Fuzzy Sets and Systems 481 (2024) 108889

Contents lists available at ScienceDirect

Fuzzy Sets and Systems

journal homepage: www.elsevier.com/locate/fss

Katětov-Tong insertion theorem for functions with values in a tensor product of complete lattices

Tomasz Kubiak [a], Iraide Mardones-Pérez [b,*,1]

[a] Wydział Matematyki i Informatyki, Uniwersytet im. Adama Mickiewicza, 61-614 Poznań, Poland
[b] Departamento de Matemáticas, Universidad del País Vasco UPV/EHU, Apdo. 644, 48080 Bilbao, Spain

ARTICLE INFO

In memoriam: Jerzy Albrycht (1924–2021)

Keywords:
Complete lattice
Tensor product
Completely distributive lattice
Supercompact
$\lhd$-separable
Katětov-Tong insertion theorem
Urysohn lemma
Tietze extension theorem
Hedgehog

ABSTRACT

It is shown that Katětov-Tong insertion theorem continues to hold for normal L-topological spaces and functions with values in appropriately L-topologized tensor product $M \otimes L$ where L is a complete lattice with an order-reversing involution and M is a completely distributive lattice with a countable join base free of supercompact elements. When the first factor is the real unit interval, the tensor product can be identified with the Hutton fuzzy unit interval. Among corollaries of our insertion theorem are Urysohn lemma and Tietze extension theorem for $(M \otimes L)$-valued functions as well as Katětov-Tong insertion theorem for M-valued functions on traditional topological spaces.

1. Introduction

The famous Katětov-Tong insertion theorem due to Katětov [8] and Tong [19] states the following: *A topological space X is normal if and only if, given an upper semicontinuous function $g : X \to [0,1]$ and a lower semicontinuous function $h : X \to [0,1]$ with $g \leq h$, there exists a continuous function $f : X \to [0,1]$ such that $g \leq f \leq h$.* If f and g are characteristic functions of closed and open sets, this powerful theorem becomes Urysohn lemma. Tietze extension theorem is another simple corollary of Katětov-Tong theorem.

In this paper we stay in the category **Top**(L) of L-topological spaces and continuous functions where L is an arbitrary complete lattice with an order-reversing involution.

Our purpose is to show that the Katětov-Tong theorem can be carried over to normal L-topological spaces and functions with values in appropriately L-topologized tensor product $M \otimes L$ where M is a completely distributive lattice with a countable join base consisting of elements which all fail to be supercompact (= completely join irreducible). When M is the real unit interval, $M \otimes L$ can be identified with the Hutton fuzzy unit interval [7] in which case the Katětov-Tong insertion theorem goes back to [9]. Further applications include, among others, simple proofs of Urysohn's type lemma and Tietze's type extension theorem for $(M \otimes L)$-valued functions on normal L-topological spaces [4] as well as the Katětov-Tong insertion theorem for M-valued functions on normal topological spaces [6].

* Corresponding author.
E-mail addresses: tkubiak@amu.edu.pl (T. Kubiak), iraide.mardones@ehu.eus (I. Mardones-Pérez).
[1] IMP acknowledges support from the Basque Government (grant IT1483-22).

https://doi.org/10.1016/j.fss.2024.108889
Received 30 August 2023; Received in revised form 5 January 2024; Accepted 26 January 2024

Available online 1 February 2024

2. Terminology and notation

Let L be a complete lattice. The set L^X of all functions from a set X into L is a complete lattice under pointwise ordering:

$$f \leq g \quad \text{in} \quad L^X \qquad \text{iff} \qquad f(x) \leq g(x) \quad \text{in} \quad L \quad \text{for all} \quad x \in X.$$

Bounds of a complete lattice are denoted 0 and 1 (unless otherwise stated). The two-point chain is denoted **2**.

2.1. Few L-topological concepts

A family $\mathcal{T} \subseteq L^X$ is an *L-topology* on X, its members are *open* in X, and $(X, \mathcal{T})$ is an *L-topological space* (usually written as X) if $\mathcal{T}$ is closed under finite meets and arbitrary joins formed in L^X. Every $A \in L^X$ has its *interior*

$$\operatorname{Int} A = \bigvee \{U \in \mathcal{T} : U \leq A\}.$$

A function $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is *continuous* if, given V in $\mathcal{U}$, the composite $V \circ f$ belongs to $\mathcal{T}$. We say $\mathcal{U}$ is *generated* by a *subbase* $\mathcal{S} \subseteq L^X$ if $\mathcal{U}$ is the intersection of all L-topologies on Y which contain $\mathcal{S}$. Because of no distributivity assumed in our L, when checking a function for continuity we shall refer to the following subbasic characterization of continuity (cf. [17, p. 282]):

Subbase lemma. *Given $\mathcal{S}$ a subbase of an L-topology $\mathcal{U}$ on Y, a function $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous if and only if $V \circ f \in \mathcal{T}$ for all $V \in \mathcal{S}$.*

If $Z \subseteq X$, the restrictions $\{U | Z : U \in \mathcal{T}\}$ form a *subspace* L-topology on Z.
If L has an order-reversing involution $(\cdot)' : L \to L$, then $K \in L^X$ is called *closed* if K' is open where

$$K'(x) = K(x)'$$

for all $x \in X$. A complete lattice L with an order-reversing involution $(\cdot)'$ is written as

$$(L, ').$$

Every $A \in L^X$ has its *closure*

$$\overline{A} = \bigwedge \{K \in L^X : A \leq K \ \text{and} \ K \text{ is closed}\} = (\operatorname{Int}(A'))'.$$

A few more concepts will be defined later in the text.

2.2. Tensor products of complete lattices

Let M and L be complete lattices. Elements of M are denoted by t, s, r, and elements of L by a, b, c. A function $\lambda : M \to L$ is *join-preserving* if

$$\lambda(\bigvee T) = \bigvee \lambda(T) \quad \text{for all} \quad T \subseteq M.$$

Thus we are in the category **Sup** of complete lattices and join-preserving functions. The category **Sup** has tensor products. One construction of a tensor product in **Sup**, which is in tune with the fuzzy unit interval of Hutton [7] (cf. [5]), has been described by Shmuely [18]. Namely, the tensor product of M and L is the complete lattice

$$M \otimes L$$

consisting of all functions $\lambda : M \to L$ which are *join-reversing* – i.e.

$$\lambda(\bigvee T) = \bigwedge \lambda(T) \quad \text{for all} \quad T \subseteq M.$$

Hence $\lambda(0) = 1$. Meets in $M \otimes L$ are computed pointwisely in L^M. The function $\otimes : M \times L \to M \otimes L$ sending $(t, a) \in M \times L$ to the function $t \otimes a : M \to L$ defined by

$$(t \otimes a)(s) = \begin{cases} 1 & \text{if} \quad s = 0 \\ a & \text{if} \quad 0 \neq s \leq t \\ 0 & \text{if} \quad s \nleq t \end{cases}$$

is a universal bimorphism. Bounds of $M \otimes L$ are the functions $\mathbf{0}, \mathbf{1} : M \to L$ defined by

$$\mathbf{0}(t) = \begin{cases} 1 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases} \qquad \text{and} \qquad \mathbf{1}(t) = 1.$$

Both M and L completely embed into $M \otimes L$ via $e_1 : M \to M \otimes L$ and $e_2 : L \to M \otimes L$ given by

$$e_1(t) = t \otimes 1 \qquad \text{and} \qquad e_2(a) = 1 \otimes a.$$

A full description of tensor products in **Sup** may be found in the book [2] (as well in [3]).

2.3. Completely distributive lattices and ⊲-separability

For the tensor product $M \otimes L$ to play the same role for L-topological spaces as the real unit interval plays in usual topology, we must assume something more about M than mere completeness. We will assume M is a completely distributive lattice endowed with an appropriate join base. To this end – instead of the usual equational definition of complete distributivity – we shall use Raney's [15] characterization of complete distributivity in terms of the totally below relation $\lhd$.

Given $s, t \in M$, we let

$$s \lhd t$$

if, whenever $t \leq \bigvee T$ with $T \subseteq M$, there exists an $r \in T$ such that $s \leq r$. Then: (1) $s \lhd t$ implies $s \leq t$, and (2) $s \leq q \lhd r \leq t$ implies $s \lhd t$. In particular, $\lhd$ is transitive.

According to [15], a complete lattice M is *completely distributive* iff each $t \in M$ has the *approximation property* – i.e.

$$t = \bigvee \{ s \in M : s \lhd t \},$$

in which case $\lhd$ has the *insertion property* – i.e.

$$s \lhd t \quad \text{implies} \quad s \lhd r \lhd t \quad \text{for some} \quad r \in M.$$

A set $Q \subseteq M$ is called a *join base* of M (in short: *base*) if each element of M is a join of a subset of Q. This is equivalent to the requirement that for all $t \in M$ one has

$$t = \bigvee \{ r \in Q : r \leq t \}.$$

We shall freely use the following equivalent properties [4]:

(1) Q is a base of M.

(2) If $t \lhd s$ in M, then there is an $r \in Q$ with $t \lhd r \lhd s$.

(3) $t = \bigvee \{ r \in Q : r \lhd t \}$ for all $t \in M$.

An element $t \in M$ is called *supercompact* (another terminology: *completely join irreducible*) if

$$t \lhd t.$$

In [6], a countable base Q of a completely distributive lattice M is called *⊲-separable* if it is free of supercompact elements – i.e. $r \not\lhd r$ for all $r \in Q$. We may say that a complete lattice M is *⊲-separable* if it is completely distributive and has a *⊲-separable* base. Observe that 0 is never supercompact and besides, if M is *⊲-separable*, also 1 fails to be supercompact.

Besides the real unit interval (in which the rationals form a ⊲-separable base), there are interesting examples of ⊲-separable lattices:

(a) The tensor product $M \otimes L$ with M a ⊲-separable lattice and L being the power set $\mathcal{P}(\mathbb{N})$ ordered by inclusion; this is in fact true for any completely distributive L with a countable base (see [4]).

(b) The Cartesian product of a countable family of ⊲-separable lattices is ⊲-separable too. In particular, this is the case of the Hilbert cube (see [6]).

3. L-topologizing $M \otimes L$

For M a completely distributive lattice, members of $M \otimes L$ can be characterized in terms of $\lhd$ (cf. [4]). Namely, $\lambda : M \to L$ is join-reversing if and only if it is *left-continuous* – i.e.

$$\lambda(t) = \bigwedge_{s \lhd t} \lambda(s) \quad \text{for all} \quad t \in M.$$

For an order-reversing function $\lambda : M \to L$, we let

$$\lambda^+(t) = \bigvee_{t \lhd s} \lambda(s) \quad \text{for all} \quad t \in M.$$

Then $\lambda^+ \le \lambda$ and λ^+ is order-reversing.

The following properties come from [4]:

3.1 Fact. *Let M be completely distributive with a base Q, and let L be complete. For each $\lambda \in M \otimes L$ we have:*

(1) $\lambda^+(t) = \bigvee_{t \lhd s,\, s \in Q} \lambda^+(s)$.

(2) $\lambda^+(t) = \bigvee_{t \lhd s,\, s \in Q} \lambda(s)$.

(3) $\lambda(t) = \bigwedge_{s \lhd t,\, s \in Q} \lambda(s)$.

(4) $\lambda(t) = \bigwedge_{s \lhd t,\, s \in Q} \lambda^+(s)$.

3.2 Remark. As already mentioned, $M \otimes L$ is ordered pointwise. By 3.1(4), if M is completely distributive, we have a useful alternative:

$$\lambda \le \mu \qquad \text{iff} \qquad \lambda^+ \le \mu^+.$$

Therefore, given $f, g \in (M \otimes L)^X$, we have

$$f \le g \qquad \text{iff} \qquad f(x)^+ \le g(x)^+ \quad \text{for all} \quad x \in X.$$

As in the case of Hutton fuzzy unit interval $[0,1](L)$ – which can be identified with $[0,1] \otimes L$ (cf. [5]) – our tensor product $M \otimes L$ carries three L-topologies.

3.3 Definition. Let M be a completely distributive lattice and $(L, ')$ be complete. For every $t \in M$, define $R_t, L_t : M \otimes L \to L$ by

$$R_t(\lambda) = \lambda^+(t) \qquad \text{and} \qquad L_t(\lambda) = \lambda(t)'.$$

The three L-topologies on $M \otimes L$ are defined as follows:

(a) the *upper* L-topology $\mathcal{R}_{M \otimes L}$ is generated by $\{R_t : t \in M\}$,

(b) the *lower* L-topology $\mathcal{L}_{M \otimes L}$ is generated by $\{L_t : t \in M\}$,

(c) the *interval* L-topology $\mathcal{I}_{M \otimes L}$ is generated by $\{R_t,\, L_t : t \in M\}$.

3.4 Remark. Zhang and Liu [20] dealt with the set of all join-preserving functions from a completely distributive M to a completely distributive $(L, ')$, which they called the *L-fuzzy modification* of M. The relationship of $M \otimes L$ to the L-fuzzy modification of M is discussed in [4].

The following properties come from [4]:

3.5 Fact. *Let M be a completely distributive lattice with a base Q, and let L be a complete lattice. For each $t \in M$ the following hold where r stands for a member of Q:*

(1) $R_t = \bigvee_{t \lhd r} R_r$.

If L has an order-reversing involution $(\cdot)'$, then:

(2) $R_t = \bigvee_{t \lhd r} L_r'$.

(3) $L_t = \bigvee_{r \lhd t} L_r$.

(4) $L_t = \bigvee_{r \lhd t} R_r'$.

3.6 Definition. Let M be completely distributive, $(L, ')$ be complete, and let X be an L-topological space. An $f : X \to M \otimes L$ is:

(1) *lower semicontinuous* if it is continuous when the set $M \otimes L$ is given the L-topology $\mathcal{R}_{M \otimes L}$,

(2) *upper semicontinuous* if it is continuous when the set $M \otimes L$ is given the L-topology $\mathcal{L}_{M\otimes L}$,

(3) *continuous* if it is continuous when the set $M \otimes L$ is given the L-topology $\mathcal{I}_{M\otimes L}$.

So, f is continuous iff it is lower and upper semicontinuous.

3.7 Remark. For M a completely distributive lattice and $(L, ')$ a complete lattice, the embedding e_1 from M into $M \otimes L$ is also an L-topological embedding of M with its traditional lower, upper, and interval topologies, respectively, into $M \otimes L$ with its lower, upper, and interval L-topologies, respectively (cf. [4]).

To avoid repetitions, we now describe a general procedure of generating continuous $(M \otimes L)$-valued functions by scales (cf. [11] and [12]).

Let L be a complete lattice and M a $\lhd$-separable lattice with base Q. Let $\{F_r\}_{r\in Q}$ be a $\lhd$-*antitone* family of element of L^X – i.e.

$$F_s \le F_r \qquad \text{if} \qquad r \lhd s.$$

Let $f : X \to M \otimes L$ be defined by

$$f(x)(t) = \bigwedge_{r\lhd t} F_r(x)$$

for all $x \in X$ and $t \in M$. The transitivity and the insertion property of $\lhd$ show that f is well defined, for $f(x)$ is left-continuous:

$$f(x)(t) = \bigwedge_{r\lhd t} F_r(x) = \bigwedge_{s\lhd t}\bigwedge_{r\lhd s} F_r(x) = \bigwedge_{s\lhd t} f(x)(s).$$

The family $\{F_r\}_{r\in Q}$ is called a *scale* generating f.

3.8 Properties. *Let $(L, ')$ be a complete lattice and M be a completely distributive lattice with a $\lhd$-separable base Q. Let $f, g : X \to M \otimes L$ be generated by scales $\{F_r\}_{r\in Q}$ and $\{G_r\}_{r\in Q}$, respectively. Then for each $t \in M$ the following hold where r and s stand for members of Q:*

(1) $L_t \circ f = \bigvee_{r\lhd t} F_r'$.

(2) $R_t \circ f = \bigvee_{t\lhd r} F_r$.

(3) $\{L_r' \circ f\}_{r\in Q}$ *and* $\{R_r \circ f\}_{r\in Q}$ *are scales generating* f.

(4) $f \le g$ *iff* $F_s \le G_r$ *whenever* $r \lhd s$.

For X an L-topological space we have

(5) f *is continuous iff* $\overline{F_s} \le \operatorname{Int} F_r$ *whenever* $r \lhd s$.

Proof. (1) restates the definition of f.

(2) By 3.5(2) and by (1) above, we have

$$R_t \circ f = \bigvee_{t\lhd s}\left(L_s' \circ f\right) = \bigvee_{t\lhd s}\bigwedge_{r\lhd s} F_r \ge \bigvee_{t\lhd s}\bigvee_{s\lhd r} F_r = \bigvee_{t\lhd r} F_r.$$

For the reverse inequality, for all s with $t \lhd s$ and for all r with $t \lhd r \lhd s$ we have

$$\bigvee_{t\lhd r} F_r \ge \bigwedge_{r\lhd s} F_r.$$

Thus

$$\bigvee_{t\lhd r} F_r \ge \bigvee_{t\lhd s}\bigwedge_{r\lhd s} F_r = R_t \circ f.$$

(3) Clearly, both $\{L_r' \circ f\}_{r\in Q}$ and $\{R_r \circ f\}_{r\in Q}$ are scales. Since $\{F_r\}_{r\in Q}$ generates f, we have by (1) that

$$\bigwedge_{r\lhd t}\left(L_r' \circ f\right)(x) = \bigwedge_{r\lhd t}\bigwedge_{s\lhd r} F_s(x) = \bigwedge_{s\lhd t} F_s(x) = f(x)(t).$$

Hence $\{L_r' \circ f\}_{r\in Q}$ generates f too. Let h be the function generated by the scale $\{R_r \circ f\}_{r\in Q}$. By (2) we have

$$R_t \circ h = \bigvee_{t\lhd r}\left(R_r \circ f\right) = \bigvee_{t\lhd r}\bigvee_{r\lhd s} F_s = \bigvee_{t\lhd s} F_s = R_t \circ f.$$

By 3.2 we obtain $h = f$.

(4) Assume $f \le g$. Let $r \lhd s$ in Q. Since scales are $\lhd$-antitone, by (1), (2), and 3.2 we have

$$F_s \le \bigwedge_{t \lhd s} F_t = L'_s \circ f \le R_r \circ f \le R_r \circ g = \bigvee_{r \lhd t} G_t \le G_r.$$

To see the reverse implication, let $s \lhd t$. Then

$$L'_t \circ f = \bigwedge_{r \lhd t} F_r \le G_s,$$

so that

$$L'_t \circ f \le \bigwedge_{s \lhd t} G_s = L'_t \circ g.$$

(5) For the *only if* part, let $r \lhd s$ in Q. Then

$$F_s \le \bigwedge_{t \lhd s} F_t = L'_s \circ f \le R_r \circ f = \bigvee_{r \lhd t} F_t \le F_r$$

with $L'_s \circ f$ closed and $R_r \circ f$ open. Hence $\overline{F_s} \le \operatorname{Int} F_r$. The argument for the *if part* is given within the proof of [4, Theorem 5.3]. We repeat it here for the sake of completeness. Namely, if $\overline{F_s} \le \operatorname{Int} F_r$ whenever $r \lhd s$, then both

$$L_t \circ f = \bigvee_{s \lhd t} F'_s = \bigvee_{s \lhd t} \overline{F_s}' \qquad \text{and} \qquad R_t \circ f = \bigvee_{t \lhd r} F_r = \bigvee_{t \lhd r} \operatorname{Int} F_r$$

are open. Hence f is continuous by Subbase lemma. □

It is sometimes convenient to identify elements of L^X with certain elements of $(M \otimes L)^X$.

3.9 Definition. Given $A \in L^X$, define $\chi_A : X \to M \otimes L$ – the characteristic function of A – by

$$\chi_A(x) = 1 \otimes A(x)$$

for all $x \in X$.

Notation. If $Z \subseteq X$, then $1_Z \in L^X$ stands for the traditional characteristic function defined by

$$1_Z(x) = \begin{cases} 1 & \text{if} \quad x \in Z \\ 0 & \text{if} \quad x \in X \setminus Z. \end{cases}$$

3.10 Proposition. *In an L-topological space X we have for each $A \in L^X$:*

(1) *A is open iff χ_A is lower semicontinuous.*

(2) *A is closed iff χ_A is upper semicontinuous.*

Proof. (1) Let $x \in X$ and $t \neq 1$. Given s with $t \lhd s$, we have $s \neq 0$, so that

$$\left(R_t \circ \chi_A\right)(x) = \chi_A(x)^+(t) = \bigvee_{t \lhd s} (1 \otimes A(x))(s) = A(x).$$

This plus the two other cases gives us

$$R_t \circ \chi_A = \begin{cases} A & \text{if} \quad t \neq 1 \;\text{ or }\; t = 1 \lhd 1 \\ 1_\varnothing & \text{if} \quad t = 1 \ntriangleleft 1. \end{cases}$$

(2) We have

$$L_t \circ \chi_A = \begin{cases} A' & \text{if} \quad t \neq 0 \\ 1_\varnothing & \text{if} \quad t = 0. \end{cases} \qquad \square$$

4. Katětov-Tong insertion theorem for $M \otimes L$-valued functions and its applications

The following lemma of [6] allows to construct scales in various situations.

4.1 Insertion lemma. *Let N be a complete lattice endowed with a relation $\Subset$ satisfying the following conditions for all $a, b, c \in N$:*

(1) *$a \Subset b$ implies $a \le b$,*

(2) *$a \le b \Subset c \le d$ implies $a \Subset d$,*

(3) $a, b \Subset c$ *implies* $a \vee b \Subset c$,

(4) $a \Subset b, c$ *implies* $a \Subset b \wedge c$,

(5) $a \Subset b$ *implies* $a \Subset d \Subset b$ *for some* $d \in N$.

Let J *be a countable set endowed with a transitive and irreflexive relation* $\prec$. *Let* $\{a_j\}_{j\in J}$ *and* $\{b_j\}_{j\in J}$ *be families of* N *satisfying the following:*

$$j \prec i \qquad \text{implies} \qquad \begin{cases} a_i \leq a_j \\ a_i \Subset b_j \\ b_i \leq b_j. \end{cases} \tag{$*$}$$

Then there exists a family $\{c_j\}_{j\in J}$ *such that*

$$j \prec i \qquad \text{implies} \qquad \begin{cases} a_i \Subset c_j \\ c_i \Subset c_j \\ c_i \Subset b_j. \end{cases}$$

We recall that an L-topological space X is *normal* [7] if, whenever K is closed in X, U is open in X, and $K \leq U$, there exists an open V in X such that

$$K \leq V \leq \overline{V} \leq U.$$

4.2 Theorem (Katětov-Tong insertion theorem for $(M \otimes L)$-valued functions). *Let* $(L, ')$ *be complete and let* M *be* $\lhd$*-separable. For* X *an* L*-topological space the following are equivalent:*

(1) X *is normal.*

(2) *If* $g : X \to M \otimes L$ *is upper semicontinuous,* $h : X \to M \otimes L$ *is lower semicontinuous, and* $g \leq h$*, then there is a continuous* $f : X \to M \otimes L$ *such that*

$$g \leq f \leq h.$$

Proof. (1) ⇒ (2): We shall use a special case of 4.1 in which: $N = L^X$, the relation $\Subset$ is defined by

$$A \Subset B \qquad \text{iff} \qquad \overline{A} \subseteq \operatorname{Int} B$$

for all $A, B \in L^X$, and $J = Q$ is a $\lhd$-separable base in which $\lhd$, when restricted to $Q \times Q$, plays the role of the transitive and irreflexive relation $\prec$. Before proceeding further, we observe that for any X the relation $\Subset$ satisfies (1)–(4) of 4.1, and $\Subset$ satisfies (5) of 4.1 if and only if X is normal.

So, let X be normal, $g, h : X \to M \otimes L$ be upper and lower semicontinuous, respectively, and let $g \leq h$. For every $r \in Q$, let

$$G_r = L'_r \circ g \qquad \text{and} \qquad H_r = R_r \circ h.$$

By 3.8(3), $\{G_r\}_{r\in Q}$ and $\{H_r\}_{r\in Q}$ are scales generating g and h, respectively. Consequently, after applying 3.8(4) to $g \leq h$, we have $G_s \leq H_r$ if $r \lhd s$. Elements of these two scales satisfy condition ($*$) of 4.1, and are closed and open by upper semicontinuity of g and lower semicontinuity of h, respectively. Hence

$$G_s \Subset H_r \qquad \text{if} \qquad r \lhd s$$

and so there exists a scale $\{F_r\}_{r\in Q}$ such that

$$r \lhd s \qquad \text{implies} \qquad \begin{cases} G_s \Subset F_r \\ F_s \Subset F_r \\ F_s \Subset H_r. \end{cases} \tag{$**$}$$

Since $\overline{F_s} \leq \operatorname{Int} F_r$ whenever $r \lhd s$, the function $f : X \to M \otimes L$ generated by $\{F_r\}_{r\in Q}$ is continuous by 3.8(5). Finally, by 3.8(4) and ($**$) it follows that $g \leq f \leq h$.

(2) ⇒ (1): If $K \leq U$ with K being closed and U being open, then $\chi_K \leq \chi_U$ where χ_K is upper and χ_U is lower semicontinuous by 3.10. So, there is a continuous $f : X \to M \otimes L$ with $\chi_K \leq f \leq \chi_U$. Then

$$K = L'_1 \circ \chi_K \leq L'_1 \circ f \leq R_0 \circ f \leq R_0 \circ \chi_U = U.$$

Select $t \in Q$ with $0 \lhd t \lhd 1$. Then $t \neq 0$ as always, and $t \neq 1$ as Q does not have supercompact elements. For any $\lambda \in M \otimes L$ we have

$$\lambda(1) \leq \lambda^+(t) \leq \lambda(t) \leq \lambda^+(0).$$

For $\lambda_x = f(x)$ with arbitrary $x \in X$, the above inequalities yield

$$K \leq L_1' \circ f \leq R_t \circ f \leq L_t' \circ f \leq R_0 \circ f \leq K.$$

Now, $V = R_t \circ f$ is open and $L_t' \circ f$ is closed, hence $K \leq V \leq \overline{V} \leq K$. $\square$

Because of 3.7, Katětov-Tong insertion theorem for $(M \otimes L)$-valued functions provides a common generalization of insertion theorems for $([0,1] \otimes L)$-valued functions [9] and $(M \otimes \mathbf{2})$-valued functions [6] (cf. [13]). An immediate corollary – stated below as 4.3 – is also the Urysohn lemma for $(M \otimes L)$-valued functions proved directly in [4]. An argument for the nontrivial part of 4.3 has already been given when proving (2) implies (1) of 4.2. Urysohn lemma with $M = [0,1]$ goes back to Hutton [7].

4.3 Theorem (Urysohn lemma for $(M \otimes L)$-valued functions). *Let $(L, ')$ be complete and let M be $\lhd$-separable. For X an L-topological space the following are equivalent:*

(1) *X is normal.*

(2) *If $K \in L^X$ is closed, $U \in L^X$ is open, and $K \leq U$, then there exists a continuous $f : X \to M \otimes L$ such that*

$$K \leq L_1' \circ f \leq R_0 \circ f \leq U.$$

Another easy corollary of 4.2 is the Tietze extension theorem for $(M \otimes L)$-valued functions proved directly in [4]. Its $([0,1] \otimes L)$-valued version is given in [9], and its $(M \otimes \mathbf{2})$-valued version is given in [6].

4.4 Theorem (Tietze extension theorem for $(M \otimes L)$-valued functions). *Let $(L,')$ be complete and let M be $\lhd$-separable. Let X be a normal L-topological space and let $Z \subseteq X$ be such that 1_Z is closed in X. Then every continuous $f : Z \to M \otimes L$ extends continuously to the whole space X.*

Proof. Let $f : Z \to M \otimes L$ be continuous. Define $g, h : X \to M \otimes L$ by

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in Z \\ \mathbf{0} & \text{if} \quad x \in X \setminus Z \end{cases} \qquad \text{and} \qquad h(x) = \begin{cases} f(x) & \text{if} \quad x \in Z \\ \mathbf{1} & \text{if} \quad x \in X \setminus Z. \end{cases}$$

Then $g \leq h$, and it is not difficult to check that g is upper semicontinuous and h is lower semicontinuous. By 4.2 there is a continuous $\hat{f} : X \to M \otimes L$ such that $g \leq \hat{f} \leq h$. Clearly, $\hat{f}$ extends f to all of X.

Let us, nevertheless, check h for lower semicontinuity. Let, for a moment, 0^L and 1^L stand for bounds of L. Since M is $\lhd$-separable, R_1 is the constant map with value 0^L and hence, $R_1 \circ h = 1_\emptyset$ is open in X. Take $t \neq 1$ in M. Since $R_t \circ f$ is open in Z, it is of the form $U_t | Z$ with U_t being open in X. Then

$$(R_t \circ h)(x) = \begin{cases} (R_t \circ f)(x) = U_t(x) & \text{if} \quad x \in Z \\ R_t(\mathbf{1}) = 1^L & \text{if} \quad x \in X \setminus Z. \end{cases}$$

In conclusion, we have the openness of $R_t \circ h = U_t \vee 1_{X \setminus Z}$. $\square$

Unlike Urysohn lemma, 4.4 with $L \neq \mathbf{2}$ does not characterize normality of L-topological spaces. According to Rodabaugh [16], an L-topological space X is called *suitable* [for extending functions from an L-topological subspace of X to the whole of X] if it has a closed 1_Z where $\emptyset \neq Z \subsetneq X$. A normal L-topological space need not be suitable. As shown in [1], $([0,1] \otimes L, \mathcal{I}_{[0,1] \otimes L})$ fails to be suitable for every completely distributive lattice $(L,')$ with $L \neq \mathbf{2}$. It would be of interest to know for which M and $L \neq \mathbf{2}$ is $(M \otimes L, \mathcal{I}_{M \otimes L})$ suitable.

Notice. Katětov-Tong insertion theorem for monotonically normal L-topological spaces and $([0,1] \otimes L)$-valued functions is discussed in [10] and an $(M \otimes \mathbf{2})$-valued version is given in [14]. A characterization of monotonically normal L-topological spaces in terms of inserting $(M \otimes L)$-valued functions is hoped to appear elsewhere.

5. Inserting hedgehog-valued functions

In this section, we apply our Katětov-Tong insertion Theorem 4.2 to obtain a characterization of normal L-topological spaces in terms of inserting a continuous hedgehog-valued function.

Let κ be a cardinal and let I be a set with cardinality κ. We first observe that our insertion theorem can be stated for functions having the L-topological product $(M \otimes L)^\kappa$ as the range space. We recall that, given Y with an L-topology $\mathcal{U}$, the Cartesian product

Y^κ of κ copies of Y is L-topologized by the subbase $\{U \circ \pi_i : U \in \mathcal{U} \text{ and } i \in I\}$ where π_i is the ith projection. Let $\mathcal{U}^\kappa$ be the product L-topology of Y^κ. Given an L-topological space X and a function $f : X \to (M \otimes L)^\kappa$, let us agree to call f:

(1) *lower semicontinuous* if it is continuous when the set $(M \otimes L)^\kappa$ is given the L-topology $\mathcal{R}^\kappa_{M\otimes L}$,

(2) *upper semicontinuous* if it is continuous when the set $(M \otimes L)^\kappa$ is given the L-topology $\mathcal{L}^\kappa_{M\otimes L}$,

(3) *continuous* if it is continuous when the set $(M \otimes L)^\kappa$ is given the L-topology $\mathcal{I}^\kappa_{M\otimes L}$.

Assume $(M \otimes L)^\kappa$ is ordered componentwise. Then, given $f, g : X \to (M \otimes L)^\kappa$, we have

$$f \le g \quad \text{iff} \quad \pi_i \circ f \le \pi_i \circ g \quad \text{for all} \quad i \in I.$$

5.1 Proposition. *Let $(L, ')$ be complete and let M be $\lhd$-separable. An L-topological space X is normal if and only if, given an upper semicontinuous $g : X \to (M \otimes L)^\kappa$ and a lower semicontinuous $h : X \to (M \otimes L)^\kappa$ with $g \le h$, there exists a continuous $f : X \to (M \otimes L)^\kappa$ such that $g \le f \le h$.*

Proof. This is obvious. Indeed, for each $i \in I$ we have $\pi_i \circ g \le \pi_i \circ h$ and so by 4.2 there is a continuous $\varphi_i : X \to M \otimes L$ such that

$$\pi_i \circ g \le \varphi_i \le \pi_i \circ h.$$

The unique function $f : X \to (M \otimes L)^\kappa$ satisfying $\pi_i \circ f = \varphi_i$ is continuous (by the Subbase lemma) and $g \le f \le h$. □

In [6], there is a version of the classical Katětov-Tong theorem for functions with values in a hedgehog identified with a subspace of the Tychonoff cube $[0,1]^\kappa$ consisting of its “coordinate axes”.

We finish this paper with a brief discussion of a fuzzy hedgehog consisting of the “coordinate axes” of the product $(M \otimes L)^\kappa$ with its interval L-topology. We recall that according to the usual definition, a *hedgehog* (having $M \otimes L$ as spines) would be the union of κ copies of the tensor product $M \otimes L$ by identifying the bottom $\mathbf{0}$ of each tensor product. We omit all the technicalities and just let

$$J_{M\otimes L}(\kappa) = \bigcup_{i \in I} \{\varphi \in (M \otimes L)^\kappa : \varphi(j) = \mathbf{0} \text{ for all } j \ne i\} \subseteq (M \otimes L)^\kappa$$

with the componentwise ordering and the subspace L-topology inherited from the Cartesian product $(M \otimes L)^\kappa$.

As earlier, given an L-topological space X, a function $f : X \to J_{M\otimes L}(\kappa)$ is called *lower semicontinuous*, *upper semicontinuous*, *continuous*, respectively, if it is continuous when the set $J_{M\otimes L}(\kappa)$ is equipped with the subspace L-topology induced from $\mathcal{R}^\kappa_{M\otimes L}$, $\mathcal{L}^\kappa_{M\otimes L}$, and $\mathcal{I}^\kappa_{M\otimes L}$, respectively. Of course, $f : X \to J_{M\otimes L}(\kappa)$ is continuous in one of those three senses if and only if so is $e \circ f : X \to (M \otimes L)^\kappa$ where e is the identity embedding of $J_{M\otimes L}(\kappa)$ into $(M \otimes L)^\kappa$.

Also, if $f, g : X \to J_{M\otimes L}(\kappa)$, then

$$f \le g \quad \text{in} \quad J_{M\otimes L}(\kappa)^X \qquad \text{iff} \qquad e \circ f \le e \circ g \quad \text{in} \quad ((M \otimes L)^\kappa)^X.$$

With all this in mind we can formulate the following:

5.2 Proposition. *Let $(L, ')$ be complete and let M be $\lhd$-separable. An L-topological space X is normal if and only if, given an upper semicontinuous $g : X \to J_{M\otimes L}(\kappa)$ and a lower semicontinuous $h : X \to J_{M\otimes L}(\kappa)$ with $g \le h$, there exists a continuous $f : X \to J_{M\otimes L}(\kappa)$ such that $g \le f \le h$.*

We omit formulations of Urysohn’s type and Tietze’s type theorems for $(M \otimes L)^\kappa$-valued and $J_{M\otimes L}(\kappa)$-valued functions.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Iraide Mardones-Perez reports travel was provided by Basque Government.

References

[1] G. Artico, R. Moresco, α^*-compactness of the fuzzy unit interval, Fuzzy Sets Syst. 25 (1988) 243–249.
[2] P. Eklund, J. Gutiérrez García, U. Höhle, J. Kortelainen, Semigroups in Complete Lattices: Quantales, Modules and Related Topics, Springer, Cham, 2018.
[3] J. Gutiérrez García, U. Höhle, T. Kubiak, Tensor products of complete lattices and their application in constructing quantales, Fuzzy Sets Syst. 313 (2017) 43–60.
[4] J. Gutiérrez García, U. Höhle, T. Kubiak, An extension of the fuzzy unit interval to a tensor product with completely distributive first factor, Fuzzy Sets Syst. 370 (2019) 63–78.
[5] J. Gutiérrez García, T. Kubiak, Forty years of Hutton fuzzy unit interval, Fuzzy Sets Syst. 281 (2015) 128–133.
[6] J. Gutiérrez García, T. Kubiak, M.A. de Prada Vicente, Insertion of lattice-valued and hedgehog-valued functions, Topol. Appl. 153 (2006) 1458–1475.
[7] B. Hutton, Normality in fuzzy topological spaces, J. Math. Anal. Appl. 50 (1975) 74–79.
[8] M. Katětov, On real-valued functions in topological spaces, Fund. Math. 38 (1951) 85–91, Correction in: Fund. Math. 40 (1953) 203–205.
[9] T. Kubiak, L-fuzzy normal spaces and Tietze extension theorem, J. Math. Anal. Appl. 125 (1987) 141–153.

[10] T. Kubiak, Monotone insertion of continuous functions, Quest. Answ. Gen. Topol. 11 (1993) 51–59.
[11] T. Kubiak, Separation axioms: extension of mappings and embedding of spaces [Chapter 6], in: U. Höhle, S.E. Rodabaugh (Eds.), Handbook on Fuzzy Set Theory, in: Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, vol. II, Kluwer, Boston, 1999, pp. 433–479.
[12] S.S. Kutateladze, Fundamentals of Functional Analysis, Kluwer, Dordrecht, 1996.
[13] Y.-M. Liu, M.-K. Luo, Lattice-valued Hahn-Dieudonné-Tong insertion theorem and stratification structure, Topol. Appl. 45 (1992) 173–188.
[14] I. Mardones-Pérez, M.A. de Prada Vicente, Monotone insertion of lattice-valued functions, Acta Math. Hung. 117 (2007) 187–200.
[15] G.N. Raney, A subdirect-union representation for completely distributive complete lattices, Proc. Amer. Math. Soc. 4 (1953) 518–522.
[16] S.E. Rodabaugh, Suitability in fuzzy topological spaces, J. Math. Anal. Appl. 79 (1981) 273–285.
[17] S.E. Rodabaugh, E.P. Klement, U. Höhle (Eds.), Applications of Category Theory to Fuzzy Subsets, Kluwer, Dordrecht, 1992.
[18] Z. Shmuely, The structure of Galois connections, Pacific J. Math. 54 (1974) 209–225.
[19] H. Tong, Some characterizations of normal and perfectly normal spaces, Duke Math. J. 19 (1952) 289–292.
[20] D. Zhang, Y.-M. Liu, L-fuzzy modification of completely distributive lattices, Math. Nachr. 168 (1994) 79–95.