^{-\tau_i} A(\tau)\delta(\tau - \tau_i)x(t+\tau)d\tau \\ &= \sum_{i=0}^{p} \int_{-\tau_i}^{-\tau_{i+1}^-} A(\tau)x(t+\tau)d\tau + \sum_{i=1}^{p} A(-\tau_i)x(t - \tau_i); \forall t \in \boldsymbol{R}_{0+}.\end{aligned} \quad (3)$$

Note that the first right-hand-side of (3) is related to the regular contribution to the Volterra integral term at any time instants. The second right-hand-side term describes the impulsive effects at the impulsive time instants, and it is zero at non-impulsive time instants.

In the case that, contrarily to Assumption 1, $A : [-h\ ,\ 0] \to \boldsymbol{R}^{n \times n}$ is not piecewise-continuous, but is still bounded discontinuous at time instant τ_i in the impulsive set, the integral at the impulsive point is calculated with the left and right limits of $A(-\tau)$ and $\tau = \tau_i$, that is, one uses

$$\int_{-\tau_i^-}^{-\tau_i} A(\tau)\delta(\tau - \tau_i)x(t+\tau)d\tau = (1/2)\left(A(-\tau_i) + A\left(-\tau_i^-\right)\right)x(t - \tau_i)$$

instead of

$$\int_{-\tau_i^-}^{-\tau_i} A(\tau)\delta(\tau - \tau_i)x(t+\tau)d\tau = A(-\tau_i)x(t - \tau_i).$$

Now, Equation (1) is rewritten, via (4) after defining $h_0 = 0$, as follows:

$$\dot{x}(t) = \sum_{i=0}^{r} A_i x(t - h_i) + \sum_{i=1}^{p} A_{-i} x(t - \tau_i) + \sum_{i=0}^{p} \int_{-\tau_i}^{-\tau_{i+1}^{-}} A(\tau) x(t + \tau) d\tau. \quad (4)$$

In this way, the system has equivalently a maximum of $q = r + p$ point delays of which r of them are generated by the impulsive effects in the distributed delay. The set of impulsive time instants is $Imp_A = \{-\tau_i : i \in \overline{p}\}$. Furthermore, $\{h_i : i \in \overline{r}\}$ and $Imp_A = \{-\tau_i : i \in \overline{p}\}$ are assumed to be disjoint sets just to facilitate the exposition. Thus, with no loss in generality, and just to simplify the subsequent analysis, the next assumption is introduced:

Assumption 2. *The assignment $\tau_0 = h$ is made to facilitate the subsequent exposition. As a result, if $(-h)$ is an impulsive time instant in Imp_A then $\tau_1 = \tau_0 = h$, otherwise $\tau_1 < \tau_0 = h$.*

The following result is concerned with the global asymptotic stability of (1).

Theorem 1. *The system (1) is globally asymptotically stable if there exist $(r+1)$ symmetric real positive definite matrices P and S_i; $i \in \overline{r}$, $(p+1)$, real numbers $q_i > 1$ for $i \in \overline{p} \cup \{0\}$, $(p+1)$ real bounded functions $\alpha_i : [-\tau_i \, , -\tau_{i+1}] \to \mathbf{R}_+ \left(\tau_0 = h \, , \, \tau_{p+1} = 0\right)$, and $(p+1)$ symmetric bounded piecewise continuous function matrices $R_i : [-\tau_i \, , -\tau_{i+1}] \to \mathbf{R}^{n \times n}$, with $R_i(\tau) \succ q_i \alpha_i(\tau) P \succ 0$; $\forall i \in \overline{p} \cup \{0\}$ such that:*

(a) The set of matrices $\overline{Q}_{\hat{\theta}}(t) \succ 0$, defined below, for each $t \in \mathbf{R}_{0+}$, and for all combinations $\hat{\theta} = \left(\theta_0 \, , \, \theta_1 \, , \cdots , \theta_p\right) \in \{-1 \, , \, 1\}^{p+1}$, where:

$$\overline{Q}_{\hat{\theta}}(t) = \begin{bmatrix} -\left(A_0^T P + P A_0 + \sum_{i=1}^{r} S_i + \sum_{i=0}^{p}\left(\int_{-\tau_i}^{-\tau_{i+1}^{-}} \theta_i(t,\tau) R_i(\tau) d\tau\right)\right) & PA_1 & \cdots & PA_r & PA_{-1} & \cdots & PA^{-p} \\ A_1^T P & S_1 & 0 & & & \cdots & 0 \\ \vdots & 0 & S_2 & & 0 & \cdots & 0 \\ A_r^T P & \vdots & & & & & \vdots \\ A_{-1}^T P & \vdots & & & \ddots & & \vdots \\ \vdots & \vdots & \cdots & & & \ddots & \vdots \\ A_{-p}^T P & 0 & & & \cdots & 0 & S_{r+p} \end{bmatrix}. \quad (5)$$

(b) The set of matrices $\Omega_{0i}(t,\tau) \preceq 0$, where:

$$\Omega_{0i}(t,\tau) = \begin{bmatrix} (q_i \alpha_i(\tau) P - R_i(\tau)) & PA(\tau) \\ A^T(\tau) P & -q_i \alpha_i(\tau) P \end{bmatrix}; \; i \in \overline{p} \cup \{0\}, \tau \in (-\tau_i \, , \, -\tau_{i+1}) \quad (6)$$

$\tau \in (-\tau_i \, , \, -\tau_{i+1})$, $i \in \overline{p} \cup \{0\}$, with $\tau_0 = h$ and $Imp_A = \{\tau_1 \, , \tau_2 \, , \cdots , \tau_p\}$ such that $\tau_1 = \tau_0 = h$ if $-h \in Imp_A$ and $\tau_1 (\neq \tau_0) < h$ if $-h \notin Imp_A$, and $\tau_{p+1} = 0$ in both cases.

Proof of Theorem 1. Consider the Lyapunov–Krasovsii functional candidate:

$$V(x_t) = V(x(t)) + \sum_{i=1}^{r} \int_{-h_i}^{0} x^T(t+\tau) S_i x(t+\tau_i) d\tau; \; V(x(t)) = x^T(t) P x(t) \quad (7)$$

where, for each $t \in \mathbf{R}_{0+}$,x_t denotes the restriction of $x : \left[-\overline{h}, \, 0\right] \to \mathbf{R}^n$ translated to $\left[t - \overline{h}, \, t\right]$. Define for any $t \in \mathbf{R}_{0+}$:

$$\overline{x}^T(t) = \left(x^T(t) \, , \, x^T(t-h_1) \, , \, \cdots , x^T(t-h_r) \, , x^T(t-\tau_1) \, , \cdots \, , x^T\left(t-\tau_p\right)\right) \quad (8)$$

$$\hat{x}^T(t+\tau) = \left(x^T(t) \, , \, x^T(t+\tau)\right). \quad (9)$$

Define $\hat{\theta}(t) = \theta(t,\tau_0) \times \theta(t,\tau_1) \times \ldots \times \theta(t,\tau_p)$ for any $t \in \boldsymbol{R}_0^+$ whose cardinal is $card\hat{\theta}(t) = 2^{p+1}$ since $\theta_i(t,\tau) \in \{-1\ ,\ 1\}$; $i \in \overline{p} \cup \{0\}$. The following cases can arise for any $t \in \boldsymbol{R}_{0+}$:

Case 1: For each $\tau \in (-\tau_i\ ,\ -\tau_{i+1})$, $V\big(x\big(t-\tau_i^- + \tau\big)\big) \le q_i V\big(x\big(t-\tau_i^-\big)\big)$ and $\theta_i(t,\tau) = 1$.

Case 2: For each $\tau \in (-\tau_i\ ,\ -\tau_{i+1})$, $V\big(x\big(t-\tau_i^- + \tau\big)\big) > q_i V\big(x\big(t-\tau_i^-\big)\big)$ and $\theta_i(t,\tau) = -1$.

Note that in both cases Case 1 and Case 2, one has for each $t \in \boldsymbol{R}_0^+$ that

$$\theta_i(t,\tau)\alpha_i(\tau)\big(q_i V\big(x\big(t-\tau_i^-\big)\big) - V\big(x\big(t-\tau_i^- + \tau\big)\big)\big) \ge 0;\ \tau \in (-\tau_i\ ,\ -\tau_{i+1}\,)$$

then,

$$\begin{aligned}&\textstyle\sum_{i=1}^{p+1} \int_{-\tau_{i-1}}^{-\tau_i^-} \alpha_i(\tau)\big|\, q_i V\big(x\big(t-\tau_i^-\big)\big) - V\big(x\big(t-\tau_i^- + \tau\big)\big)\big| d\tau \\ &\qquad = \textstyle\sum_{i=1}^{p+1} \int_{-\tau_{i-1}}^{-\tau_i^-} \theta_i(t,\tau)\alpha_i(\tau)\big(q_i V\big(x\big(t-\tau_i^-\big)\big) - V\big(x\big(t-\tau_i^- + \tau\big)\big)\big) d\tau \ge 0.\end{aligned}$$

Now, for each $t \in \boldsymbol{R}_{0+}$, if $V\big(x\big(t-\tau_i^- + \tau\big)\big) \le q_i V\big(x\big(t-\tau_i^-\big)\big)$ for $\tau \in (-\tau_i\ ,\ -\tau_{i+1}\,)$; $i \in \overline{p} \cup \{0\}$, and since the conditions a and b of the theorem statement hold,

$$\begin{aligned}
\dot{V}(x_t) &= \tfrac{d}{dt}\big(x^T(t)Px(t)\big) + \textstyle\sum_{i=1}^{r} \tfrac{d}{dt}\Big(\int_{t-h_i}^{t} x^T(\tau)S_i x(\tau)d\tau\Big) \\
&= 2\Big(x^T(t)P\dot{x}(t) + \textstyle\sum_{i=1}^{r} \int_{-h_i}^{0} x^T(t+\tau)S_i\dot{x}(t+\tau)d\tau\Big) \\
&= 2x^T(t)P\dot{x}(t) + \textstyle\sum_{i=1}^{r} \tfrac{d}{dt}\Big(\int_{t-h_i}^{t} x^T(\tau)S_i x(\tau)d\tau\Big) \\
&= 2x^T(t)P\dot{x}(t) + \textstyle\sum_{i=1}^{r}\big(x^T(t)S_i x(t) - x^T(t-h_i)S_i x(t-h_i)\big) \\
&= 2x^T(t)P\Big(\textstyle\sum_{i=0}^{r} A_i x(t-h_i) + \sum_{i=1}^{p} A_{-i}x(t-\tau_i) + \sum_{i=0}^{p} \int_{-\tau_i}^{-\tau_{i+1}^-} A(\tau)x(t+\tau)d\tau\Big) \\
&\quad + \textstyle\sum_{i=1}^{r}\big(x^T(t)S_i x(t) - x^T(t-h_i)S_i x(t-h_i)\big) \\
&\le -\overline{x}^T(t)\overline{Q}_{\hat{\theta}}(t)\overline{x}(t) - x^T(t)\Big(\textstyle\sum_{i=0}^{p} \int_{-\tau_i}^{-\tau_{i+1}^-} R_i(\tau)d\tau\Big)x(t) \\
&\quad + \textstyle\sum_{i=1}^{p+1} \int_{-\tau_{i-1}}^{-\tau_i^-} \alpha_{i-1}(\tau)\big|q_{i-1}x^T\big(t-\tau_i^-\big)Px\big(t-\tau_i^-\big) - x^T\big(t-\tau_i^- + \tau\big)Px\big(t-\tau_i^- + \tau\big)\big| d\tau \\
&= -\overline{x}^T(t)\overline{Q}_{\hat{\theta}}(t)\overline{x}(t) - x^T(t)\Big(\textstyle\sum_{i=0}^{p} \int_{-\tau_i}^{-\tau_{i+1}^-} R_i(\tau)d\tau\Big)x(t) \\
&\quad + \textstyle\sum_{i=1}^{p+1} \int_{-\tau_{i-1}}^{-\tau_i^-} \theta_{i-1}(t,\tau)\alpha_{i-1}(\tau)\big(q_i x^T\big(t-\tau_i^-\big)Px\big(t-\tau_i^-\big) - x^T\big(t-\tau_i^- + \tau\big)Px\big(t-\tau_i^- + \tau\big)\big)d\tau \\
&= -\overline{x}^T(t)\overline{Q}_{\hat{\theta}}(t)\overline{x}(t) - x^T(t)\Big(\textstyle\sum_{i=0}^{p} \int_{-\tau_i}^{-\tau_{i+1}^-} R_i(\tau)d\tau\Big)x(t) \\
&\quad + \textstyle\sum_{i=1}^{p+1} \int_{-\tau_{i-1}}^{-\tau_i^-} \theta_{i-1}(t,\tau)\alpha_{i-1}(\tau)\big(q_{i-1}V\big(x\big(t-\tau_i^-\big)\big) - V\big(x\big(t-\tau_i^- + \tau\big)\big)\big)d\tau \\
&= -\overline{x}^T(t)\overline{Q}_{\hat{\theta}}(t)\overline{x}(t) - \textstyle\sum_{i=1}^{p+1} \int_{-\tau_{i-1}}^{-\tau_i^-} \Big|\hat{x}^T\Big(t-\tau_{i-1}^- + \tau\Big)\Omega_{0,i-1}(\tau)\hat{x}\Big(t-\tau_{i-1}^- + \tau\Big)\Big| d\tau \\
&\le -\overline{x}^T(t)\overline{Q}_{\hat{\theta}}(t)\overline{x}(t) + \textstyle\sum_{i=1}^{p+1} \int_{-\tau_{i-1}}^{-\tau_i^-} \hat{x}^T\Big(t-\tau_{i-1}^- + \tau\Big)\Omega_{i-1}(\tau)\hat{x}\Big(t-\tau_{i-1}^- + \tau\Big)d\tau \\
&\le -\Big(\varepsilon_1\|\overline{x}(t)\|^2 + \varepsilon_2\|\hat{x}(t)\|^2\Big) \\
&\le -\varepsilon\|x_t\|^2;\ \forall t \in \boldsymbol{R}_{0+}
\end{aligned} \tag{10}$$

for some $\varepsilon_1\,,\varepsilon_2\,,\ \varepsilon \in \boldsymbol{R}_+$, where:

$$\Omega_i(t,\tau) = \begin{bmatrix} \theta_i(t,\tau)(q_i\alpha_i(\tau)P - R_i(\tau)) & PA(\tau) \\ A^T(\tau)P & -\theta_i(t,\tau)q_i\alpha_i(\tau)P \end{bmatrix};\ \tau \in (-\tau_i\ ,\ -\tau_{i+1}\,),\ i \in \overline{p} \cup \{0\} \tag{11}$$

since, for each pair $(t\,,\,\tau) \in \boldsymbol{R}_{0+} \times (-\tau_i\,,\,-\tau_{i+1})$; $i \in \overline{p} \cup \{0\}$, the condition $\Omega_{0i}(t,\tau) \preceq 0$, defined in (5) is achieved by the condition $q_i\alpha_i(\tau)P \preceq R_i(\tau)$.

Then, the candidate (7) is a Krasovskii–Lyapunov functional so that the system (1) is globally asymptotically stable independent of the delays for any given finite absolutely continuous function $\phi\,:\,\left[-\overline{h}\,,\,0\right] \to \boldsymbol{R}^n$. □

Remark 1. *The testing of Theorem 1 can be addressed as follows:*

(a) $\overline{Q}_{\hat{\theta}}(t) \succ 0$ *for the* 2^{p+1} *combinations of the value of* $\hat{\theta}$ *holds under the necessary condition that*

$$A_0^T P + PA_0 + \sum_{i=1}^{r+p} S_i + \sum_{i=0}^{p} \left(\int_{-\tau_i}^{-\tau_{i+1}^-} \theta_i(t,\tau) R_i(\tau) d\tau \right) \prec 0 \tag{12}$$

and the above one holds for all such combinations if

$$A_0^T P + PA_0 + \sum_{i=1}^{r+p} S_i + \sum_{i=0}^{p} \left(\int_{-\tau_i}^{-\tau_{i+1}^-} R_i(\tau) d\tau \right) \prec 0 \tag{13}$$

that is, a sufficient condition for (11) to hold is that it holds for the case when $\theta_i(t\,,\,\tau) = 1$ *for all* $i \in \overline{p} \cup \{0\}$.

Note that if all $\theta_i(t\,,\,\tau) = 1$ *then the delay free dynamic matrix* A_0 *has to be a stability matrix and of sufficiently relevant stability abscissa, related to the remaining left-hand-side terms in (12) in order that the inequality in (12) holds. However, the presence of impulses of negative sizes or alternation of negative and positive impulses can act favorably to the stabilization, irrespective of* A_0.

(b) For each pair $(t\,,\,\tau) \in \boldsymbol{R}_{0+} \times (-\tau_i\,,\,-\tau_{i+1})$; $i \in \overline{p} \cup \{0\}$, *the conditions* $\Omega_{0i}(t,\tau) \preceq 0$ *of Theorem 1, defined in (5), can be achieved if* $q_i\alpha_i(\tau)P \preceq R_i(\tau)$ *if* $\theta_i(t,\,\tau) = 1$ *for the matrix P having a sufficiently small norm so that* $\Omega_{0i}(t,\tau)$ *is diagonally dominant. This implies that if* $\theta_i(t+\tau) = 1$ *then the matrices defined in (6) fulfil* $\Omega_i(t,\tau) \preceq 0$. *If* $\theta_i(t+\tau) = -1$ *then* $\Omega_i(t,\tau) \succeq 0$. *Thus, the theorem conditions consider all the possible signs of the impulses* $\pm\delta(t)x(t)$ *being generated under the Volterra integral symbol at the impulsive time instants.*

Remark 2. *Note that for Theorem 1 to hold, the delay-free matrix of dynamics* A_0 *has to be a stability matrix since* $A_0^T P + PA_0 \prec 0$ *in order that* $\overline{Q}_{\hat{\theta}}(t) \succ 0$.

The following extension of the differential system (1) and (2) is now considered :

$$\dot{x}(t) = \sum_{i=0}^{r} A_i x(t-h_i) + \int_{-h(t)}^{0} A(\tau) v_I(\tau) x(t+\tau) d\tau;\ \forall t \in \boldsymbol{R}_{0+} \tag{14}$$

where $h : \boldsymbol{R}_{0+} \to [h_m\,,\,h_M] \subset (0\,,\,+\infty)$, and the impulsive function (2) is considered restricted to a constant subinterval $[0\,,\,h_m]$ of the time-varying distributed delay interval as follows:

$$v_i(t) = 1 + \sum_{i=1}^{p} \delta(t-\tau_i)\,;\ t \in [-h_m\,,\,0];\ \tau_i \in [-h_m\,,\,0],\ i \in \overline{p} \cup \{0\} \tag{15}$$

The above system is equivalently rewritten as follows:

$$\dot{x}(t) = \sum_{i=0}^{r} A_i x(t-h_i) + \int_{-h_m}^{0} A(\tau) v_I(\tau) x(t+\tau) d\tau + \int_{-h(t)}^{-h_m^-} A(\tau) x(t+\tau) d\tau \tag{16}$$

since it is not impulsive in the interval $[-h(t)\,,\,h_m)$ while it is impulsive in $[-h_m\,,\,0] \subset [-h(t)\,,\,0]$; $\forall t \in \boldsymbol{R}_{0+}$. The equations in the proof of Theorem 1, together with the above extra right-hand-side term, with the replacement $h_m \to h$, yield from the functional (7):

$$\dot{V}(x_t) \le -\overline{x}^T(t)\overline{Q}_{\hat{\theta}}(t)\overline{x}(t) - x^T(t)\left(\sum_{i=0}^{p}\int_{-\tau_i}^{-\tau_{i+1}^-} R_i(\tau)d\tau\right)x(t)$$
$$+\sum_{i=1}^{p+1}\int_{-\tau_{i-1}}^{-\tau_i^-}\theta_{i-1}(t,\tau)\alpha_{i-1}(\tau)\left(q_i V\left(x\left(t-\tau_i^-\right)\right) - V\left(x\left(t-\tau_i^- + \tau\right)\right)\right)d\tau \tag{17}$$
$$+\int_{-h(t)}^{-h_m^-}\alpha_{p+1}(\tau)|qV(x(t-h_m^-)) - V(x(t-h_m^- + \tau))|d\tau$$

where $\tau_{p+1} = 0$, $\tau_0 = h_m$, $\alpha_i : [-\tau_i \,, -\tau_{i+1}] \to \boldsymbol{R}_+$, $R_i : [-\tau_i \,, -\tau_{i+1}] \to \boldsymbol{R}^{n\times n}$; $i \in \overline{p} \cup \{0\}$, $R_{p+1} : [-\tau_p \,, -h_m] \to \boldsymbol{R}^{n\times n}$ such that $\overline{\overline{Q}}_{\hat{\theta}}(t) \succ 0$,where the replacement $\overline{Q}_{\hat{\theta}}(t) \to \overline{\overline{Q}}_{\hat{\theta}}(t)$ is made with $\overline{Q}_{\hat{\theta}}(t)$ defined in Equation (5) while $\overline{\overline{Q}}_{\hat{\theta}}(t)$ is defined by modifying the (1,1)-block matrix of $\overline{Q}_{\hat{\theta}}(t)$ by adding the new matrix function $R_{p+1}(\tau)$ of (16) and considering the new term in the modified corresponding sum of $(p+2)$ terms in such a block matrix. Then, $\Omega_{0i}(\tau) \preceq 0$, $\tau \in [-\tau_{i-1} \,, -\tau_i)$; $i \in \overline{p} \cup \{0\}$, Equation (5) and for some $q > 1$ and $\alpha : [-h(t) \,, -h_m] \to \boldsymbol{R}_+$. Thus, the conditions $\overline{\overline{Q}}_{\hat{\theta}}(t) \succ 0$ and $\Omega_{0i}(\tau) \le 0$, $\tau \in [-\tau_i \,, -\tau_{i+1})$; $i \in \overline{p} \cup \{0\}$ together with the extra condition

$$\Omega_{0,p+1}(\tau) = \begin{bmatrix} q\alpha_{p+1}(\tau)P - R_{p+1}(\tau) & PA(\tau) \\ A^T(\tau)P & -q_i\alpha_{p+1}(\tau)P \end{bmatrix} \preceq 0\,;\ \tau \in (-h(t)\,,\,-h_m)$$

guarantee that the candidate functional (6) is a Krasovskii–Lyapunov functional for (13) and (14) if $-\tau_0 = -h_m \notin Imp_A$ and $\tau_1 < h_m$. If $-h_m (= -\tau_0 = -\tau_1) \in Imp_A$ then the above extra condition $\Omega_{0,p+1}(\tau) \preceq 0$ is removed while $\overline{\overline{Q}}_{\hat{\theta}}(t) = \overline{Q}_{\hat{\theta}}(t)$.

The following exponential stability result is based on a "worst-case" tolerance to the sizes of the dynamics associated with the delays and impulsive effects provided that the delay-free/impulsive-free system is exponentially stable.

Theorem 2. *The system (1) is globally exponentially stable independent of the delays if*

$$\sum_{i=1}^{r}\|A_i\| + \sum_{i=1}^{p}\|A_{-i}\| < \frac{\rho}{K}\left(1 - e^{\rho h} - \rho e^{-\rho h}\right) \tag{18}$$

for some norm-dependent real constant $K \ge 1$ where $(-\rho) < 0$ is the stability abscissa of A_0.

Proof of Theorem 2. Note that (4) is identical to

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{r} A_i x(t-h_i) + \sum_{i=1}^{p} A_{-i} x(t-\tau_i) + \sum_{i=0}^{p}\int_{-\tau_i}^{-\tau_{i+1}^-} A(\tau)x(t+\tau)d\tau \tag{19}$$

so that the solution trajectory evolution on $[t-h \,,\ t]$ satisfies the constraint:

$$\|x(t)\| \le \left\|e^{A_0 h}\right\|\|x(t-h)\| + \sup_{-h\le\tau\le 0}\|A(\tau)\|\left\|\int_{-h}^{0} e^{A_0(t-h+\tau)}d\tau\right\|\sup_{t-h\le\tau\le t}\|x(\tau)\|$$
$$+\left\|\int_0^t e^{A_0(t-\tau)}\left(\sum_{i=1}^{r} A_i x(\tau-h_i) + \sum_{i=1}^{p} A_{-i}x(\tau-\tau_i)\right)d\tau\right\| \tag{20}$$
$$\le K\left(e^{-\rho h} + \frac{e^{\rho h}-1}{\rho} + \frac{1}{\rho}\left(\sum_{i=1}^{r}\|A_i\| + \sum_{i=1}^{p}\|A_{-i}\|\right)\right)\sup_{t-h\le\tau<t}\|x(\tau)\|$$

with $x(t) = \phi(t)$ for $\tau \in [-h \,,\ 0]$ with $x_0 = x(0) = \phi(0)$, for a real constant $\rho > 0$, which is the minus stability abscissa of A_0, since A_0 is a stability matrix under Theorem 1 (see Remark 1), and a (norm-dependent) constant $K \ge 1$. Then, $\|x(t)\|$ is bounded and strictly decreasing if ϕ:$[-h \,,\ 0] \to \boldsymbol{R}^n$ is finite and (1) is globally exponentially stable independent of the delays provided that (18) holds. □

Now, consider the following integro-differential Volterra-type system:

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)ds; \ \forall t \in \mathbf{R}_{0+} \tag{21}$$

where $A \in \mathbf{R}^{n\times n}$ and $B : \mathbf{R}_{0+} \to \mathbf{R}^{n\times n}$ is continuous. The solution is unique for any given finite $x(0) = x_0$ and given by

$$x(t) = e^{At}\left(x_0 + \int_0^t \int_0^\tau e^{-A\tau}B(\tau - s)x(s)dsd\tau\right); \ \forall t \in \mathbf{R}_{0+} \tag{22}$$

The following result is concerned with the positivity and negativity of the solution of (21):

Lemma 1. *Assume that $A \in M_E$ and $B \trianglerighteq 0$ for each $t \in \mathbf{R}_{0+}$. Then, $x(t) \vartriangleright 0; \forall t \in \mathbf{R}_{0+}$ if $x_0 \vartriangleright 0$ and $x(t) \vartriangleright\vartriangleright 0; \forall t \in \mathbf{R}_{0+}$ if $x_0 \vartriangleright\vartriangleright 0$. Also, $x(t) \vartriangleleft 0; \forall t \in \mathbf{R}_{0+}$ if $x_0 \vartriangleleft 0$ and $x(t) \vartriangleleft\vartriangleleft 0\,; \forall t \in \mathbf{R}_{0+}$ if $x_0 \vartriangleleft\vartriangleleft 0$.*

Proof of Lemma 1. Since e^{At} is a fundamental matrix solution of $\dot{z}(t) = Az(t)$, then non-singular for all time and, since $A \in M_E$, then $e^{At} \vartriangleright 0; \forall t \in \mathbf{R}_{0+}$. Since $B : \mathbf{R}_{0+} \to \mathbf{R}^{n\times n}$ then if $x_0 \vartriangleright 0$ then $x(t) \vartriangleright 0; \forall t \in \mathbf{R}_{0+}$. Also, the fact that $e^{At} \vartriangleright 0$ and non-singular for all time, implies that it has at least a positive entry per row. Therefore, if $x_0 \vartriangleright\vartriangleright 0$, then $x(t) \vartriangleright\vartriangleright 0; \forall t \in \mathbf{R}_{0+}$. If $x_0 \vartriangleleft 0$ then

$$(-x(t)) = e^{At}\left((-x_0) + \int_0^t \int_0^\tau e^{-A\tau}B(\tau - s)(-x(s))dsd\tau\right) \vartriangleright 0; \ \forall t \in \mathbf{R}_{0+} \tag{23}$$

so that $x(t) \vartriangleleft 0$. Similarly, $x_0 \vartriangleleft\vartriangleleft 0$ implies that $x(t) \vartriangleleft\vartriangleleft 0; \forall t \in \mathbf{R}_{0+}$. The proof is complete. □

The asymptotic stability of the above system for the case when the system solution has positivity properties and there are no sign changes between the various components of the initial conditions that follow below:

Theorem 3. *Consider the differential system (21) under the conditions of Lemma 1. Assume also that either $x_0 \trianglerighteq 0$ or $x_0 \trianglelefteq 0$ and finite. Then, $x(t) \to 0$ as $t \to \infty$ if $\left(c^T A + d^T \int_0^\infty B\,(u)du\right) \vartriangleleft 0$* and monomial, or if $\left(A + \int_0^\infty B\,(u)du\right) \vartriangleleft\vartriangleleft 0$.

Proof of Theorem 3. Take real n-vectors $c \vartriangleright\vartriangleright 0\,, d \vartriangleright\vartriangleright 0$ and, for any finite $x_0 \trianglerighteq 0$, define the Lyapunov functional candidate as follows:

$$V(x(t)\,,\,t) = c^T x(t) + d^T \int_0^t \int_t^\infty B\,(u-s)x(s)duds; \ \forall t \in \mathbf{R}_{0+} \tag{24}$$

so that $V(x(t)) > 0$ if $x(t) \neq 0; \forall t \in \mathbf{R}_{0+}$, and

$$\begin{aligned}
\dot{V}(x(t)\,,\,t) &= c^T \dot{x}(t) + d^T \textstyle\int_0^t \int_t^\infty B\,(u-s)x(s)duds \\
&= c^T\left(Ax(t) + \textstyle\int_0^t B(t-s)x(s)ds\right) + d^T\left(\textstyle\int_t^\infty B\,(u-t)x(t)du - \int_0^t B\,(t-s)x(s)ds\right) \\
&= \left(c^T A + \textstyle\int_0^\infty d^T B\,(u-t)du\right)x(t) + \left(c^T - d^T\right)\textstyle\int_0^t B\,(t-s)x(s)ds \\
&= \left(c^T A + d^T \textstyle\int_0^\infty B\,(u-t)du\right)x(t) - \left(d^T - c^T\right)\textstyle\int_0^t B\,(t-s)x(s)ds; \ \forall t \in \mathbf{R}_{0+}.
\end{aligned} \tag{25}$$

If one chooses $d \trianglerighteq c$, then $\dot{V}(x(t), t) < 0; \forall t \in \mathbf{R}_{0+}$ if $x(t) \neq 0$, so that $V(x(t), t)$ is strictly decreasing, since

$$\dot{V}(x(t),t) \leq \left(c^T A + d^T \int_0^\infty B\,(u)du\right)x(t) \leq 2d^T \left(A + \int_0^\infty B\,(u)du\right)x(t);\ \forall t \in \boldsymbol{R}_{0+} \quad (26)$$

so that $\dot{V}(x(t),t) < 0$ if $x(t) \neq 0$. Then, $x(t) \to 0$ as $t \to \infty$ for any finite $x_0 \trianglerighteq 0$ if

(a) $\left(c^T A + d^T \int_0^\infty B\,(u)du\right) \triangleleft 0$ and monomial, or

(b) $\left(A + \int_0^\infty B\,(u)du\right) \triangleleft\triangleleft 0$

(c) If $x_0 = \triangleleft\, 0$ and finite then $x(t) \to 0$ as $t \to \infty$, under the same above conditions, since from Lemma 1, $x(t) \triangleleft 0$; $\forall t \in \boldsymbol{R}_{0+}$ if $x_0 \triangleleft 0$ and $x(t) \triangleleft\triangleleft 0$; $\forall t \in \boldsymbol{R}_{0+}$ if $x_0 \triangleleft\triangleleft 0$. Then, if $x_0 \triangleleft= 0$, so that then $x|t| \triangleleft= 0$

$$\dot{V}(x(t),t) \geq -\left(c^T A + d^T \int_0^\infty B\,(u)du\right)|x(t)| \geq -2d^T \left(A + \int_0^\infty B\,(u)du\right)|x(t)|;\ \forall t \in \boldsymbol{R}_{0+} \quad (27)$$

so that $V(x(t),t) < 0$, $\dot{V}(x(t),t) > 0$ if $x(t) \neq 0$ for any $t \in \boldsymbol{R}_{0+}$ so that so that $V(x(t)\,,\,t)$ is strictly increasing with a non-positive $x(t)$. Then, $x(t) \to 0$ as $t \to \infty$ for any finite $x_0 \triangleleft= 0$ under the same conditions (a)–(c) for the case of $x_0 \trianglerighteq 0$ and the proof has been completed. □

A more general stability result for the differential system (21) in the scalar case $(n = 1)$ follows below, under non-necessarily positivity conditions:

Theorem 4. *Consider the particular differential scalar system of (21):*

$$\dot{x}(t) = Ax(t) + \int_{0^-}^t B(t-s)v_I(s)x(s)ds;\ \forall t \in \boldsymbol{R}_{0+} \quad (28)$$

with $x(0) = x_0$, subject to the Dirac distribution of the following form:

$$v_i(t) = 1 + \sum_{i=1}^{p(t)} \delta(t - t_i)\ ;\ t \in \boldsymbol{R}_{0+},\ \forall t_i \in Imp(t) \quad (29)$$

where $Imp(t) = \left\{t_1\,,\ t_2, \cdots, t_{p(t)}\right\}$, of cardinal $p(t)$, is the strictly increasing set of impulsive time instants on $[0\,,\ t]$ and the total impulsive set on $[0\,,\ \infty)$ is $Imp = Imp(\infty) = \cup_{t \in \boldsymbol{R}_{0+}} Imp(t)$.

Assume that

(a) $A < 0,\ B : \boldsymbol{R}_{0+} \to \boldsymbol{R}_+$ is a piecewise-continuous mapping with an eventual finite jump discontinuity at $t = 0$ and finite $B(0^-)$, and

(b) $\int_{0^-}^\infty B(u-t)v_I(u)du < +\infty$

(c) $A + \int_{0^-}^{t_1^-} B(u)du < -\limsup_{t\to\infty}\left(\sum_{i \in \overline{p(t)-1}} \int_{t_i}^{t_{i+1}^-} B(u)du + \int_{t_{p(t)}}^\infty B(u)du + \sum_{t_i \in Imp(t)} B(t^- - t_i)\right)$.

Then, the following properties hold:

***(i)** The system is globally asymptotically stable, i.e., $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x(t^-) = 0$, for any given finite initial condition x_0.*

***(ii)** All the system solutions are in $L_1[0\,,\ \infty)$ under finite initial conditions.*

Proof of Theorem 4. Consider the following Lyapunov functional candidate:

$$V(x(t),t) = |x(t)| + \int_{0^-}^t \int_t^\infty B(u-s)|x(s)|duds;\ \forall t \in \boldsymbol{R}_{0+} \quad (30)$$

whose time-derivative satisfies:

$$\dot{V}(x(t)\,,\,t) \leq A|x(t)| + \int_{0^-}^{t^-} B(t-s)v_I(s)|x(s)|ds + \int_{t^-}^{t} B(t-s)v_I(s)|x(s)|ds$$
$$+\int_t^{\infty} B(u-t)v_I(u)|x(t)|du - \int_0^{t^-} B(t-s)v_I(s)|x(s)|ds$$
$$= A|x(t)| + \int_0^{t^-} B(t-s)v_I(s)|x(s)|ds + \vartheta(t)B(0^-)|x(t^-)|$$
$$+\int_t^{\infty} B(u-t)v_I(s)|x(t)|du - \int_0^{t^-} B(t-s)v_I(s)|x(s)|ds$$
$$= A|x(t)| + \vartheta(t)B(0^-)|x(t^-)| + \int_t^{\infty} B(u-t)v_I(u)|x(t)|du;\ \forall t \in \mathbf{R}_{0+} \tag{31}$$

where $\vartheta : \mathbf{R}_{0+} \to \{0\,,\,1\}$ is a binary indicator function such that $\vartheta(t) = 1$ if $t \in Imp$ and $\vartheta(t) = 0$ if $t \notin Imp$ so that $\int_{t^-}^{t} v_I(s)B(t-s)|x(s^-)|ds = \vartheta(t)B(0^-)\,|x(t^-)|;\ \forall t \in \mathbf{R}_{0+}$.

On the other hand, $x(t) = x(t^-) + \vartheta(t)B(0^-)x(t^-);\ \forall t \in \mathbf{R}_{0+}$. Then,

$$\begin{aligned}
\dot{V}(x(t),t) &\leq \left(A + \tfrac{\vartheta(t)B(0^-)}{1+\vartheta(t)B(0^-)} + \int_0^{\infty} B(u-t)v_I(u)du\right)|x(t)| \\
&= \left[\left(A + \int_{0^-}^{\infty} B(u-t)v_I(u)du\right)\left(1+\vartheta(t)B(0^-)\right) + \vartheta(t)B(0^-)\right]|x(t^-)| \\
&= \Bigg[\left(A + \int_{0^-}^{t_1^-} B(u)du + \sum_{i\in\overline{p(t)-1}} \int_{t_i}^{t_{i+1}^-} B(u)du + \int_{t_{p(t)}}^{\infty} B(u)du + \sum_{t_i\in Imp(t)} B(t^- - t_i)\right) \\
&\times(1+\vartheta(t)B(0^-)) + \vartheta(t)B(0^-)\Bigg]|x(t^-)| \\
&= -\alpha(t^-)\,|\,x(t^-)| \\
&\leq -\underline{\alpha}^-|\,x(t^-)| \\
&< 0,\ \text{if } x(t^-) \neq 0;\ \forall t \in \mathbf{R}_{0+}
\end{aligned} \tag{32}$$

where $+\infty > \underline{\alpha}^- = \underset{t\in\mathbf{R}_{0+}}{inf}\ \alpha(t^-) > 0$. Note that $\underline{\alpha}^-$ is finite since $\int_{0^-}^{\infty} B(u-t)v_I(u)du < +\infty$ and $\sum_{t_i\in Imp(t)} B(t^- - t_i) \leq \sum_{t_i\in Imp} B(t^- - t_i) < +\infty$. As a result, $x(t^-) \to 0$ and $x(t) = (1+\vartheta(t)B(0^-))x(t^-) \to 0$ as $t \to \infty$ so that the system is globally asymptotically stable for any given finite initial condition. Furthermore,

$$0 \leq V(x(t)\,,\,t) \leq V(x_0,\,0) - \underline{\alpha}^- \int_0^{t^-} |x(\tau)|d\tau;\ \forall t \in \mathbf{R}_{0+} \tag{33}$$

which implies that

$$\int_0^t |x(\tau)|d\tau \leq (1+\vartheta(t)B(0^-))\int_0^{t^-}|x(\tau)|d\tau \leq (1+B(0^-))\int_0^{t^-}|x(\tau)|d\tau \leq \frac{1+B(0^-)}{\underline{\alpha}^-}V(x_0\,,\,0) < +\infty\ ;\ \forall t \in \mathbf{R}_{0+} \tag{34}$$

and

$$\underset{t\to\infty}{lim\,sup}\int_0^t |x(\tau)|d\tau \leq \frac{1+B(0^-)}{\underline{\alpha}^-}V(x_0\,,\,0) < +\infty \tag{35}$$

and the proof is complete. □

Remark 3. *Note that, since $B : \mathbf{R}_{0+} \to \mathbf{R}_+$ and all the impulsive amplitudes are positive from (29), a necessary condition for the fulfilment of the hypothesis (b) in Theorem 5 is that $\sum_{t_i\in Imp(t)} B(t-t_i) < +\infty$, that is, that the set of impulsive amplitudes on the whole infinity time interval be summable. For that concern to hold, it is necessary that $B(t-t_i) \to 0$ as $t \to \infty$ and $t_i(\in Imp) \to +\infty$. This holds, for instance, if $card Imp < \chi_0$ (that is, finite) or if $B(t-t_i) \leq C\gamma^i$ for some real constants $C > 0$ and $\gamma \in (0\,,\,1)$ so that $\sum_{t_i\in Imp(t)} B(t-t_i) \leq \frac{K}{1-\gamma} < +\infty$.*

Note also that, since $A < 0$ from the assumption (a) and $B : \mathbf{R}_{0+} \to \mathbf{R}_+$, then the assumption (c) is equivalent to

$$0 \leq \int_{0^-}^{t_1^-} B(u)du < |A| - \underset{t\to\infty}{lim\,sup}\left(\sum\nolimits_{i\in\overline{p(t)-1}}\int_{t_i}^{t_{i+1}^-} B(u)du + \int_{t_{p(t)}}^{\infty} B(u)du + \sum\nolimits_{t_i\in Imp(t)} B\left(t^- - t_i\right)\right)$$

for which, it is necessary that $|A|$ be large enough related to the whole set of impulsive contributions.

Example 1. *Assume an integro-differential system (28) and (29) with an impulse at $t = 0$ Then, one has that*

$$A + \int_{0^-}^{\infty} B(t)dt = A + B(0) + \int_{0^+}^{\infty} B(t)dt < 0$$

if $0 \leq \int_{0^+}^{\infty} B(t)dt < |A| - B(0^-)$, thus, if $A < 0$ and $B(0^-) \leq |A|$ then the conditions (a) and (b) of Theorem 4 hold. Assume further that $B(t) = K_B e^{-\beta t}$ for $t > 0$ and possibly bounded discontinuous at $t = 0$, with $K_B > 0$, $\beta > 0$. Then, the condition (c) of Theorem 4 also holds since there are no more impulses affecting to $B(t)$ than that at $t = 0$ Thus, the global asymptotic stability holds and the solution is absolutely integrable on $\mathbf{R}_{0+}$ if $A < 0$, $B(0^-) < |A|$ and $\beta > \frac{K_B}{|A|-B(0^-)}$.

Now, assume that there is an impulsive set $Imp = \{t_i\}_{i=1}^{\vartheta}$, $\vartheta < +\infty$. Then, the global asymptotic stability holds, and the solution is absolutely integrable since the conditions (a) and (b) of Theorem 4 hold while the condition (c) of Theorem 4 holds as well if

$$\sum\nolimits_{t_i\,,\,t_{i+1}\in Imp}\left[B(t_i) + \frac{K_\beta}{\beta}\left(e^{-\beta\, t_i} - e^{-\beta\, t_{i+1}}\right)\right] + \frac{K_\beta}{\beta}\left(1 - e^{-\beta t_1} + e^{-\beta t_\vartheta}\right) < |A|.$$

Theorem 4 might be generalized to a n-th order system with, in general, matrix type impulsive effects as follows:

Theorem 5. *Consider the particular n-th differential system:*

$$\dot{x}(t) = Ax(t) + \int_{0^-}^{t} B(t-s)V_I(s)x(s)ds;\ \forall t \in \mathbf{R}_{0+} \tag{36}$$

with $x(0) = x_0$, subject to the Dirac distribution matrix of the following form:

$$\begin{gathered} V_I(t) = \left(V_{I_{ij}}(t)\right) = I_n + \sum\nolimits_{i=1}^{p(t)} \Delta(t-t_i)\ ;\ \forall (i,\,j) \in \overline{n}\times\overline{n},\ t \in \mathbf{R}_{0+} \\ \Delta(t-t_i) = \left(\Delta_{ij}(t-t_i)\right);\ \Delta_{ij}(t-t_i) = \begin{cases} 1 & if\ t_i \notin Imp \\ \delta(t-t_i) & if\ t_i \in Im_{ij}p \end{cases};\ \forall t_i \in Imp(t) \end{gathered} \tag{37}$$

where $Imp(t) = \left\{t_1\,,\ t_2, \cdots, t_{p(t)}\right\}$, of cardinal $p(t)$, is the strictly increasing set of impulsive time instants on $[0\,,\ t]$ and the total impulsive set on $[0\,,\ \infty)$ is $Imp = Imp(\infty) = \cup_{t\in\mathbf{R}_{0+}} Imp(t)$ whose time-derivative and $Im_{ij}p(t) \subset Imp(t)$ and $Im_{ij}p = Im_{ij}p(\infty) = \cup_{t\in\mathbf{R}_{0+}} Im_{ij}p(t) \subset Imp$ are the sets of impulsive time instants for the (i,j) entry of the matrix function $B : \mathbf{R}_{0+} \to \mathbf{R}^{n\times n}$.

Assume that

(a) $\mu_2(A) < 0$, $B : \mathbf{R}_{0+} \to \mathbf{R}_{+}^{n\times n}$ is a piecewise-continuous mapping with finite $\|B(0^-)\|_2$, and

$$\left\|\int_{0^-}^{\infty} B_{ij}(u-t)V_{I_{ij}}(u)du\right\|_2 < +\infty;\ \forall (i,j) \in \overline{n}\times\overline{n}$$

for which a necessary condition is

$$\left\|\sum\nolimits_{t_i\in Imp} B_{ij}\left(t^- - t_i\right)\right\|_2 < +\infty;\ \forall (i,j) \in \overline{n}\times\overline{n},\ \forall t \in \mathbf{R}_{0+}$$

(b)

$$\mu_2(A)+\left\|\int_{0^-}^{t_1^-}B(u)du+\underset{t\to\infty}{limsup}\left(\sum\nolimits_{i\in\overline{p(t)-1}}\int_{t_i}^{t_{i+1}^-}B(u)du+\int_{t_{p(t)}}^{\infty}B(u)du+\sum\nolimits_{t_i\in Imp(t)}B\left(t^--t_i\right)\right)\right\|_2<0. \tag{38}$$

Then, the following properties hold:

(i) *The system is globally asymptotically stable, i.e.,* $\underset{t\to\infty}{lim}\,x(t)=\underset{t\to\infty}{lim}\,x(t^-)=0$*, for any given finite initial condition* x_0.

(ii) *All the system solutions are in* $L^1[0\,,\,\infty)$*under finite initial conditions.*

Proof of Theorem 5. Consider the Lyapunov functional candidate:

$$V(x(t)\,,\,t)=\frac{1}{2}x^T(t)x(t)+\int_{0^-}^{t}\int_{t}^{\infty}x^T(s)B(u-s)x(s)duds;\ \forall t\in\boldsymbol{R}_{0+} \tag{39}$$

is given by:

$$\begin{aligned}
\dot{V}(x(t)\,,\,t)\ &=\left(x^T(t)A^T+\textstyle\int_{0^-}^{t}x^T(s)V_I^T(s)B^T(t-s)ds\right)x(t)\\
&+\textstyle\int_{0^-}^{t^-}B(t-s)V_I(s)x^T(s)x(s)ds+\int_{t^-}^{t}x^T(s)B(t-s)V_I(s)x(s)ds\\
&+\textstyle\int_{t}^{\infty}B(u-t)V_I(u)x^T(t)x(t)du-\int_{0}^{t^-}x^T(s)B(t-s)V_I(s)\,x(s)ds\\
&=\tfrac{1}{2}x^T(t)\left(A^T+A\right)x(t)+x^T(t)\textstyle\int_{0^-}^{t}B(t-s)V_I(s)x(s)ds\\
&+\textstyle\int_{0^-}^{t^-}x^T(s)B(t-s)V_I(s)x(s)ds+\int_{t^-}^{t}x^T(s)B(t-s)V_I(s)x(s)ds\\
&+\textstyle\int_{t}^{\infty}x^T(t)B(u-t)V_I(u)x(t)du-\int_{0}^{t^-}x^T(s)B(t-s)V_I(s)\,x(s)ds\\
&=\mu_2(A)\|x(t)\|_2^2+\textstyle\int_{0^-}^{t}x^T(t)B(t-s)V_I(s)x(s)ds\\
&+\textstyle\int_{0^-}^{t^-}x^T(s)B(t-s)V_I(s)x(s)ds+\int_{t^-}^{t}x^T(s)B(t-s)\,V_I(s)x(s)ds\\
&+\textstyle\int_{t}^{\infty}x^T(t)B(u-t)V_I(u)x(t)du-\int_{0}^{t^-}x^T(s)B(t-s)V_I(s)\,x(s)ds\\
&=\mu_2(A)\|x(t)\|_2^2+\textstyle\int_{0}^{t^-}x^T(t)B(t-s)V_I(s)x(s)ds+\|\vartheta(t)B(0^-)\|_2\|x(t^-)\|_2^2\\
&+\textstyle\int_{t}^{\infty}x^T(t)B(u-t)V_I(s)x(t)du-\int_{0}^{t^-}x^T(s)B(t-s)V_I(s)x(s)ds\\
&=\mu_2(A)\|x(t)\|_2^2+\vartheta(t)B(0^-)\|x(t^-)\|_2^2+\textstyle\int_{t}^{\infty}x^T(t)B(u-t)V_I(u)x(t)du\\
&\leq\left(\mu_2(A)+\left\|\textstyle\int_{0}^{\infty}B(u-t)V_I(u)du\right\|_2\right)\|x(t)\|_2^2+\|\vartheta(t)B(0^-)\|_2\|x(t^-)\|_2^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad;\ \forall t\in\boldsymbol{R}_{0+}
\end{aligned} \tag{40}$$

where $\vartheta(t)=\left(\vartheta_{ij}(t)\right)$ of entries $\vartheta_{ij}:\boldsymbol{R}_{0+}\to\{0\,,\,1\}\,;\ \forall(i,j)\in\overline{n}\times\overline{n}$, which are binary indicator functions, such that $\vartheta_{ij}(t)=1$ if $t\in Im_{ij}p$ and $\vartheta_{ij}(t)=0$ if $t\notin Im_{ij}p$ so that

$$\int_{t^-}^{t}B(t-s)V_I(s)\left\|x\left(s^-\right)\right\|_2^2ds\leq\left\|\vartheta(t)B\left(0^-\right)\right\|\,\left\|x\left(t^-\right)\right\|_2^2;\ \forall t\in\boldsymbol{R}_{0+}.$$

On the other hand, $x(t)=\left(I_n+\vartheta(t)B(0^-)\right)x(t^-)$; $\forall t\in\boldsymbol{R}_{0+}$. Then,

$$
\begin{aligned}
\dot{V}(x(t)\,,\,t) &\leq \left(\mu_2(A) + \left\|\int_0^\infty B(u-t)V_I(u)du\right\|_2\right)\|x(t)\|_2^2 + \|\vartheta(t)B(0^-)\|_2\|x(t^-)\|_2^2 \\
&= \left[\left(\mu_2(A) + \left\|\left(\int_{0^-}^\infty B(u-t)V_I(u)du\right)\left(I_n + \vartheta(t)B(0^-)\right) + \vartheta(t)B(0^-)\right\|_2\right)\right]\|x(t^-)\|_2^2 \\
&= \left[\left(\mu_2(A) + \left\|\int_{0^-}^{t_1^-} B(u)du + \sum_{i\in\overline{p(t)-1}}\int_{t_i}^{t_{i+1}^-} B(u)du + \int_{t_{p(t)}}^\infty B(u)du + \sum_{t_i\in Imp(t)} B(t^- - t_i)\right\|_2\right)\right. \\
&\quad \times\|(I_n + \vartheta(t)B(0^-)) + \vartheta(t)B(0^-)\|_2]\|x(t^-)\|_2^2 \\
&= -\alpha(t^-)\,\|x(t^-)\|_2^2 \\
&\leq -\underline{\alpha}^-\left|\|x(t^-)\|_2^2\right| \\
&< 0
\end{aligned}
\tag{41}
$$

if $x(t^-) \neq 0$; where $+\infty > \underline{\alpha}^- = \underset{t\in\mathbf{R}_{0+}}{inf}\,\alpha(t^-) > 0$. Note that $\underline{\alpha}^-$ is finite since $\left\|\int_{0^-}^\infty B(u-t)v_I(u)du\right\|_2 < +\infty$ and $\left\|\sum_{t_i\in Imp(t)} B(t^- - t_i)\right\|_2 \leq \left\|\sum_{t_i\in Imp} B(t^- - t_i)\right\|_2 < +\infty$. As a result, $x(t^-) \to 0$ and $x(t) = (I_n + \vartheta(t)B(0^-))x(t^-) \to 0$ as $t \to \infty$ so that the system is globally asymptotically stable for any given finite initial condition. Furthermore,

$$0 \leq V(x(t)\,,\,t) \leq V(x_0,\,0) - \underline{\alpha}^-\int_0^{t^-}\|x(\tau)\|_2^2 d\tau;\ \forall t \in \mathbf{R}_{0+} \tag{42}$$

which implies that

$$\int_0^t\|x(\tau)\|_2^2 d\tau \leq \|I_n + \vartheta(t)B(0^-)\|_2\int_0^{t^-}\|x(\tau)\|_2^2 d\tau \leq \|I_n + B(0^-)\|_2\int_0^{t^-}\|x(\tau)\|_2^2 d\tau \leq \frac{\|I_n + B(0^-)\|_2}{\underline{\alpha}^-}V(x_0\,,\,0) < +\infty\,;\ \forall t \in \mathbf{R}_{0+} \tag{43}$$

and

$$\underset{t\to\infty}{lim\,sup}\int_0^t\|x(\tau)\|_2^2 d\tau \leq \frac{\|I_n + B(0^-)\|_2}{\underline{\alpha}^-}V(x_0\,,\,0) < +\infty \tag{44}$$

and the proof is complete. □

A future extension of this work is foreseen under the framework of discrete Volterra equations for non-necessarily constant sampling periods [22–27]. Future consideration of conformable fractional Volterra-type delay impulsive differential systems can also be explored by extending the non-Volterra conformable fractional model proposed in [36] and, also, the extensions to the case of stochastic delay differential systems. See, for instance, [37–41].

3. Conclusions

This paper has investigated some new Volterra-type integral differential systems together with the associated global asymptotic stability properties when the dynamics have impulsive effects at certain time instants which are characterized by Dirac distribution under the integral symbol. The differential systems include delayed dynamics originated by combined constant point delays plus Volterra-type integral terms which can be interpreted as the presence of distributed delays in the differential system. Those mentioned integral terms also include impulsive effects at certain time instants which cause finite jumps in the evaluation of the Volterra integral and, as a result, they also cause finite discontinuities in the derivative with respect to time of the solution trajectory. Further discussed results rely either on the presence of Volterra-type integral terms over time-varying integral limits or on the consideration of a positive system within the above class. This kind of analyzed extended Volterra-type differential system is the main novelty of this research with respect to the existing background results. One of the differential systems is given by delay-free dynamics together with the contributions of a finite set of constant point delay systems plus a Volterra integral term of a finite length with impulses through isolated time instants. The second differential system considers a bounded time-varying size of the distributed delay in the integral term. The basic framework for this stability study has been the use of Krasovskii–Lyapunov functionals for the differential systems under consideration, while also taking into account the influence of the impulses in the stability results.

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Appendix A

The solutions of (1) and (28)

(A) The solution of (1) is:

$$\begin{aligned} x(t) &= e^{A_0 t}x_0 + \sum_{i=1}^{r}\int_0^t e^{A_0(t-\tau)}A_i x(\tau-h_i)U(\tau\ ,\ h_i)d\tau + \int_0^t\int_{-h}^0 e^{A_0(t-\tau)}A(\tau)v_I(\tau)x(t+\tau)d\tau \\ &= e^{A_0 t}x_0 + \sum_{i=1}^{r}\int_0^t e^{A_0(t-\tau)}A_i x(\tau-h_i)U(\tau\ ,\ h_i)d\tau + \sum_{i=0}^{p}\int_0^t\int_{-\tau_i^+}^{-\tau_{i+1}^-} e^{A_0(t-\tau)}A(\tau)v_I(\tau)x(t+\tau)d\tau \\ &+e^{A_0(t-h)}A(-h)x(t^- - h)i(\tau_0) + \sum_{i=1}^{p} e^{A_0(t-\tau_i)}A(-\tau_i)x\left(t-\tau_i^-\right);\forall t\in \boldsymbol{R}_{0+} \end{aligned} \quad \text{(A1)}$$

where $x(\tau)=\phi(\tau)$ for $\tau\in[-h\ ,\ 0]$, $x_0=x(0)=\phi(0)$, $\tau_0=h$ and $\tau_{p+1}=0$; and $U(\tau\ ,\ h_i)=0$ for $\tau<h_i$ and $U(\tau,h_i)=1$ for $\tau\geq h_i;\forall i\in\overline{p}\cup\{0\}$ is the unity step function and $i(\tau_0)=1$ if $\tau_0(=h)\in Imp$ and $i(\tau_0)=0$ if $\tau_0(=h)\notin Imp$. Note that the third right-hand-side of the first and second expressions of (A1) contains Dirac distributions under the integral symbols related to the impulsive terms. The last right-hand-side expression of (A1) translates the former Dirac distributions into finite jump-type discontinuities of the solution after calculating the corresponding integrals.

(B) The solution of (28) is

$$\begin{aligned} x(t) &= e^{At}\left(x_0 + \int_{0^-}^t e^{-As}B(t-s)v_I(s)x(s)ds\right) \\ &= e^{At}x_0 + \sum_{i=1}^{p(t)}\int_{t_i}^{t_{i+1}^-} e^{A(t-t_i)}B(t-t_i)x(s)ds + \sum_{i=1}^{p(t)} e^{A(t-t_i)}B(t-t_i)x\left(t_i^-\right);\forall t\in\boldsymbol{R}_{0+} \end{aligned} \quad \text{(A2)}$$

where $x_0=x(0)$. Similar comments to those given in (A1) for Dirac distributions and associated solution with jumps apply to (A2).

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