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PhD Thesis

On Conciseness and Profinite RAAGs

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Contents

To my parents.

Acknowledgments

A PhD is not only an academic achievement, but a stage of life in which we have both great and hard times. Looking back at my time in Bilbao, I am overwhelmed by the amount of people that have helped me during these years.

First of all, the two people without whom I would not be here, my advisors. Gustavo, thank you for wanting me here, for helping me and pushing me, and for always being positive. I cannot say I'm a full believer yet, but you taught, and showed me with facts, that it's better not to be a non-believer. You were right, and I owe you a lot! To Montse, I cannot put into words how thankful I am for all the help you gave me! You stepped up in the moments I was struggling, you supported me a lot, and you have always been patient. Among all the great and amazing people I met here in Bilbao, you are the kindest one!

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Introduction

While the origin of group theory is often attributed to the work of Galois, Jordan, and Klein, all of their works were motivated by the connection that this discipline has either with number theory or geometry. The theory of abstract discrete groups obtained independent interest, without a geometrical inspiration, mainly at the beginning of the 20^{th} century, and a milestone for this is due to the work of William Burnside. In 1902 he asked whether a finitely generated torsion group is necessarily finite [15], the so-called "Burnside Problem". This question sparked interest in even deeper problems, like the study of the finiteness of finitely generated groups of finite exponent, also called "Bounded Burnside Problem". Explicitly, Grün [36] asked whether a finitely generated group G satisfying $g^n = 1$ for all $g \in G$ is necessarily finite.

We could observe that this problem can be embedded in the greater framework of one of the most natural questions that can be asked about an algebraic structure, which is "What can we say about a group if this group follows a fixed rule?"

Of course the question is extremely heuristical, but we can view a lot of the developments in earlier group theory through this approach, which can be encoded as an example of a word problem in groups.

A group word w is a finite concatenation of variables and of their inverses, which can be seen as an element of the free group generated by n variables $x_1, \ldots, x_n$. For any group G, the word w naturally gives a map from G^n to G, simply by substituting the elements of the group in the variables in every possible way. The image of this map is the set of word values in G, usually denoted by $w\{G\}$, and the subgroup $w(G)$ they generate is called verbal subgroup. Of special interest is the study of varieties of groups, that are the classes of groups in which a certain word w is a law, in the sense that it takes only the trivial value.

The "rules" mentioned in Grün's question are simply group laws, so in modern terms he asked how to study the variety of groups generated by the law x^n. Other problems that can be seen in this optics can be the study of abelian, or nilpotent and solvable groups of bounded class, which are the varieties generated by the commutator word $[x_1, x_2]$, by a lower central word or by a derived word respectively.

Rather than studying only groups in which a word w is a law, we could also wonder whether the fact that w takes finitely many values in a group G has any implication on the structure of G. It is easy to realise that any group with finitely many commutators is finite-by-abelian, or, in other words, if the set of γ_2-values is finite in a group G, then the corresponding verbal subgroup is finite. Philip Hall realized that the same is true for all power words and lower central words, not only for γ_2. As a consequence, Hall conjectured that for any group word, if the set $w\{G\}$ of word values in a certain group G is finite, then the verbal subgroup $w(G)$ is finite too. If a word satisfies this property for every group G, it is called concise and, if it does for all groups in a given class $\mathcal{C}$, it is said to be concise in $\mathcal{C}$.

Many words have been proven to be concise, moreover it was proved by Merzljakov that all words are concise in linear groups, but a counterexample for the general case was constructed by Ivanov, using small cancellation theory. Later, further counterexamples were obtained by Olshanskii and Storozhev with similar methods. The study of concise words progressed anyway, both by seeking new words that are concise in all groups, and by studying the same problem in other classes of groups. As finitely generated linear groups are residually finite, the natural candidate for the biggest class of groups in which all words are concise is the class of residually finite groups. It is interesting to notice that a word is concise in residually finite groups if and only if it is concise in profinite groups, so another important development has been recently proposed. Every profinite group of cardinality smaller than $2^{\aleph_0}$ is finite, and it was suggested that a similar phenomenon happens for word values too, leading to the conjecture that every set of word values with less than $2^{\aleph_0}$ values is finite. Joining this open problem with the conjecture that all words are concise in residually finite groups, it makes sense to define that a word is strongly concise in profinite groups if, whenever it takes less than $2^{\aleph_0}$ values, its (closed) verbal subgroup is finite.

In the first part of this thesis we discuss several contributions by the author to the theory of conciseness problems.

The first contribution concerns the most general version of the problem, which is seeking new concise words in all groups. One of the first class of words that have been proven to be concise by Philip Hall are non-commutator words, that are words not lying in the derived subgroup of the free group generated by the variables. More recently, Delizia, Shumyatsky, Tortora and Tota proved that the same is true for the word $\gamma_2(u_1, u_2)$, where u_1, u_2 are disjoint non-commutator words (i.e. in disjoint sets of variables). This result has been generalized in 2022 by Azevedo and Shumyatsky, who proved that the word $\gamma_3(u_1, u_2, u_3)$, for u_i disjoint non-commutator words, is concise.

In [34], Fernández-Alcober and the author proved that $w(u_1, \dots, u_k)$, with u_i disjoint non-commutator words, is concise in the case w is a lower central word (proving a conjecture of Azevedo and Shumyatsky), and in the case w is a derived word. The arguments involved in the aforementioned article also work, with some small modifications, when w is an outer commutator word, and we therefore fully prove this case, which includes and generalizes the case of lower central and derived words. We actually obtain a stronger property and show that all outer commutator words are concise on normal subgroups, in the sense that whenever the set of values that the word takes on a tuple $\mathbf{N}$ of normal subgroups is finite, then the subgroup they generate is also finite.

These new concise words try to approach the limit between concise and non-concise words. Indeed, there is no general condition for a word not to be concise. The techniques that are used to build up the three counterexamples that are known, by Ivanov, Olshanskii and Storozhev respectively, were developed through Small Cancellation Theory. This area of geometric group theory is based on the idea that, if the relations of a fixed presentation $G = \langle S \mid R \rangle$ of a group satisfy some additional conditions, it is possible to deduce some geometric and algebraic properties of the groups. This is done by looking at diagrams, built using the relations of G, that encode trivial words in the group. The complete construction of the three non-concise words, and of the groups in which these words are not concise, is quite technical. Because of this, we just try to give a glimpse of the general idea involved in Ivanov's result, and then highlight some differences among the three different non-concise words.

We then focus on Olshanskii's counterexample. As Shumyatsky and the author proved in [68], Olshanskii's word, that is not concise in general, is actually concise in residually finite groups. This is the first example of a word that is not concise in all groups but is concise in residually finite groups. After this, we also show that this same word is strongly concise in profinite groups, settling that these problems differ substantially from the classical questions in abstract groups.

Then, the thesis pursues the study of problems in profinite groups, beginning from some results related to strong conciseness. As we remarked, this problem could be split into two different sub-problems: proving that if $|w\{G\}| < 2^{\aleph_0}$ for a word w in a group G, then $w\{G\}$ is finite, and then proving conciseness in residually finite groups for w. For this reason, several results on strongly concise words relied on the additional hypothesis that, if a verbal subgroup of a profinite group is topologically finitely generated, then it can be generated by finitely many word values. We provide an example, with lower central words, that shows that this additional condition is not always satisfied.

We then study strong conciseness for higher order coprime commutators, that are maps strongly resembling group words. They are a useful tool to generate some important characteristic subgroups of profinite groups, like pronilpotent residuals, with an accurately chosen generating set. Similarly to usual words, we can ask whether they are (strongly) concise, in the sense that in any group with finitely many (or less than $2^{\aleph_0}$) coprime commutators, these elements generate a finite subgroup. It was shown by Acciarri, Shumyatsky and Thillaisundaram that higher order coprime commutators are concise in residually finite groups, while Detomi, Morigi and Shumyatsky proved that the basic coprime commutator map γ_2^* is strongly concise. In a joint work with de las Heras and Shumyatsky, the author proved in [39] that higher order coprime commutators γ_k^* and δ_k^* are strongly concise in profinite groups, and we provide a full detailed proof of these results.

In the second part of the thesis, we initiate the study of profinite right angled Artin groups. Abstract right angled Artin groups (RAAGs) are finitely generated groups whose only relations are commutators in the generators. These groups have a finite graph associated to their presentation, and they include, among others, free groups, free abelian groups and free or direct products of them.

The central idea in geometric group theory is to study groups via actions on spaces. For example, free action of groups on a space should provide a connection between the geometry of the space and the algebra of the group. This is the case with actions on trees: a group acts freely if and only if the group is free. If we do not require the action to be free, Bass-Serre theory gives a description of the structure of groups acting on trees through HNN extensions and amalgamated products.

If, rather than on a single tree, we require our group G to act on a direct product of two trees, then the situation is different. Indeed Burger and Mozes constructed infinite simple groups acting freely and cocompactly on them. However, Bridson, Howie, Miller and Short proved that if we require some additional residual properties, then such a group G is virtually a direct product of free groups. These results were generalised by Haglund and Wise who proved that groups acting freely, and with some additional conditions, on CAT(0) cube complexes are subgroups of RAAGs.

As profinite groups satisfy good residual properties, one can asked if no further conditions are required in this setting, namely a profinite group acts on a direct product of two profinite trees (or, even more ambitiously, on a profinite cubing) if and only if it is virtually a subgroup of a profinite RAAG. In order to approach this line of research, we must first study systematically profinite RAAGs. For a generic pseudovariety $\mathcal{C}$ of finite groups, pro-$\mathcal{C}$ RAAGs are the pro-$\mathcal{C}$ completion of abstract RAAGs and have been studied by Wilkes, Kropholler, Snopce and Zalesskii.

In accordance to the contents of the article [16], joint with Casals-Ruiz and Zalesskii and currently in preparation, we study pro-$\mathcal{C}$ RAAGs using profinite Bass-Serre theory as the main tool. This theory is an analogue of the abstract one developed mainly by Mel'nikov, Ribes and Zalesskii. We use these methods to obtain standard properties of pro-$\mathcal{C}$ RAAGs, like the structure of their centralizers, studying a Tits alternative for their subgroups, and characterizing 2-generated subgroups of pro-p RAAGs.

We then describe some properties of a pro-$\mathcal{C}$ RAAG that are immediately detectable by studying their underlying graph. For example, Krophopller and Wilkes already observed that a profinite RAAG splits as a free product if and only if the underlying graph is disconnected. We prove that pro-$\mathcal{C}$ RAAGs are directly decomposable if and only if their underlying graph is a join, and we then obtain a characterization of their splittings, as pro-$\mathcal{C}$ amalgams or HNN extensions, over abelian subgroups.

We then continue the investigation of their abelian splittings by defining JSJ decompositions. These constructions are a description of all the ways a group G can split over a certain class $\mathcal{A}$ of subgroups, and they can be either general (so $\mathcal{A}$-JSJ decompositions) or relative to another class $\mathcal{H}$ of subgroups (the so-called $(\mathcal{A}, \mathcal{H})$-JSJ decompositions), in the sense that we require all the subgroups of G in the class $\mathcal{H}$ to be elliptic.

We give a constructive proof of the existence of the $(\mathcal{A}, \mathcal{H})$-JSJ decomposition of a pro-$\mathcal{C}$ RAAG G choosing $\mathcal{A}$ to be the class of abelian subgroups, and with the assumption that canonical generators of G act elliptically. We then conclude by obtaining the general $\mathcal{A}$-JSJ decomposition of the pro-$\mathcal{C}$ RAAG G.

Structure of the Thesis

In **Chapter 1** we give an overview of the known theory of conciseness, giving a considerable importance to the historical development of the theory.

In **Chapter 2** we prove that outer commutator words are concise on normal subgroups. This will be obtained first in the case of lower central words, and we will then approach the general proof by giving an explicit description for $w = \delta_2$, and then concluding with the proof of the general case.

Chapter 3 will be devoted to the description of the counterexamples on conciseness, and then to the proof that Olshanskii's word is boundedly concise in residually finite groups and strongly concise in profinite groups. We conclude the chapter giving an example of a profinite group with procyclic derived subgroup, but whose subgroup cannot be generated by finitely many commutators.

In **Chapter 4** we prove that higher order coprime commutators γ_k^* and δ_k^* are strongly concise in profinite groups.

In **Chapter 5**, after an overview of profinite Bass-Serre theory, we focus on proving basic properties of profinite RAAGs, like the structure of their centralizers, and on characterizing their abelian splittings.

We conclude the investigation of their abelian splittings in **Chapter 6**, where we explicitly construct their general and relative abelian JSJ decompositions.

Resumen de la tesis en castellano

Aunque el origen de la teoría de grupos suele atribuirse a los trabajos de Galois, Jordan y Klein, todos estos trabajos estuvieron motivados por la conexión que esta disciplina tiene con la teoría de números o con la geometría. La teoría de grupos abstractos discretos obtuvo un interés independiente, sin inspiración geométrica, principalmente a principios del siglo XX, y un hito para ello se debe a los trabajos de William Burnside. En 1902 él preguntó si un grupo de torsión finitamente generado es necesariamente finito [15], actualmente nos referimos a esta cuestión como el "Problema de Burnside". Este trabajo despertó el interés por problemas aún más profundos, como el estudio de la finitud de los grupos finitamente generados de exponente finito, también llamado "Problema de Burnside acotado". Explícitamente, Grün [36] se preguntó si un grupo G finitamente generado que satisface $g^n = 1$ para todo $g \in G$ es necesariamente finito.

Podríamos observar que este problema se puede encuadrar en el contexto más amplio de una de las preguntas más naturales que se pueden hacer sobre una estructura algebraica, que es "¿Qué podemos decir sobre un grupo si este grupo sigue una regla fija?"

Por supuesto, la pregunta es extremadamente heurística, pero podemos ver muchos de los primeros resultados en teoría de grupos a través de este enfoque, que puede ser interpretado como un ejemplo de un problema de palabras en grupos.

Una palabra de grupo w es una concatenación finita de variables y de sus inversas, que puede verse como un elemento del grupo libre generado por n variables

$x_1, \ldots, x_n$. Para cualquier grupo G, la palabra w define naturalmente una aplicación de G^n a G, simplemente sustituyendo los elementos del grupo en las variables de todas las formas posibles. La imagen de esta aplicación es el conjunto de valores de la palabra en G, normalmente denotado por $w\{G\}$, y el subgrupo $w(G)$ que generan se llama subgrupo verbal. De especial interés es el estudio de las variedades de grupos, que son las clases de grupos en las que una determinada palabra w es una ley, en el sentido de que toma sólo el valor trivial.

Las "reglas"mencionadas en la pregunta de Grün son simplemente leyes en el grupo, así que en términos modernos la cuestión es cómo estudiar la variedad de grupos generada por la ley x^n. Otros problemas que pueden verse desde esta óptica son el estudio de los grupos abelianos, nilpotentes y resolubles de clase acotada, que son las variedades generadas por la palabra conmutador $[x_1, x_2]$, por una palabra central inferior o por una palabra derivada respectivamente.

En lugar de estudiar sólo los grupos en los que una palabra w es una ley, también podríamos preguntarnos si el hecho de que w tome un número finito de valores en un grupo G tiene alguna implicación en la estructura de G. Es fácil darse cuenta de que cualquier grupo con un número finito de conmutadores es finito-por-abeliano, o, en otras palabras, si el conjunto de valores de γ_2 es finito en un grupo G, entonces el subgrupo verbal correspondiente es finito. Philip Hall se dio cuenta de que lo mismo es cierto para todas las palabras potencia x^n, y las centrales inferiores γ_k, no sólo para γ_2. Como consecuencia, Hall conjeturó que para cualquier palabra de grupo, si el conjunto $w\{G\}$ de valores en un cierto grupo G es finito, entonces el subgrupo verbal $w(G)$ también es finito. Si una palabra satisface esta propiedad para cualquier grupo G, se llama concisa y, si lo hace para todos los grupos de una clase dada $\mathcal{C}$, se dice que es concisa en $\mathcal{C}$.

Se ha demostrado que muchas palabras son concisas, además Merzljakov demostró que todas las palabras son concisas en grupos lineales, pero Ivanov construyó un contraejemplo para el caso general utilizando la Teoría de la cancelación pequeña. Más tarde, Olshanskii y Storozhev obtuvieron otros contraejemplos con métodos similares. El estudio de las palabras concisas progresó de todos modos, tanto buscando nuevas palabras que fueran concisas en todos los grupos, como estudiando el mismo problema en otras clases de grupos. Como los grupos lineales finitamente generados son residualmente finitos, el candidato natural para la mayor clase de grupos en los que todas las palabras son concisas es la clase de los grupos residualmente finitos. Es interesante observar que una palabra es concisa en grupos residualmente finitos si y sólo si es concisa en grupos profinitos, por lo que recientemente se ha propuesto otro avance importante.

Cada grupo profinito de cardinalidad menor que $2^{\aleph_0}$ es finito, y se sugirió que un fenómeno similar ocurre también para los valores de las palabras, llevando a la conjetura de que cada conjunto de valores de palabras con menos de $2^{\aleph_0}$ valores es finito. Uniendo este problema abierto con la conjetura de que todas las palabras son concisas en grupos residualmente finitos, tiene sentido definir que una palabra es fuertemente concisa en grupos profinitos si, siempre que tome menos de $2^{\aleph_0}$ valores, su subgrupo (cerrado) verbal es finito.

En la primera parte de esta tesis se discuten varias contribuciones que el autor ha aportado a la teoría de los problemas de concisión.

La primera contribución se refiere a la versión más general del problema, que consiste en buscar nuevas palabras concisas en todos los grupos. Una de las primeras clases de palabras que Philip Hall demostró que son concisas son las palabras no conmutadoras, es decir, las palabras que no se encuentran en el subgrupo derivado del grupo libre generado por las variables. Más recientemente, Delizia, Shumyatsky, Tortora y Tota demostraron que lo mismo es cierto para la palabra $\gamma_2(u_1, u_2)$, donde u_1, u_2 son palabras no conmutadoras disjuntas (es decir, en conjuntos disjuntos de variables). Este resultado fue generalizado en 2022 por Azevedo y Shumyatsky, quienes demostraron que la palabra $\gamma_3(u_1, u_2, u_3)$, para u_i palabras no conmutadoras disjuntas, es concisa.

En [34], Fernández-Alcober y el autor demostraron que $w(u_1, \ldots, u_k)$, con u_i palabras no conmutadoras disjuntas, es concisa en el caso de que w sea una palabra central inferior (demostrando una conjetura de Azevedo y Shumyatsky), y en el caso de que w sea una palabra derivada. Los argumentos del artículo mencionado también funcionan, con algunas pequeñas modificaciones, cuando w es un conmutador externo, por lo que probamos completamente este caso, que incluye y generaliza el caso de las palabras centrales inferiores y derivadas. En realidad obtenemos una propiedad más fuerte y demostramos que todos los conmutadores externos son concisos en subgrupos normales, en el sentido de que siempre que el conjunto de valores que toma la palabra en una tupla $\mathbf{N}$ de subgrupos normales sea finito, entonces el subgrupo que generan también lo es.

Estas nuevas palabras concisas intentan acercarse al límite entre las palabras concisas y las que no lo son. De hecho, actualmente se desconocen condiciones generales para que una palabra no sea concisa. Las técnicas utilizadas para construir los tres contraejemplos conocidos, el de Ivanov, de Olshanskii y de Storozhev respectivamente, han sido desarrolladas dentro de la Teoría de la cancelación pequeña. Esta área de la teoría geométrica de grupos se basa en la idea de que, si las relaciones de una presentación fija $G = \langle S \mid R \rangle$ de un grupo satisfacen algunas condiciones adicionales, es posible deducir algunas propiedades geométricas y algebraicas de los grupos. Esto se consigue observando diagramas, construidos utilizando las relaciones de G, que representan elementos triviales en el grupo. La construcción completa de las tres palabras no concisas, y de los grupos en los que estas palabras no son concisas, es bastante técnica. Por ello, sólo trataremos de dar una idea general del resultado de Ivanov y, a continuación, destacaremos algunas diferencias entre las tres palabras no concisas.

A continuación, nos centramos en el contraejemplo de Olshanskii. Como demostraron Shumyatsky y el autor en [68], la palabra de Olshanskii, que no es concisa en general, es en realidad concisa en grupos residualmente finitos. Este es el primer ejemplo de una palabra que no es concisa en todos los grupos, pero es concisa en grupos residualmente finitos. Luego mostramos también que esta misma palabra es fuertemente concisa en grupos profinitos, estableciendo que estos problemas difieren sustancialmente de las cuestiones clásicas en grupos abstractos.

La tesis prosigue con el estudio de problemas en grupos profinitos, partiendo de algunos resultados relacionados con la concisión fuerte. Como comentamos, este problema podría dividirse en dos subproblemas diferentes: probar que si $|w\{G\}| < 2^{\aleph_0}$ para una palabra w en un grupo G, entonces $w\{G\}$ es finito, y luego probar la concisión en grupos residualmente finitos para w. Por esta razón, varios resultados sobre palabras fuertemente concisas se basaban en la hipótesis adicional de que, si un subgrupo verbal de un grupo profinito es topológicamente finitamente generado, entonces puede ser generado por un número finito de valores de la palabra. Aportamos un ejemplo, con palabras centrales inferiores, que muestra que esta condición adicional no siempre se cumple.

A continuación, estudiamos la concisión fuerte para conmutadores coprimos de orden superior, que son aplicaciones muy similares a las palabras de grupo. Son una herramienta útil para generar algunos subgrupos característicos importantes de los grupos profinitos, como los residuales pronilpotentes, con un conjunto generador elegido con cuidado. De forma similar a las palabras usuales, podemos preguntarnos si son (fuertemente) concisas, en el sentido de que en cualquier grupo con un número finito (o menor que $2^{\aleph_0}$) de conmutadores coprimos, estos elementos generan un subgrupo finito. Acciarri, Shumyatsky y Thillaisundaram demostraron que los conmutadores coprimos de orden superior son concisos en grupos residualmente finitos, mientras que Detomi, Morigi y Shumyatsky demostraron que el conmutador coprimo básico γ_2^* es fuertemente conciso. En un trabajo conjunto con de las Heras y Shumyatsky, el autor demostró en [39] que los conmutadores coprimos de orden superior γ_k^* y δ_k^* son fuertemente concisos en grupos profinitos, y nosotros proporcionamos una demostración detallada completa de estos resultados.

En la segunda parte de la tesis, iniciamos el estudio de los grupos de Artin de ángulos rectos profinitos. Los grupos abstractos de Artin de ángulos rectos (RAAGs) son grupos finitamente generados cuyas únicas relaciones son conmutadores en los generadores. Estos grupos tienen un grafo finito asociado a su presentación, e incluyen, entre otros, los grupos libres, los grupos abelianos libres y los productos libres o directos de ellos.

La idea central de la teoría geométrica de grupos es estudiar los grupos mediante acciones en espacios. Por ejemplo, la acción libre de grupos en un espacio debería proporcionar una conexión entre la geometría del espacio y el álgebra del grupo. Éste es el caso de las acciones en árboles: un grupo actúa libremente en un árbol si y sólo si el grupo es libre. Si no exigimos que la acción sea libre, la teoría de Bass-Serre proporciona una descripción de la estructura de los grupos que actúan en árboles en términos de extensiones HNN y productos amalgamados.

En lugar de en un único árbol, si requerimos que nuestro grupo G actúe en un producto directo de dos árboles, entonces la situación es diferente. En efecto, Burger y Mozes construyeron grupos simples infinitos que actúan libre y cocompactamente en ellos. Sin embargo, Bridson, Howie, Miller y Short demostraron que si exigimos algunas propiedades residuales adicionales, entonces tal grupo G es virtualmente un producto directo de grupos libres. Estos resultados fueron generalizados por Haglund y Wise, quienes demostraron que los grupos que actúan libremente, y con algunas condiciones adicionales, en complejos cúbicos CAT(0) son subgrupos de los RAAG.

Como los grupos profinitos satisfacen buenas propiedades residuales, cabe preguntarse si no se requieren más condiciones en este contexto, a saber, que un grupo profinito actúa en un producto directo de dos árboles profinitos (o, aún más ambicioso, en una cubicación profinita) si y sólo si es virtualmente un subgrupo de un RAAG profinito. Para abordar esta línea de investigación, primero debemos estudiar sistemáticamente los RAAG profinitos. Para una pseudovariedad genérica $\mathcal{C}$ de grupos finitos, los RAAG pro-$\mathcal{C}$ son la compleción pro-$\mathcal{C}$ de los RAAG abstractos y han sido estudiados por Wilkes, Kropholler, Snopce y Zalesskii.

De acuerdo con el contenido del artículo [16], conjunto con Casals-Ruiz y Zalesskii y actualmente en preparación, estudiamos RAAGs pro-$\mathcal{C}$ utilizando la teoría profinita de Bass-Serre como herramienta principal. Esta teoría es un análogo de la abstracta desarrollada principalmente por Mel'nikov, Ribes y Zalesskii. Utilizaremos estos métodos para obtener propiedades estándar de los RAAGs pro-$\mathcal{C}$, como la estructura de sus centralizadores, estudiando una alternativa de Tits para sus subgrupos, y caracterizando subgrupos 2-generados de RAAGs pro-p.

A continuación, describiremos algunas propiedades de un RAAG pro-$\mathcal{C}$ que se pueden detectar inmediatamente a partir de su grafo subyacente. Por ejemplo, Krophopller y Wilkes ya observaron que un RAAG profinito se descompone como producto libre si y sólo si el grafo subyacente es disconexo. De manera dual, demostraremos que un RAAG pro-$\mathcal{C}$ se descompone como producto directo si y sólo si su grafo subyacente es una suma de grafos, y a continuación obtendremos una caracterización de sus decomposiciones, como amalgamas pro-$\mathcal{C}$ o extensiones HNN, sobre subgrupos abelianos.

Posteriormente, continuamos con la investigación de las decomposiciones abelianas de un RAAG pro-$\mathcal{C}$, esta vez en el contexto de las decomposiciones JSJ. Estas construcciones son una descripción de todas las formas en que un grupo G puede decomponerse sobre una cierta clase $\mathcal{A}$ de subgrupos, y pueden ser generales (por tanto descomposiciones $\mathcal{A}$-JSJ) o relativas a otra clase $\mathcal{H}$ de subgrupos (las llamadas descomposiciones $(\mathcal{A}, \mathcal{H})$-JSJ), en el sentido de que requerimos que todos los subgrupos de G en la clase $\mathcal{H}$ sean elípticos.

Daremos una prueba constructiva de la existencia de la decomposición $(\mathcal{A}, \mathcal{H})$-JSJ de un RAAG G pro-$\mathcal{C}$ eligiendo $\mathcal{A}$ como la clase de subgrupos abelianos, y con el supuesto de que los generadores canónicos de G actúen elípticamente. Concluiremos obteniendo la descomposición general $\mathcal{A}$-JSJ del pro-$\mathcal{C}$ RAAG G.

1
Problems on group words

In this chapter we set the foundations of the theory of concise words.

Initially we give the basic definitions of word maps and verbal subgroups. We then describe varieties of groups, that are one of the main motivations driving the development of word problems in groups.

In Section 3, we give the formulation of three conjectures of Philip Hall. We briefly analyse the partial answer to the first two of them, and then we discuss the follow-up of the third problem in the fourth section. Indeed, the last question of Hall consisted in proving that, if a word w takes finitely many values in a group G, the associated verbal subgroup is finite. A word satisfying this is said to be concise. We describe the partial positive answers and then mention the counterexamples to Hall conjecture.

In Section 5, we describe the more recent driving areas in conciseness, namely the study of words that are concise in residually finite groups. A further investigation is due to the conjecture that every word w is strongly concise in profinite groups, meaning that whenever the set of w-values has less than $2^{\aleph_0}$ elements, then the verbal subgroup is finite.

In Section 6 we glide over all the results of conciseness, addressing in which threads of investigation the mathematical community was able to make improvements, and then conclude with a summary of the best results obtained so far in each direction.

1.1 Words and verbal subgroups

Consider the free group $F(X_\infty)$ of countable rank over the set $X_\infty = \{x_i \mid i \in \mathbb{N}\}$. We will say that any element of this set is a *group word*. Of course any such element can be written using finitely many variables, so we will denote a generic element by $w(x_1, \dots, x_n)$ where $n \in \mathbb{N}$ is the number of indeterminates involved in the word (as up to reordering we can always assume that, if $w \in F(X_\infty)$, then in w appear exactly the variables $x_1, \dots x_n$).

Fix now an arbitrary abstract group G. We can associate to w a valuation map on G obtained by substituting a tuple $\mathbf{g} = (g_1, \dots, g_n)$ of elements of G for the indeterminates $x_1, \dots, x_n$, explicitly

$$\begin{array}{rccc} \nu_w : & G^n & \to & G \\ & \mathbf{g} = (g_1, \dots, g_n) & \mapsto & w(\mathbf{g}) \end{array}$$

Definition 1.1 (Word values and verbal subgroup)**.** The set $w\{G\} = \{w(\mathbf{g}) \mid \mathbf{g} \in G^n\}$ is the set of *w-values*.

The subgroup $w(G) = \langle w\{G\} \rangle$ is the *verbal subgroup* of w.

Obviously in general the set of word values is not a subgroup. As for each homomorphism $\phi : G \to H$ we have that $\phi(w(g_1, \dots, g_n)) = w(\phi(g_1), \dots \phi(g_n))$, we immediately get that $w\{G\}N/N = w\{G/N\}$ and therefore $w(G)N/N = w(G/N)$, and also that verbal subgroups are fully invariant subgroups, and in particular they are characteristic and normal.

Example 1.2.

- For each integer $n \in \mathbb{Z}$ we can consider the *power word* $w(x_1) = x_1^n$. The set $w\{G\}$ is the set of elements of G that are n-th powers, and we will denote the corresponding verbal subgroup as G^n.
- The *commutator word* $w(x_1, x_2) = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. The verbal subgroup $w(G)$ is the derived subgroup G'.
- The *lower central words* (or *simple commutators*) γ_k in k variables, defined inductively as $\gamma_1 = x_1$, $\gamma_k = [\gamma_{k-1}, x_k]$. The corresponding verbal subgroups are the subgroups of the lower central series of G.
- The *derived words* δ_k in 2^k variables, defined as $\delta_0 = x_1$ and

  $$\delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})].$$

  The corresponding verbal subgroups are the subgroups of the derived series.

- The *outer commutator words*, defined recursively. Any single variable $x_i \in X_\infty$ is an outer commutator, and any commutator $[u, v]$ where u and v are outer commutators in disjoint sets of variables, is an outer commutator. This class includes γ_k, δ_k and several other words, like $[[x_1, [x_2, x_3]], [x_4, x_5]]$.

- The *Engel words* e_n, $n \in \mathbb{N}$, defined recursively as $e_1 = [x_1, x_2]$ and $e_n = [e_{n-1}, x_2]$.

Together with the verbal subgroup, there is another important subgroup connected to it, the one consisting of the elements that can be freely removed from the word.

Definition 1.3 (Marginal subgroup)**.** The *marginal subgroup* of a word w in a group G is defined as

$$w^*(G) = \big\{a \in G \,| w(g_1, \ldots, g_i, \ldots, g_n) = w(g_1, \ldots, ag_i, \ldots, g_n)$$
$$\text{for all } g_i \in G, i = 1, \ldots, n\big\}.$$

Example 1.4. Let $w = [x_1, x_2]$. Any element in the center $Z(G)$ is obviously in the marginal subgroup, moreover if $y \in w^*(G)$, for every $g \in G$ we have $[y, g] = [1, g] = 1$, so $y \in Z(G)$.

1.2 Varieties and relatively free groups

If $\mathcal{V}$ is a set, possibly infinite, of group words, then we will denote by $\mathcal{V}(G)$ the subgroup $\langle v(G) \mid v \in \mathcal{V}\rangle$. We will say that a word v is a *law* in a group G if $v(G) = 1$, and extend this definition naturally to sets of groups words.

Definition 1.5 (Closed sets of words)**.** A set of words $\mathcal{V} \subseteq F(X_\infty)$ is *closed* if and only if it is closed by inverses, products and if for every $v \in \mathcal{V}$ word in n variables and any tuple $\mathbf{u} = (u_1, \ldots, u_n) \in X_\infty^n$ we have $v(\mathbf{u}) \in \mathcal{V}$.

For each set of words $\mathcal{V}$, we can define its closure $\overline{\mathcal{V}}$ as the smallest closed set of words containing it. It is possible to see that $\mathcal{V}(G) = \overline{\mathcal{V}}(G)$ for any group G, or in other words the verbal subgroup defined by $\mathcal{V}$ is the same as the one defined by its closure.

Definition 1.6 (Varieties)**.** The class of groups satisfying a certain set of laws $\mathcal{V}$ is called the *variety* defined by $\mathcal{V}$.

By the previous discussions, each variety corresponds to a single closed set of laws. Let F_n and F_∞ be the free groups on n and countably many generators respectively. In each variety there is an infinitely generated group that satisfies exactly all the laws of the closed set $\overline{\mathcal{V}}$, which is precisely $F_\infty/\mathcal{V}(F_\infty)$, and all groups of the variety defined by $\mathcal{V}$ are quotients of this group. In general, every d-generated group in the variety generated by $\mathcal{V}$ is a quotient of $F_d/\mathcal{V}(F_d)$. We will say that these groups are the *relatively free* groups in the variety generated by $\mathcal{V}$. They all have a generating set S such that every mapping of this set into the group itself can be extended to a homomorphism.

Even if, for a generic group, not every fully invariant subgroup is verbal, this is true for relatively free groups.

Theorem 1.7 (Theorem 13.31 [58])**.** *Every fully invariant subgroup of a relatively free group G is of the form $\mathcal{W}(G)$ for a (possibly infinite) set $\mathcal{W}$ of words.*

An important theorem of Birkhoff characterizes precisely which classes of groups are varieties.

Theorem 1.8 (Birkhoff)**.** *A class of groups is a variety if and only if it is closed by subgroups, quotients and (unrestricted) cartesian products.*

Example 1.9.

- The variety generated by $w(x_1, x_2) = [x_1, x_2]$ is the variety of abelian groups. The relatively free d-generated group of this variety is $F_d/F_d' \cong \mathbb{Z}^d$.

- The variety generated by $w(x) = x^n$ is the Burnside variety. The d-generated relatively free group in this variety is denoted by $\mathcal{B}(d, n)$. The Bounded Burnside problem asked for which integers n the *free Burnside group* $\mathcal{B}(d, n)$ is finite. These groups are finite for $n = 2, 3, 4, 6$ ($n = 2$ is an easy exercise, the other results are of Burnside, Sanov and M. Hall respectively). Novikov and Adian proved in 1968 [61][62][63] that for big odd integers n and $d \geq 2$ these groups are infinite. The best bound that guarantees the infiniteness of free Burnside groups of odd exponents is $n > 665$ [6]. The infiniteness of $\mathcal{B}(d, n)$ for even numbers $n > 2^{48}$ was proved by Ivanov [44], with an improvement to $n > 8000$ obtained by Lysenok [55].

- With his solution of the Restricted Burnside Problem, Zelmanov proved in [86][87] that the locally finite groups of finite exponent form a variety, and in particular for each positive integer d, n each finite d-generated group of exponent n is a quotient of of the finite relatively free group $\mathcal{Z}(d, n)$.

Of course this variety satisfies the law $w(x) = x^n$, but it is unclear which additional laws have to be added in order to restrict the Burnside variety to locally finite groups.

The version of small cancellation theory that was developed by Olshanskii was constructed to obtain a more accessible proof of the aforementioned Novikov-Adian theorem. This proof is much more accessible, but settles the problem only for odd $n > 10^{10}$.

1.3 On three questions of P.Hall

A classical result that Schur proved in 1904 is the following:

Theorem 1.10 (Schur)**.** *If $[G : Z(G)] = m$, then G' is finite and of exponent dividing m.*

The converse of this theorem is not true, as we can consider a countable product of the quaternion group Q_8 and amalgamating all the centers. In this group the commutator subgroup has order two, but the center has infinite index.

All the counterexamples to the converse of Schur's Theorem require the group G to be infinitely generated. Indeed, if $G = \langle g_1, \dots, g_k \rangle$ and the set of commutators $\gamma_2\{G\}$ is finite, the index $[G : C_G(g_i)]$ is finite (as the set of right cosets of $C_G(g_i)$ is in bijection with the set $\{[g, g_i] \mid g \in G\}$). This implies that the center $Z(G) = \bigcap_{i=1}^{k} C_G(g_i)$ has finite index in G too.

In the previous paragraph, in order to prove that $[G : Z(G)] < \infty$, we have not used the finiteness of the whole subgroup G', but only the finiteness of the set of commutators $\gamma_2\{G\}$. If we then apply Schur's Theorem we have actually proved that the finiteness of $\gamma_2\{G\}$ implies the finiteness of $\gamma_2(G) = G'$. Notice that we could reach the same conclusion, under the hypothesis of $|\gamma_2\{G\}| < \infty$, even if G is not finitely generated. Indeed we could find a finite set S of elements of G such that $\gamma_2\{G\} = \{[s_1, s_2] \mid s_1, s_2 \in S\}$, then apply the previous reasoning to $\langle S \rangle$ and obtain that $G' = \langle S \rangle'$ is finite.

In the 50's Philip Hall asked whether all the interlacing among finiteness of $w\{G\}, w(G)$ and $[G : w^*(G)]$ is valid for every group word w rather than only for $w = \gamma_2$.

(Q1) If $[G : w^*(G)]$ is finite and a π-number, is $|w(G)|$ finite and a π-number?

(Q2) If $w(G)$ is finite and G satisfies the maximal condition on subgroups, is $[G : w^*(G)]$ finite?

(Q3) If $w\{G\}$ is finite, is $w(G)$ finite too?

We recall that a group satisfies the maximal condition on subgroups if and only if every ascending chain of subgroups stabilizes or, equivalently, every subgroup is finitely generated.

The previous questions appeared for the first time in the article [80] of Turner-Smith, but were all attributed to Philip Hall. As we have already discussed, if $w = \gamma_2$ all these questions have a positive answer.

The answer to Question 1 is positive for outer commutator words (Baer [10]). Moreover Stroud [78] proved that the same is true if w is a word such that every group in which w is a law is locally residually finite.

In [81], Turner-Smith proved that the answer to Question 2 is also positive whenever w is an outer commutator word. He attributed this result to P. Hall again, but also proved a slightly stronger version of this statement for outer commutator words.

It is interesting to point out that the answer to Question 2 is negative if we remove the requirement of G satisfying the maximal condition on subgroups. One example are all infinite extraspecial groups, like the one obtained by amalgamating the centers of an infinite product of copies of Q_8.

If w is not an outer commutator word, the answer to both questions is negative. Indeed Kleiman [49] constructed a word w and a group G with $[G : w^*(G)] = p^2$ for a prime $p \neq 2$, but such that $|w(G)| = 2$. We could anyway still ask whether the finiteness of $[G : w^*(G)]$ implies the finiteness of $w(G)$. Of course we have that if w is a word in n variables and $[G : w^*(G)] = m < \infty$, then $|G_w| \leq m^n$, but the finiteness of the verbal subgroup has yet to be proved or disproved.

A counterexample to Question 2 was provided by Ashmanov and Olshanskii in [7]. It is important to notice that both the counterexample of Kleiman and the counterexample of Ashmanov and Olshanskii are obtained through the version of Small Cancellation Theory developed by Olshanskii. We will talk in further detail about this theory in Chapter 3.

The first two questions of Hall have not had a huge follow-up, but the third question led to the development of a whole theory in order to tackle the problem.

1.4 Conciseness

The third problem of Philip Hall has been intensively studied, giving rise to several methods in order to tackle problems of conciseness of words.

Definition 1.11 (Concise words)**.** A group word w is *concise word*concise if $w(G)$ is finite for every group G such that $w\{G\}$ is finite.

We begin with some general reductions. The following result is classical, but was first explicitly given in a quantitative way in Lemma 4 of [26]. We will anyway provide a proof, due to the importance of this lemma.

Lemma 1.12. *Let w be a group word and G be a group such that $|w\{G\}| \leq m$. Then $|w(G)'| \leq f(m)$ for a function $f : \mathbb{N} \to \mathbb{N}$.*

Proof. The set $w\{G\}$ is closed by conjugation, therefore for each $g \in w\{G\}$ we have $|G : C_G(g)| = |g^G| \leq m$. This implies that $|G : C_G(w(G))| \leq m^m$, so by Schur's Theorem $w(G)$ is finite and of exponent dividing m^m. Notice that $w(G)$ is generated by the finite normal set $\big\{[h_1, h_2] \mid h_1, h_2 \in w\{G\}\big\}$, so $w(G)'$ is a subgroup generated by at most m^2 elements and of exponent m^m. By Dietzman's Lemma (Lemma 14.5.7 of [73], the proof gives a bound) there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $|w(G)'| \leq f(m)$. □

This basic result implies that, whenever we are trying to study if a group word w is concise, we can always assume $w(G)$ to be abelian, and therefore just study whether $w(G)$ is periodic. An immediate application of this is the following Lemma, which Turner-Smith attributes to P. Hall in [80], of which we will give a proof because some of the ideas involved in this short proof are commonly used in more recent results. We will say that a word $w \in F(X_\infty)$ is a *non-commutator* word if $w \notin F(X_\infty)'$.

Lemma 1.13. *Every non-commutator word is concise.*

Proof. Let w be a non-commutator word in n variables $x_1, \ldots, x_n$. By applying the classical commutator calculus formula $ab = ba[a, b]$, we can rewrite w as

$$w(x_1, \ldots, x_n) = x_1^{e_1} \cdots x_n^{e_n} v(x_1, \ldots, x_n)$$

with $v(x_1, \ldots, x_n) \in F(X_\infty)'$. As w is a non-commutator word, there exists at least an index $i \in \{1, \ldots, n\}$ such that $e_i \neq 0$. By reordering the indices, we can assume $i = 1$.

Let G be a group with $|w\{G\}| \leq \infty$ and choose $g \in G$. By substituting g to x_1 and the identity to each x_i, $i = 2, \ldots, n$, we have that $w(g, 1 \ldots, 1) = g^{e_1}$ and in particular the set $\{g^{e_1} \mid g \in G\}$ is finite. This implies that G is a group of finite exponent, hence the abelian quotient $w(G)/w(G)'$ is finite, and by Lemma 1.12 this is sufficient to conclude. □

We can therefore restrict our search to commutator words. In [81], Turner-Smith proved that lower central words γ_k are concise (this was already known to P. Hall), moreover he extended the result to derived words δ_k, but the arguments in this case are already more advanced. For several years the problem was untouched, until Wilson proved in [82] that all outer commutator words are concise.

The dreams of obtaining an affirmative answer to conciseness problems were shattered by a counterexample, obtained by Ivanov in 1989 [43]. We will give some ideas of the construction of this counterexample in Section 3.2.

Still, many more words have been proved to be concise. Even further, many words have been proved to be *boundedly concise*.

Definition 1.14 (Boundedly concise words)**.** A word w is *boundedly concise* in a class $\mathcal{C}$ of groups if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, if there is a group $G \in \mathcal{C}$ with $|w\{G\}| \leq m$, then $|w(G)| \leq f(m)$.

In 2009, Fernández-Alcober and Morigi obtained a different proof of conciseness of outer commutator words in [31]. In the same article, there are two proofs of the following result, one by the authors and one that was communicated to them by Mann.

Theorem 1.15. *Any word w that is concise is boundedly concise.*

1.5 Conciseness in other classes of groups

The counterexample of Ivanov did not impede pursuing better and further results on conciseness. In particular, a huge development of the theory shifted toward proving in which classes of groups every word is concise.

Definition 1.16 (Verbal conciseness)**.** We will say that a class of groups $\mathcal{C}$ is *verbally concise* if, for every group $G \in \mathcal{C}$ and any group word w we have that, if $w\{G\}$ is finite, then $w(G)$ is finite too.

Some classes of groups that are obviously verbally concise are abelian groups (because, if G is abelian, $w(G) = w\{G\}$) or finite groups. Turner-Smith proved that each word is concise in residually finite groups such that all of their quotients are residually finite [81].

The most important open conjecture regarding conciseness is the following. This conjecture was discussed by several authors, but it is usually attributed to either Jaikin-Zapirain or Segal.

Conjecture 1.17. The class of residually finite groups is verbally concise.

Studying conciseness in residually finite groups involves a different machinery compared to the analogous problem in general abstract groups. These additional tools made it possible to prove that some words, which are unknown to be concise or not in general, are actually concise in residually finite groups. Consider, as an example, Engel words, that are defined iteratively as $e_1(x,y) = [x,y]$ and $e_n = [e_{n-1}, y] = [x, y, \overset{n}{\ldots}, y]$. It is known that these words are concise only in the cases of $n \leq 4$ (see [1] [32]), but it is unknown whether they are in general. However, all these words are concise in residually finite groups ([26]).

Any residually finite group embeds in its profinite completion, so it is a natural question whether the study of conciseness in profinite groups can yield an affirmative answer to the previous conjecture. An important remark is that in profinite groups we will denote by $w(G)$ the closure of the abstract subgroup generated by the set $w\{G\}$. In this setting, it is actually possible to prove that it is equivalent to formulate Conjecture 1.17 for profinite groups.

Proposition 1.18. *A word w is concise in all residually finite groups if and only if it is concise in all profinite groups.*

Proof. Let w be a word that is concise in residually finite groups and suppose that $w\{G\}$ is finite in a profinite group G. As G is residually finite, the abstract subgroup generated by $w\{G\}$ is finite too, but finite subsets are closed, and therefore $w(G)$ is finite too.

Suppose now that w is concise in profinite groups and assume that it takes finitely many values in a residually finite group G. Each residually finite groups embeds in its profinite completion $\widehat{G}$. The first step is to prove that w takes finitely many values in $\widehat{G}$. Let $g_1, ..., g_k \in \widehat{G}$. For each $j = 1, \ldots, k$ we can find a net of elements $g_{j,i} \in G$, indexed by a set I, such that $\lim_{i \in I} g_{j,i} = g_j$ and therefore $w(g_1, ..., g_k) = \lim_{i \in I} w(g_{1,i}, ..., g_{k,i}) \in \overline{w\{G\}} = w\{G\}$, where the last equality is true because $w\{G\}$ is finite hence closed. By hypothesis $w(\widehat{G}) \leq \widehat{G}$ is finite and so $w(G)$ is finite too. $\square$

Any profinite group is either finite or uncountable. Detomi, Morigi and Shumyatsky realized that a similar duality could be valid also for word maps. For this reason they conjectured in [25] that any word taking countably many values in a profinite group has a finite verbal subgroup, proving the conjecture for outer commutators and other specific words. An improvement of this was obtained in [24], where the authors managed to avoid the dependence on the continuum hypothesis.

Definition 1.19 (Strongly concise words)**.** A word w is said to be *strongly concise* if, whenever $|w\{G\}| < 2^{\aleph_0}$ in a profinite group G, then $w(G)$ is finite.

Detomi, Klopsch and Shumhyatsky proved that outer commutators and other specific words are indeed strongly concise, leading to a strengthening of Conjecture 1.17.

Conjecture 1.20. Every word is strongly concise.

In view of Theorem 1.15, we could ask whether words that are concise in residually finite groups are also boundedly concise in residually finite groups. This is currently unknown, because one essential tool in the proofs of Fernández-Alcober and Morigi or Mann in [31] was constructing an ultraproduct of groups. We cannot generalize their proof to residually finite groups because the ultraproduct of residually finite groups is not necessarily residually finite. For this reason, this is currently an open problem.

Conjecture 1.21. Every word that is concise in residually finite groups is also boundedly concise in residually finite groups.

1.6 A comprehensive list of known concise words

We will give a comprehensive list of all results regarding conciseness of words.

As already mentioned, the first article that mentioned the problem is by Turner-Smith [80] in 1964, in which he proved that non-commutator words, lower central words and derived words are concise. Wilson proved that all outer commutator words are concise in [82] in 1974, but the proof is already more convoluted. It is important to mention that Fernández-Alcober and Morigi gave a different proof of this last result in [31]. This last proof developed new methods in the study of outer commutator words, by applying proofs by induction on the *height* and *defect* of these words, by representing them as finite trees.

Apart from outer commutator words, the first type of words for which conciseness problems were extensively studied are Engel words. Indeed, in 2011 both Abdollahi and Russo [1] and Fernández-Alcober, Morigi and Traustason [32] proved that Engel words $e_n = [x,_n y]$ are concise for $n \leq 4$. These results rely heavily on the fact that any group in which e_4 is a law is locally nilpotent, whereas it is unknown if the same is true for the general n-Engel word e_n. The proof of Fernández-Alcober, Morigi and Traustason obtains some structural results for groups G such that $e_n\{G\}$ is finite for a certain positive integer n. Indeed, they proved that in this case $[e_n(G), G]$ is a finite subgroup.

Another class of words that was studied are words obtained by nesting non-commutator words into outer commutator words. We will say that some words $u_1, \ldots, u_n$ are *disjoint* if the sets of variables appearing in each of them are pairwise disjoint. In 2019 Delizia, Shumyatsky, Tortora and Tota proved in [22] that the word $[u_1, u_2]$ is concise for u_1, u_2 disjoint non-commutator words. This result was generalized by Azevedo and Shumyatsky in [9] to commutators $[u_1, u_2, u_3]$ for u_1, u_2, u_3 disjoint non-commutator words. In the same article, Azevedo and Shumuyatsky proved that, if $u_1, \ldots, u_k$ are disjoint copies of the same non-commutator word u and v is another non-commutator word disjoint from $u_1, \ldots, u_k$, then both $[u_1, \ldots, u_s]$ and $[v, u_1, \ldots, u_s]$ are concise. Lastly, they proved that if u is an outer commutator word and v is a disjoint non-commutator word, then $[u, v]$ is concise.

In Chapter 2 we will give a full proof of a result that generalizes all of these. In [34], Fernández-Alcober and the author proved a stronger version of a conjecture of Azevedo and Shumyatsky, showing that, whenever $u_1, \ldots, u_k$ are non-commutator words, then the words $\gamma_k(u_1, \ldots, u_k)$ and $\delta_k(u_1, \ldots, u_{2^k})$ are concise.

Theorem 1.15 assures that any word that is concise is also boundedly concise. However, some results proved that some sets of words $\mathcal{W}$ are *uniformly boundedly concise*, which means that for every $w \in \mathcal{W}$ the same function f gives a bound as in Definition 1.14. In [13] Brazil, Krasilnikov and Shumyatsky proved that all lower central words and derived words are uniformly boundedly concise. This result was generalized to all outer commutator words by Fernández-Alcober and Morigi in [31].

Moving towards conciseness in some restricted classes of groups, we already mentioned that Turner-Smith proved that every word is concise in residually finite groups all whose quotients are residually finite. In 1967 an extremely important result of Merzljakov in [57] extended verbal conciseness to the class of groups such that, for each integer $m \in \mathbb{N}$, there exists a finite index normal subgroup $N(m)$ such that $N(m)$ is residually (finite of order coprime to m). This result was used in Merzljakov's article to prove that every finitely generated linear group is verbally concise. In this direction, a recent result of Zozaya [89] proved that the class of compact R-analytic groups is also verbally concise.

In a similar way, there are other classes of groups that are verbally concise simply because no word can take finitely many values, like the class of groups that do not satisfy any law. This class of groups contains for example free groups and, as shown by Abért in [2], Thompson's group F, weakly branch groups or profinite groups with alternating composition factors of unbounded degree. Conciseness for this class of groups follows from this easy lemma.

Lemma 1.22. *If a word w takes finitely many values in a group G, then G satisfies a law.*

Proof. Assume $|w\{G\}| \leq m$ and that w is a word in n variables. Consider $n \times (m+1)$ variables, that we denote by x_i^j, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m+1\}$ and define $w_a = w(x_1^a, \dots, x_n^a)$, $w_{a,b} = w_a^{-1} w_b$. If γ_t is the simple commutator of length $t = m(m-1)/2$, the word

$$\gamma_t(w_{1,2}, \dots, w_{1,m+1}, w_{2,3} \dots, w_{m-1,m})$$

obtained by computing γ_t on all couples $(a, b) \in \{1, \dots, m+1\}^2$ with $a < b$ is a law, because at least two of the w_i, $i \in \{1 \dots, m+1\}$ must be equal. $\square$

We will now discuss conciseness in residually finite and profinite groups.

The first words that were proven to be concise in the class of residually finite groups, but which are not known to be concise in all groups, are words of the type w^q for w an outer commutator word and q a prime power. This was proved by Acciarri and Shumyatsky in [3], where they also showed that if w is a lower central word, then w^q is boundedly concise in residually finite groups.

In 2015 Guralnick and Shumyatsky proved that *weakly rational words* are concise in residually finite groups [38]. A word w is weakly rational if, for all finite groups G and every integer e coprime to $|G|$ the set $w\{G\}$ is closed by taking e-th powers.

Burns and Medvedev in [14] defined that a word w *implies virtual nilpotency* if every finitely generated metabelian group in which w is a law has a nilpotent subgroup of finite index. The authors proved that w implies virtual nilpotency if and only if for all primes p, w is not a law in the wreath product $C_p \wr C_\infty$. Some examples of words that imply virtual nilpotency are uv^{-1} for u, v semigroup words in finitely many generators, Engel words and some generalizations of Engel words.

In a series of two articles [26] and [27], Detomi, Morigi and Shumyatsky proved bounded conciseness in residually finite groups for words implying virtual nilpotency and several words of Engel type $[w,_n y]$ for n positive integer and w an outer commutator word. For $w = \gamma_k^n$ for n positive integer they showed that both $[w,_n y]$ and $[y,_n w]$ are boundedly concise. If w is a prime power of an outer commutator word, they proved that $[w,_n y]$ is concise in residually finite groups, but it is unknown whether it is boundedly concise too. The best result in this direction was recently obtained by Acciarri and Shumyatsky in [4], showing that w and $[w,_n y]$ are concise in residually finite groups for w an arbitrary power of an outer commutator word and y a variable not appearing in w.

A more recent result of Azevedo and Shumyatsky [9] states that whenever u, v are two disjoint words, if u is concise in residually finite groups and v is a non-commutator word, then $[u, v]$ is concise in residually finite groups. Moreover, if u is boundedly concise, then the same is true for $[u, v]$.

In [25] Detomi, Morigi and Shumyatsky proved that if $w\{G\}$ is countable in a profinite group G for $w = x^2$, $w = [x^2, y]$ or w an outer commutator word, then $w(G)$ is finite. All these results were generalized to the case $|w\{G\}| < 2^{\aleph_0}$ by Detomi, Klopsch and Shumyatsky in [24], where they obtained the same result also for the words $w = x^2$, $w = x^3$, $w = x^6$, $w = [x^3, y]$, $w = [x, y, y]$, $w = [x, y, y, z_1, \ldots, z_r]$, $w = [x^2, z_1, \ldots, z_r]$ and $w = [x^3, z_1, \ldots, z_r]$ where x, y, z_i are different variables. In [48], Khukhro and Shumyatsky obtained strong conciseness for all Engel words $w = [x,_n y]$ in finitely generated profinite groups. We can also extend the notion of strong conciseness to some maps that are not word maps, like coprime and anti-coprime commutators. We will discuss these maps in detail in Chapter 4.

We also mention some results on strong conciseness under the additional hypothesis that $w(G)$ is generated by finitely many w-values. In [24] the authors proved that in this case, weakly rational words and words implying virtual nilpotency are strongly concise. Under the same hypothesis, Azevedo and Shumyatsky proved in [8] that $[y,_n v^q]$ and $[v^q,_n, y]$ are strongly concise for $v = \gamma_k(x_1, \ldots, x_k)$, and extended this result to some additional specific words under more conditions.

Overall, Conjectures 1.17 and 1.20 are still widely open, but they have been partially settled for some specific subclasses of profinite groups. Indeed in [23] Detomi proved that every word is strongly concise in virtually nilpotent profinite groups, whereas in [4] Acciarri and Shumuyatsky proved that Conjecture 1.17 reduces to proving conciseness in the class of virtually pro-p groups for an arbitrary prime p.

1.7 Tables of concise words

We conclude the chapter with some tables summarizing the results we described, highlighting only the most general results.

Concise words		
Words	References	Notes
Non-commutators	[81] (P. Hall)	
Outer commutators	[82], [31]	Uniformly concise [31]
Engel words e_n, $n \leq 4$	[1], [32]	$[e_n(G), G]$ is finite for every n [32]
$\gamma_k(u_1, \ldots, u_k)$, $\delta_k(u_1, \ldots, u_{2^k})$ u_i disjoint non-commutators	[34]	

Verbally concise classes of groups	
Class of groups	References
Res. finite with all quotients res. finite	[81]
Linear groups	[57]
Compact R-analytic	[89]
Groups without any law	Lemma 1.22

We also remark that every word is strongly concise in virtually nilpotent profinite groups ([23]).

Words concise in residually finite groups		
Words	References	Notes
w^q, $[w^q,_n y]$ w outer comm., $q \in \mathbb{N}$	[4]	
Weakly rational	[38]	
Words implying virtual nilpotency	[26]	
$[w^q,_n y]$, $[y,_n w^q]$ w outer comm., $q \in \mathbb{N}$	[27]	boundedly concise for $[\gamma_k^q,_n y]$, $[y,_n \gamma_k^q]$
$[u, v]$, u, v disjoint, u concise in res. finite v non-commutator	[9]	boundedly concise if u boundedly concise

We will write (FG) for "$w(G)$ is generated by finitely many w-values".

Strongly concise words		
Words	References	Notes
Outer commutator	[24]	
$w = x^q$, $q = 2, 3, 6$ and some specific words	[24]	
Engel words e_n, $n \in \mathbb{N}$	[48]	For finitely generated profinite groups
Coprime commutators γ_k^*, δ_k^*	[39]	Not group words see Chapter 4
Strongly concise words under additional conditions		
Weakly rational, implying virtual nilpotency	[24]	Condition (FG)
$[y,_n \gamma_k^q]$, $[\gamma_k^q,_n y]$ y, γ_k disjoint, $q \in \mathbb{N}$ and some specific words	[8]	Condition (FG)

2

Conciseness on normal subgroups

In this chapter we describe some contributions to the list of known concise words. Delizia, Shumyatsky, Tortora, and Tota proved in [22] that, if u_1 and u_2 are non-commutator words in disjoint sets of variables, then $[u_1, u_2]$ is concise too. This result has been extended to the case when u_1 is an outer commutator word and u_2 is a non-commutator and to commutators $[u_1, u_2, u_3]$ of non-commutators in [9]. For longer commutators, the only partial result was obtained by Azevedo and Shumyatsky in [9], who proved that if $u_1, \dots, u_k$ are copies of the same non-commutator word in different variables, then $[u_1, \dots, u_k]$ is concise.

Azevedo and Shumyatsky conjectured that, if u_i are non-commutator words in disjoint sets of variables and $w = \gamma_k$, then $w(u_1, \dots, u_k)$ is concise. The aim of this chapter is to prove this conjecture, and moreover to extend it to the case of a generic outer commutator word w. We will roughly follow the article [34] of Fernández-Alcober and the author, where we proved these results for lower central words and derived words.

In the first section we will develop some preliminary lemmas. These will be sufficient to settle the conjecture of Azevedo and Shumyatsky, for $w = \gamma_k$, in the second section. The main idea of the proof is to find a series of verbal subgroups such that each section of this series has some linearity properties. This could be obtained as a corollary of the case of generic outer commutator words, but the proof in this case is more straightforward and easier, so it makes sense to have a section dedicated to it.

The proof for a generic outer commutator word w, however, involves some further technicalities. The focal point of the arguments is producing a series of subgroups similarly to the case $w = \gamma_k$, but this series is more complicated than the lower central words case. In order to illustrate the main steps of the proofs, we will first give an explicit construction of such a series in the case $w = \delta_2$ in Section 3, and we will formally prove the conjecture for a generic w, in Section 4.

2.1 Preliminaries

The aim of this chapter is to prove the following

Theorem 2.1. *Let $w = w(x_1, \dots, x_r)$ be an outer commutator word. If $u_1, \dots, u_r$ are non-commutator words in disjoint sets of variables, then the word $w(u_1, \dots, u_r)$ is concise. In particular, the word $w(x_1^{n_1}, \dots, x_r^{n_r})$ is concise whenever $n_1, \dots, n_r \in \mathbb{Z} \setminus \{0\}$.*

In order to obtain this result, we will need to work on studying the values of an outer commutator words with some additional restrictions on the subgroups where the variables take values from. Similarly to how we defined usual word values in a group G, if $\mathbf{S} = (S_1, \dots, S_r)$ is an r-tuple of subsets of G, we can consider the set of values

$$w\{\mathbf{S}\} = \{w(\mathbf{g}) \mid \mathbf{g} \in S_1 \times \cdots \times S_r\},$$

and the corresponding verbal subgroup on $\mathbf{S}$, namely $w(\mathbf{S}) = \langle w\{\mathbf{S}\}\rangle$. Of special interest is the case when $\mathbf{S}$ is a tuple $\mathbf{N} = (N_1, \dots, N_r)$ of normal subgroups of G. We then say that $w(\mathbf{N})$ is the $\mathbf{N}$-*verbal subgroup* of w and that it is a verbal subgroup on normal subgroups. We can also say that w is *concise on normal subgroups* if $w(\mathbf{N})$ is finite whenever $|w\{\mathbf{N}\}| < \infty$ for any tuple $\mathbf{N}$ of normal subgroup.

The other main result of this chapter will be the following:

Theorem 2.2. *Let $w = w(x_1, \dots, x_r)$ be an outer commutator word in r variables. Assume that $\mathbf{N} = (N_1, \dots, N_r)$ is a tuple of normal subgroups of a group G such that $w\{\mathbf{N}\}$ is finite. Then the subgroup $w(\mathbf{N})$ is also finite.*

We first need a few results regarding word values and verbal subgroups on normal subgroups or on normal subsets, in the case of outer commutator words. If $w = w(x_1, \dots, x_r)$ is an outer commutator word that is not a variable, then we can write $w = [\alpha, \beta]$, where α and β are again outer commutator words. Without loss of generality, after renaming variables if necessary, we may assume that $\alpha = \alpha(x_1, \ldots, x_q)$ and $\beta = \beta(x_{q+1}, \ldots, x_r)$, with $1 \leq q < r$.

Lemma 2.3. *Let $w = w(x_1, \ldots, x_r)$ be an outer commutator word, and let $\mathbf{N} = (N_1, \ldots, N_r)$ be an r-tuple of normal subgroups of a group G.*

1. *Assume that $w = [\alpha, \beta]$, with $\alpha = \alpha(x_1, \ldots, x_q)$ and $\beta = \beta(x_{q+1}, \ldots, x_r)$. If we set $\mathbf{N}_1 = (N_1, \ldots, N_q)$ and $\mathbf{N}_2 = (N_{q+1}, \ldots, N_r)$, then $w(\mathbf{N}) = [\alpha(\mathbf{N}_1), \beta(\mathbf{N}_2)]$.*

2. *Assume that $N_i = \langle S_i \rangle$ for every $i = 1, \ldots, r$, where each S_i is a normal subset of G. If we set $\mathbf{S} = (S_1, \ldots, S_r)$, then the subgroup $w(\mathbf{N})$ is generated by $w\{\mathbf{S}\}$.*

Proof. Both (i) and (ii) follow immediately from the simple fact that if S and T are two normal subsets of a group G then

$$[\langle S \rangle, \langle T \rangle] = \langle [s, t] \mid s \in S, \ t \in T \rangle,$$

where for part (ii) we use (i) and induction on the number of variables. □

We are interested in words of the form $w(u_1, \ldots, u_r)$, where $u_1, \ldots, u_r$ are non-commutator words that involve different variables. Let us introduce the following concept.

Definition 2.4 (Disjoint words)**.** Let $u_1, \ldots, u_r$ be group words. We say that these words are *disjoint* if the sets of variables that they involve are pairwise disjoint.

If w is a word in r variables and $u_1, \ldots, u_r$ are disjoint words, then the set of values of the word $w^* = w(u_1, \ldots, u_r)$ in a group G can be written as $w\{\mathbf{S}\}$, where

$$\mathbf{S} = (u_1\{G\}, \ldots, u_r\{G\}).$$

Since every $u_i\{G\}$ is a normal subset of G, we get the following consequence of the previous lemma.

Corollary 2.5. *Let $w = w(x_1, \ldots, x_r)$ be an outer commutator word and let $u_1, \ldots, u_r$ be arbitrary disjoint words. If $w^* = w(u_1, \ldots, u_r)$ then for every group G we have*

$$w^*(G) = w(u_1(G), \ldots, u_r(G)).$$

Now we want to make part (ii) of Lemma 2.3 quantitative. If we take a standard generator $w(n_1, \ldots, n_r)$ of $w(\mathbf{N})$, with $n_i \in N_i$, how can we estimate the number of factors from $w\{\mathbf{S}\}^{\pm 1}$ that are needed to write it? We need to introduce the following notation.

Definition 2.6 (Sets S^{*n})**.** Let G be a group and let S be a subset of G. For every $n \in \mathbb{N}$, we define S^{*n} to be the set of all products of elements of $S \cup S^{-1}$ of length at most n.

In other references, S^{*n} is defined as the set of products of exactly n elements of S. Since we can always replace S with $S \cup S^{-1} \cup \{1\}$, both definitions are basically equivalent. We prefer the definition above because it suits better the description of the elements of the subgroup $\langle S \rangle$. Also, with this definition, we have $S^{*k} \subseteq S^{*n}$ whenever $n \geq k$. Note that if $|S| \leq m$ then $|S^{*n}| \leq (2m+1)^n$ for every $n \in \mathbb{N}$. Let us connect this concept with values of outer commutator words.

Lemma 2.7. *Let $w = w(x_1, \ldots, x_r)$ be an outer commutator word, and let S be a normal subset of a group G. Suppose that $\mathbf{t} = (t_1, \ldots, t_r)$ is a tuple of elements of G, one of whose components belongs to S. Then $w(\mathbf{t}) \in S^{*2^{r-1}}$.*

Proof. The result is obvious for $r = 1$, so we assume $r > 1$. Then we can write $w(\mathbf{t}) = [\alpha(\mathbf{t}'), \beta(\mathbf{t}'')]$, where α and β are outer commutator words, and the tuples $\mathbf{t}'$ and $\mathbf{t}''$ form a partition of $\mathbf{t}$. Assume without loss of generality that $\mathbf{t}'$ contains an entry from S. By induction on r, we have $\alpha(\mathbf{t}') \in S^{*2^{r-2}}$. Consequently,

$$w(\mathbf{t}) = \alpha(\mathbf{t}')^{-1} \alpha(\mathbf{t}')^{\beta(\mathbf{t}'')} \in S^{*2^{r-1}},$$

since S is a normal subset of G. $\square$

On the other hand, by Lemma 2.8 of [40], if $w = w(x_1, \ldots, x_r)$ is an outer commutator and $g_1, \ldots, g_r, h$ are elements of a group G, then for every $i = 1, \ldots, r$ we have

$$\begin{aligned} w(g_1, \ldots, g_{i-1}, g_i h, g_{i+1}, \ldots, g_r) = w(g_1^*, \ldots, g_{i-1}^*, g_i^*, g_{i+1}^*, \ldots, g_r^*) \\ \cdot\, w(g_1, \ldots, g_{i-1}, h, g_{i+1}, \ldots, g_r), \end{aligned}$$

where g_j^* is a conjugate of g_j in G for every $j = 1, \ldots, r$. The following lemma follows easily from this result by induction on $m_1 \cdots m_r$.

Lemma 2.8. *Let $w = w(x_1, \ldots, x_r)$ be an outer commutator word, and let $\mathbf{S} = (S_1, \ldots, S_r)$ be a tuple of normal subsets of a group G. If $\mathbf{t} = (t_1, \ldots, t_r)$ with $t_i \in S_i^{*m_i}$ for every $i = 1, \ldots, r$, then*

$$w(\mathbf{t}) \in w\{\mathbf{S}\}^{*m_1 \ldots m_r}.$$

In an abelian group G, the word map $(g_1, \ldots, g_r) \mapsto w(g_1, \ldots, g_r)$ is a group homomorphism for every word w, and consequently $w(G) = w\{G\}$. Of course, this is far from being true in arbitrary groups. Outer commutator words, although they are also called multilinear words because the same type of commutator arrangements yields multinear words in Lie rings, are not multilinear in groups. However, our approach to proving Theorems 2.1 and 2.2 relies on showing that, in suitable sections that cover the section $w(\mathbf{N})/w(\mathbf{N})'$, outer commutator words are linear in one specific variable (which depends on the section). To this purpose, we give the following definition.

Definition 2.9 (Linearity)**.** Let $w = w(x_1, \ldots, x_r)$ be a word and let $\mathbf{N} = (N_1, \ldots, N_r)$ be an r-tuple of normal subgroups of a group G. We say that w is *linear in position i of the tuple* $\mathbf{N}$ provided that, for all $g_j \in N_j$ for $j = 1, \ldots, r$ and $h_i \in N_i$, we have

$$w(g_1, \ldots, g_{i-1}, g_i h_i, g_{i+1}, \ldots, g_r) = w(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_r) \cdot w(g_1, \ldots, g_{i-1}, h_i, g_{i+1}, \ldots, g_r). \quad (2.1)$$

Typically, we will search for linearity in a normal section K/L of the ambient group G that is generated by the image of $w\{\mathbf{N}\}$, so that condition (2.1) above is required to hold modulo L. Obviously, this type of linearity is inherited by sections of the form KN/LN for a given $N \trianglelefteq G$. Next we show that it is also preserved under taking suitable commutators.

Lemma 2.10. *Let $w = [\alpha, \beta]$ be an outer commutator word, with $\alpha = \alpha(x_1, \ldots, x_q)$ and $\beta = \beta(x_{q+1}, \ldots, x_r)$. Assume that $\mathbf{N} = (N_1, \ldots, N_r)$ is a tuple of normal subgroups of a group G, and set $\mathbf{N}_1 = (N_1, \ldots, N_q)$ and $\mathbf{N}_2 = (N_{q+1}, \ldots, N_r)$. Then the following hold:*

1. *If K/L is a normal section of G generated by the image of $\alpha\{\mathbf{N}_1\}$ and α is linear in component i of $\mathbf{N}_1$ modulo L, then the section U/V, where $U = [K, \beta(\mathbf{N}_2)]$ and $V = [w(\mathbf{N}), \alpha(\mathbf{N}_1)][L, \beta(\mathbf{N}_2)]$, is generated by the image of $w\{\mathbf{N}\}$ and w is linear in component i of $\mathbf{N}$ modulo V.*

2. *If K/L is a normal section of G generated by the image of $\beta\{\mathbf{N}_2\}$ and β is linear in component i of $\mathbf{N}_2$ modulo L, then the section U/V, with $U = [\alpha(\mathbf{N}_1), K]$ and $V = [w(\mathbf{N}), \beta(\mathbf{N}_2)][\alpha(\mathbf{N}_1), L]$, is generated by the image of $w\{\mathbf{N}\}$ and w is linear in component $q + i$ of $\mathbf{N}$ modulo V.*

Proof. We only prove part (i). To start with, we have

$$[K, \beta(\mathbf{N}_2)] = [\alpha(\mathbf{N}_1)L, \beta(\mathbf{N}_2)] = [\alpha(\mathbf{N}_1), \beta(\mathbf{N}_2)]\,[L, \beta(\mathbf{N}_2)] = w(\mathbf{N})\,[L, \beta(\mathbf{N}_2)],$$

where the last equality follows from (i) of Lemma 2.3. Thus the section U/V is generated by the image of $w\{\mathbf{N}\}$.

As for the assertion about linearity, let us consider the general congruence stating that α is linear in component i of $\mathbf{N}_1$ modulo L. This can be written in the form $x \equiv yz \pmod{L}$, where x, y, and z are like the three elements appearing in (2.1) (with α playing the role of w). In particular, $x, y, z \in \alpha\{\mathbf{N}_1\}$. Standard commutator identities then yield that, for every $n \in \beta\{\mathbf{N}_2\}$, we have

$$[x, n] \equiv [y, n][z, n] \pmod{[\alpha(\mathbf{N}_1), \beta(\mathbf{N}_2), \alpha(\mathbf{N}_1)]\,[L, \beta(\mathbf{N}_2)]}.$$

This proves the result. □

2.2 Lower central words

We can use the previous lemma to determine, for every lower central word γ_r and every r-tuple $\mathbf{N}$ of normal subgroups, a series from $[\gamma_r(\mathbf{N}), \gamma_r(\mathbf{N})]$ to $\gamma_r(\mathbf{N})$ that is linear for γ_r at every section.

Theorem 2.11. *Let $r \in \mathbb{N}$. Assume that $\mathbf{N} = (N_1, \ldots, N_r)$ is a tuple of normal subgroups of a group G, and define*

$$\mathbf{N}_i = (N_1, \ldots, N_{i-1}, \gamma_i(N_1, \ldots, N_i), N_{i+1}, \ldots, N_r)$$

for every $i = 1, \ldots, r$. Then there is a series

$$[\gamma_r(\mathbf{N}), \gamma_r(\mathbf{N})] = P^r_{r+1} \leq P^r_r \leq \cdots \leq P^r_i \leq \cdots \leq P^r_1 = \gamma_r(\mathbf{N})$$

such that, for every $i = 1, \ldots, r$, the section P^r_i/P^r_{i+1} is generated by the image of $\gamma_r\{\mathbf{N}_i\}$ and the word γ_r is linear in component i of $\mathbf{N}_i$ modulo P^r_{i+1}.

Proof. Set $Q^r_i = \gamma_r(\mathbf{N}_i)$ for every $i = 1, \ldots, r$, and $Q^r_{r+1} = [\gamma_r(\mathbf{N}), \gamma_r(\mathbf{N})]$. Obviously, the conditions that $P^r_{r+1} = [\gamma_r(\mathbf{N}), \gamma_r(\mathbf{N})]$ and that P^r_i/P^r_{i+1} is generated by the image of Q^r_i mean that we need to choose $P^r_i = Q^r_i Q^r_{i+1} \ldots Q^r_{r+1}$, for $i = 1, \ldots, r+1$.

Let us then prove the linearity of γ_r in component i of $\gamma_r(\mathbf{N}_i)$ modulo P^r_{i+1}. We argue by induction on $r - i$. The basis of the induction, $i = r$, follows from the congruence $[g, x_r y_r] \equiv [g, x_r][g, y_r] \pmod{P_{r+1}}$ for all $g \in G$ and $x_r, y_r \in \gamma_r(\mathbf{N})$, which holds because $P^r_{r+1} = [\gamma_r(\mathbf{N}), \gamma_r(\mathbf{N})]$.

Let us now assume that $1 \leq i < r$ and that the result is true for differences less than $r - i$. For every $i = 1, \ldots, r$, let Q^{r-1}_i and P^{r-1}_i be defined from the tuple

$\mathbf{N}^* = (N_1, \dots, N_{r-1})$ in the same way as we defined P_i^r and Q_i^r from $\mathbf{N}$. Then linearity holds in position i of

$$\mathbf{N}_i^* = (N_1, \dots, N_{i-1}, \gamma_i(N_1, \dots, N_i), N_{i+1}, \dots, N_{r-1})$$

modulo P_{i+1}^{r-1}. Now we apply Lemma 2.10 by taking $K = P_i^{r-1}$, $L = P_{i+1}^{r-1}$, $\alpha = \gamma_{r-1}$ and $\beta = x_r$. Thus γ_r is linear in component i of $\mathbf{N}_i$ modulo the subgroup

$$[\gamma_r(\mathbf{N}), \gamma_{r-1}(\mathbf{N}^*)]\,[P_{i+1}^{r-1}, N_r]. \tag{2.2}$$

Observe that

$$[\gamma_r(\mathbf{N}), \gamma_{r-1}(\mathbf{N}^*)] = [N_1, \dots, N_{r-1}, \gamma_r(N_1, \dots, N_r)] = Q_r^r \leq P_{i+1}^r,$$

since $r - i \geq 1$. On the other hand,

$$[P_{i+1}^{r-1}, N_r] = \left(\prod_{j=i+1}^{r-1} [Q_j^{r-1}, N_r] \right) \cdot [Q_r^{r-1}, N_r] = \left(\prod_{j=i+1}^{r-1} Q_j^r \right) \cdot [Q_r^{r-1}, N_r],$$

and

$$\begin{aligned} [Q_r^{r-1}, N_r] &= [\gamma_{r-1}(N_1, \dots, N_{r-1}), \gamma_{r-1}(N_1, \dots, N_{r-1}), N_r] \\ &\leq [N_1, \dots, N_{r-1}, \gamma_r(N_1, \dots, N_r)] = Q_r^r, \end{aligned}$$

where the inclusion follows from P. Hall's Three Subgroup Lemma. Hence the subgroup in (2.2) is contained in P_{i+1}^r, and the result follows. □

We are now in a position to prove the key theorem that will provide both Theorems 2.1 and 2.2 for lower central words. We need the following version for normal subgroups of a well-known lemma in the theory of concise words (see, for example, [26, Lemma 4]). The proof is exactly the same, based on Schur's Theorem, and taking into account also part (ii) of Lemma 2.3 in this case, so we omit it. In the remainder of the chapter, for a tuple S of parameters, we use the expression S-*bounded* to mean "bounded by a function of S".

Lemma 2.12. *Let $w = w(x_1, \dots, x_r)$ be an arbitrary word and consider an r-tuple $\mathbf{N} = (N_1, \dots, N_r)$ of normal subgroups of a group G. Suppose that $N_i = \langle S_i \rangle$, where S_i is a normal subset of G for every $i = 1, \dots, r$, and set $\mathbf{S} = (S_1, \dots, S_r)$. If $w\{\mathbf{S}\}$ is finite of order m then $w(\mathbf{N})'$ is finite of m-bounded order.*

Theorem 2.13. *Let $r \in \mathbb{N}$ and let $\mathbf{N} = (N_1, \dots, N_r)$ be a tuple of normal subgroups of a group G. Assume that $N_i = \langle S_i \rangle$ for every $i = 1, \dots, r$, where:*

1. S_i is a normal subset of G.

2. There exists $n_i \in \mathbb{N}$ such that all n_ith powers of elements of N_i are contained in S_i.

If for the tuple $\mathbf{S} = (S_1, \ldots, S_r)$ the set of values $\gamma_r\{\mathbf{S}\}$ is finite of order m, then the subgroup $\gamma_r(\mathbf{N})$ is also finite, of $(m, r, n_1, \ldots, n_r)$-bounded order.

Proof. We follow the notation $\mathbf{N}_i$ and P_i^r, introduced in the statement of Theorem 2.11. We are going to prove that P_i^r is finite of bounded order for $i = 1, \ldots, r+1$ by reverse induction on i. Since $P_1^r = \gamma_r(\mathbf{N})$, this proves the result.

The basis of the induction follows from Lemma 2.12, since we have that $P_{r+1}^r = [\gamma_r(\mathbf{N}), \gamma_r(\mathbf{N})]$. Let us assume that P_{i+1}^r is finite of bounded order and prove that the same holds for P_i^r. Recall that the quotient P_i^r/P_{i+1}^r is the image of $\gamma_r(\mathbf{N}_i)$, and then, by a suitable application of Lemma 2.3, it can be generated by the images of the set $\mathbf{T}$ of commutators

$$[s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_r],$$

with $s_j \in S_j$ for $1 \leq j \leq r$, $j \neq i$, and $x_i \in \gamma_i\{\mathbf{S}_i\}$, where $\mathbf{S}_i = (S_1, \ldots, S_i)$. By Lemma 2.7, we have $\gamma_i\{\mathbf{S}_i\} \subseteq S_i^{*2^{i-1}}$, and then Lemma 2.8 implies that

$$[s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_r] \in \gamma_r\{\mathbf{S}\}^{*2^{i-1}} \subseteq \gamma_r\{\mathbf{S}\}^{*2^{r-1}}.$$

From the assumption that $|\gamma_r\{\mathbf{S}\}| = m$, we get

$$|\mathbf{T}| \leq (2m+1)^{2^{r-1}},$$

and consequently P_i^r/P_{i+1}^r can be generated by an (m, r)-bounded number of elements. Since P_i^r/P_{i+1}^r is abelian, the proof will be complete once we show that all elements in $\mathbf{T}$ have bounded finite order modulo P_{i+1}^r.

By Theorem 2.11, the word γ_r is linear in position i of the tuple $\mathbf{N}_i$ modulo P_{i+1}^r. In particular,

$$[s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_r]^{\lambda n_i} \equiv [s_1, \ldots, s_{i-1}, x_i^{\lambda n_i}, s_{i+1}, \ldots, s_r] \pmod{P_{i+1}^r}, \tag{2.3}$$

for every $\lambda \in \mathbb{Z}$. Since $x_i \in \gamma_i(N_1, \ldots, N_i) \leq N_i$, it follows from (ii) in the statement of the theorem that $x_i^{\lambda n_i} \in S_i$ for all $\lambda \in \mathbb{Z}$. Thus we get

$$[s_1, \ldots, s_{i-1}, x_i^{\lambda n_i}, s_{i+1}, \ldots, s_r] \in \gamma_r\{\mathbf{S}\}.$$

Since $\gamma_r\{\mathbf{S}\}$ is finite of order m, it follows that there exist $\lambda, \mu \in \{0, \ldots, m\}$, $\lambda \neq \mu$, such that

$$[s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_r]^{\lambda n_i} \equiv [s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_r]^{\mu n_i} \pmod{P_{i+1}^r}.$$

This implies that $[s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_r]$ has (m, n_i)-bounded finite order modulo P_{i+1}^r, as desired. □

If we take $S_i = N_i$, we get Theorem 2.2 for the lower central words.

Corollary 2.14. *Let $r \in \mathbb{N}$ and let $\mathbf{N} = (N_1, \ldots, N_r)$ be a tuple of normal subgroups of a group G. If $\gamma_r\{\mathbf{N}\}$ is finite of order m, then the subgroup $\gamma_r(\mathbf{N})$ is also finite, of (m, r)-bounded order.*

Now we deduce Theorem 2.1 for lower central words.

Corollary 2.15. *Let $r \in \mathbb{N}$ and let $u_1, \ldots, u_r$ be disjoint non-commutator words. Then the word $\gamma_r(u_1, \ldots, u_r)$ is boundedly concise. In particular, $\gamma_r(x_1^{n_1}, \ldots, x_r^{n_r})$ is boundedly concise for all $n_i \in \mathbb{Z} \setminus \{0\}$.*

Proof. Let us consider the word $w = \gamma_r(u_1, \ldots, u_r)$, and let G be a group in which w takes finitely many values, say $|w\{G\}| = m$. By Corollary 2.5, we have $w(G) = \gamma_r(u_1(G), \ldots, u_r(G))$. Note that $u_i(G) = \langle S_i \rangle$, where $S_i = u_i\{G\}$, and that $w\{G\} = \gamma_r\{\mathbf{S}\}$, where $\mathbf{S} = (S_1, \ldots, S_r)$. Now observe that S_i is a normal subset of G and that, since u_i is a non-commutator word, for some $n_i \in \mathbb{Z} \setminus \{0\}$ we have $\{g^{n_i} \mid g \in G\} \subseteq u_i\{G\}$. Hence $w(G)$ is finite of $(m, r, n_1, \ldots, n_r)$-bounded order by Theorem 2.13. □

2.3 An example: the word δ_2

We now want to prove Theorems the analogues of Theorems 2.1 and 2.2 for a generic outer commutator word w. The general strategy is still the same as for lower central words: we are going to obtain a suitable series of normal subgroups of G, going from $[w(\mathbf{N}), w(\mathbf{N})]$ to $w(\mathbf{N})$, with the property that each of the factors of the series can be generated by a verbal subgroup on a tuple of normal subgroups that is closely related to $w(\mathbf{N})$ and linear in one component. This is basically Theorem 2.20 below. For simplicity, let us refer to such a series as a linear series. The argument needed to obtain a linear series for derived words presents difficulties and subtleties that did not arise with lower central words, and is also significantly more technical. For the convenience of the reader and in order to make the procedure for a general w more understandable, first of all we are going to provide a sketch of it in the particular case of δ_2.

Of course, $\delta_1 = \gamma_2$ and, according to Theorem 2.11, we have the following linear series for $\delta_1(N_1, N_2)$:

$$\begin{tikzcd}
{[\boxed{N_1}, N_2]} \arrow[d, dash] \\
{\big[N_1, \boxed{[N_1, N_2]}\big]} \arrow[d, dash] \\
{\big[[N_1, N_2], [N_1, N_2]\big]}
\end{tikzcd}$$

Figure 2.1: Series of $[N_1, N_2]$

In this and in the next diagrams, a red box indicates the component in which we have linearity.

Let us see how we can construct a linear series for $\delta_2(N_1, N_2, N_3, N_4)$ from the series above for δ_1. To this purpose, we will use Lemma 2.10, which ensures that linearity is preserved after taking suitable commutators, and also the remark made before that lemma, saying that linearity is preserved after multiplying by a normal subgroup. To start with, we take the commutator of the terms of the previous series with $[N_3, N_4]$, obtaining the series

$$\begin{tikzcd}
{\big[[\boxed{N_1}, N_2], [N_3, N_4]\big]} \arrow[d, dash] \\
{\Big[\big[N_1, \boxed{[N_1, N_2]}\big], [N_3, N_4]\Big]} \arrow[d, dash] \\
{\Big[\big[[N_1, N_2], [N_1, N_2]\big], [N_3, N_4]\Big]}
\end{tikzcd}$$

Now we multiply this series by $\big[[N_1, N_2], [[N_1, N_2], [N_3, N_4]]\big]$, which contains the subgroup $\big[[N_1, N_2], [N_1, N_2], [N_3, N_4]\big]$ by P. Hall's Three Subgroup Lemma, and we get the following diagram:

$$
\begin{array}{c}
\big[[\boxed{N_1}, N_2], [N_3, N_4]\big] \\
\Big| \\
\Big[\big[N_1, \boxed{[N_1, N_2]}\big], [N_3, N_4]\Big] \\
\Big| \\
\Big[[N_1, N_2], \big[[N_1, N_2], [N_3, N_4]\big]\Big]
\end{array}
$$

Figure 2.2: First diagram for $\big[[N_1, N_2], [N_3, N_4]\big]$

Here, and in the remaining diagrams, instead of the subgroups of the series, we are showing verbal subgroups on normal subgroups whose images coincide with the corresponding factors of the series. After all, it is in these subgroups where we are going to obtain the linearity conditions. Be aware then that vertical lines in the diagrams do not denote inclusions from this point onwards.

By swapping the roles of (N_1, N_2) and (N_3, N_4), we can obtain this other diagram:

$$
\begin{array}{c}
\big[[N_1, N_2], [\boxed{N_3}, N_4]\big] \\
\Big| \\
\Big[[N_1, N_2], \big[N_3, \boxed{[N_3, N_4]}\big]\Big] \\
\Big| \\
\Big[\big[[N_1, N_2], [N_3, N_4]\big], [N_3, N_4]\Big]
\end{array}
$$

Figure 2.3: Second diagram for $\big[[N_1, N_2], [N_3, N_4]\big]$

Now we take the commutator of $[N_1, N_2]$ with the terms of this last diagram, and we add the extra term $\delta_2(N_1, N_2, N_3, N_4)'$ at the bottom:

$$\Big[[N_1, N_2], \big[[N_1, N_2], [\boxed{N_3}, N_4]\big]\Big]$$

$$\Big[[N_1, N_2], \big[[N_1, N_2], \big[N_3, \boxed{[N_3, N_4]}\big]\big]\Big]$$

$$\Big[[N_1, N_2], \big[\boxed{\big[[N_1, N_2], [N_3, N_4]\big]}, [N_3, N_4]\big]\Big]$$

$$\Big[\big[[N_1, N_2], [N_3, N_4]\big], \big[[N_1, N_2], [N_3, N_4]\big]\Big]$$

Figure 2.4: Series of $\Big[[N_1, N_2], \big[[N_1, N_2], [N_3, N_4]\big]\Big]$

Finally, by gluing the diagrams in Figures 2 and 4 together, we obtain a linear series for the subgroup $\delta_2(N_1, N_2, N_3, N_4)$.

Of course, this is simply a sketch without proofs, but we are going to follow the same procedure in the proof of Theorem 2.20, in order to get a linear series for $w = [\alpha, \beta]$ from the series for the outer commutator words α and β. At this point, it is worth noting an important difference with the situation for a lower central word γ_r. In that case, every factor of the linear series is of the following form (again we show the linear component in red):

$$\big[N_1, \ldots, N_{i-1}, \boxed{[N_1, \ldots, N_i]}, N_{i+1}, \ldots, N_r\big].$$

We observe that this subgroup is of the form $\gamma_r(\mathbf{M})$, where the jth component M_j of $\mathbf{M}$ is either N_j or a commutator of the terms of $\mathbf{N}$ that involves N_j, and the linearity happens in M_i. However, if we look at the series for δ_2 obtained above, the first two subgroups in Figure 2.4 are

$$\delta_2(N_1; N_2; [N_1, N_2]; [\boxed{N_3}, N_4]) \tag{2.4}$$

and

$$\delta_2(N_1; N_2; [N_1, N_2]; [N_3, \boxed{[N_3, N_4]}]), \tag{2.5}$$

which are not of the form $\delta_2(\mathbf{M})$ with every M_j a commutator from $\mathbf{N}$ involving N_j, as we can see by looking at the third component of δ_2. Also, the linearity does not happen in a component of δ_2, but in a more interior position. Nevertheless, we can write these subgroups as verbal subgroups on normal subgroups for outer commutator words different from δ_2. More specifically, if

$$v(x_1, x_2, x_3, x_4, y_1, y_2) = \big[[x_1, x_2], [[y_1, y_2], [x_3, x_4]]\big]$$

then the subgroups in (2.4) and (2.5) are $v(\mathbf{M}_1)$ and $v(\mathbf{M}_2)$, where

$$\mathbf{M}_1 = (N_1, N_2, \boxed{N_3}, N_4, N_1, N_2) \text{ and } \mathbf{M}_2 = (N_1, N_2, N_3, \boxed{[N_3, N_4]}, N_1, N_2), \quad (2.6)$$

where again we have marked the linear components in red.

2.4 Outer commutator words

After having illustrated the procedure with the case of δ_2, let us proceed to systematically develop the tools that are necessary for the proof of Theorem 2.20.

We start by introducing a special type of words that we can derive from a given outer commutator word w, which we call extended words of w. Before giving the definition, we show the idea behind extended words with an example. Consider the word $\delta_2 = [[x_1, x_2], [x_3, x_4]]$. This is formed by taking the commutator of x_1 and x_2, taking the commutator of x_3 and x_4, and then taking the commutator of these two commutators. Now suppose that on some occasions, before performing one of these commutators, we introduce a change by taking first the commutator of one (or both) of the components with an outer commutator word not involving the variables $x_1, \dots, x_4$ appearing in δ_2. For example, before producing $[x_1, x_2]$, we take the commutator $[[y_1, y_2], x_1]$ and now we follow as in δ_2 taking the commutator with x_2, obtaining $\big[[[y_1, y_2], x_1], x_2\big]$. We could continue with the process of taking commutators without making any other changes, so getting

$$\big[\big[[[y_1, y_2], x_1], x_2\big], [x_3, x_4]\big],$$

but we could also make some similar changes in the process, as in the words

$$\Big[\big[[[y_1, y_2], x_1], x_2\big], \big[x_3, [y_3, x_4]\big]\Big]$$

and

$$\Big[\ \big[[[y_1, y_2], x_1], x_2\big], \big[\ \big[x_3, [y_3, x_4]\big], y_4\big]\ \Big].$$

Another possibility is to make a commutator at the very end, after having completed δ_2, as in

$$\big[y_1, [[x_1, x_2], [x_3, x_4]]\big].$$

Observe that all these extended words are again outer commutator words, because we never repeat a variable when we make changes in the construction of δ_2.

Let us now give the formal definition of extended words. Notice that this definition differs from the one of extensions of outer commutator words given in Definition 3.1 of [33].

Definition 2.16 (Extended words)**.** Let $w = w(x_1, \dots, x_r)$ be an outer commutator, and let $Y = \{y_n\}_{n\in\mathbb{N}}$ be a set of variables that are disjoint from X. For every $k \in \mathbb{N} \cup \{0\}$, we define recursively the set $\mathrm{ext}_k(w)$ of *kth extended words* of w as follows:

1. $\mathrm{ext}_0(w) = \{w\}$.

2. For $k \geq 1$, $\mathrm{ext}_k(w)$ consists of the set

$$\begin{aligned}&\{[p, q], [q, p] \mid p \text{ outer commutator in } Y,\ q \in \mathrm{ext}_{k-1}(w),\ p \text{ and } q \text{ disjoint}\}\\ &= \{[p, q] \mid p \text{ outer commutator in } Y,\ q \in \mathrm{ext}_{k-1}(w),\ p \text{ and } q \text{ disjoint}\}^{\pm 1},\end{aligned}$$

   and, if $w = [\alpha, \beta]$, also of the set

$$\bigcup_{\ell+m=k} \{[p, q] \mid p \in \mathrm{ext}_\ell(\alpha),\ q \in \mathrm{ext}_m(\beta),\ p \text{ and } q \text{ disjoint}\}.$$

If $v \in \mathrm{ext}_k(w)$ then we say that w is an *extended word* of degree k of w by outer commutators.

For brevity, in the remainder we will simply speak of extended words when we mean extended words by outer commutators. Observe that an extended word v of an outer commutator $w = w(x_1, \dots, x_r)$ is again an outer commutator, in the variables $\{x_1, \dots, x_r\} \cup Y$. Whenever it is convenient we will assume, after renaming the variables, that $v = v(x_1, \dots, x_r, y_{r+1}, \dots, y_s)$.

Next we generalize Lemma 2.8 to extended words of an outer commutator word.

Lemma 2.17. *Let $v = v(x_1, \dots, x_r, y_{r+1}, \dots, y_s)$ be an extended word of degree k of an outer commutator word $w = w(x_1, \dots, x_r)$. Assume that $\mathbf{S} = (S_1, \dots, S_r)$ is a tuple of normal subsets of a group G. If $\mathbf{t} = (t_1, \dots, t_s)$ is a tuple of elements of G such that $t_i \in S_i^{*m_i}$ for every $i = 1, \dots, r$, then*

$$v(\mathbf{t}) \in w\{\mathbf{S}\}^{*m_1 \dots m_r 2^k}.$$

Proof. We use induction on $k+r$. If $k=0$ then $v=w$ and the result is Lemma 2.8. This gives in particular the basis of the induction. Suppose now that the result holds for smaller values of $k+r$, and that $k \geq 1$. According to Definition 2.16, we may assume that $v(\mathbf{t}) = [p(\mathbf{t}'), q(\mathbf{t}'')]$, where p and q are disjoint and

1. either p is an outer commutator word in Y and $q \in \mathrm{ext}_{k-1}(w)$,

2. or $p \in \mathrm{ext}_\ell(\alpha)$, $q \in \mathrm{ext}_m(\beta)$, with $w = [\alpha, \beta]$ and $\ell + m = k$.

In case (i), all elements $t_1, \ldots, t_r$ appear in the vector $\mathbf{t}''$, and by the induction hypothesis we have $q(\mathbf{t}'') \in w\{\mathbf{S}\}^{*m_1 \ldots m_r 2^{k-1}}$. Then the result follows by applying Lemma 2.7 to the commutator word $[x_1, x_2]$ and the normal subset $w\{\mathbf{S}\}^{*m_1 \ldots m_r 2^{k-1}}$.

Suppose now that we are in case (ii), and assume without loss of generality that $\alpha = \alpha(x_1, \ldots, x_q)$ and $\beta = \beta(x_{q+1}, \ldots, x_r)$. Set $\mathbf{S}' = (S_1, \ldots, S_q)$ and $\mathbf{S}'' = (S_{q+1}, \ldots, S_r)$. Since α and β involve less variables than w, the result is true for p and q, and so

$$p(\mathbf{t}') \in \alpha\{\mathbf{S}'\}^{*m_1 \ldots m_q 2^\ell} \quad \text{and} \quad q(\mathbf{t}'') \in \beta\{\mathbf{S}''\}^{*m_{q+1} \ldots m_r 2^m}.$$

Now the result follows by applying Lemma 2.8 to the commutator word $[x_1, x_2]$ and the pair of normal subsets $(\alpha\{\mathbf{S}'\}, \beta\{\mathbf{S}''\})$. □

We also need to define a type of extensions of tuples of normal subgroups and of verbal subgroups on normal subgroups. The idea behind the definition is to be able to deal with tuples like the ones appearing in (2.6) and with the corresponding verbal subgroups on normal subgroups in that paragraph.

Definition 2.18 (Outer commutator extension)**.** Let G be a group and consider two tuples $\mathbf{N} = (N_1, \ldots, N_r)$ and $\mathbf{M} = (M_1, \ldots, M_s)$ of normal subgroups of G. We say that $\mathbf{M}$ is an *outer commutator extension* of $\mathbf{N}$ if the following conditions hold:

1. $s \geq r$.

2. For every $i = 1, \ldots, s$, we have $M_i = w_i(\mathbf{N}_i)$, where w_i is an outer commutator word and all components of $\mathbf{N}_i$ belong to $\mathbf{N}$.

3. For every $i = 1, \ldots, r$, the subgroup N_i is a component of $\mathbf{N}_i$, and consequently $M_i \leq N_i$.

Definition 2.19 (Extensions of $w(\mathbf{N})$)**.** Let $w = w(x_1, \dots, x_r)$ be a word and let $\mathbf{N}$ be an r-tuple of normal subgroups of a group G. An *extension* of degree k of $w(\mathbf{N})$ by outer commutators is a subgroup of the form $v(\mathbf{M})$, where v is an extended word of degree k of w and $\mathbf{M}$ is an outer commutator extension of $\mathbf{N}$.

For example, we can see the subgroup in (2.5) as an extension of $\delta_2(\mathbf{N}) = \delta_2(N_1, N_2, N_3, N_4)$ by taking $v = \big[[x_1, x_2], [[y_1, y_2], [x_3, x_4]]\big]$ and the tuple $\mathbf{M} = (N_1, N_2, N_3, [N_3, N_4], N_1, N_2)$. Note that $v(\mathbf{M})$ is linear in the fourth component modulo the subgroup that appears below it in Figure 4.

We now prove the existence of a linear series for outer commutator words. We recall that the *height* of an outer commutator word $w = [\alpha, \beta]$ is defined inductively, with a single variable having height 0, and with the height of w being $1 + \max\{\text{height}(\alpha), \text{height}(\beta)\}$. Notice that the height of an outer commutator word in s variables will always be at most $s - 1$.

Theorem 2.20. *Let $r \in \mathbb{N}$ and let $\mathbf{N} = (N_1, \dots, N_r)$ be a tuple of normal subgroups of a group G. Consider an outer commutator word $w = [\alpha, \beta]$ in r variables, say of height h. Then there exists a series*

$$[w(\mathbf{N}), w(\mathbf{N})] = V_0 \leq V_1 \leq \cdots \leq V_t = w(\mathbf{N})$$

of normal subgroups of G such that, for every $i = 1, \dots, t$, the following hold:

1. *The section V_i/V_{i-1} is the image of an extension $v_i(\mathbf{M}_i)$ of $w(\mathbf{N})$ of degree at most $h - 1$.*

2. *In the section V_i/V_{i-1}, the word v_i is linear in one component of the tuple $\mathbf{M}_i$.*

Furthermore, the words v_i and the words appearing in the outer commutator extensions $\mathbf{M}_i$ depend only on w and r, and not on the group G or on the tuple $\mathbf{N}$.

Proof. We prove the theorem by induction on the height of the outer commutator word w, with the base case being a single variable, which is obvious. We can then assume that there exist two series of subgroups satisfying the conditions of the theorem for the outer commutator words of smaller height α and β. Assume that $x_1, \dots, x_q$ and $x_{q+1}, \dots, x_{q+m}$ are the variables involved in α and β respectively, and in particular $r = q + m$.

Set $\mathbf{N}_1 = (N_1, \dots, N_q)$ and $\mathbf{N}_2 = (N_{q+1}, \dots, N_{q+m})$. By the induction hypothesis, there exist two series of length s and r respectively

$$A_0 = [\alpha(\mathbf{N}_1), \alpha(\mathbf{N}_1)] \leq \cdots \leq A_i \leq \cdots \leq A_s = \alpha(\mathbf{N}_1) \tag{2.7}$$

and

$$B_0 = [\beta(\mathbf{N}_2), \beta(\mathbf{N}_2)] \leq \cdots \leq B_i \leq \cdots \leq B_r = \beta(\mathbf{N}_2) \tag{2.8}$$

such that, for every $i = 1, \ldots, s$, the factors A_i/A_{i-1} and B_i/B_{i-1} are the images of $v_i^\alpha(\mathbf{M}_i^\alpha)$ and $v_i^\beta(\mathbf{M}_i^\beta)$, respectively, where:

(a) v_i^α and v_i^β are extended words of α and β respectively, each of degree at most $h-2$.

(b) $\mathbf{M}_i^\alpha$ is an outer commutator extension of $\mathbf{N}_1$.

(c) $\mathbf{M}_i^\beta$ is an outer commutator extension of $\mathbf{N}_2$.

(d) In the sections A_i/A_{i-1} and B_i/B_{i-1}, the words v_i^α and v_i^β are linear in one component of the tuples $\mathbf{M}_i^\alpha$ and $\mathbf{M}_i^\beta$, respectively.

Let us now see how to obtain the series for w and for the tuple $\mathbf{N}$ from the two series (2.7) and (2.8). We will have that the length t of the series we are looking for depends on the length of these two series, in the form that $t = r + s + 1$. We start by taking the commutator of all terms of the series (2.7) with $\beta(\mathbf{N}_2)$. This way we obtain the series

$$[A_0, \beta(\mathbf{N}_2)] \leq \cdots \leq [A_i, \beta(\mathbf{N}_2)] \leq \cdots \leq [\alpha(\mathbf{N}_1), \beta(\mathbf{N}_2)] = w(\mathbf{N}). \tag{2.9}$$

By P. Hall's Three Subgroup Lemma, we have

$$\begin{aligned}[A_0, \beta(\mathbf{N}_2)] &= [\alpha(\mathbf{N}_1), \alpha(\mathbf{N}_1), \beta(\mathbf{N}_2)] \\ &\leq [\alpha(\mathbf{N}_1), \beta(\mathbf{N}_2), \alpha(\mathbf{N}_1)] = [\alpha(\mathbf{N}_1), w(\mathbf{N})].\end{aligned}$$

Now we multiply all terms of the series (2.9) by $[\alpha(\mathbf{N}_1), w(\mathbf{N})]$, and this is the rightmost part of the series we are seeking (where $t = r + s + 1$, as above):

$$\begin{aligned}V_{t-s} = [\alpha(\mathbf{N}_1), w(\mathbf{N})] \leq \cdots \leq V_{t-s+i} = [A_i, \beta(\mathbf{N}_2)]\,[\alpha(\mathbf{N}_1), w(\mathbf{N})] \\ \leq \cdots \leq V_t = w(\mathbf{N}).\end{aligned} \tag{2.10}$$

Note that $t - s = r + 1$. The factors in this series are the images of the subgroups

$$[v_i^\alpha(\mathbf{M}_i^\alpha), \beta(\mathbf{N}_2)],$$

which can be represented in the form $v_i(\mathbf{M}_i)$ by taking

$$v_i = [v_i^\alpha, \beta(x_{q+1}, \ldots, x_{q+m})]$$

and defining $\mathbf{M}_i$ to be the concatenation of $\mathbf{M}_i^\alpha$ and $\mathbf{N}_2$, where the elements of $\mathbf{N}_2$ occupy the positions $q+1,\dots,q+m$ (which are the positions corresponding to the variables $x_{q+1},\dots,x_{q+m}$). Note that $\mathbf{M}_i$ is an outer commutator extension of $\mathbf{N}$.

In a symmetric way, by first taking the commutator of $\alpha(\mathbf{N}_1)$ with all terms of the series (2.8) and then multiplying by $[w(\mathbf{N}),\beta(\mathbf{N}_2)]$, we get the series

$$U_{t-r}=[w(\mathbf{N}),\beta(\mathbf{N}_2)]\le\cdots\le U_{t-r+i}=[\alpha(\mathbf{N}_1),B_i]\,[w(\mathbf{N}),\beta(\mathbf{N}_2)]\\ \le\cdots\le U_t=w(\mathrm{N}).\quad(2.11)$$

In this series, the factors are given by the images of the subgroups $u_i(\mathbf{L}_i)$, where

$$u_i=[\alpha(y_{2^h+1},\dots,y_{2^h+q}),v_{i,2}^\beta],$$

$v_{i,2}^\beta$ being the same word as v_i^β, with $x_1,\dots,x_m$ replaced with $x_{q+1},\dots,x_{q+m}$, and $\mathbf{L}_i$ being the concatenation of $\mathbf{M}_i^\beta$ and $\mathbf{N}_1$, where we put the components of the second tuple after the components of the first. Note that u_i is an extended word of $\beta(x_{q+1},\dots,x_{q+m})$ of degree at most $h-1$ that only depends on β.

Now we take the commutator of $\alpha(\mathbf{N}_1)$ with the terms of the last series, and subtract s to all indices, getting

$$Z_1=\big[\alpha(\mathbf{N}_1),[w(\mathbf{N}),\beta(\mathbf{N}_2)]\big]\le\cdots\le Z_{t-r-s+i}=[\alpha(\mathbf{N}_1),U_{t-r+i}]\\ \le\cdots\le Z_{t-s}=[\alpha(\mathbf{N}_1),w(\mathbf{N})],\quad(2.12)$$

since $t-r-s=1$. Finally, we define $V_0=[w(\mathbf{N}),w(\mathbf{N})]$ and multiply all terms of (2.12) by this subgroup, setting $V_i=Z_iV_0$ for $i=1,\dots,t-s$. Since $V_0\le[\alpha(\mathbf{N}_1),w(\mathbf{N})]$, we get the series

$$V_0=[w(\mathbf{N}),w(\mathbf{N})]\le\cdots\le V_{t-r-s+i}=Z_{t-r-s+i}[w(\mathbf{N}),w(\mathbf{N})]\\ \le\cdots\le V_{t-s}=[\alpha(\mathbf{N}_1),w(\mathbf{N})].\quad(2.13)$$

In this series, the factors V_i/V_{i-1} for $i=2,\dots,t-s$ are given by the images of the subgroups $v_i(\mathbf{M}_i)$, where

$$v_i=[\alpha(x_1,\dots,x_q),u_{i+s}]$$

is an extended word of w of degree at most $h-1$, and $\mathbf{M}_i$ is the concatenation of $\mathbf{N}_1$ and $\mathbf{L}_{i+s}$, with the components of the second tuple after the components of the first. On the other hand, the quotient V_1/V_0 is given by the image of $v_1(\mathbf{M}_1)$, where

$$v_1 = [\alpha(x_1, \ldots, x_q), [y, \beta(x_{q+1}, \ldots, x_{q+m})]]$$

and $\mathbf{M}_1 = (\mathbf{N}, w(\mathbf{N}))$.

Now the concatenation of (2.10) and (2.13) is the desired series for w and $\mathbf{N}$. The discussion of the previous paragraphs shows that $v_i(\mathbf{M}_i)$ is an extension of $w(\mathbf{N})$ for every $i = 1, \ldots, t$. Thus we only need to check linearity of every word v_i in one component of the vector $\mathbf{M}_i$. For $i = t - s + 1, \ldots, t$, if v_i^α is linear in component j of $\mathbf{M}_i^\alpha$ of the initial series (2.7), then combining this fact with Lemma 2.10, it follows that v_i is linear in the same component of $\mathbf{M}_i$. For $i = 1, \ldots, t - s$, we can use similarly the linearity of the series (2.8). Finally for $i = 1$, since $V_0 = [w(\mathbf{N}), w(\mathbf{N})]$ we have linearity in the component corresponding to y, that takes values in $w(\mathbf{N})$. $\square$

Remark 2.21. Suppose $w = [\alpha, \beta]$ is an outer commutator word of height $h \in \mathbb{N}$. Notice that the number t of terms of the series associated to w in 2.20 is $1 + r + s$, where r, s are the lengths of the series of α and β respectively, with the case of a single variable having length one. In particular, it is immediate to prove by induction that the length t is always at most the length of the series for the derived word δ_h. This length can be explicitly computed, being equal to one if $h = 0$, equal to two if $h = 1$ (this can be obtained from $\delta_1 = \gamma_2$) and, using the recursion formula, the length of the series for δ_h is equal to $2^h + 2^{h-1} - 1$.

We can now prove the corresponding version of Theorem 2.13 for the a generic outer commutator word w.

Theorem 2.22. *Let w be an outer commutator word of height h in r variables and let $\mathbf{N} = (N_1, \ldots, N_r)$ be a tuple of normal subgroups of a group G. Assume that $N_i = \langle S_i \rangle$ for every $i = 1, \ldots, r$, where:*

1. *S_i is a normal subset of G.*
2. *There exists $n_i \in \mathbb{N}$ such that all n_ith powers of elements of N_i are contained in S_i.*

If for the tuple $\mathbf{S} = (S_1, \ldots, S_r)$ the set of values $w\{\mathbf{S}\}$ is finite of order m, then the subgroup $w(\mathbf{N})$ is also finite and of $(m, r, n_1, \ldots, n_r)$-bounded order.

Proof. Let us consider the series

$$[w(\mathbf{N}), w(\mathbf{N})] = V_0 \leq V_1 \leq \cdots \leq V_t = w(\mathbf{N})$$

of Theorem 2.20. By Remark 2.21, $t \leq 2^h + 2^{h-1} - 1$, where h is the height of the outer commutator word w. We prove that every V_i is finite of bounded order by induction on i. The result for $i = 0$ follows from Lemma 2.12. Assume now that $i \geq 1$ and that the result is true for V_{i-1}. By Theorem 2.20, the section V_i/V_{i-1} coincides with the image of a subgroup $v_i(\mathbf{M}_i)$ that is an extension of $w(\mathbf{N})$ of degree at most $h-1$.

Let $\mathbf{M}_i = (M_1, \ldots, M_s)$, which is an outer commutator extension of $\mathbf{N}$. Hence $s \geq r$ and for every $j = 1, \ldots, s$ we have $M_j = w_j(\mathbf{N}_j)$, where w_j is an outer commutator word, all components in $\mathbf{N}_j$ belong to $\mathbf{N}$, and one of these components must be N_j for $j = 1, \ldots, r$.

Let $T_j = w_j\{\mathbf{S}_j\}$, where $\mathbf{S}_j$ is obtained from $\mathbf{N}_j$ by replacing each subgroup N_ℓ with its given generating set S_ℓ. Hence $T_j \subseteq M_j$. Recall from Theorem 2.20 that the word v_i (and hence also the number s of variables of v_i) and the words $w_1, \ldots, w_s$ only depend on w, and not on G or on $\mathbf{N}$. From this fact, and since $\mathbf{S}_j$ consists of normal subsets of G, it follows from Lemma 2.7 that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $T_j \subseteq S_j^{*f(r)}$ for every $j = 1, \ldots, r$. If we set $\mathbf{T} = (T_1, \ldots, T_s)$ then, by Lemma 2.17, we get $v\{\mathbf{T}\} \subseteq w\{\mathbf{S}\}^{*n}$, where

$$n = f(r)^r 2^{h-1} \leq f(r)^r 2^{r-1}.$$

Consequently

$$|v_i\{\mathbf{T}\}| \leq (2m+1)^n, \tag{2.14}$$

and $v_i\{\mathbf{T}\}$ is finite of (m,r)-bounded cardinality. On the other hand, it follows from Lemma 2.3 that $v_i(\mathbf{M}_i)$ can be generated by the set of values $v_i\{\mathbf{T}\}$.

From Theorem 2.20, we know that the word v_i is linear in some position $j \in \{1, \ldots, s\}$ of the tuple $\mathbf{M}_i$ modulo V_{i-1}. Since $M_j = w_j(\mathbf{N}_j)$ is as above, we have $M_j \leq N_\ell$ for some $\ell \in \{1, \ldots, r\}$, and actually $\ell = j$ if $j \in \{1, \ldots, r\}$. Now, from linearity, for every tuple $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbf{T}$ and every $\lambda \in \mathbb{Z}$, we have

$$v_i(\mathbf{t})^{\lambda n_\ell} = v_i(t_1, \ldots, t_j, \ldots, t_s)^{\lambda n_\ell} \equiv v_i(t_1, \ldots, t_j^{\lambda n_\ell}, \ldots, t_s) \pmod{V_{i-1}}. \tag{2.15}$$

We have $t_j^\lambda \in M_j \leq N_\ell$ and then, by (ii) in the statement of the theorem, $t_j^{\lambda n_\ell} \in S_\ell$. So we get

$$v_i(t_1, \ldots, t_j^{\lambda n_\ell}, \ldots, t_s) \in v_i\{\mathbf{T}_j\},$$

where $\mathbf{T}_j$ is the tuple obtained from $\mathbf{T}$ after replacing T_j with S_ℓ. Similarly to (2.14) and taking into account that $\ell = j$ if $j \in \{1, \ldots, r\}$, it follows that the set $v_i\{\mathbf{T}_j\}$ is finite of (m,r)-bounded cardinality. Hence there exist (m,r)-bounded integers λ and μ, with $\lambda \neq \mu$, such that

$$v_i(\mathbf{t})^{\lambda n_\ell} \equiv v_i(\mathbf{t})^{\mu n_\ell} \pmod{V_{i-1}}.$$

This implies that $v_i(\mathbf{t})$ has finite order modulo V_{i-1}, bounded in terms of m, r and n_ℓ.

Summarizing, the abelian quotient V_i/V_{i-1} is the image of the verbal subgroup $v_i(\mathbf{M}_i)$, which is generated by the set $v_i\{\mathbf{T}\}$ of (m,r)-bounded cardinality, and each element of $v_i\{\mathbf{T}\}$ has (m,r,n_ℓ)-bounded order. We conclude that the order of V_i/V_{i-1} is (m,r,n_ℓ)-bounded, which completes the proof. □

Exactly as in the case of lower central words, we obtain Theorems 2.1 and 2.2 as special cases of this last result.

Corollary 2.23. *Let w be an outer commutator word in r variables and let $u_1,\dots,u_r$ be non-commutator words. Then the word $w(u_1,\dots,u_r)$ is boundedly concise. In particular, $w(x_1^{n_1},\dots,x_r^{n_r})$ is boundedly concise for all $n_i \in \mathbb{Z} \setminus \{0\}$.*

Corollary 2.24. *Let w be an outer commutator word in r variables and let $\mathbf{N} = (N_1,\dots,N_r)$ be a tuple of normal subgroups of a group G. If $w\{\mathbf{N}\}$ is finite of order m, then the subgroup $w(\mathbf{N})$ is also finite, of (m,r)-bounded order.*

3

Counterexamples to Hall's conjecture

In this chapter we present some counterexamples to Hall's conjecture on the conciseness of words in groups.

In the first section, we describe the main ideas of small cancellation theory, which is the key technical tool to build counterexamples to Hall's conjecture. This theory utilises geometric diagrams on surfaces in order to obtain information on the structure of the groups.

In the second and third sections, we give a sketch of Ivanov's, Olshanskii's and Storozhev's counterexamples to Hall's conjecture that all words are concise in every group. All of these examples exhibit a specific word and a specific group where the word takes a single non-trivial value, but the associated verbal subgroup is infinite cyclic.

In the fourth section we obtain some original results regarding Olshanskii's word, following the preprint [68] of Shumyatsky and the author. We prove that this word is actually boundedly concise in residually finite groups. This is the first example of a word that is not concise in general, but is concise in residually finite groups. We then show that this word is also strongly concise in profinite groups.

In Section 5, we provide an answer to a question of [24] on generation of verbal subgroups in profinite groups. We construct a group with derived subgroup that is topologically finitely generated, but that cannot be generated by a finite set of commutators.

3.1 Elements of small cancellation theory

In this section we introduce some basics in small cancellation theory, which are crucial for the construction of the counterexamples to Hall's conjecture. We start with some notation.

Given a surface or a polygon X, we denote by $\partial(X)$ the boundary of X and by $\iota(X) = X \smallsetminus \partial(X)$ the interior of X. If we view an edge X of a polygon as a polygon itself, then $\partial(X)$ consists of the two endpoints.

When defining diagrams over groups, if a group G has a set S of generators, it will be useful to consider the set S^* of abstract words in the alphabet $S \cup S^{-1}$. In accordance to the notation introduced by Olshanskii, in this chapter we will denote words in S^* by capital letters C, L, M, X, Y, Z. We will write $|X|$ to denote the length of the word $X \in S^*$ and, if $X, Y \in S^*$, we will write $X \equiv Y$ (and say that X and Y are *visually equal*) if $|X| = |Y|$ and we have a letter-by-letter equality.

Definition 3.1 (Cells)**.** Consider a n-gon P in a plane with edges $e_1, \dots, e_n$. Consider a map $f : P \to X$, where X is any surface, such that:

- $f|_{\iota(P)}$ is an embedding;
- $f|_{\iota(e_i)}$ is an embedding for each $i \in \{1, \dots, n\}$;
- if $a, b \in P$ are distinct points with $f(a) = f(b)$, then $a, b \in \partial(P)$. If a is a vertex, so is b, otherwise if $a \in e_i$, $b \in e_j$, then $f(e_i) = f(e_j)$.

The image $f(P)$ of such a map is called a *cell* on X.

Informally, a cell is an identification of the n-agon P in X, but we allow vertices and edges to be pasted together by f, still preserving the structure of open disc of $\iota(P)$.

Definition 3.2 (Cell decomposition)**.** A *cell decomposition* of a surface X is a finite set $\{(P_i, f_i) \mid i = 1, \dots, m\}$ of cells such that $X = \bigcup_{i=1}^{m} f_i(P_i)$ and such that $f_i(P_i) \cap f_j(P_j)$, $i \neq j$ is either empty or it is a set of vertices and/or edges.

A cell decomposition can be thought as a partition of X into a finite set of cells, but allowing these cells to intersect in edges and/or vertices. The images of vertices or edges of any of the P_i will be called vertices and edges of the cell decomposition. Normally, we will denote a cell $f_i(P_i)$ with a single letter $\mathcal{C}$.

Even if the theory can be developed for arbitrary surfaces, we will only work with orientable surfaces. It will be useful to give an orientation to edges of a cell decomposition, by assuming that any edge e admits an inverse e^{-1}, which geometrically corresponds to the same element, but with inverse orientation.

Fix now an alphabet S and assign to each oriented edge e of the cell decomposition a label $\phi(e) \in S \cup S^{-1}$. If these labels are chosen such that $\phi(e^{-1}) = \phi(e)^{-1}$ for each edge e of the cell decomposition, we will moreover say that the decomposition is a *diagram*. If p is a path, obtained by concatenating some oriented edges $p = e_1 \cdots e_k$ the label of p is the word $\phi(p) = \phi(e_1) \cdots \phi(e_k) \in S^*$, where the endpoint of e_i coincides with the beginning of e_{i+1}.

Whenever the surface X underlying a diagram is a disc, it will be called a *circular diagram*. Notice that if X has a boundary, then it must consist of edges and vertices of the diagram, and therefore each of its connected components will have a label as a path. We will say that any connected component of the boundary ∂X of X is a *contour* of X. Moreover any cell $\mathcal{C}$ of a diagram can be seen as a disc (possibly with some parts of the boundary pasted together), and in this case the contour is equal to the boundary, and thus we will use the same notation $\partial \mathcal{C}$. We will use the convention that the label of the contour of every cell of a diagram over an orientable surface will be read clockwise, and similarly for the label of the contour of a circular diagram.

If we have a cell $\mathcal{C}$ in a diagram, the boundary $\partial \mathcal{C}$ is a path (induced by the orientation of the polygons), so we can always consider the label $\phi(\partial \mathcal{C})$ of the contour. The length of the contour of a cell or of a surface X is the number of edges of ∂X as a finite path and will be denoted by $|\partial X|$.

Normally, we simply study groups given by a presentation $G = \langle S \mid \mathcal{R} \rangle$. In our case, we will need to consider groups with graded relations, in the sense that we partition the set $\mathcal{R}$ of relations as $\mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{R}_i$ in such a way that no relator in $\mathcal{R}_i$ can coincide with a cyclic conjugate of a word in $\mathcal{R}_j$ or its inverse if $j \neq i$. In this setting, we will consider the graded presentation

$$G = \Big\langle S \mid \mathcal{R} = \bigcup_{i=1}^{\infty} \mathcal{R}_i \Big\rangle. \tag{3.1}$$

A cell in a diagram Δ is an $\mathcal{R}$-cell if its label is visually equal (up to cyclic conjugation) to a relator or an inverse of a relator in $\mathcal{R}$. If such relator is in the set $\mathcal{R}_i$, we will say that the cell is an i-cell, or a cell of rank i. Moreover we will say that it is a 0-cell (or a cell of rank 0) if its label is visually equal to a word ss^{-1} or $s^{-1}s$ for $s \in S$.

Definition 3.3 (Diagram over a group)**.** If G is a group given by the presentation (3.1), a *diagram over* G is a diagram Δ over the alphabet S such that all cells are either $\mathcal{R}$-cells or 0-cells. The rank of the diagram is the maximum among the ranks of its cells.

Notice that this definition depends on the presentation (3.1) chosen for G, not only on the group G itself. When studying diagrams over a group G, we want to study the simplest possible version of them. In our case, we will say that a circular subdiagram of rank j can be simplified if we can substitute it with a subdiagram of smaller rank with the same countour label. As it is shown in Section 13.2 of [66], a sequence of these operations can always lead to a *reduced* diagram, that is a diagram without subgraphs that can be simplified.

In 1933 van Kampen proved that some fundamental problems in group theory, like understanding if a word in the generators is the trivial element in the group, can be solved through the use of diagrams of groups. We will give a version of his result for reduced diagrams over graded groups, and refer to Theorem 13.1 of [66] for the proof.

Theorem 3.4 (van Kampen)**.** *Let W be a non-empty word in the alphabet $S \cup S^{-1}$. Then $W = 1$ in a group G with graded presentation (3.1) if and only if there exists a reduced circular diagram over G such that the label of its contour is visually equal to W.*

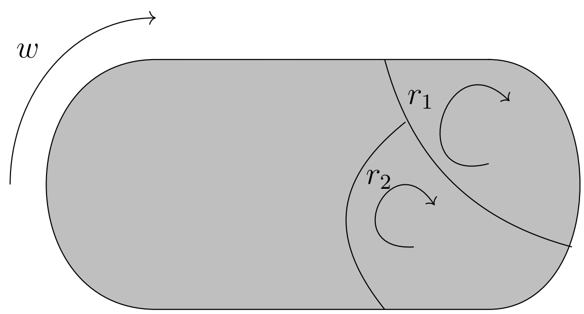


Figure 3.1: Van Kampen's Lemma

The name "small cancellation theory" is due to the fact that we often require that different relators have a small overlapping. This can be made more precise with the following definition.

Definition 3.5 (Pieces and Condition $C'(\lambda)$)**.** Let G be a group with graded presentation (3.1), and let R_1, R_2, $R_1 \neq R_2$, be two cyclic conjugates of two relators or inverses of relators in $\mathcal{R}$. A word X in the alphabet $S \cup S^{-1}$ is a *piece* if R_1 and R_2 are visually equal to words of the form XY_1 and XY_2 respectively.

The presentation (3.1) satisfies *small cancellation condition* $C'(\lambda)$ for a number $0 < \lambda \leq 1$ if, whenever R is a cyclic conjugate of a relator, or of an inverse of a relator, such that it is visually equal to XY for a piece X, then $|X| < \lambda |R|$.

Example 3.6. The group $\langle a, b \mid aba^{-1}b^{-1} \rangle$ satisfies $C'(\lambda)$ for all $\lambda > 1/4$. Pieces consist of single letters or their inverses.

The surface group $\langle a, b, c, d \mid [a, b][c, d] \rangle$ satisfies $C'(\lambda)$ for all $\lambda > 1/8$, and as before pieces consist of single letters or their inverses.

It is possible to see that any group can have a presentation satisfying condition $C'(\lambda)$ for $\lambda > \frac{1}{5}$ (Gol'berg, see Section 12.4 of [66]), but if we ask λ to be smaller, it allows us to obtain interesting conditions on the groups.

Theorem 3.7 (Greendlinger's, Theorem 12.1 [66])**.** *Let Δ be a reduced circular diagram over a presentation of a group G that satisfies $C'(\lambda)$ for $\lambda \leq \frac{1}{6}$ with at least one $\mathcal{R}$-cell. Suppose that the label $\phi(\partial\Delta)$ is cyclically reduced and has no proper subword equal to the identity. Then there exists an exterior arc p (i.e. a path $p \in \partial C \cap \partial\Delta$) of some $\mathcal{R}$-cell C satisfying $|p| > \frac{1}{2}|\partial C|$.*

This theorem has crucial implications in combinatorial group theory because, if there exists a presentation (3.1) of a group G satisfying $C'(\lambda)$ for $\lambda \leq \frac{1}{6}$, it is possible to define an algorithm (called *Dehn's algorithm*) that in a finite number of steps can recognize if a word $W \in S^*$ is equal to the identity in G. Moreover this combinatorial condition has strong geometric implications, namely the group G is an hyperbolic group.

Even if the presentations we will use will not satisfy any $C'(\lambda)$ condition, we will find an an analogue of Greendlinger's Theorem in groups satisfying weaker small cancellation conditions.

3.2 Ivanov's counterexample

In 1989 Ivanov obtained the first counterexample to P.Hall's conjecture stating that all words are concise in the class of all groups. More precisely, he provided a word w_I and a group I in which w_I is not concise.

Theorem 3.8 ([43])**.** *The word*

$$w_I(x, y) = [[x^{pn}, y^{pn}]^n, y^{pn}]^n$$

for $n > 10^{10}$ odd and $p > 5000$ prime, takes only two values $\{1, z\}$ in a torsionfree 2-generated group I but the verbal subgroup $w_I(I) = \langle z \rangle$, that corresponds to the center of the group I, is infinite cyclic.

This group is a central extension of an infinite two generated group $G_I(\infty)$ of bounded exponent, which is constructed using small cancellation theory.

We first need a crucial result in central extensions. We recall that a set $\mathcal{R}$ of relations for a group $G = \langle S \mid \mathcal{R}\rangle$ is *independent* if no proper set $\mathcal{R}' \subseteq \mathcal{R}$ of relations gives the same group G (with the identity map on S).

Theorem 3.9. *Suppose that the group $G = \langle S \mid \mathcal{R}\rangle$ can be considered as $G \cong F/N$, with F being the free group with basis S and $N = \langle \mathcal{R}\rangle$. Then*

- $\overline{G} = F/[F,N]$ *is a central extension of $G = F/N$, i.e. $\overline{N} = N[F,N]$ is contained in the center of $\overline{G}$. Moreover, if G is centerless, then $\overline{N} = Z(\overline{G})$;*
- $\overline{G} = \langle S \mid [r,s] \text{ for } r \in \mathcal{R}, s \in S\rangle$;
- *if $\mathcal{R}$ is an independent set of relations for G, then $\overline{N}$ is a free abelian group with basis $\overline{\mathcal{R}} = \mathcal{R}[F,N]$.*

For the proof, we refer to Chapter 31 of [66], in particular to Theorem 31.1 and the discussion above.

In the following we construct the group I, giving an idea of the arguments involved in the proof that w_I takes a single non-trivial value in I. In order to do so, we first construct the group $G_I(\infty)$, which is a variation of the free Burnside group constructed by Olshanskii in [64] by inductively imposing (possibly different) torsion to elements of a free group, and by obtaining a torsion group as the limit of all of these quotients.

Let $F_2 = F(a,b)$ the free group in two letters and let $V, W \in F_2$ be elements of the free group. Denote by $|V|$ the minimal length of V as a word in the alphabet $\{a, b, a^{-1}, b^{-1}\}$. Fix an ordering in F_2 such that if $|V| < |W|$, then $V < W$ (but we do not necessarily have to choose lexicographic order for words of the same length). For each $i \geq 1$ we inductively construct the groups $G_I(i)$. Define $G_I(0) = F_2$, then assume we already constructed $G_I(i-1)$. Let $C_i \in F_2$ be the smallest word (with respect to the fixed ordering of F_2) corresponding to an element of infinite order in $G_I(i-1)$, such a word will be called *period of rank i*. Define

$$G_I(i) = G_I(i-1)/\langle C_i^{n_i}\rangle^{G_I(i-1)},$$

where $\langle C_i^{n_i}\rangle^{G_I(i-1)}$ is the normal closure of $C_i^{n_i}$ and n_i is a odd number greater than $n = 10^{10}$. The limit of these quotients is the group

$$G_I(\infty) = F_2/\langle C_i^{n_i} \mid i \in \mathbb{N}\rangle^{F_2}. \tag{3.2}$$

When $n_i = n$ for every i we obtain the free Burnside group $B(2,n)$. In this construction, by using diagrams on groups, Olshanskii proved that the set $\{C_i^{n_i} | i \in$

$\mathbb{N}\}$ is an independent set of relations (i.e. no proper subsets of relations defines the same group $G_I(\infty)$), that every word of finite order in $G_I(i)$ is conjugate to a power of a period C_j for $j \leq i$ (so in the torsion group $G_I(\infty)$ all words are conjugate to a power of a period), and most importantly that the group obtained in this way is infinite and with trivial center.

Ivanov's group I will be obtained as a central extension of $G_I(\infty)$ for some specific choices of the exponents n_i. In this case, we must choose some relators in a different way and we will impose some specific periods to have different orders. In detail, for $n > 10^{10}$ odd and $p > 5000$ prime we require that the words C_i satisfy that:

- the smallest word of length 1 is $C_1 = B_1 = a$ and we impose it to have order $p^2 n$;
- the smallest word C_i of length $4(pn+1)$ is $B_2 = [ba^{pn}b^{-1}, a^{pn}]$ and we impose it to have order pn;
- the 8 smallest words C_i of length $8n(pn+1)$ will be $B_3, \dots, B_{10}$
$$[[b^{\varepsilon_1} a^{\varepsilon_2 pn} b^{-\varepsilon_1}, a^{\varepsilon_3 pn}]^n, a^{\varepsilon_3 pn}]$$
for $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$. We impose these 8 words to have order n;
- all the other words C_i will have order n.

In Lemma 2 of [43] it is proved that the word C_i has infinite order in $G_I(i-1)$ (and hence it can be chosen to be a period of appropriate rank) and that the group $G_I(\infty)$ with presentation (3.2) obtained by imposing these restrictions is infinite.

The group I is obtained as the quotient

$$I = F_2 / \langle [C_i^{n_i}, T], C_i^{n_i} = C_j^{n_j} \mid i, j \in \mathbb{N}, T \in F_2 \rangle^{F_2}.$$

In accordance to Theorem 3.9, if we considered only the first set of relators $\{[C_i^{n_i}, T]\}$ we would obtain a central extension of $G_I(\infty)$. The center of such a group would be a free abelian group with infinite basis $\{C_i^{n_i} \mid n \in \mathbb{N}\}$, but by adding the relators $C_i^{n_i} = C_j^{n_j}$ for all $i, j \in \mathbb{N}$, we obtain a cyclic center, generated by a single element $z = C_i^{n_i}$ for every $i \in \mathbb{N}$. Now we want to show that the only non-trivial value taken by the word w_I in the group I is exactly z.

Following the steps of a proof in [64], Ivanov proved the following result (Lemma 1 of [43]), which has a clear analogy to Theorem 3.7.

Lemma 3.10. *Let Δ be a reduced annular diagram or diagram on a disc with two holes over $G_I(\infty)$ (with presentation 3.2, C_i as in the previous paragraph) such that the labels of the contour segments are cyclically unshortenable. If Δ contains at least an $\mathcal{R}$-cell, then there exists a cell C with $\partial C \cap \partial\Delta = p$ for a path p such that $|p| \geq 10^{-4}|\partial C|$.*

To show that z is indeed the only value assumed by the word, we need to "funnel" the values of some subwords of the word w_I. We will explicitly explain the first steps to show the use of diagrams over groups and to understand why we need the diversification of the exponents.

Suppose first that X and Y are two words in the alphabet $S = \{a, b\}$. We first assume that $[X^{pn}, Y^{pn}]^n = 1$ in $G_I(\infty)$, in which case $[X^{pn}, Y^{pn}]^n$ is in the center of I so $v(X, Y) = 1$. As all words are conjugate to a power of a period in $G_I(\infty)$, if X or Y are conjugate to a period different than $C_1 = a$, then either X^{pn} or Y^{pn} are equal to the identity in $G_I(\infty)$ (as all the periods different from $C_1 = a$ have order dividing pn) and we are in the case $[X^{pn}, Y^{pn}]^n = 1$ in $G_I(\infty)$, that we already considered. We can therefore assume that $X = L^{-1}a^{t_1}L$ and $Y = M^{-1}a^{t_2}M$ for some words $L, M \in S^*$, $t_1, t_2 \in \mathbb{Z}$, and that $[X^{pn}, Y^{pn}]^n \neq 1$ in $G_I(\infty)$.

In this case, the word $[X^{pn}, Y^{pn}]$ must be conjugate in $G_I(\infty)$ to a power of $B_1 = a$ or B_2, being the only periods with order not dividing n in $G_I(\infty)$. We want to prove that it cannot be conjugate to a power of a. Suppose by contradiction it is possible. Then, for a certain $N \in S^*$, we would have

$$[L^{-1}a^{t_1pn}L, M^{-1}a^{t_2pn}M] = N^{-1}a^{t_3}N.$$

By Theorem 3.4, and then pasting the paths of the contour with label N and N^{-1}, we can construct a diagram Δ over $G_I(\infty)$ on a disc with an hole such that the exterior contour has label a^{t_3} and the interior contour, read clockwise, has label $[L^{-1}a^{t_1pn}L, M^{-1}a^{t_2pn}M]$ (see Figure 3.2). We can now paste together the two segments of the internal contour with label $LM^{-1}a^{t_2pn}ML^{-1}$ and its inverse respectively, so we get a diagram on a disc with two holes and the contours are a^{t_3}, a^{t_1pn} and a^{-t_1pn} (Figure 3.3). By refining it if necessary, we can assume the diagram to be reduced.

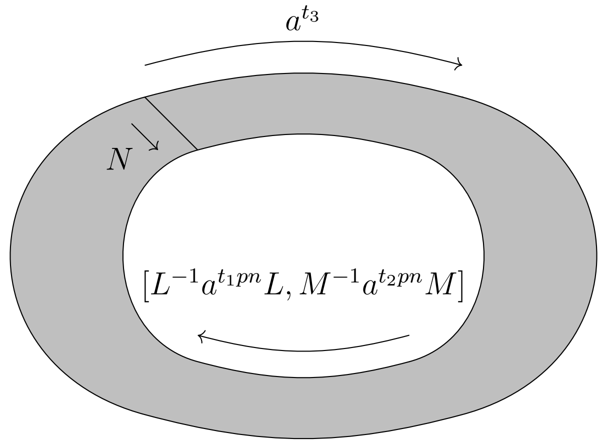


Figure 3.2: Pasting contours with label N

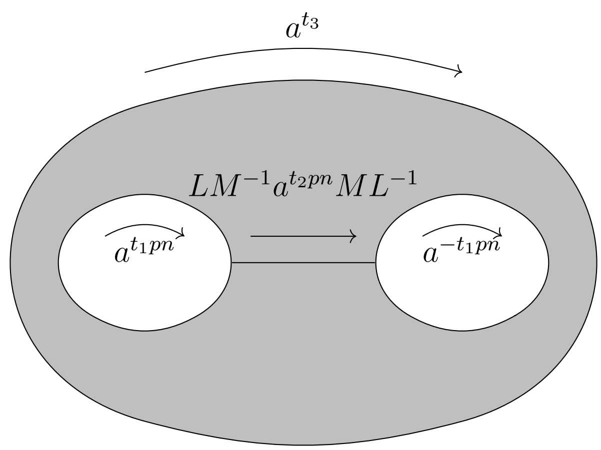


Figure 3.3: Pasting contours with label $LM^{-1}a^{t_2pn}ML^{-1}$

Now use small cancellation theory: by Lemma 3.10, if Δ contains at least an $\mathcal{R}$-cell, there exists a cell $\mathcal{C}$ with contour label $C_j^{n_j}$ for a certain $j \in \mathbb{N}$ such that it has a boundary arc p of length at least $10^{-4}|C_j^{n_j}|$. As $10^{-4}n_i \geq 2$ for every $i \in \mathbb{N}$, C_j^2 must be a subword of a^k for $k = \pm\, t_1pn$ or $k = \pm\, t_3$, so $C_j = C_1 = a$. We can now excise the cell $\mathcal{C}$, in the sense that we remove $\mathcal{C}$ and, if $\partial\mathcal{C} = pq$ with p being the boundary arc $\mathcal{C} \cap \delta\Delta$, the new contour of Δ will follow the path q in place of the previous boundary arc p (Figure 3.4).

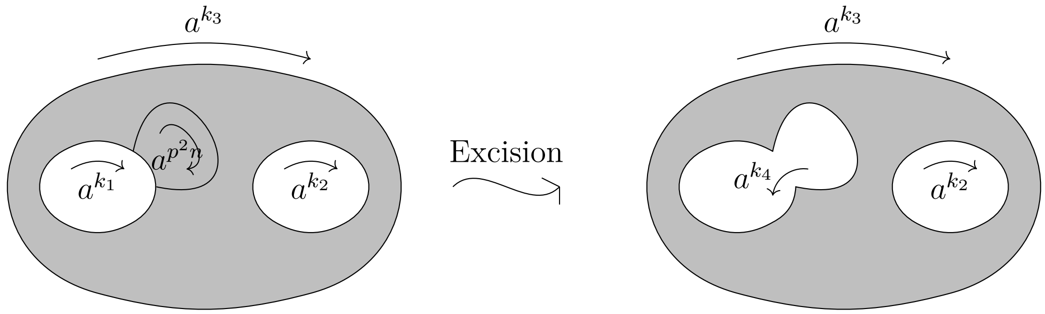


Figure 3.4: Excision of a cell

By excising all the cells of this type, with labels $a^{\pm p^2 n}$, we change the exponents of some labels of the contour, but not their residual class modulo $p^2 n$. After having excised all these cells, we have a diagram on a disc with two holes and by Lemma 3.10 it cannot contain any $\mathcal{R}$-cell (and in particular the disc with two holes is degenerate, with no interior, Figure 3.5). Looking at the final diagram, we can notice that the label of the exterior boundary is equal to the label of the path obtained by concatenating the two interior boundaries. This implies that $t_3 \equiv pn(t_1 - t_1) \equiv 0 \pmod{p^2 n}$ so $[L^{-1}a^{t_1 pn}L, M^{-1}a^{t_2 pn}M] = N^{-1}a^{t_3}N = 1$ in $G_I(\infty)$, and this contradicts our assumptions.

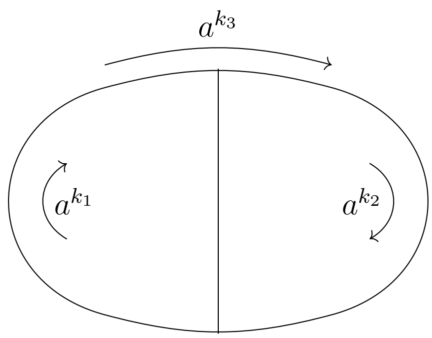


Figure 3.5: Final diagram, after exicisions

With similar arguments, by means of congruences preserved by cell excision in diagrams, Ivanov proved that if $[X^{pn}, Y^{pn}]^n \neq 1$, then this word is conjugate to either B_2 or B_2^{-1}, not to a proper power of them. Still applying the same ideas, but with more complicate congruences, he also proved that $[[X^{pn}, Y^{pn}]^n, Y^{pn}]$ must be a conjugate of exactly one of the words $B_3, \ldots, B_{10}$. We refer to the last part of

Lemma 3 in [43] for the explicit computations. This is sufficient to conclude that the only non-identical value of the word must be z.

A further interesting remark is that, as it is written in the acknowledgements of [43], the anonymous referee claimed that, using Adian's arguments of [6], the word $w_A = [x^r, y^r]^n$ takes exactly two values in a central extension (with cyclic center) of the free Burnside group $B(2, n)$ for odd $n = 3r \geq 1005$. This claim has not been proved, but it would provide the first example of a word that is concise (in this case $[x^r, y^r]$, see [22], or Theorem 2.13) but such that its power is not concise. Notice that if such a claim was true, using that each inverse of a w_A-value is still a w_A-value, the word w_A would have to take at least three values (the identity, a non-trivial element of infinite order and its inverse). Even with this correction, this remains only a claim and the proof is not a straightforward adaptation of Ivanov's methods.

In view of Conjectures 1.17 and 1.20, it is natural to ask whether the counterexample obtained by Ivanov provides a negative answer to these questions too. However, it is well known that the group I is not residually finite, thus cannot be used to obtain a counterexample to the aforementioned conjectures in a straightforward way. We now provide a proof of this fact.

Lemma 3.11. *Let G be a residually finite group and let N be the marginal subgroup of a word $w(x_1, \dots, x_n)$. Then G/N is residually finite.*

Proof. Let $g \in G$ such that $g \notin N$, in particular there exists an index $i \in \{1, \dots, n\}$ and some elements $h_1, \dots, h_n \in G$ such that

$$t = w(h_1, \dots, h_i g, \dots, h_n) w(h_1, \dots, h_i, \dots, h_n)^{-1} \neq 1$$

By residually finiteness there exists a normal subgroup M of finite index in G such that $t \notin M$. We claim that $g \notin MN$. If it was, let $m \in M, n \in N$ such that $g = mn$. As N is marginal for w, we would have that

$$\begin{aligned} t =& w(h_1, \dots, h_i mn, \dots, h_n) w(h_1, \dots, h_i, \dots, h_n)^{-1} = \\ & w(h_1, \dots, h_i m, \dots, h_n) w(h_1, \dots, h_i, \dots, h_n)^{-1} \end{aligned}$$

but this would imply that $t \equiv 1 \pmod{M}$, contradicting our choice of M. This proves that for every $g \notin N$ there exists a finite index subgroup MN such that $g \notin MN$, as desired. □

Corollary 3.12. *Any finitely generated group which is central-by-(infinite group of finite exponent) cannot be residually finite. In particular, Ivanov's group I is not residually finite.*

Proof. By Lemma 3.11 if I was residually finite, using that the center is the marginal subgroup for the commutator word, then also the quotient $I/Z(I)$ would be residually finite. By Zelmanov's solution of the Restricted Burnside problem (see [86], [87]), finitely generated residually finite groups of finite exponent are finite, obtaining a contradiction with the fact that $I/Z(I)$ is infinite. □

3.3 Olshanskii's counterexample

In his article [65], and subsequently in his book [66], Olshanskii studied the words

$$v_O(x,y) = [[x^d, y^d]^d, [y^d, x^{-d}]^d]$$

and

$$w_O(x,y) = [x,y]v_O(x,y)^n[x,y]^{\varepsilon_1}v_O(x,y)^{n+1}\cdots[x,y]^{\varepsilon_{h-1}}v_O(x,y)^{n+h-1} \qquad (3.3)$$

where

$$\varepsilon_{10k+1} = \varepsilon_{10k+2} = \varepsilon_{10k+3} = \varepsilon_{10k+5} = \varepsilon_{10k+6} = 1$$

$$\varepsilon_{10k+4} = \varepsilon_{10k+7} = \varepsilon_{10k+8} = \varepsilon_{10k+9} = \varepsilon_{10k+10} = -1$$

for $k = 0, 1, \dots, (h-1)/10$; $n > 10^{10}$ odd and $h \equiv 1 \mod 10$, $h > 50000$, d and n are integers "big enough". The minimal bounds for d and n are not immediate to get but can be recovered from the article [65] or from Chapters 29 and 30 of [66].

The choice of the ε_i is due to the need of a tuple of numbers that is not equal to its opposite, mirror image or cyclic shift. In the construction of $w_O(x,y)$, the word $v_O(x,y)$ takes the role of a "disturbing noise", in the sense that w_O is substantially different from a commutator word, but when we remove the disturbance, $w_O(x,y) = [x,y]$. This immediate fact that can be directly observed from the definition of w_O.

Lemma 3.13. *In every group where $v_O(x,y)$ is a law, and in particular in metabelian groups, the values of the word $w_O(x,y)$ coincide exactly with the values of the commutator word $[x,y]$.*

By using that every finite non-abelian group contains a non-abelian metabelian group (Corollary 6.1 of [66]), we can recover this straightforward result.

Theorem 3.14 (Lemma 29.1 of [66])**.** *Every finite group where w_O is a law is abelian.*

Proof. Consider a non-abelian metabelian subgroup H of a group G with $w_O(G) = 1$. As H is metabelian, $v_O(H) = 1$, and in that case, by Lemma 3.13, $1 = w_O(H) = H'$, contradicting that H is not abelian. $\square$

In Theorem 30.1 of the same book, Olshanskii proved that there are non-abelian infinite groups in the variety generated by w_O, like the relatively free group in two generators $G_O(\infty)$. This was used to prove that the variety generated by w_O cannot be generated by finite groups, as otherwise it would be a sub-variety of the abelian one. This provides a negative answer to a classical question of H. Neumann in [58], who asked whether any variety can be generated by its finite elements. However, our main interest lies in a different result appearing in the same book, precisely Theorem 39.7 of [66].

Theorem 3.15 (Theorem 39.7 of [66])**.** *There is a group O where w_O takes a single value, but $w_O(G)$ is infinite.*

The main idea of the proof of Olshanskii is similar to the proof of Ivanov, but he makes use of some results that he proved in the construction of the infinite non-abelian group in the variety generated by w_O, which is actually the aforementioned relatively free group $G_O(\infty)$ on a set of two generators $S = \{a, b\}$. The main difference between Ivanov's and Olshanskii's proofs is that in the former the quotient $I/Z(I)$ is torsion, and the torsion is used in the "funneling" of the values, whereas in the latter the quotient $O/Z(O) \cong G_O(\infty)$ is torsionfree.

The group $G_O(\infty)$ will be obtained with a graded presentation like (3.1), but relators will be values of the word w_O rather than power words. This is another difference compared to Ivanov's constructions of $G_I(\infty)$, as in that case the relators $\mathcal{R}_i$ were powers words $C_i^{n_i}$, that are not values of v_I. In order to obtain this, he partitioned the set of pairs $(X, Y) \in S^* \times S^*$, with each equivalence class represented by a pair $(\overline{X}, \overline{Y})$. We will call these representatives *O-pairs*. For each of these pairs, he added a single appropriately chosen relation $\mathcal{R}(\overline{X}, \overline{Y})$, which is a value of the word w_O (in the book, the role of O-pairs is taken by the so-called "generalized (A, j)-triples"). It must be noted that he used again an inductive construction, in the sense that the final presentation of $G_O(\infty)$ will be graded like in (3.1), and the equivalence class of a certain O-pair $(\overline{X}_i, \overline{Y}_i)$ is chosen depending on periods in $G_O(i-1) = \langle S \mid \bigcup_{j=1}^{i-1} \mathcal{R}_j \rangle$. The set $\mathcal{R}_i$ will consist of the relation $\mathcal{R}(\overline{X}, \overline{Y})$ associated to this O-pair.

If $G_O(\infty) = \langle S \mid \mathcal{R} \rangle$, we can define again the maximal central extension $\overline{O}$ as

$$\overline{O} = F_2/\langle [R, T] \mid R \in \mathcal{R}, T \in F_2 \rangle^{F_2},$$

where $F_2 = F(a,b)$ is the free group generated by S. As usual, we use capital letters X, Y to denote elements of S^*, that can be therefore considered as elements both of F_2 or of some appropriate quotients.

As w_O is a law in $G_O(\infty)$, the values of w_O in $\overline{O}$ are contained in the center and by construction $\overline{O}/Z(\overline{O}) \cong G_O(\infty)$. In Lemma 39.11 of [66], the author proved that if $[X_1, Y_1] = [X_2, Y_2]$ in $G_O(\infty)$, then $[X_1, Y_1] = [X_2, Y_2]$ in $\overline{O}$. Subsequently, he proved that every couple (X, Y) is conjugate in $G_O(\infty)$ to an O-pair, and hence it suffices to study $w_O(X, Y)$ when (X, Y) runs over the set of O-pairs used in the construction of $G_O(\infty)$. By Theorem 3.9, this set of w_O-values generates $w_O(\overline{O}) = Z(\overline{O})$, and each O-pair gives rise to a different w_O-value because the sets $\mathcal{R}_i$ of relators were independent. Thus $Z(\overline{O}) \cong \mathbb{Z}^{\mathbb{N}}$. The group $O = \overline{O}/N$ is obtained by quotienting out $\overline{O}$ by the normal subgroup N generated by all but one of the generators of $Z(\overline{O}) \cong \mathbb{Z}^{\mathbb{N}}$.

Both in Ivanov's and Olshanskii's examples, the way to pass from $\overline{I}$ to I and from $\overline{O}$ to O was by quotienting out a subgroup of the center, but there are several ways to choose such subgroup. In [43], the author wondered whether it was possible to construct an analogous counterexample, but in a relatively free group in a variety. This question has been answered by Storozhev, providing a third counterexample to Hall's conjecture. He proved in [77] that taking

$$v_S(x, y) = [(x^d y^d)^d x^d, x^d]$$

and w_S defined as 3.3 with v_S in place of v_O, there exists a relatively free group S where w_S takes only one non-trivial value. In Olshanskii's example, we could determine the value of $w_O(X, Y)$ from the behaviour of $[X, Y]$ in the quotient $G_O(\infty)$. In this case, we can obtain some information on $w_S(X, Y)$ by looking at the words X, Y in the quotient F_2/F_2' instead.

We can define a central extension $\overline{S}$ of a torsion group exactly as we did for $\overline{O}$, but with w_S taking the role of w_O. Storozhev then defined the subgroup $V \leq F_2$ generated by $w(X, Y)$ for all couples (X, Y) such that X, Y do not form a basis of F_2/F_2' and by all $w(X_1, Y_1)w(X_2, Y_2)^{-1}$ for all couples $(X_1, Y_1), (X_2, Y_2)$ that are both a basis of F_2/F_2'. He then proved that if two different w_S-values $w_S(X_1, Y_1)$ and $w_S(X_2, Y_2)$ are equal in $\overline{S}$, then $X_1 \equiv X_2 \pmod{F_2'}$ and $Y_1 \equiv Y_2 \pmod{F_2'}$. This implies that the subgroup V is fully invariant and hence, by Theorem 1.7, it is a verbal subgroup of F_2. If $\overline{S} = F_2/N$ (noticing that it is the 2-generated relatively free group in the variety corresponding to the word $[w_S, z]$), the group $S = F_2/NV$ is relatively free and by definition of V, $w_S\{S\} = \{1, z\}$ with $z = w_S(X, Y)$ for a pair $(X, Y) \in S^*$ that is a basis of the abelian quotient F_2/F_2'.

As the quotients $O/Z(O)$ and $S/Z(S)$ are torsionfree, we cannot conclude that O and S are not residually finite as we did with Ivanov's group I. We will obtain the non-residually finiteness of O as a consequence of w_O being concise in residually finite groups in the next section, but it is currently unknown whether S is residually finite.

We finish this section by summarizing the main differences of the three counter-examples we discussed.

	Ivanov	Olshanskii	Storozhev
Is $G/Z(G)$ torsion?	Yes	No	No
How to funnel the values?	Using different orders in $I/Z(I)$	Using O-pairs, through $O/Z(O)$	Using bases (X, Y) of F_2/F_2'
Is the group residually finite?	No (3.12)	No (3.16)	Unknown

3.4 Olshanskii's word in profinite groups

First of all, we prove that the word w_O defined by Olshanskii is boundedly concise in residually finite groups. We recall that Theorem 1.15 cannot be used in this setting, as it is currently unknown whether words that are concise in the class of residually finite groups are also boundedly concise within that class.

Theorem 3.16. *The word w_O is boundedly concise in residually finite groups.*

Proof. Let m be a positive integer and G a residually finite group in which w_O takes m values. In view of Lemma 1.12 there is a number $f_1(m)$ depending only on m such that $|w_O(G)'| \leq f_1(m)$. If Q is a finite homomorphic image of G, observe that the quotient $Q/w_O(Q)$ is abelian by Theorem 3.14. Hence $Q/w_O(Q)'$ is metabelian, so Lemma 3.13 implies that $Q/w_O(Q)'$ has at most m commutators. Note that the commutator word is boundedly concise (see for example [74] for an explicit bound), so $|w_O(Q/w_O(Q)')| \leq f_2(m)$, for an integer $f_2(m)$ depending only on m. Hence $|w_O(Q)| \leq f_1(m)f_2(m)$. This holds for every finite homomorphic image Q of G so we deduce that $|w_O(G)| \leq f_1(m)f_2(m)$ too. □

It is immediate that the group O in Theorem 3.15 is not residually finite.

We will now prove that the word w_O is strongly concise in profinite groups, but we will first need a classical result on conjugacy classes.

Lemma 3.17 ([24] Lemma 2.2)**.** *Let G be a profinite group and $g \in G$ be an element whose conjugacy class g^G contains less than $2^{\aleph_0}$ elements. Then g^G is finite.*

In order to prove the general result, we first have to consider two cases: first we will assume that G is a cartesian product of finite simple groups and then we will study the problem when G is prosolvable.

Lemma 3.18. *Let G be a profinite group topologically isomorphic to a Cartesian product of finite non-abelian simple groups. If the word w_O takes less than $2^{\aleph_0}$ values in G, then G is finite.*

Proof. Write $G = \prod_{i \in I} S_i$, where the factors S_i are finite non-abelian simple groups. By Theorem 3.14, $w(S_i)$ is nontrivial for every $i \in I$. Now we only need to show that the index set I is finite.

Assume by contradiction that I is infinite and choose a nontrivial w_O-value $c_i \in S_i$ for each $i \in I$. Observe that for each subset $J \subseteq I$ the product $c_J = \prod_{i \in J} c_i$ is a w_O-value. If $J_1 \neq J_2$, then $c_{J_1} \neq c_{J_2}$ and therefore G contains at least $2^{\aleph_0}$ distinct w_O-values, a contradiction. □

Lemma 3.19. *Let G be a prosolvable group. If the word w_O takes less than $2^{\aleph_0}$ values in G, then the commutator subgroup G' is finite.*

Proof. By Lemma 3.13, $w_O(G/G'') = G'/G''$. Moreover, as the commutator word is strongly concise in profinite groups (see Theorem 1.1 in [24]), G'/G'' is finite and therefore there exists a finite set T of w_O-values such that $G' = \langle T \rangle G''$. Note that by Lemma 3.17 each element of T has finitely many conjugates in G. So we can choose T in such a way that the subgroup $\langle T \rangle$ is normal in G. Set $\overline{G} = G/\langle T \rangle$. Observe that $\overline{G}$ is a prosolvable group with the property that $\overline{G}' = \overline{G}''$. It follows that $\overline{G}$ is abelian and so $G' = \langle T \rangle$.

Now by Lemma 3.17, for each $t \in T$ we have that $[\langle T \rangle : C_G(t)] < \infty$, whence $[\langle T \rangle : Z(\langle T \rangle)] < \infty$. By Schur's Theorem, the commutator subgroup $\langle T \rangle'$ is finite. This implies that G is finite-by-metabelian. Factoring out G'' we can assume that G is metabelian and apply again Lemma 3.13. Since the commutator word is strongly concise, we conclude that G' is finite and generated by finitely many w_O-values, as required. □

By Hall-Higman theory, if K is a finite group, then there exists a series

$$1 = K_0 \leq K_1 \leq \cdots \leq K_{2h+1} = K \tag{3.4}$$

of normal subgroups of K such that K_{i+1}/K_i is either solvable (possibly trivial) if i is odd, or a cartesian product of non-abelian simple groups if i is even. The number of non-solvable factors in this series is called the insoluble length $\lambda(K)$ of K. Theorem 1.4 of [46] implies that if the Sylow 2-subgroup of K is solvable with derived length l, then $\lambda(K)$ is bounded in terms of l only. We are now ready to complete the proof that the word w_O is strongly concise in profinite groups.

Theorem 3.20. *Let G be a profinite group in which the word w_O takes less than $2^{\aleph_0}$ values. Then the verbal subgroup $w_O(G)$ is finite.*

Proof. Choose a 2-Sylow subgroup P of G. In view of Lemma 3.19 observe that P is solvable, say of derived length l. It follows that if Q is any finite homomorphic image of G, the insoluble length $\lambda(Q)$ is bounded in terms of l only. Let $\mathcal{C}$ be the class of finite groups with a series as in (3.4) of fixed length. Lemma 2 of [83] states that any pro-$\mathcal{C}$ group has a series of normal subgroups

$$1 = G_0 \leq G_1 \leq \cdots \leq G_{2h+1} = G \tag{3.5}$$

of the same length such that G_{i+1}/G_i is prosolvable (possibly trivial) if i odd, or an inverse limit of (finite direct products of non-abelian simple groups) if i is even. Lemma 3 of [83] assures that, in the second case, such a profinite group is a cartesian product of finite non-abelian simple groups.

As $\lambda(Q)$ is l-bounded for each finite quotient Q of G, we obtain that G has a normal series like (3.5) of l-bounded length.

Lemma 3.18 shows that the non-prosolvable factors of this series are finite. Moreover, as in each simple group w_O is non-trivial, we have that $w_O(G_{i+1}/G_i) = G_{i+1}$ whenever i is even. In this case, we can obtain a finite normal set $T_i \subseteq w_O\{G\}$ such that $\langle T_i \rangle G_i = G_{i+1}$. On the other hand, if i is odd, by Lemma 3.19 there exists a finite normal set $T_i \subseteq w_O\{G\}$ such that $G_{i+1}/\langle T_i \rangle G_i$ is abelian. Let $\overline{T} = \bigcup_{i=1}^{2h+1} T_i$. Overall, $G/\langle \overline{T} \rangle$ is a prosolvable group, and applying again Lemma 3.19, we conclude that it is finite-by-abelian, with derived subgroup generated by a finite normal set $\widetilde{T}$ of w_O-values.

Let $T = \overline{T} \cup \widetilde{T}$. As we did in Lemma 3.19, we can obtain that the center of $\langle T \rangle$ has finite index and by Schur's Theorem T is finite-by-abelian. In particular G is finite-by-metabelian and applying again Lemma 3.19, we conclude that G is finite-by-abelian. As $w_O(G) \leq G'$, w is strongly concise in profinite groups.

□

In [24], the authors proved strong conciseness of several classes of group words, like words implying virtual nilpotency or weakly rational words, under the additional assumption that, if the verbal subgroup $w(G)$ is finitely generated, then it can be generated by finitely many w-values. If $w(G)$ is a pro-p group, then by looking at the quotient $w(G)/\Phi(w(G))$ and using Burnside basis theorem, it is immediate to see that $w(G)$ is finitely generated if and only if it is generated by finitely many w-values. The authors asked whether this is always true.

Conjecture 3.21. Let G be a profinite group and w be a word. If $w(G)$ is topologically finitely generated, it can be generated by finitely many word values.

We will now show that this question has a negative answer for lower central words $w = \gamma_k$.

Theorem 3.22. *There is a profinite group G such that the subgroup $\gamma_k(G)$ is procyclic for every k, but it cannot be generated by finitely many γ_k-values.*

Clearly the group G in our construction cannot be finitely generated otherwise, as Nikolov and Segal proved in [59][60], all the abstract subgroups of the lower central series would be closed. In that case, whenever a verbal subgroup $w(G)$ is finitely generated, it is also generated by finitely many w-values: since it coincides with an abstract subgroup that is finitely generated, each generator is a finite word in the alphabet $w\{G\}$, so the subgroup itself is also generated by finitely many w-values.

A special case of the question we are interested in is when the derived subgroup is procyclic. Under this more restrictive hypothesis, is it true that it is generated by a single commutator? This question was studied, in the setting of abstract groups and cyclic subgroups, by Macdonald in [56]. He proved the following result.

Theorem 3.23 (Macdonald)**.** *Let G be an abstract group and assume G' is cyclic. If either G is nilpotent or G' is infinite, then G' is generated by a suitable commutator. In general, for any given positive integer k, there is a finite group M_k such that M_k' is cyclic but it cannot be generated by less than k commutators.*

The main tool in our proof is the group M_k in the second part of the proposition, so we will give an idea of its structure. Fixed k, set $m = 2^{2k} - 1$ and pick a set of different odd primes $p_1, \ldots, p_m$, chosen arbitrarily. The group M_k will be a semidirect product $C \rtimes C_2^{2k}$, where $C_2^{2k} = \langle a_1, \ldots, a_{2k} \rangle$ is the direct product of

$2k$ copies of the cyclic group of order 2 and $C = \langle c \rangle$ is a cyclic group of order $p_1 p_2 \cdots p_m$.

Assume $[c, a_i] = c^{\alpha_i}$ for some integer α_i. Macdonald proved, with the use of some accurately chosen congruences, that it is possible to select the integers α_i in the construction of M_k in such a way that the derived subgroup is the whole C and that for any set of $k-1$ commutators $g_1, \ldots, g_{k-1}$ there is a prime $p_j \in \{p_1, \ldots, p_m\}$ such that

$$\langle g_1, \ldots, g_{k-1} \rangle = \langle c^{p_j} \rangle.$$

In [45], Kappe observed that $\gamma_2\{M_k\} = \gamma_j\{M_k\}$ for all $j \geq 2$, hence the following result is a direct consequence of Macdonald's theorem.

Corollary 3.24 (Kappe). *For any given positive integer k and any $j \geq 2$, there is a finite group M_k such that $\gamma_j(M_k)$ is cyclic but it cannot be generated by less than k commutators.*

Proof of Theorem 3.22. We will prove the result for the commutator subgroup. The same construction works for all lower central words by Corollary 3.24.

For every k, let M_k be the group constructed by Macdonald in Theorem 3.23. In the choices of the group M_k, we had to choose some odd primes $p_1^k, \ldots, p_{2^{2k}-1}^k$, we can require all of them to be different both pairwise and from all primes p_l^j for $1 < j < k$ and $1 < l < 2^{2j} - 1$.

Define $B_i = \prod_{k=1}^{i} M_k$. By construction, B_i' is a direct product of cyclic subgroups of coprime order, so it is cyclic too. As M_i' cannot be generated by less than i commutators, the same is true for B_i'. Moreover, by construction the groups B_i, for $i \in \mathbb{N}$, form an inverse system of finite groups, so we can define the profinite group $G = \varprojlim B_i$.

Clearly G' is procyclic as it is the inverse limit of B_i'. It cannot be generated by a finite set X of commutators, say of cardinality n, because otherwise the images of X in the quotient B_{n+1} would generate B_{n+1}' too, contradicting the previous paragraph. $\square$

In the way the authors use this property in [24], it is still relevant to ask whether the same phenomena can happen under the assumption that $|w\{G\}| < 2^{\aleph_0}$. Of course, in view of Conjecture 1.20, it is possible that no word can take countably many values in a group unless it takes finitely many, so this could be considered as an intermediate step towards the proof of the strong conciseness conjecture.

Conjecture 3.25. Let G be a profinite group and w be a word. If $w(G)$ is topologically finitely generated and $|w\{G\}| < 2^{\aleph_0}$, then $w(G)$ can be generated by finitely many word values.

Clearly the example in Theorem 3.22 cannot be used in order to contradict this conjecture because lower central words are strongly concise (see [24]).

4

Coprime commutators

In this chapter we prove strong conciseness of coprime commutators γ_k^* and δ_k^*, following the article [39] by de las Heras, Shumyatsky and the author.

We will first define coprime commutators and give an overview of their history. Coprime commutators are not word maps, but behave in a similar way, and they constitute a good generating set of the pronilpotent residual (for γ_k^*, $k \geq 2$) or of the k-th pronilpotent residual (for δ_k^*, $k \geq 1$).

In the second section we will discuss some basic lemmas that are necessary to develop our results. Some of them were already present in the literature and others are original.

In the third section we will outline the structure of the proofs, with a description of an interesting set of pronilpotent subgroups that is present in prosolvable groups.

We will then prove the main theorems, of strong conciseness of coprime commutators, both in the meta-pronilpotent case for γ_k^* in Section 4, and in the (prosolvable of Fitting heigth $k+1$) case for δ_k^* in Section 5. The general statements will be then proved jointly in Section 6.

4.1 History of coprime commutators

Higher order coprime commutators were introduced by Pavel Shumyatsky in [75] as a way to obtain a smaller natural set of generators for some classical subgroups.

Given a profinite group G and an element $x \in G$, we denote by $|G|$ (respectively $|x|$) the order of G (respectively x) as a supernatural number and $\pi(G)$ (respectively $\pi(x)$) will stand for the set of prime numbers dividing $|G|$ (respectively $|x|$). We will say that an element $g \in G$ is a *simple coprime commutator* if and only if it can be written as $g = [g_1, g_2]$ for $g_1, g_2 \in G$ with $(|g_1|, |g_2|) = 1$.

It was already well-known that the set of simple coprime commutators in a finite group G generates the nilpotent residual $\gamma_\infty(G)$, that is, the smallest normal subgroup N such that G/N is nilpotent (see Theorem 2.1 of [75]). Of course, in profinite groups the pronilpotent residual $\gamma_\infty(G) = \bigcap_i \gamma_i(G)$ is the intersection of the terms of the lower central series of G.

Coprime commutators of higher order were defined in [75] for finite groups, but the definition naturally extends to the profinite case.

Definition 4.1 (Higher order coprime commutators)**.** Let

$$\gamma_1^*\{G\} = \delta_0^*\{G\} = G$$

and, for every positive integer i define inductively the sets

$$\gamma_i^*\{G\} = \left\{[x^\lambda, g] \mid x \in \gamma_{i-1}^*\{G\}, \lambda \in \widehat{\mathbb{Z}}, g \in G, (|x^\lambda|, |g|) = 1\right\}$$

$$\delta_i^*\{G\} = \left\{[x^{\lambda_1}, y^{\lambda_2}] \mid x, y \in \delta_{i-1}^*\{G\}, \lambda_1, \lambda_2 \in \widehat{\mathbb{Z}}, (|x^{\lambda_1}|, |y^{\lambda_2}|) = 1\right\}.$$

Moreover, for the generated subgroups we will write $\gamma_i^*(G) = \langle \gamma_i^*\{G\} \rangle$ and $\delta_i^*(G) = \langle \delta_i^*\{G\} \rangle$.

Even if coprime commutators are not word maps, the analogy with classical word maps is clear. Indeed, $\gamma_i^*(G)$ and $\delta_i^*(G)$ are fully invariant subgroups because the order of $f(x)$ always divides the order of x for every homomorphism f and every $x \in G$. For this reason, it is interesting to describe the subgroups generated by coprime commutators, and the main results of the article [75] completely solve this natural question.

Theorem 4.2 ([75] Theorems 2.1, 2.7)**.** *Let G be a profinite group.*

If $k \geq 2$, the subgroup $\gamma_k^*(G)$ is trivial if and only if G is pronilpotent.

The subgroup $\delta_k^*(G)$ is trivial if and only if G is prosolvable of Fitting height at most k.

It is interesting to point out that a consequence of Theorem 4.2 is that there exists no word $w \in F(X_\infty)$ such that $w(G) = \gamma_i^*(G)$ $(i = 2, 3, \ldots)$ or $w(G) = \delta_i^*(G)$ (i positive integer) for every profinite group G because nilpotent groups of unbounded class do not form a variety of groups.

Several problems, that were classical for usual commutators, were then adapted to coprime commutators. An example is Ore's Conjecture, which stated that every element of a finite simple group is a commutator, and was solved in [53]. In [75], the author conjectured that every element of a finite simple group can be realized as a *coprime* commutator and proved the conjecture for the class of alternating groups. The same conjecture was later settled for $\mathrm{PSL}_2(q)$ for every prime power q in [67] and for Suzuki groups ${}^2B_2(q)$ for every odd q in [88].

Another natural consequence of the analogy between coprime commutators and usual commutators was the study of conciseness problems for them. Of course we will say that γ_i^* (resp δ_i^*) is *concise* if $\gamma_i^*(G)$ (resp. $\delta_i^*(G)$) is finite whenever $\gamma_i^*\{G\}$ (resp. $\delta_i^*\{G\}$) is finite.

In [5] the authors proved that, if there exists a positive integer m such that the word γ_i^* or δ_i^* takes at most m values in a finite group G, then the generated subgroup has m-bounded order. The bound does not depend on i, so that coprime commutators are *uniformly concise* in the class of finite groups. A straightforward consequence is that coprime commutators of higher order are concise in residually finite groups.

In the article [24], that began the investigation in strong conciseness, the authors noticed that the concept of strong conciseness can be applied in a wider context. Suppose $\mathcal{C}$ is a class of profinite groups and $\phi\{G\}$ is a subset of G for every $G \in \mathcal{C}$. Is the subgroup generated by $\phi\{G\}$ finite whenever $|\phi\{G\}| < 2^{\aleph_0}$? Such map ϕ is said to be *strongly concise in the class* $\mathcal{C}$ if the answer is positive. This question is interesting whenever $\phi\{G\}$ is defined in some natural way and/or properties of the subgroup $\langle \phi\{G\} \rangle$ have strong impact on the structure of G. For this reason, in [28] the authors examined strong conciseness for coprime commutators and managed to set that the map γ_2^* is strongly concise in profinite groups. In this chapter, which roughly follows the article [39], we will prove strong conciseness of γ_i^* and δ_i^* for every positive integer i.

Theorem 4.3. *A profinite group G is finite-by-pronilpotent if and only if there is k such that the set of γ_k^*-values in G has cardinality smaller than $2^{\aleph_0}$.*

Theorem 4.4. *A profinite group G is finite-by-(prosolvable of Fitting height at most k) if and only if the set of δ_k^*-values in G has cardinality smaller than $2^{\aleph_0}$.*

Of course there are results of strong conciseness because by Theorem 4.2 the values of the words γ_k^* and δ_k^* generate the finite subgroups of Theorems 4.3 and 4.4.

4.2 Preliminaries

We will first list some results that were present in the literature, or some small variations of them, that will be useful in the proofs of Theorems 4.3 and 4.4.

The first one is a fundamental result in the study of strong conciseness. A direct application of this result is that conjugacy classes in profinite groups are either finite or of cardinality at least $2^{\aleph_0}$ (see Lemma 3.17).

Proposition 4.5 ([24] Lemma 2.1)**.** *Let $\varphi : X \to Y$ be a continuous map between two non-empty profinite spaces that is nowhere locally constant (i.e. there is no non-empty open subset $U \subseteq_o X$ where $\varphi|_U$ is constant). Then $|\varphi(X)| \geq 2^{\aleph_0}$.*

A classical result in the theory of coprime automorphisms is the following.

Lemma 4.6 ([42], Lemma 4.29)**.** *Let A be a group of automorphisms of a finite group G with $(|A|, |G|) = 1$. Then, $[G, A] = [G, A, A]$.*

The following lemma is a stronger version of this result for the case where G is a pronilpotent group.

Lemma 4.7 ([47] Lemma 4.6)**.** *Let φ be an automorphism of a pronilpotent group G with $(|\varphi|, |G|) = 1$. Define the set the set $S = \{[g, \varphi] \mid g \in G\}$. Then the map $\theta : S \to S$ defined as*

$$\theta : x \to [x, \varphi]$$

is bijective.

The following is a profinite version of Lemma 2.4 in [75].

Lemma 4.8. *Let G be a profinite group and let $g_1, \ldots, g_k$ be δ_{k-1}^*-values in G. Suppose that $g_1, \ldots, g_k \in N_G(H)$ for a subgroup $H \leq G$ with $(|H|, |g_i|) = 1$ for every $i \in \{1, \ldots, k\}$. Then, for every $h \in H$, the element $[h, g_1, \ldots, g_k]$ is a δ_k^*-value.*

Using the previous two lemmas together, we will be able to guarantee that some special types of long commutators are also values of δ_k^*.

Lemma 4.9. *Let $G_1, \ldots, G_k$ be pronilpotent subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for all $j \leq i$. Let $x_i \in G_i$ for every i and assume that $(|x_i|, |x_{i+1}|) = 1$ for all $i = 1, \ldots, k$. Then the element $g = [x_1, \ldots, x_k]$ is in $\delta^*_{k-1}\{G\}$ and $\pi(g) \subseteq \pi(x_k)$.*

Proof. We will prove by induction on i that $g_i := [x_1, \ldots, x_i] \in \delta^*_{i-1}\{G\}$ for every $i \in \{1, \ldots, k\}$ and that $\pi(g_i) \subseteq \pi(x_i)$. The statement of the lemma corresponds to the case $i = k$. If $i = 1$ the result is obvious, so assume $i > 1$ and that g_{i-1} is a δ^*_{i-2}-value with $\pi(g_{i-1}) \subseteq \pi(x_{i-1})$, so in particular $(|g_{i-1}|, |x_i|) = 1$. If H is the minimal Hall subgroup of the pronilpotent group G_i containing x_i, then g_{i-1} acts as a coprime automorphism of H. By Lemma 4.7, there exists $y_i \in H$ such that

$$[x_i, g_{i-1}] = [y_i, g_{i-1}, \overset{i-1}{\ldots}, g_{i-1}],$$

and Lemma 4.8 shows that $g_i = [x_i, g_{i-1}]$ is a δ^*_{i-1}-value, as desired. As $g_i \in H$, we immediately have that $\pi(g_i) \subseteq \pi(x_i)$. □

The next result is a profinite version of Lemma 2.4 in [5]. We recall that by "meta-pronilpotent" group we mean a profinite group G having a normal pronilpotent subgroup N such that G/N is pronilpotent.

Lemma 4.10. *Let G be a meta-pronilpotent group. Then $\gamma_\infty(G) = \prod_p [K_p, H_{p'}]$, where K_p is a Sylow p-subgroup of $\gamma_\infty(G)$ and $H_{p'}$ is a Hall p'-subgroup of G.*

For a general group word w, the set $w\{G\}$ of w-values of a profinite group G is always closed in G. We will show that the same is true for the sets of γ^*_k and δ^*_k-values.

Proposition 4.11. *Let $S_1, \ldots, S_k$ be closed subsets of a profinite group G. Then the set*

$$C = \{(g_1, \ldots, g_k) \in S_1 \times \cdots \times S_k \mid (|g_i|, |g_{i+1}|) = 1 \textit{ for all } i = 1, \ldots, k-1\}$$

is closed in $S_1 \times \cdots \times S_k$. Furthermore, the sets $\gamma^*_k\{G\}$ and $\delta^*_k\{G\}$ are closed in G.

Proof. Let $\mathcal{P}$ be the set of all primes and $p \in \mathcal{P}$. First notice that for every closed subset S of G the set

$$S_{p'} = \{g \in S \mid p \notin \pi(g)\}$$

is closed. Indeed $S_{p'} = \bigcap_{N \trianglelefteq_o G} S_{p'}N$ because $p \in \pi(g)$ if and only if there is a normal subgroup N such that gN has order divided by p in G/N. Also, the set

$S^{\widehat{\mathbb{Z}}} = \{g^\lambda \mid g \in S, \lambda \in \widehat{\mathbb{Z}}\}$ is the image under the continuous map $f(g,\lambda) = g^\lambda$ of the compact set $S \times \widehat{\mathbb{Z}}$, so it is closed too.

Let now A, B be subsets of G. We claim that the set

$$R_{A,B} = \bigcap_{p \in \mathcal{P}} \big((A \times B_{p'}) \cup (A_{p'} \times B)\big) \tag{4.1}$$

is exactly the set of elements $(a,b) \in A \times B$ with $|a|$ and $|b|$ coprime. On the one hand, if $|a|$ and $|b|$ are coprime then $(a,b) \in (A \times B_{p'}) \cup (A_{p'} \times B)$ for every $p \in \mathcal{P}$, because, if $b \in B \smallsetminus B_{p'}$, then $a \in A_{p'}$ necessarily. On the other hand, if $(a,b) \in R_{A,B}$ and a prime p divides $|a|$, then $(a,b) \in A \times B_{p'}$ so p does not divide $|b|$, and the claim follows. Notice now that if A and B are closed, the set $R_{A,B}$ is an intersection of closed subsets of $G \times G$ so it is closed too.

It is now easy to prove by induction on k that the sets $\gamma_k^*\{G\}$, $\delta_k^*\{G\}$ are closed: just note that $\gamma_k^*\{G\}$ is exactly the set $R_{A,B}$ in (4.1) with $A = (\gamma_{k-1}^*\{G\})^{\widehat{\mathbb{Z}}}$, $B = G$, whereas $\delta_k^*\{G\}$ is the set $R_{A,B}$ in (4.1) with $A = B = (\delta_{k-1}^*\{G\})^{\widehat{\mathbb{Z}}}$.

To prove that the set C is closed in $S_1 \times \cdots \times S_k$, it suffices to notice that by the above arguments the set

$$C_i = S_1 \times \cdots \times S_{i-1} \times R_{S_i, S_{i+1}} \times S_{i+2} \times \cdots \times S_k$$

is closed for every $i \in \{1, \ldots, k-1\}$ and $C = \bigcap_{i=1}^{k-1} C_i$. $\square$

As we showed in Lemma 1.12, whenever a group word w takes finitely many values in a group G, the subgroup $w(G)$ is finite if and only if $w(G)/w(G)'$ is finite. If w takes less than $2^{\aleph_0}$ values in G we cannot obtain the same conclusion in general, but with some slightly stronger hypothesis we can anyway obtain a similar result.

Lemma 4.12. *Let ϕ be a map that associates to every group G a normal subset $\phi\{G\} \subseteq G$. Let G be a profinite group with $|\phi\{G\}| < 2^{\aleph_0}$ and let K be a pronilpotent subgroup of $\langle \phi\{G\} \rangle$ generated by a subset of $\phi(G)$. If K/K' is finite, then K is finite.*

Proof. Since K is pronilpotent, we have $K' \leq \Phi(K)$, where $\Phi(K)$ stands for the Frattini subgroup of K. Thus $K/\Phi(K)$ is finite, and hence we can find a finite subset S of $\phi\{G\}$ generating K. Since $\phi(G)$ is a normal subset of G, by Lemma 3.17 each of these generators has finitely many conjugates in G, so in particular $|G : C_G(s)| < \infty$ for every $s \in S$. Since $C_G(K) = \bigcap_{s \in S} C_G(s)$, this implies that $Z(K) = K \cap C_G(K)$ has finite index in K, and by Schur's theorem K' is finite. $\square$

We will use Lemma 4.12 for $\phi = \gamma_k^*$ or $\phi = \delta_k^*$, but it could be applied to other cases, such as any group word map or uniform (anti-coprime) commutators (see [28] or [29]).

4.3 Introduction to the proofs

In order to fully understand the proofs of Theorems 4.3 and 4.4, we have to begin from the proof of Detomi, Morigi and Shumyatsky in [28] that settled the analogous result for γ_2^*.

In the aforementioned article, the authors first proved that γ_2^* is strongly concise in meta-pronilpotent groups and then used this partial result to settle the general case. We will similarly split our proof: first we will prove strong conciseness of γ_k^* in meta-pronilpotent groups (Proposition 4.20 in Section 4.4), then we will settle the problem for δ_k^* in prosolvable groups of Fitting height $k+1$ (Proposition 4.33 in Section 4.5) and we will use these partial results in the proof of Theorems 4.3 and 4.4, that will be proved jointly in Section 4.6.

The proof of Proposition 4.20 consists of extending the reasoning that was used in [28] for γ_2^*, with a focal use of Lemma 4.7. The proof of the general case also partially follows [28], with some complications in the arguments.

The case of δ_k^* in prosolvable groups of Fitting height $k+1$, however, involved a lot of technical problems and is surely the more complex part of this chapter. For this reason, in this case we give a deeper analysis and motivation of the ideas involved.

An essential tool of the proof is the following collection of subgroups.

Definition 4.13 (Sylow basis)**.** A *Sylow basis* of a profinite group G is a family $\{P_i\}$ of Sylow subgroups of G, one for each prime in $\pi(G)$, such that $P_iP_j = P_jP_i$ for every i, j. The *normalizer of a Sylow basis* is $T = \bigcap_i N_G(P_i)$.

Basic properties of Sylow bases for finite groups can be found in Section 9.2 of [73] and they extend naturally to profinite groups.

Lemma 4.14. *Any prosolvable group admits a Sulow basis and any two Sylow bases are conjugate. In this case, the Sylow basis normalizer T is pronilpotent and $G = T\gamma_\infty(G)$. Moreover, if G is meta-pronilpotent, $\gamma_\infty(G) = [T, \gamma_\infty(G)]$.*

Proof. The first statement is a classical result, see for example Proposition 2.3.9 of [72], whereas the fact that $G = T\gamma_\infty(G)$ is Lemma 5.6 of [69]. If G is meta-pronilpotent, we have that

$$\gamma_\infty(G) = [G, \gamma_\infty(G)] = [T\gamma_\infty(G), \gamma_\infty(G)] = [T, \gamma_\infty(G)]$$

where the last equality follows from $\gamma_\infty(G)' \le \Phi(\gamma_\infty(G))$. $\square$

As a consequence of Theorem 4.2, in every profinite group G the subgroup $\delta_k^*(G)$ coincides with the k-th nilpotent residual $\gamma_\infty \overset{k}{\cdots} \gamma_\infty(G)$ (i.e. with γ_∞ repeated k times) and therefore $\delta_k^*(G) = \gamma_\infty(\delta_{k-1}^*(G))$. Let $\{P_i\}$ be a Sylow basis of G and observe that $\{P_i \cap \delta_j^*(G)\}$ is a Sylow basis of $\delta_j^*(G)$ for every $j \ge 1$. Let T_j be the normalizer in $\delta_j^*(G)$ of the Sylow basis $\{P_i \cap \delta_j^*(G)\}$, so that $G = T_0 T_1 \cdots T_k \delta_{k+1}(G)$ for every $k \ge 0$. We then have $T_j \le N_G(T_i)$ for every $j \le i$ because, for every $P \in \{P_i\}$, every $t \in T_j$ normalizes both $P \cap \delta_j^*(G)$ and $\delta_j^*(G)$, so n^t normalizes $P \cap \delta_i^*(G)$ for all $n \in T_i$. In particular, if G is a prosolvable group of Fitting height $k+1$, then $\delta_{k+1}^*(G) = 1$, and therefore $G = T_0 \cdots T_k$. We want to refine this series to a similar one with some additional properties.

Proposition 4.15. *Let G be a prosolvable group of Fitting height $k+1$. There exist pronilpotent subgroups $U_0, \dots, U_k$ satisfying the following properties, where we denote by $P_i(p)$ and $H_i(p')$ the Sylow p-subgroup and Hall p'-subgroup of U_i respectively.*

- *$U_j \le N_G(U_i)$ for every $j \le i$;*
- *$G = U_0 \cdots U_j \delta_{j+1}^*(G)$ for every $j \in \{0, \dots, k\}$;*
- *$U_k = \delta_k^*(G)$;*
- *$P_j(p) = [P_j(p), H_{j-1}(p')]$ for every $j \in \{0, \dots, k\}$, $p \in \pi(G)$.*

Proof. Let $U_0 = T_0$; for $j \ge 1$ we construct inductively the subgroups $U_j \le T_j$ in the following way. Let $H_{j-1}(p')$ and $Q_j(p)$ be, respectively, the Hall p'-subgroup of U_{j-1} and the Sylow p-subgroup of T_j, and define

$$U_j = \prod_{p \in \pi(G)} [H_{j-1}(p'), Q_j(p)].$$

Notice that we can write the direct product because by induction $U_{j-1} \le T_{j-1} \le N_G(T_j)$ and by pronilpotency $U_{j-1} \le N_G(Q_j(p))$ too. This is sufficient to notice that $[H_{j-1}(p'), Q_j(p)] \le Q_j(p)$. As $T_j \le N_G(T_i)$ for every $j \le i$, we also have by pronilpotency that $U_j \le N_G(U_i)$ for every $j \le i$. Denote by $P_j(p) = [H_{j-1}(p'), Q_j(p)]$ the Sylow p-subgroup of U_j.

We claim that $Q_j(p) \equiv P_j(p) \pmod{\delta_{j+1}^*(G)}$ for every $p \in \pi(G)$ and every $j \in \{0, \dots, k\}$, and therefore $T_j \equiv U_j \pmod{\delta_{j+1}^*(G)}$. Notice that this shows that $U_k = T_k = \delta_k^*(G)$ and that $G = U_0 \cdots U_j \delta_{j+1}^*(G)$ for every $j \in \{0, \dots, k\}$.

The case $j = 0$ follows trivially, so let $j \geq 1$ and assume by induction that the congruences hold for $j-1$. Denote by $H_j(p')$ the Hall p'-subgroup of T_j and consider $K_{j-1}(p') = H_{j-1}(p')H_j(p')$, which is a Hall p'-subgroup of $T_{j-1}T_j$. Since $\gamma_\infty(T_{j-1}T_j) = T_j \pmod{\delta^*_{j+1}(G)}$, Lemma 4.10 yields

$$Q_j(p) \equiv [K_{j-1}(p'), Q_j(p)] \equiv [H_{j-1}(p'), Q_j(p)] = P_j(p) \pmod{\delta^*_{j+1}(G)},$$

The second congruence holds because $T_{j-1}T_j \equiv U_{j-1}T_j \pmod{\delta^*_{j+1}(G)}$ by induction, and hence $K_{j-1}(p') \equiv H_{j-1}(p')H_j(p') \pmod{\delta^*_{j+1}(G)}$. Of course we have $[H_j(p'), Q_j(p)] = 1$ because U_j is pronilpotent.

Furthermore, as $(|Q_j(p)|, |H_{j-1}(p')|) = 1$, by Lemma 4.6 we have

$$P_j(p) = [Q_j(p), H_{j-1}(p')] = [Q_j(p), H_{j-1}(p'), H_{j-1}(p')] = [P_j(p), H_{j-1}(p')]. \quad (4.2)$$

□

In view of the series of subgroups of Proposition 4.15, we will often work with subgroups $G_1, \ldots, G_t$, for a positive integer t, of a profinite group G such that $G_j \leq N_G(G_i)$ for every $j \leq i$. We will obtain some results on coprime commutators of length t for an arbitrary positive integer t, and in the end of Section 4.5 we will apply these lemmas to the case $t = k+1$, with the series $U_0, \ldots, U_k$ mentioned above. In this setting, we will write

$$\begin{array}{rccc} \varphi: & G_1 \times \cdots \times G_t & \longrightarrow & G_t \\ & (g_1, \ldots, g_t) & \longmapsto & [g_1, \ldots, g_t] \end{array}$$

where $\varphi(G_1, \ldots, G_k) \subseteq G_k$ because $G_j \leq N_G(G_i)$ for every $j \leq i$. Consider the sequences of coprime elements

$$\mathcal{C} = \{(g_1, ..., g_t) \in G_1 \times \cdots \times G_t \mid (|g_i|, |g_{i+1}|) = 1\}; \quad (4.3)$$

for $S_i \subseteq G_i$, $i \in \{1, \ldots, t\}$, we define the set

$$\varphi^*(S_1, \ldots, S_t) = \varphi((S_1 \times \cdots \times S_t) \cap \mathcal{C}).$$

It is important to point out that in general $\varphi^*(S_1, \ldots, S_k)$ is different from the set $\gamma^*_k\{S_1, \ldots, S_k\}$ of coprime commutators with variables restricted in $(S_1, \ldots, S_k)$. Indeed, the former requires that two subsequent entries have coprime orders, whereas the latter requires the i-th entry to be coprime with a power of a value of $\gamma^*_{i-1}\{S_1, \dots, S_{i-1}\}$.

We will also often need to consider the maps φ or φ^*, but with entries chosen in two different tuples of sets. For this reason we introduce this compact notation, which is consistent with the one already present in the literature for usual outer commutator maps (see for example [24]).

For $i \in \{1, \dots, t\}$, let $X_i, Y_i \subseteq G_i$. For $J \subseteq \{1, \dots, t\}$ we can define the set

$$\varphi_J(X_i; Y_i) = \varphi(Z_1, \dots, Z_t) \quad \text{with} \quad Z_i = \begin{cases} X_i & \text{if } i \in J, \\ \\ Y_i & \text{if } i \notin J. \end{cases}$$

Notice that in order for φ_J to be well-defined, we just need the subsets X_i where $i \in J$ and the subsets Y_i where $i \notin J$, so we will often use the same notation when X_i are defined only for $i \in J$ and Y_i only for $i \notin J$. In a similar way, define the set

$$\varphi^*_J(X_i; Y_i) = \varphi((Z_1 \times \cdots \times Z_t) \cap \mathcal{C}).$$

If $J = \{1, \dots, t\}$, in accordance to the initial definitions of φ and φ^*, we will just write $\varphi_J(X_i; Y_i) = \varphi(X_i)$ and $\varphi^*_J(X_i; Y_i) = \varphi^*(X_i)$.

Remark 4.16. Notice that whenever we have subgroups $G_1, \dots, G_\ell$ of a profinite group G with $G_j \le N_G(G_i)$ for every $j \le i$, and we take an open subgroup $U \trianglelefteq_o G_\ell$, there exists an open normal subgroup $V \trianglelefteq_o G$ such that $V \cap G_\ell \le U$. This implies that $V \cap G_\ell \trianglelefteq G_1 \cdots G_\ell$ and $V \cap G_\ell \trianglelefteq_o G_\ell$.

We are now ready to explain the main ideas of the proof of Proposition 4.33. Let G be a prosolvable group of Fitting height k and consider a tuple $(U_0, \dots, U_k)$ of subgroups as in Proposition 4.15. We will prove Proposition 4.33 by funnelling all values of $\varphi^*(U_0, \dots, U_k)$ into a finite normal subgroup. Using Lemma 4.10 it will be possible to prove that these values generate $\delta^*_k(G)$.

Denote by $P_i(p)$ the p-Sylow of U_i for any prime p. A further simplification, through Lemma 4.22, will allow us to prove that we can recover the whole $\varphi^*(U_i)$ just from studying the sets $\{\varphi^*(P_i(p_i)) \mid p_i \in \mathcal{P}\}$. We want to reduce our study to values of this type because they are either trivial (if $p_j = p_{j+1}$ for a certain j) or they coincide with the usual commutators $\gamma_k\{P_i(p_i)\}$, for which classical properties of standard commutators apply.

If $|\pi(U_j)| < \infty$ for every $j = 1, \dots, k$, we can simply study finitely many sets of the form $\varphi\{P_i(p_i)\}$, but if there exists at least an index j such that $|\pi(U_j)| = \infty$, we would have to study infinitely many of these sets. For this reason the proof of Proposition 4.33 will be by induction on the number of factors U_j with $|\pi(U_j)| = \infty$. This reduction will be done in Lemma 4.30 using some subgroups N_σ defined in 4.27, and all the first part of Section 4.5 will be devoted to obtaining results that will be mainly used in the proof of this lemma.

Some themes used in the proof of Lemma 4.30 can be retraced to the article of Detomi, Klopsch and Shumyatsky that outer commutator words are strongly concise. However, our case has some complications. First of all, in [24] the authors consider commutator words where each entry could be chosen in the whole group G, whereas we allow the i-th entry of γ_t to be only in U_{i-1}. Moreover, when we work with the map φ^*, we must be careful to preserve coprimality in the factors. As an example, one tool that was often used in [24] was reducing, under suitable hypothesis, a coset identity of the type $\varphi(x_iU_i) = 1$ to a coset/subgroup identity $\varphi_J(U_i, x_iU_i) = 1$ for $J \subsetneq \{1\dots, k\}$. If we are considering coprime maps φ^*, we must be extremely cautious when we remove a coset representative from a coset identity. For example, if $t = 2$ and we pick two coset representatives x_1, x_2 of $U \leq G$ that are not coprime it could happen that $\varphi^*(x_1U, x_2U) = \varnothing$, but $\varphi^*(x_1U, U)$ is non-empty, as it would contain at least the trivial element. The statements of the first lemmas in Section 4.5 require several specific hypotheses in order to account for similar issues in preserving coprimality.

4.4 The meta-pronilpotent case for γ_k^*

In this section we will prove Theorem 4.3 in the case when the profinite group G is meta-pronilpotent.

We first require some results that are present in [28].

Lemma 4.17 ([28] Lemma 2.4)**.** *Let H, Q be subgroups of a group G with Q normal in QH and such that $Q = [Q, H]$. Any normal subgroup $N \trianglelefteq Q$ such that $[N, H] = 1$ is contained in the center $Z(QH)$.*

Lemma 4.18 ([28] Lemma 2.6)**.** *Let G be a finite group, where $H, Q \leq G$. Suppose that Q is a normal nilpotent subgroup of QH such that $(|Q|, |H|) = 1$ and $|Q : C_Q(h)| \leq m$ for all $h \in H$. Then the order of $[Q, H]$ is m-bounded.*

The next lemma is a modification of Lemma 3.1 of [28].

Lemma 4.19. *Let G be a profinite group that is the product of a subgroup H and a normal pronilpotent subgroup Q with $(|H|, |Q|) = 1$. Suppose that*

$$|\{[h, q] \mid h \in H, q \in Q\}| < 2^{\aleph_0}.$$

Then $[H, Q]$ is finite.

Proof. Lemma 4.6 implies that $[Q, H, H] = [Q, H]$, so replacing Q with $[Q, H]$ we can assume that $Q = [Q, H]$ and we can write G as a product $G = [Q, H]H$. For every $h \in H$ the set of cosets of $C_Q(h)$ in Q is a profinite space in bijection with h^Q, so in bijection with $\{[h, q] \mid q \in Q\}$ too. By hypothesis this space has less than $2^{\aleph_0}$ elements, hence it must be finite (we can follow the proof of Proposition 2.3.1 of [72] for a profinite set of cosets rather than a profinite group). This implies that $|h^Q|$ is finite for every $h \in H$. For each integer $j \in \mathbb{N}$ we can consider the sets

$$C_j = \{h \in H \mid |h^Q| \leq j\},$$

that are closed by Lemma 5 of [52]. As their union is H, we can apply Baire category theorem and hence there exists an integer ℓ such that C_ℓ has non-empty interior (in the subspace topology of H). In particular there exist $h \in H, U \trianglelefteq_o H$ with $|(hu)^Q| \leq \ell$ for every $u \in U$.

For every $u \in U, q \in Q$ we can write $u^q = (h^{-1})^q (hu)^q$, but we chose h and U so that $|(h^{-1})^Q|, |(hu)^Q| \leq \ell$. This proves that $|u^Q| \leq \ell^2$, so by Lemma 4.18 the subgroup $[Q, U]$ is finite.

We can then factor out $[Q, U]$ and replace Q with $Q/[Q, U]$. Now $U \leq C_G(Q)$, so we can factor out U too and assume that $H = H/U$ is finite. As $C_Q(H) = \bigcap_{h \in H} C_Q(h)$ and H is finite, $|Q : C_Q(H)|$ is finite too. Consider now the normal core N of $C_Q(H)$ in G, which has still finite index, both in Q and in G. By Lemma 4.17 N is an open subgroup contained in the center of G, so by Schur's Theorem G' is finite. $\square$

We are now ready to prove the strong conciseness of γ_k^* in meta-pronilpotent groups.

Proposition 4.20. *Let G be a meta-pronilpotent group with $|\gamma_k^*\{G\}| < 2^{\aleph_0}$. Then $\gamma_\infty(G)$ is finite.*

Proof. Let $g \in G$ and $h \in \gamma_\infty(G)$ such that $(|g|, |h|) = 1$, and let H be the minimal Hall subgroup of $\gamma_\infty(G)$ containing h. Since H is pronilpotent, Lemma 4.7 shows that there exists $h' \in H$ such that $[h, g] = [h', g, \overset{k-1}{\ldots}, g]$, and therefore $[h, g] \in \gamma_k^*(G)\{G\}$. Hence, we have

$$|\{[g, h] \mid g \in G, h \in \gamma_\infty(G), (|g|, |h|) = 1\}| < 2^{\aleph_0}.$$

We first prove that for every $p \in \pi(\gamma_\infty(G))$, the p-Sylow P of $\gamma_\infty(G)$ is finite. Denote by $H(p')$ a p'-Hall subgroup of G; by Lemma 4.10, $P = [P, H(p')]$ and by Lemma 4.19 it is finite.

In order to conclude, we now have to prove that $\pi(\gamma_\infty(G))$ is finite. By contradiction, let $\pi(\gamma_\infty(G)) = \{p_i \mid i \in I\}$ be an infinite set and denote by P_i the p_i-Sylow of $\gamma_\infty(G)$ for every $i \in I$. Let T be the normalizer of a Sylow basis of G, so that $G = T\gamma_\infty(G)$ with both T and $\gamma_\infty(G)$ pronilpotent. By Lemma 4.10 $P_i = [P_i, A_i]$ for A_i the group of automorphisms induced by the p_i'-Hall subgroup of T on P_i; let $\sigma_i = \pi(A_i)$. As we have proved that all P_i are finite, σ_i is finite too.

Let Q be a q-Sylow of T for $q \in \pi(T)$; applying Lemma 4.19 we have that $[Q, O_{q'}(\gamma_\infty(G))]$ is finite. In particular, q acts non-trivially on finitely many P_i, so it is in finitely many of the sets σ_i. Define $\tau_i = \sigma_i \cup \{p_i\}$, by the previous discussion, for every fixed index $i \in I$ there are only finitely many indices $j \in I$ such that $\tau_i \cup \tau_j \neq \emptyset$. We can iteratively construct a set $J \subseteq I$ such that $\tau_i \cup \tau_j = \emptyset$ for every $i, j \in J$, $i \neq j$. Let Q_i a q_i-Sylow subgroup of T for any $q_i \in \sigma_i$. By construction $[P_i, Q_i] \neq 1$ and $[P_i, Q_j] = 1$ whenever $i \neq j$, so for every $i \in J$ there exist two elements $g_i \in P_i$ and $h_i \in Q_i$ such that $[h_i, q_i] \neq 1$, and in particular it is a coprime commutator. By Lemma 4.7 the element $c_i = [h_i, q_i, \overset{k-1}{\ldots}, q_i]$ is also a non-trivial γ_k^*-value. As all the sets τ_i are disjoint and every Q_j acts trivially on P_i whenever $j \neq i$, the element

$$c_{J'} = \prod_{i \in J'} c_i = \Big[\prod_{i \in J'} h_i, \prod_{i \in J'} q_i, \overset{k-1}{\ldots}, \prod_{i \in J'} q_i\Big]$$

is a nontrivial γ_k^*-value for every $J' \subseteq J$. As all these elements are different for every subset $J' \subseteq J$, we would have at least $2^{\aleph_0}$ different γ_k^*-values, contradicting the hypothesis. This proves that $\pi(\gamma_\infty(G))$ is finite and the proposition follows. □

4.5 The poly-pronilpotent case for δ_k^*

The next two lemmas are useful applications of basic commutator calculus. The first one follows the ideas of Lemma 2.8 of [40] while Lemma 4.22 is an application of Lemma 4.21 to coprime commutators.

Lemma 4.21. *Let $G_1, \ldots, G_t$ be subgroups of a group G such that $G_j \leq N_G(G_i)$ for every $j \leq i$. For every $i \in \{1, \ldots, t\}$ let $g_i \in G_i$, and for a fixed $\ell \in \{1, \ldots, t\}$, let $g_\ell' \in G_\ell$. Then*

$$\varphi_{\{\ell\}}(g_\ell' g_\ell; g_i) = [g_1, \ldots, g_{\ell-1}, g_\ell', g_{\ell+1}^{h_\ell}, \ldots, g_t^{h_{t-1}}]^{h_t} \varphi(g_i),$$

where $h_i \in G_\ell \cdots G_i$ for $i \in \{\ell, \ldots, t\}$. In particular $g_{i+1}^{h_i} \in G_{i+1}$ for all $i \in \{\ell, \ldots, t-1\}$

Proof. Assume first that $\ell \neq 1$, and proceed by induction on $t-\ell$. If $t-\ell=0$, then

$$[g_1,\ldots,g_{t-1},g_t'g_t]=[g_1,\ldots,g_t']^{g_t[g_1,\ldots,g_t]^{-1}}[g_1,\ldots,g_t],$$

and the result follows. Assume $t-\ell>0$, and we write, for the sake of brevity, $y=[g_1,\ldots,g_{\ell-1}]$. By induction, we have

$$[y,g_\ell'g_\ell,g_{\ell+1},\ldots,g_t]=[[y,g_\ell',g_{\ell+1}^{h_\ell},\ldots,g_{t-1}^{h_{t-2}}]^{h_{t-1}}[g_1,\ldots,g_{t-1}],g_t]$$

with $h_i \in G_\ell \cdots G_i$ for $i \in \{\ell,\ldots,t-1\}$. Now,

$$\begin{aligned}
&[[y,g_\ell',g_{\ell+1}^{h_\ell},\ldots,g_{t-1}^{h_{t-2}}]^{h_{t-1}}[g_1,\ldots,g_{t-1}],g_t]\\
&\qquad=[[y,g_\ell',g_{\ell+1}^{h_\ell},\ldots,g_{t-1}^{h_{t-2}}]^{h_{t-1}},g_t]^{[g_1,\ldots,g_{t-1}]}[g_1,\ldots,g_t]\\
&\qquad=[y,g_\ell',g_{\ell+1}^{h_\ell},\ldots,g_{t-1}^{h_{t-2}},g_t^{(h_{t-1})^{-1}}]^{h_{t-1}[g_1,\ldots,g_{t-1}]}[g_1,\ldots,g_t],
\end{aligned}$$

and the lemma follows. If $\ell=1$, a similar argument applies. □

Lemma 4.22. *Let $G_1,\ldots,G_t$ be subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for every $j \leq i$. Let $\ell \in \{1,\ldots,t\}$ and $Y_1,Y_2 \subseteq G_\ell$ be such that $\pi(y_1),\pi(y_2) \subseteq \pi(y_1y_2)$ for every $y_1 \in Y_1$, $y_2 \in Y_2$. Let $X_i \subseteq G_i$ for $i \in \{1,\ldots,\ell-1\}$, and for $i \in \{\ell+1,\ldots,t\}$ denote $X_i=G_i$. Then:*

1. *If $\varphi^*_{\{\ell\}}(Y_1;X_i)=\varphi^*_{\{\ell\}}(Y_2;X_i)=1$, then $\varphi^*_{\{\ell\}}(Y_1Y_2;X_i)=1$.*
2. *If $\varphi^*_{\{\ell\}}(Y_j;X_i)=\varnothing$ for some $j \in \{1,2\}$, then $\varphi^*_{\{\ell\}}(Y_1Y_2;X_i)=\varnothing$.*

Proof. Since $\pi(y_1),\pi(y_2) \subseteq \pi(y_1y_2)$ for every $y_1 \in Y_1$, $y_2 \in Y_2$, the second statement is straightforward. Moreover, if $\varphi_{\{\ell\}}(y_1y_2;g_i) \in \varphi^*_{\{\ell\}}(Y_1Y_2;X_i)$, then for $j \in \{1,2\}$ we have $\varphi_{\{\ell\}}(y_j;g_i) \in \varphi^*_{\{\ell\}}(Y_j;X_i)$. The result follows now directly from Lemma 4.21. □

In view of the preceding lemma, we now introduce a convenient way to choose coset representatives of normal subgroups. These will play an important role throughout the chapter.

Definition 4.23 (Good representatives)**.** Let G be a profinite group and $U \trianglelefteq G$. An element $g \in G$ is a *good representative* of the coset gU if $\pi(g),\pi(u) \subseteq \pi(gu)$ for every $u \in U$.

Lemma 4.24. *Let U be an open normal subgroup of a pronilpotent group G. Let g be a representative of the coset gU and write $g=\prod_{p\in\pi(G)} g_p$ with g_p a p-element of G. Then the following are equivalent:*

(i) g is a good representative of the coset gU;

(ii) $g_p = 1$ whenever $g_p \in U$ for $p \in \pi(G)$;

(iii) $\pi(g)$ is minimal among all representatives of the coset gU.

In this case, if $\sigma = \pi(G/U)$, then $\pi(g) \subseteq \sigma$.

Proof. We first prove (i)$\Rightarrow$ (ii). Assume g is a good representative and suppose that $g_p \in U$. If $g_p \neq 1$, then $\pi(g \cdot g_p^{-1})$ does not contain p, contradicting that $\pi(g) \subseteq \pi(gu)$ for all $u \in U$.

(ii)$\Rightarrow$ (i). Write $u = \prod_{p \in \pi(G)} u_p$ for a certain $u \in U$ and suppose $g_p = 1$ whenever $g_p \in U$. Then, if either $g_p \neq 1$ or $u_p \neq 1$, then $g_p u_p \neq 1$, that is exactly the condition of being a good representative.

(ii)$\Leftrightarrow$(iii) is immediate, and the last remark follows from (ii).

□

The following lemma is an application of Proposition 4.5 to a special type of coprime commutators.

Lemma 4.25. *Let $G_1, \ldots, G_t$ be pronilpotent subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for all $j \leq i$, and $|\delta^*_{t-1}\{G\}| < 2^{\aleph_0}$. For every $i \in \{1, \ldots, t\}$, let S_i be a closed subset of G_i. If $\varphi^*(S_i) \neq \varnothing$, then, there exist elements $x_i \in G_i$ and open subgroups $U_i \trianglelefteq_o G_i$ such that $|\varphi^*(x_i U_i \cap S_i)| = 1$.*

Proof. Let

$$\mathcal{C} = \left\{ (x_1, \ldots, x_t) \in S_1 \times \cdots \times S_t \;\middle|\; (|x_i|, |x_{i+1}|) = 1 \text{ for all } i = 1, \ldots, t \right\}.$$

As $\varphi(\mathcal{C}) = \varphi^*(S_i)$, we have $\mathcal{C} \neq \varnothing$. Note that $\mathcal{C}$ is closed in $G_1 \times \cdots \times G_t$ by Lemma 4.11.

Fix $(x_1, \ldots, x_t) \in \mathcal{C}$. By Lemma 4.9 the element $g_k := [x_1, \ldots, x_t]$ is in $\delta^*_{t-1}\{G\}$. Hence, $|\mathrm{Imm}(\varphi)| < 2^{\aleph_0}$, and by Proposition 4.5, it follows that there exist elements $x_i \in G_i$ and open normal subgroups $U_i \trianglelefteq G_i$ such that

$$\mathcal{C} \cap (x_1 U_1 \times \cdots \times x_t U_t) \neq \varnothing$$

and $|\varphi^*(x_i U_i \cap S_i)| = 1$. □

Lemma 4.25 will often provide some cosets of open subgroups of G in which coprime commutators are trivial. Lemmas 4.26 and 4.29 below will allow us to relate coprime commutators of these cosets with coprime commutators of the open subgroups themselves.

Lemma 4.26. *Let $G_1, \dots, G_t$ be subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for every $j \leq i$, and for every $i \in \{1, \dots, t\}$, let $x_i \in G_i$ and $U_i \trianglelefteq G_i$. Assume also that $G_j \leq N_G(U_i)$ for every $j \leq i$. Fix $j \in \{1, \dots, t\}$ and write $J = \{1, \dots, j-1\}$, then:*

(i) If $\varphi(x_iU_i) = 1$ then $\varphi_J(x_iU_i; U_i) = 1$.

(ii) If $\varphi_J(x_iU_i; U_i) = 1$ then

$$\varphi_{J\cup\{j\}}(x_iU_i; U_i) = \varphi(x_1U_1, \dots, x_{j-1}U_{j-1}, x_j, U_{j+1}, \dots, U_t).$$

Proof. (i) We will proceed by reverse induction on $j \in \{1, \dots, t+1\}$, where the base case $j = t+1$ translates to $\varphi(x_iU_i) = 1$, which is true by hypothesis. Let thus $j < t+1$ and assume that $\varphi_{J\cup\{j\}}(x_iU_i; U_i) = 1$.

Let $C_t = 1$ and for every $i \in \{j+1, \dots, t-1\}$ define $C_i = C_{U_i}(U_{i+1}/C_{i+1})$. Note that C_i is well-defined, since using that for every ℓ the subgroup U_ℓ is normal in $G_1 \cdots G_\ell$, one can easily show by induction that $C_\ell \trianglelefteq G_1 \cdots G_\ell$.

If $j \geq 2$, let

$$Y = \{[x_1u_1, \dots, x_{j-1}u_{j-1}] \mid u_i \in U_i,\ i = 1, \dots, j-1\}.$$

Then, we can rewrite $\varphi_{J\cup\{j\}}(x_iU_i; U_i) = 1$ as

$$[Y, x_jU_j] \subseteq C_{G_j}(U_{j+1}/C_{j+1}).$$

For every $i \in \{1, \dots, j\}$, fix $u_i \in U_i$ and shorten $y = [x_1u_1, \dots, x_{j-1}u_{j-1}]$. Then we have $[y, x_ju_j] = [y, u_j][y, x_j]^{u_j}$, and since $C_{G_j}(U_{j+1}/C_{j+1})$ is a normal subgroup of G_j containing $[y, x_ju_j]$ and $[y, x_j]$, it follows that $[y, u_j] \in C_{G_j}(U_{j+1}/C_{j+1})$. This shows that $\varphi(x_1U_1, \dots, x_{j-1}U_{j-1}, U_j, U_{j+1}, \dots, U_t) = 1$, as we wanted.

For the case $j = 1$, note that both x_1 and x_1U_1 lay in $C_{G_1}(U_2/C_2)$, so that $U_1 \leq C_{G_1}(U_2/C_2)$.

(ii) For every $i \in \{j+1, \dots, t\}$ we define C_i as in (i). For $i \in \{1, \dots, t\}$, let $u_i \in U_i$ and shorten $y = [x_1u_1, \dots, x_{j-1}u_{j-1}]$. Then,

$$[y, x_ju_j] = [y, u'x_j] = [y, x_j][y, u']^{x_j} = [y, u']^{x_i[x_j,y]}[y, x_j]$$

for some $u' \in U_j$, and note that $z := [y, u']^{x_j[x_j,y]} \in C_{G_j}(U_{j+1}/C_{j+1})$. Then $[z, u'_{j+1}, \dots, u'_t] = 1$ for every $u'_i \in U_i$, $i \in \{j+1, \dots, t\}$, so that

$$[y, x_ju_j, u_{j+1}, \dots, u_t] = [z[y, x_j], u_{j+1}, \dots, u_t] = [y, x_j, u_{j+1}, \dots, u_t],$$

where the last equality follows from Lemma 4.21. The lemma follows. $\square$

Definition 4.27 (Subgroup N_σ)**.** Let $G_1, \dots, G_t$ be pronilpotent subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for all $j \leq i$. Let σ be a finite set of primes. We define the normal subgroup

$$N_\sigma = \langle \varphi^*_{\{j\}}(H_i; G_i) \mid j \text{ is such that } |\pi(G_j)| = \infty \rangle^G,$$

where H_i is the Hall σ-subgroup of G_i for every i. If $|\pi(G_i)| < \infty$ for all i, then $N_\sigma = \langle \varnothing \rangle^G = 1$ for every σ.

The subgroups $G_1, \dots, G_t$ of G for which the definition of N_σ applies will be clear from the context. Notice that for any finite sets of primes σ_1 and σ_2 such that $\sigma_1 \subseteq \sigma_2$ we have

$$N_{\sigma_1} \leq N_{\sigma_2}. \tag{4.4}$$

Lemma 4.28. *Let $G_1, \dots, G_t$ be pronilpotent subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for all $j \leq i$. Fix $\ell \in \{1, \dots, t\}$ and $x_\ell \in G_\ell$. For $i \in \{1, \dots, \ell - 1\}$, let $X_i \subseteq G_i$, and for $i \in \{\ell, \dots, t\}$ let $U_i \trianglelefteq_o G_i$ be such that $G_j \leq N_G(U_i)$ for $j \leq i$. Suppose that $(|x_\ell|, |x_{\ell-1}|) = (|x_\ell|, |U_{\ell+1}|) = 1$ for every $x_{\ell-1} \in X_{\ell-1}$. If $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, U_{\ell+1}, \dots, U_t) = 1$, then we have $\varphi^*(X_1, \dots, X_{\ell-1}, U_\ell, \dots, U_t) = 1$.*

Proof. First of all, observe that since $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, U_{\ell+1}, \dots, U_t) \neq \varnothing$, there are $y_1, \dots, y_{\ell-1}$ such that $y_i \in X_i$ and

$$(|y_j|, |y_{j+1}|) = 1 \tag{4.5}$$

for all $j \in \{1, \dots, \ell - 2\}$. Note that the tuple $(y_1, \dots, y_{\ell-1}, 1, \dots, 1)$ is in $\mathcal{C}$ and then $\varphi^*(X_1, \dots, X_{\ell-1}, U_\ell, \dots, U_t) \neq \varnothing$.

Fix then a tuple $(x_1, \dots, x_{\ell-1}, u_\ell, \dots, u_t) \in \mathcal{C}$ with $x_j \in X_j$ and $u_j \in U_j$. In order to conclude we want to prove that $\varphi^*(x_1, \dots, x_{\ell-1}, u_\ell, \dots, u_t) = 1$. For $i \in \{\ell, \dots, t\}$, let H_i be the minimal Hall subgroup of U_i containing u_i, and notice that we have

$$(|x_{\ell-1}|, |H_\ell|) = (|H_j|, |H_{j+1}|) = 1 \tag{4.6}$$

for all $j \in \{\ell, \dots, t-1\}$. Since G_ℓ is pronilpotent, we have $\pi(x_\ell h) \subseteq \pi(x_\ell) \cup \pi(h)$ for all $h \in H_\ell$, and hence, as $(|x_\ell|, |x_{\ell-1}|) = (|x_\ell|, |U_{\ell+1}|) = 1$, we have $\varphi(x_1, \dots, x_{\ell-1}, x_\ell H_\ell, H_{\ell+1}, \dots, H_t) \subseteq \varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, U_{\ell+1}, \dots, U_t)$, and it is then equal to the trivial subgroup.

Lemma 4.26(i) now gives $\varphi(x_1, \dots, x_{\ell-1}, H_\ell, \dots, H_t) = 1$, and therefore we have $\varphi^*(x_1, \dots, x_{\ell-1}, u_\ell, \dots, u_t) = 1$. $\square$

Lemma 4.29. *Let G_i, ℓ, X_i, U_i be as in Lemma 4.28.*

(i) For $i \in \{\ell, \dots, t\}$, suppose that either $|\pi(G_i)| = \infty$, in which case we write $Y_i = G_i$, or $|\pi(G_i)| = 1$, in which case we write $Y_i = U_i$. Assume moreover that if $\pi(G_i) = \{p\}$ consists of a single prime, then $p \notin \pi(G_{i-1}) \cup \pi(G_{i+1})$.

Suppose we also have that $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, \dots, x_t U_t) = 1$ for some $x_\ell \in G_\ell$ such that $(|x_\ell|, |x_{\ell-1}|) = 1$ for every $x_{\ell-1} \in X_{\ell-1}$. Then, there exists a finite set of primes σ such that $\varphi^*(X_1, \dots, X_{\ell-1}, Y_\ell, \dots, Y_t) \subseteq N_\sigma$ (cf. Definition 4.27).

(ii) Suppose that we fix $x_i \in G_i$, $i = \ell, \dots, t$, such that $(|x_i|, |x_{i+1}|) = 1$ for all $i \in \{\ell, \dots, t-1\}$ and $(|x_\ell|, |x_{\ell-1}|) = 1$ for all $x_{\ell-1} \in X_{\ell-1}$. If the set $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, \dots, x_t U_t)$ is empty, then we also have that $\varphi^*(X_1, \dots, X_{\ell-1}, G_\ell, \dots, G_t) = \varnothing$.

Proof. (i) Write $L = \{\ell, \dots, t\}$, and for $i \in L$, define

$$\sigma_i = \begin{cases} \pi(G_i/U_i) & \text{if } |\pi(G_i)| = \infty, \\ \\ \pi(G_i) & \text{if } |\pi(G_i)| = 1. \end{cases}$$

Let $\sigma = \sigma_\ell \cup \dots \cup \sigma_t$. Up to changing the representative, we can assume that every x_i is a good representative of $x_i U_i$, and in particular that they are all σ-elements by Lemma 4.24. Furthermore, since $\varphi^*_L(x_i U_i; X_i) \neq \varnothing$ and $\pi(x_j) \subseteq \pi(x_j u_j)$ for every $u_j \in U_j$, it follows that $(|x_i|, |x_{i+1}|) = 1$ for all $i \in \{\ell, \dots, t-1\}$.

For $i \in L$ with $|\pi(G_i)| = \infty$, let V_i be the Hall σ'-subgroup of G_i, and for $i \in L$ with $|\pi(G)| = 1$, set $V_i = U_i$ (notice that $V_i \le U_i$ if $|\pi(G_i)| = \infty$). We want to apply Lemma 4.28 $t - \ell + 1$ times, first to the index t, then decreasing until we reach the index ℓ, with the V_i taking the role of the U_i. Say we are applying it to the index $\ell \le j \le t$ and let us check that the two coprimality conditions of Lemma 4.28 are satisfied. We first check the hypothesis $(|x_j|, |V_{j+1}|) = 1$. If $|\pi(G_{j+1})| = \infty$, then $\pi(V_j) \subseteq \sigma'$ and the hypothesis is satisfied. If $\pi(G_{j+1}) = \{p\}$, then $p \notin \pi(G_j)$ and in particular $p \notin \pi(x_j)$. As for the other condition, if $j = \ell$, it is simply one of the hypotheses of the lemma. If $\ell + 1 \le j \le t$, we have that $(|x_j|, |x_{j-1}|) = 1$ and $(|x_j|, |v_{j-1}|) = 1$ for all $v_{j-1} \in V_{j-1}$, either because V_{j-1} is a σ'-subgroup if $|\pi(G_{j-1})| = \infty$ or by hypothesis if $|\pi(G_{j-1})| = 1$.

At the end of this process we obtain $\varphi^*_L(V_i; X_i) = 1$. Now, if $|\pi(G_i)| = 1$, then $Y_i = U_i = V_i$. If $|\pi(G_i)| = \infty$, writing H_j for the Hall σ-subgroup of

G_j, then $\varphi^*(X_1, \dots, X_\ell, G_{\ell+1}, \dots, G_{j-1}, H_j, G_{j+1}, \dots, G_t) \subseteq N_\sigma$ by definition and by Lemma 4.22(i) we obtain that $\varphi^*_L(Y_i; X_i) \subseteq N_\sigma$.

(ii) If $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell U_\ell, \dots, x_t U_t) = \varnothing$ then in particular we have that $\varphi^*(X_1, \dots, X_{\ell-1}, x_\ell, \dots, x_t) = \varnothing$. The only way for this to happen is that there exists an index $j \in \{1, \dots, l-2\}$ such that $(|x_j|, |x_{j+1}|) \neq 1$ for all $x_j \in X_j$, $x_{j+1} \in X_{j+1}$, and the lemma follows.

$\square$

The following lemma is the focal point of the proof of Proposition 4.33, as it will allow us to funnel some values of certain coprime commutators into an accurately chosen subgroup.

Lemma 4.30. *Let $G_1, \dots, G_t$ be pronilpotent subgroups of a profinite group G such that $G_j \leq N_G(G_i)$ for all $j \leq i$, and $|\delta^*_{t-1}\{G\}| < 2^{\aleph_0}$. Then, there exist a finite set $W \subseteq \varphi^*(G_i)$ and a finite set σ of primes such that $\varphi^*(G_i) \subseteq N_\sigma \langle W \rangle^G$.*

As this is the most technical proof, we will first give an example of the procedure for a specific case to clarify the main ideas.

Example 4.31. We restrict to the case $t = 2$, so we are studying $\varphi^*(G_1, G_2)$, in the specific case when $|\pi(G_1)| = 1$, $|\pi(G_2)| = \infty$ and $\pi(G_1) \cap \pi(G_2) = \varnothing$. Notice that for $t = 2$ some easier reasoning could lead to an analogous result, but we will follow the algorithm beneath the proof of Lemma 4.30 in order to illustrate it.

By Lemma 4.25, for $i \in \{1, 2\}$, we obtain $U_i \trianglelefteq_o G_i$ and $x_i \in G_i$ such that $\varphi^*(x_1 U_1, x_2 U_2) = \{w\}$ consists of a single value. Set $W = \{w\}$, we will work in $G/\langle W \rangle^G$ and assume $w = 1$. We recall that by Remark 4.16, we can always refine an open normal subgroup $U_2 \trianglelefteq G_2$ with another normal open subgroup which is normalized by G_1 too, so we will always assume that $G_1 \leq N_G(U_2)$.

Lemma 4.29 (with $\ell = 1$) gives a set $\sigma(\varnothing)$ of primes such that $\varphi^*(U_1, G_2) \subseteq N_{\sigma(\varnothing)}$. We can factor out this subgroup and assume $\varphi^*(U_1, G_2) = 1$. Fix now a set $S = \{s_1 = 1, \dots, s_m\}$ of coset representatives of U_1 in G_1. As $1 \in S$ and $\pi(G_1) = 1$, every element of S is a good representative for U_1.

Set now $V_0 = G_2$. For every $\ell \in \{1, \dots, m\}$, if $\varphi^*(s_\ell, V_{\ell-1}) = \varnothing$, then set $V_\ell = V_{\ell-1}$, otherwise Lemma 4.25 gives a coset $V_\ell \subseteq V_{\ell-1}$, such that $\varphi^*(s_\ell, V_\ell) = 1$. Notice that each V_ℓ is a coset of an open subgroup of G_2. Repeating this procedure m times we get $V_m = gV$ for $V \trianglelefteq_o G_2$, $g \in G_2$ such that $\varphi^*(s_\ell, gV)$ is either empty or consists of the trivial element for every $\ell = 1, \dots, m$. Notice that, being $1 \in S$, the set $\varphi^*(S, gV)$ is non-empty.

Applying now Lemma 4.29, this time with $\ell = 2$, we can obtain a finite set of primes σ satisfying $\varphi^*(S, G_2) \subseteq N_\sigma$. Now, if we work in G/N_σ, we can apply Lemma 4.22 and obtain that $\varphi^*(s_\ell U_1, G_2)$ is either empty or trivial for every $\ell \in \{1, \dots, m\}$. As S was a set of coset representatives of U_1 in G_1, we have that $\varphi^*(G_1, G_2) = 1$. Since the beginning of the proof, we have factored out the normal subgroups $\langle W\rangle^G$ and $N_{\sigma(\varnothing)\cup\sigma}$, settling Lemma 4.30 in our case.

Overall with several subgroups $G_1, \dots G_t$ some additional steps might be necessary, but this case exemplifies the main ideas of the proof.

Proof of Lemma 4.30. Let

$$\mathcal{I} = \{i \in \{1, \dots, t\} \mid |\pi(G_i)| = \infty\}.$$

It suffices to prove the theorem in the case when $|\pi(G_i)| = 1$ for all G_i with $i \notin \mathcal{I}$. The general case, where each G_i, $i \notin \mathcal{I}$, is the product of its Sylow subgroups follows by applying Lemma 4.22.

For $i \notin \mathcal{I}$, let p_i be a prime such that $\pi(G_i) = \{p_i\}$. Then we have

$$\varphi^*(G_i) = \varphi^*(G_1, \dots, G_{i-2}, H_{i-1}, G_i, H_{i+1}, G_{i+2}, \dots, G_t),$$

where H_{i-1} and H_{i+1} are the Hall p_i'-subgroups of G_{i-1} and G_{i+1}, respectively. We can therefore assume, again by Lemma 4.22(i), that for all $i \notin \mathcal{I}$ we have

$$p_i \notin \pi(G_{i-1}) \cup \pi(G_{i+1}). \tag{4.7}$$

We claim that that for every $J \subseteq \{1, \dots, t\} \setminus \mathcal{I}$ there exist a finite set $W_J \subseteq \varphi^*(G_i)$, a finite set of primes $\sigma(J)$ and subgroups $U_i^J \trianglelefteq_o G_i$ with $i \notin \mathcal{I} \cup J$ such that $\varphi^*_{\mathcal{I}\cup J}(G_i; U_i^J) \subseteq N_{\sigma(J)}\langle W_J\rangle^G$.

We proceed by induction on $|J|$. Assume first $J = \varnothing$. By Lemma 4.25, for every $i \in \{1, \dots, t\}$ there exist elements $x_i \in G_i$ and subgroups $U_i^\varnothing \trianglelefteq_o G_i$ such that $\varphi^*(x_i U_i^\varnothing) = \{w_\varnothing\}$ for a suitable $w_\varnothing \in G$. Moreover, by Remark 4.16, we may assume that $G_j \leq N_G(U_i^\varnothing)$ for every $j \leq i$. Hence, Lemma 4.29 produces a finite set $\sigma(\varnothing)$ of primes such that $\varphi^*_{\mathcal{I}}(G_i; U_i^\varnothing) \subseteq N_{\sigma(\varnothing)}\langle w_\varnothing\rangle^G$, so the claim follows for $|J| = 0$.

Assume now that $|J| \geq 1$ and that for every $J^- \subsetneq J$ there exist a finite set $W_{J^-} \subseteq \varphi^*(G_i)$, a finite set of primes $\sigma(J^-)$ and subgroups $U_i^{J^-} \trianglelefteq_o G_i$, $i \notin \mathcal{I} \cup J^-$, such that $\varphi^*_{\mathcal{I}\cup J^-}(G_i; U_i^{J^-}) \subseteq N_{\sigma(J^-)}\langle W_{J^-}\rangle^G$. For convenience, we also set $U_i^{J^-} = G_i$ if $i \in J^-$, so that $U_i^{J^-}$ is defined for all $i \notin \mathcal{I}$. Let $W_J = \bigcup_{J^-} W_{J^-}$, $\rho = \bigcup_{J^-} \sigma(J^-)$ and $V_i = \bigcap_{J^-} U_i^{J^-}$ for all $i \notin \mathcal{I}$, so that, by (4.4), we have

$\varphi^*_{\mathcal{I}\cup J^-}(G_i; V_i) \subseteq N_\rho\langle W_J\rangle^G$ for every $J^- \subsetneq J$. Furthermore, by factoring out $N_\rho\langle W_J\rangle^G$, we may assume that

$$\varphi^*_{\mathcal{I}\cup J^-}(G_i; V_i) = 1 \tag{4.8}$$

for every $J^- \subsetneq J$. Moreover, taking into account Remark 4.16 we may further assume that V_i is invariant under the conjugacy action of G_j for every $j \leq i$.

Write $J = \{j_1, \dots, j_n\}$ with $j_1 < \cdots < j_n$, and for every $i \in J$, fix a set S_i of coset representatives for V_i in G_i containing the identity. Write

$$S_{j_1} \times \cdots \times S_{j_n} = \{\mathbf{s}_1, \dots, \mathbf{s}_m\}$$

with $\mathbf{s}_\ell = (s_{\ell,j_1}, \dots, s_{\ell,j_n})$ for $\ell \in \{1, \dots, m\}$. Denote $V_i = G_i$ for $i \in \mathcal{I}$. Since $1 \in S_i$ for every i, we have $\varphi^*_J(S_i; V_i) \neq \varnothing$, so applying Lemma 4.25 we obtain elements $x_i \in V_i$ and subgroups $U_i \trianglelefteq_o V_i$ such that $\varphi^*_J(S_i; x_iU_i)$ takes a single value. Actually, since $1 = \varphi^*_J(1; x_iU_i) \subseteq \varphi^*_J(S_i; x_iU_i)$, we have $\varphi^*_J(S_i; x_iU_i) = 1$. Thus, for every $\ell \in \{1, \dots, m\}$, we either have

$$\varphi^*_J(s_{\ell,i}; x_iU_i) = \varnothing \quad \text{ or } \quad \varphi^*_J(s_{\ell,i}; x_iU_i) = 1. \tag{4.9}$$

We may assume x_i to be a good representative of the coset x_iU_i and therefore, if J does not contain neither i nor $i+1$, then $(|x_i|, |x_{i+1}|) = 1$. Also, by Remark 4.16 we may further assume that U_i is invariant under the conjugacy action of G_j for every $j \leq i$.

Let $J_0 = \varnothing$, and for $r \in \{1, \dots, n\}$, let $J_r = \{j_1, \dots, j_r\}$. We also write $j_0 = 0$ for convenience. We will show that for every $r \in \{0, \dots, n\}$, there exists a finite set of primes $\tau(r)$ such that $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r)}) \subseteq N_{\tau(r)}$ for every $\ell \in \{1, \dots, m\}$, where

$$Y_i^{(r)} = \begin{cases} G_i & \text{if } \ i \geq j_r,\ i \in \mathcal{I} \cup J, \\ U_i & \text{if } \ i > j_r,\ i \notin \mathcal{I} \cup J, \\ x_iU_i & \text{if } \ i < j_r. \end{cases}$$

Notice that right now we are not using $Y_{j_r}^{(r)}$, but it will be convenient to have it defined for later. We argue by reverse induction on $r \in \{0, \dots, n\}$; assume first $r = n$. Since $j_r \notin \mathcal{I}$, we deduce from (4.7) that $(|s_{\ell,j_r}|, |G_{j_r-1}|) = (|s_{\ell,j_r}|, |x_{j_r+1}|) = 1$. Thus, for all $\ell \in \{1, \dots, m\}$, we obtain from (4.9) and Lemma 4.29 a finite set of primes $\tau(r,\ell)$ such that $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r)}) \subseteq N_{\tau(r,\ell)}$. Defining $\tau(r) = \bigcup_{\ell=1}^m \tau(r,\ell)$, we obtain $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r)}) \subseteq N_{\tau(r)}$ for every $\ell \in \{1,\dots,m\}$.

Hence, we assume $r \leq n-1$. By induction, we know that there exists a finite set of primes $\tau(r+1)$ such that

$$\varphi^*_{J_{r+1}}(s_{\ell,i}; Y_i^{(r+1)}) \subseteq N_{\tau(r+1)} \tag{4.10}$$

for every $\ell \in \{1,\dots,m\}$.

The inductive step will be divided in two phases. We will first show that $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r+1)}) \subseteq N_{\tau(r+1)}$ (meaning that the only difference from (4.10) is position j_{r+1}). In order to obtain this, we have to substitute in the j_{r+1}-th position first $s_{\ell,j_{r+1}}$, and then $s_{\ell,j_{r+1}}U_{j_{r+1}}$ for all $\ell \in \{1,\dots,m\}$. We will then conclude the inductive step by proving that there exists a finite set $\tau(r)$ of primes such that $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r)}) \subseteq N_{\tau(r)}$ for every $\ell \in \{1,\dots,m\}$.

We begin by noting that $Y_i^{(r+1)} \leq V_i$ for every $i \notin \mathcal{I} \cup J$ and that $U_{j_{r+1}} \leq V_{j_{r+1}}$, so (4.8) yields

$$\varphi^*_{J_r}(s_{\ell,i}; \widetilde{Y}_i) \subseteq N_{\tau(r+1)}, \tag{4.11}$$

where $\widetilde{Y}_i = Y_i^{(r+1)}$ if $i \neq j_{r+1}$ and $\widetilde{Y}_{j_{r+1}} = U_{j_{r+1}}$. As we chose the sets of representatives S_j in such a way that the identity is contained in them, for every $\ell \in \{1,\dots,m\}$, either $s_{\ell,j_{r+1}}$ is trivial or $|\pi(s_{\ell,j_{r+1}})| = 1$, so in particular $s_{\ell,j_{r+1}}$ is a good representative. Thus, by (4.10) and (4.11), we deduce from Lemma 4.22 that $\varphi^*_{J_r}(s_{\ell,i}; \overline{Y}_i) \subseteq N_{\tau(r+1)}$, where $\overline{Y}_i = Y_i^{(r+1)}$ if $i \neq j_{r+1}$ and $\overline{Y}_{j_{r+1}} = s_{\ell,j_{r+1}}U_{j_{r+1}}$. Since this holds for every $\ell \in \{1,\dots,m\}$, and since $G_{j_{r+1}} = \bigcup_{s \in S_{j_{r+1}}} sU_{j_{r+1}}$, we obtain $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r+1)}) \subseteq N_{\tau(r+1)}$, as we wanted.

Now using (4.7) and Lemma 4.29, we conclude exactly as in the case $r = n$ that there exists a finite set $\tau(r)$ of primes such that $\varphi^*_{J_r}(s_{\ell,i}; Y_i^{(r)}) \subseteq N_{\tau(r)}$ for every $\ell \in \{1,\dots,m\}$.

This completes the reverse induction on r. In particular, for $r = 0$, it follows that $\varphi^*_J(G_i; U_i) \subseteq N_{\tau(0)}$, so this, in turn, concludes the inductive step on $|J|$, and the claim is proved.

Finally, taking J in such a way that $\mathcal{I} \cup J = \{1,\dots,t\}$, we obtain a finite set of primes $\sigma(J)$ and a finite set $W \subseteq \varphi^*(G_i)$ such that $\varphi^*(G_1,\dots,G_t) \subseteq N_{\sigma(J)}\langle W\rangle^G$, as desired. □

Recall that if G is a prosolvable group of Fitting height $k+1$, there exist some pronilpotent subgroups $U_0,\dots,U_k$ satisfying Proposition 4.15.

We remark that φ and φ^* were defined with variables $\{x_i \mid i = 1,\dots,t\}$ for a generic positive integer t. Since we now want to apply the previous results to the subgroups $U_0, \dots, U_k$, we will set $t = k+1$ and we will write $\varphi(U_{i-1})$ for $\varphi(U_0, \dots, U_k)$ and $\varphi^*(U_{i-1})$ for $\varphi((U_0 \times \cdots \times U_k) \cap \mathcal{C})$, where $\mathcal{C}$ is defined as in (4.3).

Lemma 4.32. *Let $G = U_0 \cdots U_k$ be as in Proposition 4.15 with $U_k = \delta_k^*(G)$ abelian, and assume $|\delta_k^*\{G\}| < 2^{\aleph_0}$. Let $g \in \varphi^*(U_{i-1})$. Then, there exists a finite normal subgroup $N \trianglelefteq G$ such that $g \in N$.*

Proof. Write $g = [x_0, \dots, x_k]$, where $x_j \in U_j$ for all j and $(|x_\ell|, |x_{\ell+1}|) = 1$ for all $\ell \in \{0, \dots, k-1\}$. By Lemma 4.9, $[x_0, \dots, x_j]$ is a δ_j^*-value for every $j \in \{0, \dots, k\}$.

In particular $x := [x_0, \dots, x_{k-1}]$ is a δ_{k-1}^*-value. Let H be the minimal Hall subgroup of $\delta_k^*(G)$ containing x_k, so that $(|x|, |H|) = 1$, again by Lemma 4.9. Since, again, $[x, h]$ is a δ_k^*-value for every $h \in H$, the set $K := \{[x, h] \mid h \in H\}$ has less than $2^{\aleph_0}$ values, and, since H is abelian and normal in G, it follows that K is actually a closed subgroup of G. In particular, K is finite, so every element of K has finite order. Thus, we deduce from Lemma 3.17 that the set $S := \bigcup\{k^G \mid k \in K\}$ is finite, and therefore $N = \langle S \rangle$ is finite by Dietzmann's Lemma (see Lemma 14.5.7 of [73]). □

We are now ready to prove the strong conciseness of δ_k^* in prosoluble groups of Fitting height $k+1$.

Proposition 4.33. *Let G be a prosoluble group of Fitting height $k+1$. Assume that $|\delta_k^*\{G\}| < 2^{\aleph_0}$. Then $\delta_k^*(G)$ is finite.*

Proof. In view of Lemma 4.12, we may assume that $\delta_k^*(G)$ is abelian. Thus, we can take $U_0, \dots, U_k \leq G$ as in Proposition 4.15, so that $G = U_0 \cdots U_k$ with $U_k = \delta_k^*(G)$ abelian.

We claim that for every family of subgroups $G_{i-1} \leq U_{i-1}$ with $i \in \{1, \dots, k+1\}$ such that $G_j \leq N_G(G_i)$ for $j \leq i$, we have $|\varphi^*(G_{i-1})| < \infty$. We argue by induction on $|\mathcal{I}|$, where

$$\mathcal{I} = \{i \in \{1, \dots, k+1\} \mid |\pi(G_{i-1})| = \infty\}.$$

If $|\mathcal{I}| = 0$, then Lemma 4.30 gives the result since for every finite set $W \subseteq \varphi^*(G_{i-1})$, the normal subgroup $\langle W \rangle^G$ is finite by Lemma 4.32, and since, by definition, $N_\sigma = 1$ for every finite set of primes σ. Suppose thus $|\mathcal{I}| \geq 1$. Then, Lemma 4.30 produces a finite set of primes σ and a finite set $W \subseteq \varphi^*(G_{i-1})$ such that $\varphi^*(G_{i-1}) \subseteq N_\sigma \langle W \rangle^G$. Observe that by induction, for every $j \in \mathcal{I}$, we have $|\varphi^*_{\{j\}}(H_{i-1}; G_{i-1})| < \infty$, where H_{i-1} is the Hall σ-subgroup of G_{i-1}, and therefore N_σ is finite by Lemma 4.32. Again by Lemma 4.32, $\langle W \rangle^G$ is also finite, and the claim follows.

In particular, we have shown that $|\varphi^*(U_{i-1})| < \infty$.

Denote by $P_i(p)$ the Sylow p-subgroup of U_i for every $p \in \pi(G)$, $i \in \{0, \dots, k\}$. As each U_i is a pronilpotent subgroup, for every $j \in \{1, \dots, k\}$ and every $p \in \pi(G)$ Proposition 4.15 yields

$$P_j(p) = \prod_{\substack{q \in \pi(G) \\ q \neq p}} [P_{j-1}(q), P_j(p)].$$

Therefore, for every $q_k \in \pi(G)$,

$$P_k(q_k) = \prod_{(q_0, \dots, q_{k-1}) \in S_{q_k}} [P_0(q_0), \dots, P_k(q_k)],$$

where

$$S_{q_k} = \{(q_0, \dots, q_{k-1}) \in \pi(G)^k \mid q_j \neq q_{j+1} \text{ for every } j = 0, \dots, k-1\}.$$

By Lemma 4.21, this implies that $P_k(q_k) \leq \langle \varphi^*(U_{i-1}) \rangle$ for every $q_k \in \pi(G)$, and so

$$\delta_k(G) = U_k = \prod_{p \in \pi(G)} P_k(p) \leq \langle \varphi^*(U_{i-1}) \rangle.$$

The proposition follows from Lemma 4.32, as we already proved that $|\varphi^*(U_{i-1})| < \infty$. $\square$

4.6 Strong conciseness of coprime commutators

We recall that a *minimal simple group* is a finite non-abelian simple group all of whose proper subgroups are soluble. These groups have been classified by Thompson in [79]. By applying induction on the order of the group, it is immediate to see that every finite simple group has a section that is a minimal simple group.

Lemma 4.34. *In every minimal simple group there exist an involution e and an element h of odd order such that $h^e = h^{-1}$. Moreover, for every positive integer k, the element*

$$g_k = [h, e, \overset{k-1}{\dots}, e]$$

is both a non-trivial γ_k^*-value and a non-trivial δ_{k-1}^*-value.

Proof. The first claim follows from Theorem 2.13 of [42] and the fact that non-abelian simple groups are of even order. Notice that $g_k = h^{(-2)^{k-1}}$, so $g_k \neq 1$ for every positive integer k. Clearly g_i and e are coprime for every $i \in \{1, \dots, k-1\}$, so g_k is a γ_k^*-value. Thus, it suffices to prove that the same is true for δ_{k-1}^*. By Proposition 25 of [11], every involution of a minimal simple group is a δ_ℓ^*-value for every $\ell \in \mathbb{N}$. Hence, we can use Lemma 4.8 with $g_1 = \dots = g_{k-1} = e$ and $H = \langle h \rangle$ and conclude the proof. $\square$

We are now ready to prove our main results. As in [28], we start showing that the Fitting subgroup of any infinite profinite group G with $|\gamma_k^*\{G\}| < 2^{\aleph_0}$ or $|\delta_k^*\{G\}| < 2^{\aleph_0}$ is infinite.

Proposition 4.35. *Let G be an infinite profinite group and let $w^* = \delta_k^*$ or $w^* = \gamma_k^*$. Suppose that $1 \neq |w^*\{G\}| < 2^{\aleph_0}$. Then the Fitting subgroup F of G is infinite.*

Proof. We first show that F is non-trivial. Assume by contradiction that $F = 1$. For every non-trivial w^*-value x of G, the normal closure $\langle x^G \rangle$ is finite. Indeed, by Lemma 3.17, x^G is finite, so in particular $|G : C_G(\langle x^G \rangle)| < \infty$. As a consequence, the index $|\langle x^G \rangle : Z(\langle x^G \rangle)|$ is also finite, but $Z(\langle x^G \rangle)$ is contained in $F = 1$. This implies that G possesses finite minimal normal subgroups, so let N be the product of all of the subgroups obtained in this way from the set of w^*-values.

If N is finite, then there exists a normal open subgroup $K \trianglelefteq_o G$ such that $K \cap N = 1$. Such a subgroup cannot contain any non-trivial w^*-value since otherwise, repeating the same argument as before, we would obtain a minimal normal subgroup of G contained in K, contradicting that $K \cap N = 1$. If $w^*(K) = 1$, then by Theorem 4.2 K is either pronilpotent (if $w^* = \gamma_k^*$) or prosoluble of Fitting height k (if $w^* = \delta_k^*$), and this contradicts the fact that $F \cap K = 1$. This proves that N is an infinite subgroup of G.

None of the infinitely many minimal normal subgroups of G contained in N is abelian because $F = 1$, so each of these minimal normal subgroups contains a section isomorphic to a minimal simple group. For each minimal normal subgroup N_i, with $i \in I$, choose a section isomorphic to a minimal simple group S_i. We remark that by the previous discussion I is an infinite set and so the Cartesian product of S_i is a section of G. By Lemma 4.34 in each of these groups S_i there exist an involution $e_i \in S_i$ and an element $h_i \in S_i$ of odd order with $h_i^{e_i} = h_i^{-1}$ such that $g_i := [h_i, e_i, \overset{k-1}{\dots}, e_i]$ is a non-trivial w^*-value. Now, using the structure of Cartesian product, for each subset $J \subseteq I$ the element $c_J = \prod_{j \in J} g_j$ can be written as

$$c_J = \Big[\prod_{j \in J} h_j, \prod_{j \in J} e_j, \dots, \prod_{j \in J} e_j \Big].$$

Clearly $\prod_{j\in J} e_j$ is an involution normalizing the cyclic subgroup generated by the element of odd order $\prod_{j\in J} h_j$, and hence it is a w^*-value (if $w^* = \delta_k^*$ we also need to use Lemma 4.8). However, there exist at least $2^{\aleph_0}$ distinct subsets $J \subseteq I$ that give rise to different c_J, against the assumption that $|w^*\{G\}| < 2^{\aleph_0}$, so $F \neq 1$.

If we assume by contradiction that the Fitting subgroup F is finite, then there would be a subgroup $K \trianglelefteq_o G$ with $K \cap F = 1$, so that K has trivial Fitting subgroup. By the previous argument, this can happen only if $w^*(K) = 1$, so K is either pronilpotent or prosoluble of Fitting height k, contradicting that $K \cap F = 1$ and proving the proposition. □

Proofs of Theorems 4.3 and 4.4. In view of Theorem 4.2 it is sufficient to show that if $|w^*\{G\}| < 2^{\aleph_0}$, then G is finite-by-pronilpotent in the case $w^* = \gamma_k^*$ or finite-by-(prosoluble of Fitting height at most k) if $w^* = \delta_k^*$. We can assume G to be infinite, otherwise the theorem is trivially true.

We will denote the Fitting subgroup of G by F, and for $i \geq 2$, let F_i be the i-th Fitting subgroup of G. By Proposition 4.35, F is infinite (and hence the same is true for all F_k). Let $n = 2$ if $w^* = \gamma_k^*$ and $n = k+1$ if $w^* = \delta_k^*$. By Propositions 4.20 and 4.33, $w^*(F_n)$ is finite. Therefore there exists an open normal subgroup $R \trianglelefteq_o F_n$ with $R \cap w^*(F_n) = 1$. Theorem 4.2 implies that the Fitting height of R is at most $n-1$, and hence R is contained in F_{n-1}. However, since F_n/F_{n-1} is the Fitting subgroup of G/F_{n-1}, it follows that G/F_{n-1} has finite Fitting subgroup, and by Proposition 4.35 this can only happen if G/F_{n-1} is finite.

Thus, we will prove the result by induction on $|G : F_{n-1}(G)|$, with the base case $G = F_{n-1}(G)$ being trivial by Theorem 4.2. Assume then that $|G : F_{n-1}(G)| > 1$ and suppose first that G/F_{n-1} has a nontrivial proper normal subgroup N. The inductive hypothesis yields $|w^*(N)| < \infty$, and working in $G/w^*(N)$, we obtain by Theorem 4.2 that $N/w^*(N)$ is prosoluble of Fitting height at most $n-1$. This implies that $N/w^*(N)$ is contained in the $(n-1)$-th Fitting subgroup of $G/w^*(N)$ and by inductive hypothesis $w^*(G/w^*(N))$ must be finite, so $w^*(G)$ is finite too.

We can hence assume that G/F_{n-1} is a simple group. Notice that if G/F_{n-1} is abelian, then we can conclude simply by applying Proposition 4.20 or Proposition 4.33. Thus, the only case left is when G/F_{n-1} is a finite non-abelian simple group. By Theorem 4.2 we have $w^*(G/F_{n-1}) = G/F_{n-1}$, so there is a finite set S consisting of w^*-values such that $G = \langle S\rangle F_{n-1}$. By Lemma 3.17 the set $T := \bigcup\{s^G \mid s \in S\}$ is finite, so the index $|G : C_G(T)|$ is also finite. This implies that the center of $\langle T\rangle$ has finite index in $\langle T\rangle$, so by Schur's theorem $\langle T\rangle'$ is finite too. Note that $\langle T\rangle'$ is normal in G. Factoring out $\langle T\rangle'$, we can assume G/F_{n-1} to be abelian, and we conclude the proof as before. □

5 Profinite right-angled Artin groups

In this chapter we will begin developing the theory of pro-$\mathcal{C}$ right-angled Artin groups. This chapter and the next are part of a preprint, which is currently in preparation, joint with M. Casals Ruiz and P. Zalesskii.

The first section is devoted to describing the profinite version of Bass-Serre theory developed mainly by Melnikov, Ribes and Zalesskii. This theory aims at understanding the structure of groups by their action on a profinite tree. One of the main reasons to use this approach is that the action of a profinite group on a profinite tree naturally gives a description of the structure of its subgroups, that can be directly obtained by looking at the action of the subgroup on the tree.

In the second section we define profinite right-angled Artin groups and we develop their basic properties. These groups are defined by a finite graph, choosing vertices as generators and setting as relations that two adjacent vertices commute. We then obtain some results on standard subgroups, which are the subgroups generated by a subset of the canonical generators of a profinite RAAG, and we show that all these groups are torsionfree.

In Section 3 we prove that a profinite RAAG splits as a direct product if and only if its underlying graph is a join. We then obtain a description of centralisers of elements analogous to the one of abstract RAAGs obtained by Baudisch in [12], proving that they split as a direct product of a standard subgroup and of some projective groups. We will then use this characterization to conclude that, in pro-p RAAGs, centralisers are retracts.

In the last section, we conclude by proving that a pro-$\mathcal{C}$ RAAG either contains free pro-$\mathcal{C}$ groups or it is solvable, and then give a description of two-generated subgroup of pro-p RAAGs.

5.1 Profinite groups acting on profinite trees

In this section, we describe the analogue results of Bass-Serre theory for profinite groups acting on profinite trees. A deeper description can be found in [70] or, for pro-p groups, in [72] and [71].

In the current and next chapter, we assume $\mathcal{C}$ to be a class of finite groups closed under taking subgroups, homomorphic images, direct product and extensions with abelian kernel. The *primes involved in* $\mathcal{C}$ is the set of primes that divide the order of a group $G \in \mathcal{C}$ and will be denoted as $\pi(\mathcal{C})$. As usual, if π is a set of primes, a pro-π group is an inverse limit of finite groups of π-order.

Definition 5.1 (Profinite graph)**.** A *profinite graph* is a profinite space Γ with a distinguished non-empty closed subset $V(\Gamma)$ and two continuous maps (called *incidence maps*) $d_0, d_1 : \Gamma \to V(\Gamma)$ which restrict to the identity on $V(\Gamma)$.

The elements of $V(\Gamma)$ are the *vertices* of the profinite graph, whereas the elements of $E(\Gamma) := \Gamma \smallsetminus V(\Gamma)$ are the *edges*. A *morphism* $\alpha : \Gamma \to \Delta$ is a map of profinite spaces respecting incidence maps, so $\alpha d_i = d_i \alpha$ for $i \in \{0, 1\}$. A profinite graph is the inverse limit of its finite quotients graphs (see Proposition 1.5 of [71]) and we say that Γ is *connected* if all of these finite quotient graphs are connected (as abstract finite graphs).
For each profinite graph Γ, we define $(E^*(\Gamma), *) = (\Gamma/V(\Gamma), *)$ the pointed profinite quotient space, where the distinguished point is the representative of $V(\Gamma)$. For every prime p, we have a complex of free profinite $\mathbb{F}_p$-modules

$$0 \longrightarrow \mathbb{F}_p[[E^*(\Gamma), *]] \xrightarrow{\delta} \mathbb{F}_p[[V(\Gamma)]] \xrightarrow{\epsilon} \mathbb{F}_p \longrightarrow 0 \tag{5.1}$$

where the maps are defined as $\delta(e) = d_1(e) - d_0(e)$ for every $e \in E^*(\Gamma)$ and $\epsilon(v) = 1$ for every $v \in V(\Gamma)$.

Definition 5.2 (Profinite tree)**.** A profinite graph Γ is a *pro-p tree* if the associated chain complex (5.1) is an exact sequence. A profinite graph is a *pro-$\mathcal{C}$ tree* if the associated chain complex (5.1) is an exact sequence for each prime $p \in \pi(\mathcal{C})$.

In particular, a pro-$\mathcal{C}$ tree is a pro-π tree with $\pi = \pi(\mathcal{C})$ (see Proposition 2.4.2 of [70]). As every pro-$\mathcal{C}$ group is a pro-$\pi(\mathcal{C})$ group, we can often reduce the study of pro-$\mathcal{C}$ groups acting on pro-$\mathcal{C}$ trees to the study of pro-π groups acting on pro-π trees. For this reason, many theorems we refer to are originally stated in the pro-π version in the sources we cite but they are still valid for pro-$\mathcal{C}$ groups and trees, and we state them in this form.

All the subtrees of a pro-$\mathcal{C}$ tree Γ are partially ordered by inclusion and the minimal subtree containing two vertices $v, w \in V(\Gamma)$, which we denote as $[v, w]$, is called a *geodesic*.

Some results that are valid for abstract trees are true for pro-$\mathcal{C}$ trees too. For example, we will make use of Helly's Theorem for pro-$\mathcal{C}$ trees.

Lemma 5.3. *Let $S = \{T_i, i \in I\}$ be an arbitrary family of non-empty pro-$\mathcal{C}$ subtrees of a pro-$\mathcal{C}$ tree T and suppose $T_i \cap T_j \neq \varnothing$ for every $i, j \in I$. Then $\bigcap_{i \in I} T_i \neq \varnothing$.*

Proof. As the pro-$\mathcal{C}$ tree T is compact, it suffices to prove that every finite subset of S has a non-empty intersection, so we prove the result for $S_k = \{T_i, i = 1, \dots, k\}$ by induction on k.

If $k = 1$ or $k = 2$, the statement holds trivially from the assumption that the trees pairwise intersect. We treat the case $k = 3$ separately as it is going to be used in the inductive step. Let $v_{12}, v_{13} \in V(T)$ be vertices in $T_1 \cap T_2$ and $T_1 \cap T_3$ respectively. The geodesic $[v_{12}, v_{13}]$ is contained in T_1, and by Lemma 2.8 of [71] we have that $[v_{12}, v_{13}] \cap T_2 \cap T_3 \neq \varnothing$, hence $T_1 \cap T_2 \cap T_3 \neq \varnothing$.

Suppose by induction that the result holds for every set with less than k trees and consider the set $S_k = \{T_i \mid i = 1, \dots, k\}$. Define $\overline{T} = T_{k-1} \cap T_k$. Notice that by Proposition 2.4.9 of [70], the intersection of any family of pro-$\mathcal{C}$ subtrees is still a pro-$\mathcal{C}$ subtree (possibly empty) and so $\overline{T}$ is a pro-$\mathcal{C}$ tree. By induction (using the case $k = 3$) we have that $\overline{T} \cap T_i \neq \varnothing$ for all $i \in \{1, \dots, k-2\}$, hence we can apply the inductive hypothesis to the family $\overline{S}_k = \{\overline{T}, T_1, \dots, T_{k-2}\}$, which has by definition the same intersection as the family S_k and the result follows. $\square$

A pro-$\mathcal{C}$ group G acts on a pro-$\mathcal{C}$ tree Γ if it respects the incident maps, i.e. $gd_i = d_i g$ for $i \in \{0, 1\}$, $g \in G$, and the action is continuous. If an element $g \in G$ fixes at least a point of Γ we say that g is *elliptic*, on the other hand, if g does not fix any point, then g is a *hyperbolic* element. We moreover say that a subgroup $H \leq G$ is elliptic if the whole subgroup fixes a point of Γ.
Whenever an element $g \in G$ (or a subgroup $H \leq G$) is elliptic, we can consider the set of fixed points T^g (respectively T^H), that is a pro-$\mathcal{C}$ tree by Theorem 4.1.5 of [70].

When studying actions of groups on trees, we often need to restrict to minimal invariant subtrees, whose existence is guaranteed by the following lemma, which is Proposition 2.4.12 of [70].

Lemma 5.4. *If G is a pro-$\mathcal{C}$ group acting on a pro-$\mathcal{C}$ tree Γ, then there exists a minimal G-invariant pro-$\mathcal{C}$ subtree Δ of Γ. If Δ contains more than one vertex, then it is unique.*

The action of G on a pro-$\mathcal{C}$ tree Γ is *irreducible* if Γ has no proper G-invariant subtrees. From now on, for every subset S of a group G acting on a pro-$\mathcal{C}$ tree T, we denote by T_S the minimal pro-$\mathcal{C}$ subtree on which $\langle S\rangle \leq G$ acts. Similarly to the abstract case, when elements commute we can obtain some additional information on their action.

Lemma 5.5. *Let G be a pro-$\mathcal{C}$ group acting faithfully on a pro-$\mathcal{C}$ tree T.*

1. *Let $g, h \in G$ be such that h normalises $\langle g\rangle$, then h leaves T_g invariant and in particular, if $[g, h] = 1$ then $T_g = T_h$.*
2. *Let $S = \{g_1, \ldots, g_k\}$ be a set of elements such that the action of each g_i, $i \in \{1, \ldots, k\}$, is elliptic. If $[g_i, g_j] = 1$ for every $i, j \in \{1, \ldots, k\}$, then there exists a vertex of T fixed by the whole set S.*

Proof. Part (1) follows immediately by observing that $h \cdot (T_g) = T_{hgh^{-1}} \subseteq T_g$.
We first prove part (2) for two elements $g_1, g_2 \in G$. If both g_1 and g_2 are elliptic, consider the subtrees T^{g_1} and T^{g_2} fixed by g_1 and g_2 respectively; by (1) we have that T^{g_1} is a non-empty pro-$\mathcal{C}$ subtree invariant under the action of g_2. By Corollary 4.1.9 of [70], g_1 fixes a vertex of T^{g_2}, hence $T^{g_2} \cap T^{g_1}$ is not trivial.
Applying the case $k = 2$ to each pair, we have that $T^{g_i} \cap T^{g_j} \neq \varnothing$ and g_i and g_j fix pointwise the intersection for every $i, j \in \{1, \ldots, k\}$ so we can apply Lemma 5.3 to the set $\{T^{g_1}, \ldots, T^{g_k}\}$ and conclude that $\bigcap_{i\in I} T_i \neq \varnothing$ and each g_i fixes this intersection, thus (2) follows. □

Let $\Delta = (V(\Delta), E(\Delta))$ be a graph. We set $m \in \Delta$ if $m \in V(\Delta)$ or $m \in E(\Delta)$.

A finite *graph of pro-$\mathcal{C}$ groups* $(\mathcal{G}, \Delta)$ over a finite abstract graph Δ is a collection of pro-$\mathcal{C}$ groups $\mathcal{G}(m)$ for each $m \in \Delta$, and continuous monomorphisms $\partial_i : \mathcal{G}(e) \longrightarrow \mathcal{G}(d_i(e))$ for each edge $e \in E(\Delta)$, $i \in \{0, 1\}$. We only work with finite graphs of pro-$\mathcal{C}$ groups, in the sense that the graph Δ is finite, but it is possible to define an analogous concept for graphs of pro-$\mathcal{C}$ groups over profinite graphs Δ (see Chapter 6 of [70]). A graph of groups is *reduced* if edge groups corresponding to edges that are not loops are properly contained in adjacent vertex groups.

Definition 5.6 (Pro-$\mathcal{C}$ fundamental group)**.** Given a finite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Delta)$, we define its *pro-$\mathcal{C}$ fundamental group* $G = \Pi_1(\mathcal{G}, \Delta)$ as follows. Fix a maximal subtree D of Δ; then G is a pro-$\mathcal{C}$ group, together with a collection of continuous homomorphisms

$$\nu_m : \mathcal{G}(m) \longrightarrow G \quad (m \in \Delta)$$

and a continuous map $E(\Delta) \longrightarrow G$, denoted $e \mapsto t_e$ $(e \in E(\Delta))$, such that $t_e = 1$ if $e \in E(D)$, and such that

$$(\nu_{d_0(e)}\partial_0)(x) = t_e(\nu_{d_1(e)}\partial_1)(x)t_e^{-1} \quad \forall x \in \mathcal{G}(e),\ e \in E(\Delta);$$

that satisfies the following universal property:
whenever we have

- a pro-$\mathcal{C}$ group H,
- a collection of continuous homomorphisms $\beta_m : \mathcal{G}(m) \longrightarrow H$, $(m \in \Delta)$,
- a map $e \mapsto s_e$ $(e \in E(\Delta))$ with $s_e = 1$ if $e \in E(D)$, and
- $(\beta_{d_0(e)}\partial_0)(x) = s_e(\beta_{d_1(e)}\partial_1)(x)s_e^{-1} \quad \forall x \in \mathcal{G}(e),\ e \in E(\Delta)$,

then there exists a unique continuous homomorphism $\delta : G \longrightarrow H$ with $\delta(t_e) = s_e$ $(e \in E(\Delta))$ such that for each $m \in \Delta$ the diagram

$$\begin{tikzcd}
 & G \arrow[dd, "\delta"] \\
\mathcal{G}(m) \arrow[ur, "\nu_m"] \arrow[dr, "\beta_m"'] & \\
 & H
\end{tikzcd}$$

commutes.

It was proven in [84] that this definition does not depend on the choice of the maximal subtree D, moreover the existence and uniqueness of this group is proven in Proposition 6.2.1 and Theorem 6.2.4 of [70].

One can construct the fundamental group of a graph of pro-$\mathcal{C}$ groups by iterating two operations, namely pro-$\mathcal{C}$ amalgamated products and pro-$\mathcal{C}$ HNN extensions, denoted by $G_1 \amalg_H G_2$ and $HNN(G_1, H, f)$ respectively, and where G_1 and G_2 are pro-$\mathcal{C}$ groups, $H \leq G_1$, and $f : H \to H' \leq G_1$ is an isomorphism. Both of these constructions are defined by means of a universal property and can be obtained as a certain pro-$\mathcal{C}$ completion of the abstract amalgamated product and HNN extension of the corresponding groups. We refer to Sections 9.2 and 9.4 of [72] for the precise definitions and basic properties.

It is important to remark that, contrary to the abstract case, the factors G_1 and G_2 (resp. the base group G_1) do not necessarily embed into $G_1 \amalg_H G_2$ (resp. $HNN(G_1, H, f)$). Whenever they embed, the amalgamated product (resp. HNN extension) is said to be *proper*. Some necessary and sufficient conditions for pro-$\mathcal{C}$ amalgamated products and HNN extensions to be proper were described in Theorem 9.2.4 and Proposition 9.4.3 of [72]. We remark that properness is assured if if the amalgamated subgroup H is a virtual retract of G_1 and G_2 (as G_1 and G_2 would induce the full pro-$\mathcal{C}$ topology on H and the hypothesis of Thm 9.2.4 in [72] hold in this case).

Abstract Bass-Serre theory relates fundamental groups of graphs of groups with groups acting on trees. Such a relation is true for the pro-$\mathcal{C}$ case assuming that the action on a pro-$\mathcal{C}$ tree is cofinite and not true in general. Namely given a fundamental group of a graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Delta)$, there is a natural pro-$\mathcal{C}$ tree T on which it acts. The construction of this tree, called the *standard pro-$\mathcal{C}$ tree*, is described in Chapter 6 of [70]. The converse is true only for the cofinite action.

If the fundamental group of the graph of pro-$\mathcal{C}$ groups is a pro-$\mathcal{C}$ amalgamated product $G = G_1 \amalg_H G_2$ or a pro-$\mathcal{C}$ HNN extension $G = HNN(G_1, H, f)$, then each vertex stabiliser G_v of a vertex v is a conjugate of G_1 or G_2 (or of G_1 if $G = HNN(G_1, H, f)$) and each edge stabiliser G_e is a conjugate of H.

Abstract Bass-Serre theory is extremely useful for studying the structure of subgroups of fundamental groups of graphs of groups. The same is true for the pro-$\mathcal{C}$ version of Bass-Serre theory, and the main tool is Theorem 7.1.7 of [70]. We state the applications of these results to the case when the group acting on the pro-$\mathcal{C}$ tree is a pro-$\mathcal{C}$ amalgamated product or HNN extension. As usual, we denote by $\widehat{\mathbb{Z}}_{\mathcal{C}} = \prod_{p\in\pi(\mathcal{C})} \mathbb{Z}_p$ the pro-$\mathcal{C}$ completion of $\mathbb{Z}$ for any set of primes $\pi(\mathcal{C})$.

Theorem 5.7. *Let K be a subgroup of a proper free amalgamated pro-$\mathcal{C}$ product $G = G_1 \amalg_H G_2$ of pro-$\mathcal{C}$ groups. Then one of the following holds:*

1. *$K \leq gG_ig^{-1}$ for $g \in G$ and $i \in \{1, 2\}$;*
2. *K has a non-abelian free pro-p subgroup P for a certain $p \in \pi(\mathcal{C})$ such that $P \cap gG_ig^{-1} = 1$ for all $g \in G$ and $i \in \{1, 2\}$;*
3. *there exists a subgroup $H_0 \trianglelefteq K$ (which is the kernel of the action of K on T_K) that is contained in a conjugate of H and such that K/H_0 is solvable*

and isomorphic to a projective group $\mathbb{Z}_\sigma \rtimes \mathbb{Z}_\rho$ ($\sigma, \rho \subseteq \pi(\mathcal{C})$ with $\sigma \cap \rho = \varnothing$) or $\mathbb{Z}_\sigma \rtimes C_n$ (with $\sigma \subseteq \pi(\mathcal{C})$ and C_n a finite cyclic group). In the last case, it can be a profinite Frobenius group or, if $C_n = C_2$ and $2 \in \sigma$, an infinite dihedral pro-σ group.

Theorem 5.8. *Let K be a subgroup of a proper pro-$\mathcal{C}$ HNN extension $G = HNN(G_1, H, f)$. Then one of the following holds:*

1. *$K \leq gG_1g^{-1}$ for $g \in G$;*
2. *K has a non-abelian free pro-p subgroup P for $p \in \pi(\mathcal{C})$ such that $P \cap gG_1g^{-1} = 1$ for all $g \in G$;*
3. *there exists a subgroup $H_0 \trianglelefteq K$ (which is the kernel of the action of K on T_K) that is contained in a conjugate of H and such that K/H_0 is solvable and isomorphic to a projective group $\mathbb{Z}_\sigma \rtimes \mathbb{Z}_\rho$ ($\sigma, \rho \subseteq \pi(\mathcal{C})$ with $\sigma \cap \rho = \varnothing$) or $\mathbb{Z}_\sigma \rtimes C_n$ (with $\sigma \subseteq \pi(\mathcal{C})$ and C_n a finite cyclic group). In the last case, it can be a profinite Frobenius group or, if $C_n = C_2$ and $2 \in \sigma$, an infinite dihedral pro-σ group.*

A useful remark is that, in the third case of the previous theorems, H/H_0 is torsionfree if and only if it is isomorphic to $\mathbb{Z}_\sigma \rtimes \mathbb{Z}_\rho$. In this case, as this is a projective group, we have that $H \cong H_0 \rtimes (\mathbb{Z}_\sigma \rtimes \mathbb{Z}_\rho)$.

Finally, we record the following observation.

Lemma 5.9. *Let $G = G_1 \amalg_H G_2$ be a proper amalgamated pro-$\mathcal{C}$ product of two pro-$\mathcal{C}$ groups G_1 and G_2 and let T be the standard pro-$\mathcal{C}$ tree associated with this splitting. Let $g_1, \ldots, g_k$ be a sequence of elliptic elements such that $[g_i, g_{i+1}] = 1$ for all $i \in \{1, \ldots, k-1\}$. Then there are some vertices $v_1, \ldots, v_k \in V(T)$ (not necessarily distinct) such that $g_1 \in G_{v_1}$ and $g_i \in G_{t_i}$ for each $t_i \in [v_{i-1}, v_i]$.*

Proof. By Lemma 5.5 there exists a vertex v_i stabilized by every pair of commuting elements g_i, g_{i+1} for every $i \in \{1, \ldots, k-1\}$. Define v_k to be any vertex stabilized by g_k. In this setting, g_i stabilizes both v_{i-1} and v_i, hence it stabilizes the whole subtree $[v_{i-1}, v_i]$ by Corollary 4.1.6 of [70]. □

5.2 Basics on pro-$\mathcal{C}$ RAAGs

The aim of this section is to describe basic properties pro-$\mathcal{C}$ RAAGs. The abstract version of the definitions and results that we discuss can be found, for example, in [18].

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an undirected finite graph without double edges or loops, where $V(\Gamma)$ and $E(\Gamma)$ are the set of vertices and edges respectively. A subgraph $\Delta < \Gamma$ is called *full* if for all $e \in \Gamma$ with $d_0(e), d_1(e) \in \Delta$ we have that $e \in \Delta$. Notice that full subgraphs are uniquely determined by the subset of vertices $V(\Delta)$ of $V(\Gamma)$.

Definition 5.10 (Right-angled Artin pro-$\mathcal{C}$ groups)**.** The *right-angled Artin pro-$\mathcal{C}$ grouppro-$\mathcal{C}$ RAAG* (pro-$\mathcal{C}$ RAAG for short) G_Γ is the pro-$\mathcal{C}$ group given by the pro-$\mathcal{C}$ presentation

$$G_\Gamma = \langle V(\Gamma) | [u, v] = 1 \text{ if and only if } u \text{ and } v \text{ are adjacent in } \Gamma \rangle.$$

We recall some standard terminology.

Definition 5.11 (Canonical Generators)**.** The generators associated with the vertices of Γ are called *canonical generators* and, abusing the notation, we denote them with the same letter as the corresponding vertex.

Definition 5.12 (Standard subgroups)**.** A subgroup of G_Γ is called a *standard subgroup* if it is the subgroup generated by a subset $V' \subseteq V(\Gamma)$. If $\Gamma = \varnothing$, by convention we set G_Γ to be the trivial subgroup.

Abusing the notation, if $S \subseteq V(\Gamma)$, we denote by G_S the standard subgroup generated by the full subgraph generated by S. We begin by stating some properties of standard subgroups.

Lemma 5.13. *Let G_Γ be a pro-$\mathcal{C}$ RAAG. Then:*

1. *G_Γ is the pro-$\mathcal{C}$ completion of the abstract RAAG $G(\Gamma)$;*
2. *the standard subgroup generated by a subset of vertices $V' \subseteq V(\Gamma)$ is the pro-$\mathcal{C}$ RAAG G_Δ generated by the full subgraph $\Delta \subseteq \Gamma$ determined by V';*
3. *the standard subgroups of G_Γ are retracts;*
4. *the intersection of standard subgroups is a (possibly trivial) standard subgroup.*

Proof. For every group G, we denote by $\widehat{G}$ its pro-$\mathcal{C}$ completion.

1. Follows from the pro-$\mathcal{C}$ presentation (see Definition 5.10).
2. In the abstract case, the subgroup of $G(\Gamma)$ generated by V' is exactly $G(\Delta)$, see for example Corollary 2.11 of [50]. As this subgroup is a retract of $G(\Gamma)$, the pro-$\mathcal{C}$ topology of $G(\Gamma)$ induces on it the full pro-$\mathcal{C}$ topology, so the pro-$\mathcal{C}$ subgroup $\langle V' \rangle \leq G_\Gamma$ is $\widehat{G(\Delta)}$, that by (1) coincides with G_Δ.

3. The map $\mathrm{pr}_\Delta : G_\Gamma \to G_\Delta$ whose restriction to G_Δ is the identity and such that $\mathrm{pr}_\Delta(v) = 1$ for every $v \in V(\Gamma) \smallsetminus V'$ is surjective. Since by (2) G_Δ is a subgroup of G_Γ, we have that pr_Δ is a retraction onto G_Δ.

4. Consider two standard subgroups G_Δ, G_Λ of G_Γ. By (3), a non-trivial element g of G_Γ is in $G_\Delta \cap G_\Lambda$ if and only if $\mathrm{pr}_\Delta(\mathrm{pr}_\Lambda(g)) = g$, but this composition of maps corresponds exactly to $\mathrm{pr}_{\Delta\cap\Lambda}(g)$, and therefore $G_\Delta \cap G_\Lambda = G_{\Delta\cap\Lambda}$.

□

It follows from the pro-$\mathcal{C}$ version of Theorem 9.2.4 of [72] that, if H is a retract of two groups G_1 and G_2, then a pro-$\mathcal{C}$ $G_1 \amalg_H G_2$ is a proper pro-$\mathcal{C}$ amalgamated product. Similarly, it follows from Theorem 9.4.3 that pro-$\mathcal{C}$ HNN-extension $HNN(G_1, H, f)$ is proper if H is a retract of G_1. As standard subgroups of a RAAG are retracts, we deduce the following.

Corollary 5.14. *Let G_Γ be a pro-$\mathcal{C}$ RAAG. If G_Γ is a pro-$\mathcal{C}$ amalgamated product $G_1 \amalg_H G_2$ or a pro-$\mathcal{C}$ HNN extension $HNN(G_1, H, f)$ with $G_1, G_2, H, f(H)$ standard subgroups of G_Γ, then the free product with amalgamation or HNN extension is proper.*

We now want to define the notion of support of an element, but we first begin by proving that this concept is well-defined.

Lemma 5.15. *Let G_Γ be a pro-$\mathcal{C}$ RAAG and let $g \in G_\Gamma$. Then there exists a unique minimal standard subgroup containing g. Moreover there exists an element h in the conjugacy class of g whose corresponding minimal standard subgroup is contained in each standard subgroup containing conjugates of g.*

Proof. The unique minimal standard subgroup containing g is the intersection of all the standard subgroups containing it, and this intersection is still a standard subgroup by Lemma 5.13. Suppose now that Δ_1, Δ_2 are full subgroups of Γ such that $g \in G_{\Delta_1}$ and $g^t \in G_{\Delta_2}$ for $t \in G_\Gamma$. We claim that there exists $s \in G_\Gamma$ such that $g^s \in G_{\Delta_1\cap\Delta_2}$. Indeed let pr_{Δ_1} be the retraction of G_Γ to G_{Δ_1} and define $s = \mathrm{pr}_{\Delta_1}(t)$. Then $g^s = \mathrm{pr}_{\Delta_1}(g^t) \in G_{\Delta_1\cap\Delta_2}$. In order to prove the lemma it suffices to apply this observation to the lattice of full subgraphs of Γ containing a conjugate of g. Notice that if $g = 1$ we have that $g \in G_\varnothing$ and by convention, the standard subgroup generated by the empty set is the trivial group. □

Definition 5.16 (Support of an element)**.** Let g be an element of a (pro-$\mathcal{C}$) RAAG G_Γ. The *support* $\alpha(g)$ of g is the set of canonical generators of the unique minimal standard subgroup of G_Γ containing g.

In view of Lemma 5.15, in any conjugacy class there exists an element g such that $\alpha(g) \subseteq \alpha(g^t)$ for every $t \in G$, in this case we say that g^t is an *element of minimal support among its conjugates.*

Definition 5.17 (Links and stars)**.** Let g be an element of a (pro-$\mathcal{C}$) RAAG G_Γ. The *link* $\mathrm{Link}(g)$ of g is the set of vertices of $\Gamma \setminus \alpha(g)$ that are adjacent to each of the vertices in $\alpha(g)$.
If v is a canonical generator, we denote by $\mathrm{Star}(v)$ the full subgraph generated by $\mathrm{Link}(v) \cup v$.

Remark 5.18. If $v \in V(\Gamma)$, we can split G_Γ as a pro-$\mathcal{C}$ HNN extension as

$$G_\Gamma = HNN(G_{\Gamma \setminus \{v\}}, G_{\mathrm{Link}(v)}, id) \tag{5.2}$$

with stable letter v, and by Corollary 5.14 this is a proper pro-$\mathcal{C}$ HNN extension. It follows that if g is an element with minimal support among its conjugates and $v \in \alpha(g)$, then Theorem 5.8 guarantees that its action on the standard pro-$\mathcal{C}$ tree T associated with this splitting is hyperbolic.

Abstract right-angled Artin groups are torsion-free, but the pro-$\mathcal{C}$ completion of torsion-free groups is not always torsion-free (even the profinite completion as shown in [54],[19]). However, in the case of pro-$\mathcal{C}$ RAAGs this is true.

Theorem 5.19. *Pro-$\mathcal{C}$ RAAGs are torsion-free profinite groups.*

Proof. A pro-$\mathcal{C}$ RAAG is the pro-$\mathcal{C}$ completion of the corresponding (abstract) RAAG. In [30], the authors proved that abstract RAAGs are residually (finitely generated torsion-free nilpotent), and hence the pro-$\mathcal{C}$ completion of a RAAG embeds in a direct product of the pro-$\mathcal{C}$ completions of finitely generated torsion-free nilpotent groups. By Theorem 4.7.10 of [72] the profinite completion $\widehat{N}$ of a finitely generated torsion-free nilpotent group N is torsion-free. But $\widehat{N} = \prod_p \widehat{N}_p$ is the direct product of the pro-p completions and the pro-$\mathcal{C}$ completion of N is the direct product $\prod_{p \in \pi(\mathcal{C})} N_p$. Hence the pro-$\mathcal{C}$ completion of N is torsion-free. $\square$

5.3 Direct product decomposition of pro-$\mathcal{C}$ RAAGs

Our goal is to show that the direct product decomposition of a pro-$\mathcal{C}$ RAAG is determined by the defining graph. More precisely $G_\Gamma \simeq A_1 \times A_2$, where A_1 and A_2 are non-trivial pro-$\mathcal{C}$ groups, if and only if Γ is a join, see Theorem 5.22.

Lemma 5.20. *Let G_Γ be a pro-$\mathcal{C}$ RAAG and let $g \in G_\Gamma$ be an element with minimal support among its conjugates. Then, the centraliser of g is contained in the standard subgroup generated by* $\mathrm{Link}(g) \cup \alpha(g)$*. In particular, if $g = v$ is a standard generator, then $C_G(v) = G_{\mathrm{Star}(v)} = \langle v \rangle \times G_{\mathrm{Link}(v)}$.*

Proof. Suppose towards contradiction that there is an element h commuting with g whose support is not contained in $\mathrm{Link}(g) \cup \alpha(g)$. Then there exists $v \in \alpha(h)$ such that $v \notin \mathrm{Link}(g) \cup \alpha(g)$. Denoting by $G_0 = G_{\Gamma \setminus \{v\}}$ and by $A = G_{\mathrm{Link}(v)}$ and using Remark 5.18, we have that the group G_Γ splits as a proper HNN extension of the form

$$G_\Gamma = HNN(G_0, A, id)$$

where the action by conjugation of v on A is trivial. Notice that from the assumption on h, we have that $h \notin G_0$. We next study the action of g and h on the standard pro-$\mathcal{C}$ tree T associated with this splitting.

Notice that $g \in G_0$ and so g is elliptic. However, g cannot belong to any edge stabiliser. Indeed, otherwise, there would exist an element $t \in G_\Gamma$ such that $g^t \in A$ and in this case, since g has by assumption minimal support, it would follow from Lemma 5.15 that $\alpha(g) \subseteq \alpha(g^t) \subseteq \mathrm{Link}(v)$ and so $v \in \mathrm{Link}(g)$ contradicting the choice of v. Since g cannot be in any edge stabiliser, we conclude that g only fixes the vertex v stabilised by G_0, i.e. $T_g = \{v\}$. From Lemma 5.5 (1), h has to leave $T_g = \{v\}$ invariant and, in particular, h fixes v. Then h belongs to G_0, a contradiction. □

Lemma 5.21. *Suppose a pro-$\mathcal{C}$ RAAG $G = G_\Gamma$ decomposes as a direct product $G_\Gamma = A_1 \times A_2$ of non-trivial groups. Then for each canonical generator $v \in \Gamma$, at least one factor A_i is contained in $G_{\mathrm{Star}(v)}$.*

Proof. Let v be a canonical generator. Since $\alpha(v) = \{v\}$, by Lemma 5.20 we have that $C_G(v) = G_{\mathrm{Star}(v)} = \langle v \rangle \times G_{\mathrm{Link}(v)}$.

Suppose that $v = a_1 \cdot a_2$ where $a_i \in A_i$, $i = 1, 2$. Since $C_G(v) = C_{A_1}(a_1) \times C_{A_2}(a_2)$ and $a_i \in C_{A_i}(v)$, from the description of the centraliser $C_G(v)$, we deduce that $a_i = v^{e_i} a_i'$ for $e_i \in \mathbb{Z}_{\pi(\mathcal{C})}$ and $a_i' \in G_{\mathrm{Link}(v)}$. Since $v = a_1 \cdot a_2$, we have that $e_i \neq 0$ for either $i = 1$ or $i = 2$; without loss of generality assume $e_1 \neq 0$. Let t be an element such that $t^{-1} a_1 t$ has minimal support among its conjugates, we can assume $t \in A_1$ because $A_2 \subseteq C_G(a_1)$. Applying Lemma 5.20 we have

$$tA_2t^{-1} = A_2 \subseteq C_G(a_1) = tC_G(t^{-1}a_1t)t^{-1} \subseteq t\big(G_{\alpha(t^{-1}a_1t)} \times G_{\mathrm{Link}(t^{-1}a_1t)}\big)t^{-1}.$$

Notice that by Lemma 5.15 $\alpha(t^{-1}a_1t) \subseteq \alpha(a_1) \subseteq \mathrm{Star}(v)$ and, since $v \in \alpha(t^{-1}a_1t)$, the definition of link implies that $\mathrm{Link}(t^{-1}a_1t) \subseteq \mathrm{Star}(v)$. Overall, we conclude that $A_2 \subseteq G_{\mathrm{Star}(v)}$.

$\square$

We are now ready to fully characterize when a pro-$\mathcal{C}$ RAAG splits as a direct product. We recall that a graph is a *join* if and only if there is a non-empty subgraph $\Delta \lneq \Gamma$ such that for each $v \in \Delta$ and each $w \in \Gamma \smallsetminus \Delta$, v, w are adjacent.

Theorem 5.22. *Let G_Γ be a pro-$\mathcal{C}$ RAAG. Then G_Γ has a non-trivial direct product decomposition if and only if Γ is a join. In particular, each factor in a direct product decomposition of G_Γ is a standard subgroup.*

Proof. The analogous result for abstract RAAGs is classical (see for example Corollary 2.15 in [50]). From the abstract result and Lemma 5.13, it is straightforward that whenever Γ is a join, then G_Γ splits as a direct product.

We now want to prove the converse implication. By Lemma 5.21 for each canonical generator v, at least one among A_1 or A_2 is contained in $G_{\mathrm{Star}(v)}$.

Let $\Gamma_1 \subseteq V(\Gamma)$ be the set of canonical generators v such that $A_1 < \mathrm{Star}(v)$ and $\Gamma_2 = \Gamma \smallsetminus \Gamma_1$. Then, for each canonical generator $v \in \Gamma_2$, since by definition of Γ_2 we have that $A_1 \not< \mathrm{Star}(v)$, by Lemma 5.21 again we conclude that $A_2 \leq G_{\mathrm{Star}(v)}$.

For $i = 1, 2$ define $\Delta_i \subseteq \Gamma$ such that $G_{\Delta_i} = \bigcap_{v \in \Gamma_i} G_{\mathrm{Star}(v)}$; by Lemma 5.13 G_{Δ_i} is a standard subgroup and by definition it contains A_i and each $v \in \Gamma_i$ is connected to each $w \in \Delta_i$. In particular Δ_i are non-empty graphs. Notice that if there is a canonical generator $w \in \Delta_1 \cap \Delta_2$, then w is by definition in the star of each vertex in Γ_i and so $\Gamma_1, \Gamma_2 < \mathrm{Star}(w)$. Hence such a canonical generator $w \in \Delta_1 \cap \Delta_2$ would be central and Γ would decompose as a join. For this reason we can assume that G_{Δ_1} and G_{Δ_2} are disjoint and since A_1 and A_2 generate G, so do G_{Δ_1} and G_{Δ_2}.

Hence, we can decompose $V(\Gamma)$ as the disjoint union of the (possibly empty) sets $\Gamma_2 \cap \Delta_1$, $\Gamma_1 \cap \Delta_2$ and $\Lambda = (\Gamma_1 \cap \Delta_1) \cup (\Gamma_2 \cap \Delta_2)$.

Since Δ_i is non-empty for $i = 1, 2$, then either $\Lambda \neq \emptyset$ or $\Gamma_2 \cap \Delta_1$ and $\Gamma_1 \cap \Delta_2$ are non-empty. If at least two of the sets are non-empty, then they define a join, because each vertex in a set is connected to each vertex in the other set, because each element in Γ_i is connected to each element in Δ_i for $i = 1, 2$.

We are left to consider the case when only Λ is non-empty, so that $\Lambda = V(\Gamma)$. In this case, each vertex in Γ_i is in Δ_i too and in particular they are connected to each other. It follows that $\Gamma_i \cap \Delta_i = \Gamma_i = \Delta_i$ is a complete graph for $i = 1, 2$. Since $A_i \leq G_{\Delta_i}$ and G_{Δ_i} is abelian, so is A_i. Hence $G = A_1 \times A_2$ is abelian and Γ is a complete graph and a join. $\square$

These results are in line with other properties of pro-$\mathcal{C}$ RAAGs that can be recognized from the abstract graph. For example, abstract RAAGs split as a free product if and only if the underlying graph is disconnected, and Wilkes and Kropholler proved that the same is true for profinite RAAGs in [51]. Similarly, both abstract and pro-p RAAGs are coherent if and only if the underlying graph is chordal, see [76].

5.4 Centralisers and normalisers of elements

In this section, we describe explicitly the structure of centralisers of elements in pro-$\mathcal{C}$ RAAGs, obtaining a description similar to the one that Baudisch proved for abstract RAAGs in [12]. In a free pro-p group, centralisers of elements are cyclic. However, in the pro-$\mathcal{C}$ case, the situation is substantially different as the centraliser of an element does not need to be cyclic. Indeed, for example, the projective group $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$, with the generator of $\mathbb{Z}_2$, say a, acting on $\mathbb{Z}_3$ by inversion, embeds in a free profinite group, so the centraliser of a^2 contains this solvable projective group.

Theorem 5.23. *Let $G = G_\Gamma$ be a pro-$\mathcal{C}$ RAAG and let $g_0 \in G$. Then there is an element g in the conjugacy class of g_0 such that its centraliser is of the form*

$$C_G(g) = H_1 \times \cdots \times H_s \times \overline{\langle \mathrm{Link}(g) \rangle}$$

where:

1. $\alpha(H_i),\ \alpha(H_j),\ \mathrm{Link}(g)$ *are all disjoint for* $i \neq j$*;*
2. $G_{\alpha(g)} = G_{\alpha(H_1)} \times \cdots \times G_{\alpha(H_s)}$*;*
3. H_i *are projective pro-*$\mathcal{C}$ *groups;*
4. *if G is pro-p, $H_i = \overline{\langle h_i \rangle}$ and $g = h_1^{k_1} \cdots h_s^{k_s}$, for some $k_i \in \mathbb{Z}_p$.*

Proof. We begin the proof with some reductions.

If g is trivial, then $V(\Gamma) = \mathrm{Link}(g)$ and the result holds trivially, so we further assume $g \neq 1$.

Among the conjugates of g_0, we choose an element g of minimal support among its conjugates, so that by Lemma 5.20 $C_G(g)$ is contained in the standard subgroup generated by $\mathrm{Link}(g) \cup \alpha(g)$. Hence, we can assume that $V(\Gamma) = \mathrm{Link}(g) \cup \alpha(g)$. In this case, we have from Theorem 5.22 that $G = G_{\alpha(g)} \times G_{\mathrm{Link}(g)}$. Clearly $G_{\mathrm{Link}(g)} \leq C_G(g)$, so it suffices studying the centraliser in the standard subgroup $G_{\alpha(g)}$ and then

$$C_G(g) = C_{G_{\alpha(g)}}(g) \times G_{\mathrm{Link}(g)}.$$

We further assume that $\alpha(g) = V(\Gamma)$. If G is decomposable as a direct product $G_\Gamma = G_1 \times \cdots \times G_s$, then $g = g_1 \times \cdots \times g_s$ for $g_i \in G_i$, $i \in \{1, \ldots, s\}$, and the centraliser $C_G(g)$ decomposes as $C_G(g) = C_{G_1}(g_1) \times \cdots \times C_{G_s}(g_s)$. Moreover, by Theorem 5.22, each G_i is a standard subgroup. As g was chosen to be an element of minimal support among its conjugates, every g_i has also minimal support among its conjugates, so we have reduced the problem to studying centralisers when $G = G_{\alpha(g)}$ is directly indecomposable.

Our goal is to show that if $G = G_{\alpha(g)}$ is directly indecomposable, then $C_G(g)$ is a projective group. Fix any vertex v of Γ and denote by $G_0 = G_{\Gamma \smallsetminus \{v\}}$ and by $A = G_{\mathrm{Link}(v)}$. Consider the decomposition as an HNN extension

$$G = HNN(G_0, A, id);$$

as described in Remark 5.18, the action of g on the standard pro-$\mathcal{C}$ tree T associated with this splitting is hyperbolic.
We first claim that no nontrivial element $h \in C_G(g)$ is contained in a conjugate of A. Indeed, take any element $h \in C_G(g)$ and assume that $t \in G$ is an element such that h^t has minimal support among the G-conjugates of h, so that by Lemma 5.15 we have $\alpha(h^t) \subseteq \Gamma \smallsetminus \{v\}$. Then $h^t \in C_G(g^t)$ and by Lemma 5.20 we have $\alpha(g^t) \subseteq \alpha(h^t) \times \mathrm{Link}(h^t)$. By Lemma 5.15, $V(\Gamma) = \alpha(g) \subseteq \alpha(g^t)$, but then $G = G_{\alpha(h^t)} \times G_{\mathrm{Link}(h^t)}$. As G is directly indecomposable and $\alpha(h^t)$ is a proper subset of $V(G)$, h must be trivial.

Let T_g be the minimal g-invariant subtree of g. From the preceding paragraph we deduce that $C_G(g)$ acts faithfully on T_g and so by Lemma 4.2.6 of [70] it is projective. In particular, if G is a pro-p group, then $C_G(g)$ must be isomorphic to $\mathbb{Z}_p$. $\square$

Our next goal is to prove that centralisers of elements are virtual retracts in pro-p groups.

Lemma 5.24. *Let G be a pro-p group acting without fixed points on a pro-p tree T. Assume that H is a procyclic subgroup, generated by a hyperbolic element g. Then H is a virtual retract of G.*

Proof. For each subgroup K of G, define

$$\widetilde{K} = \overline{\langle K \cap G_t | t \in V(T) \rangle}$$

to be the subgroup generated by the intersections of K with all vertex stabilisers. Notice that $\widetilde{H} = 1$ since H is procyclic generated by a hyperbolic element. As H is closed, it is the intersection of all open subgroups $\{U_i, i \in I\}$ containing it and then also $\bigcap_{i \in I} \widetilde{U_i} = \widetilde{H} = 1$. This implies that there must be an open $U \trianglelefteq_o G$

such that $g \notin \widetilde{U}$. But $U/\widetilde{U}$ is a free pro-p group by Corollary 3.6 of [71] hence by the profinite version of Marshall Hall Theorem (see Theorem 9.1.19 of [72]) the procyclic subgroup $H\widetilde{U}/\widetilde{U}$ of $U/\widetilde{U}$ is a free factor of a finite index subgroup of $U/\widetilde{U}$. As $H \cap \widetilde{U}$ is trivial, we can lift the retracts to U and we have that $H \cong H\widetilde{U}/\widetilde{U}$ is a virtual retract of $U/\widetilde{U}$, hence a virtual retract of G too. □

Theorem 5.25. *Let $G = G_\Gamma$ be a pro-p RAAG and let H be the centraliser of an element h. Then H is a virtual retract of G.*

Proof. We can assume that h has minimal support among its conjugates. By Theorem 5.23, H is contained in the standard subgroup generated by $\alpha(h) \cup \mathrm{Link}(h)$, which is a retract of G. As a virtual retract of a standard subgroup of G can be lifted to a virtual retract of G, we restrict to the case that $V(\Gamma) = \alpha(h) \cup \mathrm{Link}(h)$. Suppose then that $G = G_1 \times \cdots \times G_k \times G_{\mathrm{Link}(h)}$ is the direct product decomposition of the standard subgroup G. By Theorem 5.23, $H = H_1 \times \cdots \times H_k \times G_{\mathrm{Link}(h)}$, where H_i is a procyclic subgroup of G_i for every $i \in \{1, \dots, k\}$. By Lemma 5.24 and Remark 5.18, every H_i is a virtual retract of G_i, hence the direct product of all of them is a virtual retract of G and the result follows. □

5.5 Subgroups of pro-$\mathcal{C}$ and pro-p RAAGs

We now aim to describe the structure of some subgroups of pro-$\mathcal{C}$ and pro-p RAAGs. We recall that a subgroup $H \leq G$ is *isolated* (or isolated in G) if whenever $g^k \in H$ for a certain $g \in G$ $k \in \mathbb{Z}_{\mathcal{C}}$, then $g \in H$.

Lemma 5.26. *Standard subgroups of pro-$\mathcal{C}$ RAAGs are isolated.*

Proof. Let $G = G_\Gamma$. The theorem is equivalent to the statement that for every $g \in G$ and $k \in \mathbb{Z}_{\mathcal{C}}$, $\alpha(g) = \alpha(g^k)$. Suppose this is not true, so that there exists a vertex $v \in \alpha(g) \smallsetminus \alpha(g^k)$. Then, consider the pro-$\mathcal{C}$ tree T associated to the splitting (5.2) and notice that $g^k \in G_{\Gamma \smallsetminus \{v\}}$ stabilizes the vertex t of T, which is stabilized by the standard subgroup $G_{\Gamma \smallsetminus \{v\}}$. Suppose first that g acts hyperbolically on T. By Theorem 5.8, there exists a subgroup $H_e \trianglelefteq \langle g \rangle$ stabilizing an edge such that $\langle g \rangle \cong H_e \rtimes \mathbb{Z}_{\mathcal{C}}$. As g acts hyperbolically, the whole $\langle g \rangle$ is not contained in the edge group, contradicting that g^k acts elliptically.

Suppose then that both g and g^k act elliptically. In this case, we can argue by induction on the number of generators of G_Γ and suppose that standard subgroups of pro-$\mathcal{C}$ RAAGs with at most $|V(\Gamma)| - 1$ generators are isolated. By induction, and using that isolation is invariant by conjugation, in the pro-$\mathcal{C}$ tree T associated to the splitting in (5.2), edge stabilizers are isolated in adjacent vertex stabilizers. Denote by f the projection onto the standard subgroup $G_{\Gamma\smallsetminus\{v\}}$, and set $x = f(g)$.

Notice that the set of edges of a standard pro-$\mathcal{C}$ tree associated to a splitting is compact by construction, and therefore for any vertex of any of its pro-$\mathcal{C}$ subtrees containing at least an edge, there always exists an edge of the subtree adjacent to it. As now $x^k = g^k$ fixes both the vertex fixed by g and the vertex t stabilized by $G_{\Gamma\smallsetminus\{v\}}$, there exists a stabilizer of an edge adjacent to t that contains x^k. By inductive hypothesis on isolation, such edge contains x too. In particular, this edge is conjugated to the one stabilized by $G_{\mathrm{Link}(v)}$. By applying the conjugation map that fixes t and sends the vertex stabilized by x onto the vertex stabilized by $G_{\mathrm{Link}(v)}$, we obtain an element g', conjugated to g, fixing a vertex different from t, but such that g'^k fixes t. Then, $f(g') = y$ is contained in $G_{\mathrm{Link}(v)}$ and $y^k = g'^k$.

Denoting by $\langle\!\langle v\rangle\!\rangle$ the normal closure of v in G_Γ, we have that $g' \in \langle y\rangle\langle\!\langle v\rangle\!\rangle$, as standard subgroups are retracts. By Theorem B of [85], using that $\langle\!\langle v\rangle\!\rangle$ does not intersect any conjugate of $G_{\mathrm{Link}(v)}$, we obtain that $\langle\!\langle v\rangle\!\rangle$ is a free pro-$\mathcal{C}$ group $F(v^S)$, with basis v^S, where S is the image of a continuous section $\sigma : \big(G_{\Gamma\smallsetminus\{v\}}/G_{\mathrm{Link}(v)}\big) \to G_{\Gamma\smallsetminus\{v\}}$. The element y acts on the set v^S by permuting the sections of S and, letting $R = \sigma\big(C_G(y)/G_{\mathrm{Link}(v)}\big)$, y fixes v^r for all $r \in R$.

Let $S^* = (S/\langle y\rangle)\smallsetminus R$ be the quotient of all sections of S modulo the conjugating action of y, excluding the ones of R. We claim that $\langle x\rangle$ acts freely on $F(v^{S^*})$. As the action on the set $\{v^S\}$ is by permutation, we only have to check that if there is a non-trivial $\overline{y} \in \langle y\rangle$ such that, for a certain $s \in S$ we have $v^{s\cdot\overline{y}} = v^s$, then $s \in R$. Indeed, this means that $s\overline{y}s^{-1} \in G_{\mathrm{Link}(v)}$. Now, if f_v is the projection onto $G_{\mathrm{Link}(v)}$, we have that $y^s = y^{s^{-1}f_v(s)}$, or analogously that $s^{-1}f_v(s) \in C_G(y)$. This implies that $s \in C_G(y)f_v(s) \subseteq C_G(y)G_{\mathrm{Link}(v)}$, and therefore $s \in R$.

Overall, we have proved that that

$$\langle y\rangle\langle\!\langle v\rangle\!\rangle = \langle y\rangle F(v^S) = \big(\langle y\rangle \times F(v^R)\big)\coprod F(v^{S^*})$$

Now g' centralizes $g^p = y^p$ and y^p clearly lies in the factor $\langle y\rangle \times F(v^R)$ of the free product. By Theorem B of [41], the element g must lie in $\langle y\rangle \times F(v^R)$ too. Let $c \in F(v^R)$ be such that $g' = y^\ell \cdot c$, then $y^k = g'^k = y^{k\ell} \cdot c^k$. Using that G is torsion-free by Proposition 5.19, $c = 1$ and $\ell = 1$. This would imply that $g' = y$ and that g' fixes the vertex t of the tree T, and this is a contradiction.

$\square$

The next result proves that the only subgroups of a pro-$\mathcal{C}$ RAAGs that do not contain free pro-p subgroups are metabelian. This can be seen as an analogous of Tits alternative for pro-$\mathcal{C}$ RAAGs.

Theorem 5.27. *Let H be a subgroup of a pro-$\mathcal{C}$ RAAG G_Γ that does not contain a free non-abelian pro-q subgroup for any prime q. Then H is metabelian and polycyclic. Moreover, if H is pro-p, then H is abelian.*

Proof. We use induction on the number of vertices of Γ. If Γ consists of a single vertex, then any subgroup of G_Γ is pro-$\mathcal{C}$ cyclic and the result follows. As we observed in Remark 5.18, $G_\Gamma = HNN(G_{\Gamma\setminus\{v\}}, G_{\mathrm{Link}(v)}, id)$ for an arbitrarily chosen vertex v. If H is conjugate to a subgroup of $G_{\Gamma\setminus\{v\}}$ then we deduce the result from the induction hypothesis. Otherwise, by Theorem 5.8 there exists a normal subgroup $H_0 \trianglelefteq H$ contained in some conjugate of Link(v) such that H/H_0 is metacyclic. Lemma 5.26 guarantees that $G_{\mathrm{Link}(v)}$ is an isolated subgroup of G_Γ and so H_0 is isolated in H. It follows that the only possibility for H/H_0 in Theorem 5.8 is the projective group $\mathbb{Z}_\sigma \rtimes \mathbb{Z}_\rho$, so that $H = H_0 \rtimes P$, with $P \cong \mathbb{Z}_\sigma \rtimes \mathbb{Z}_\rho$ and we may assume without loss of generality that $[P, P] \cong \mathbb{Z}_\sigma$.

Consider the projection $f : G_\Gamma \longrightarrow G_{\Gamma\setminus\{v\}}$ and let K be the kernel of f. The image $f([H, H])$ is abelian and finitely generated by the induction hypothesis. Since $K \cap H$ and H_0 are normal in H and do not intersect, as $H_0 \subseteq G_{\mathrm{Link}(v)}$, we have that $(K \cap H)H_0 = (K \cap H) \times H_0$. Then the commutator subgroup $[H, H] = [H_0, H_0][H_0, P][P, P]$, and $[H_0, H_0][H_0, P]$ is abelian and polycyclic because it is in the image of $f([H, H])$. Thus we just need to show that $[P, P]$ centralizes $[H_0, H_0][H_0, P]$. To see this observe that $f([P, P])$ is torsion-free and so $[P, P] \cong (K \cap [P, P]) \times f([P, P])$. Now K centralizes H_0 and $f([P, P])$ centralizes $f([H_0, H_0][H_0, P]) = [H_0, H_0][H_0, P]$ by inductive hypothesis, so we deduce that $[H, H]$ is abelian.

Suppose now H is pro-p. Then $P \cong \mathbb{Z}_p$. But the action of P on H_0 is the same as the action of $f(P)$ on H_0 which is trivial since $f(H)$ is abelian by the inductive hypothesis. Therefore $H = H_0 \times P$ is abelian. □

The next result describes two-generated subgroups of pro-p RAAGs. The pro-$\mathcal{C}$ case is necessarily more complicate, as metabelian pro-$\mathcal{C}$ groups could appear, as we already discussed, but also pro-$\mathcal{C}$ groups of the form $(\mathbb{Z}_{\sigma_1} \coprod \mathbb{Z}_{\tau_1}) \times \cdots \times (\mathbb{Z}_{\sigma_\ell} \coprod \mathbb{Z}_{\tau_\ell})$ for $\sigma_i, \tau_i \subseteq \pi(\mathcal{C})$, with σ_i pairwise disjoint and τ_i pairwise disjoint.

Theorem 5.28. *Let H be a two-generated subgroup of a pro-p RAAG G_Γ. Then H is either free pro-p or free abelian.*

Proof. We use induction on the number of vertices of Γ. If Γ is a vertex, then G_Γ is abelian and so is each subgroup so the statement holds. As we noticed in Remark 5.18, chosen an arbitrary vertex v, we can obtain a decomposition $G_\Gamma = HNN(G_{\Gamma\setminus\{v\}}, G_{\mathrm{Link}(v)}, id)$. If H is conjugate to a subgroup of $G_{\Gamma\setminus\{v\}}$

then we deduce the result from the induction hypothesis. Otherwise, H acts non-trivially on the standard pro-p tree associated with HNN extension $G_\Gamma = HNN(G_{\Gamma\setminus\{v\}}, G_{\mathrm{Link}(v)}, id)$. Considering the projection $f : G_\Gamma \longrightarrow G_{\Gamma\setminus\{v\}}$, by induction hypothesis we can deduce that $f(H)$ is either free or abelian. If $f(H)$ is free, then we are done because by the Hopfian property (see Proposition 2.5.2 of [72]) the projection must be an isomorphism.

Suppose suppose now that $f(H)$ is abelian. By the induction hypothesis, we can assume that every stabiliser of a vertex in H is abelian of rank at most two. By Lemmas 5.13 and 5.26, $G_{\mathrm{Link}(v)}$ is an isolated subgroup of G_Γ and so every edge stabiliser of H is isolated in the vertex stabiliser of its incident vertex. As the only isolated proper subgroups of an abelian pro-p group of rank two are procyclic, it follows that such are the edge stabilisers. Then by Theorem 6.8 of [20] H is the fundamental group of a finite graph of pro-p groups $(\mathcal{H}, \Delta)$ whose vertex and edge groups are isomorphic to vertex and edge stabilisers in H respectively. Assuming without loss of generality that this graph of groups is reduced, we deduce that isolation of edge groups in the incident vertex groups implies that either the edge groups have strictly smaller rank than the incident vertex groups, or they coincide (in the case when Δ contains loops).

As H is two generated and edge groups are isolated, Δ cannot have more than two vertices. For the same reason, if $|V(\Delta)| = 2$, as the graph of groups is reduced, the edge group between the two vertices can only be trivial. Another remark is that the stable letter of each HNN extension corresponding to a loop must be one of the two generators of the group.

If at least an edge group is trivial, then H splits as a free pro-p product (see Proposition 2.16 of [17]) and so, being 2-generated, it has to be a free pro-p product of torsion-free cyclic groups, hence it is free pro-p. We only have to analyse the case when no edge group is trivial.

If Δ has a single vertex, we have to analyse the case when there are zero, one, or two loops. If there are two loops, as each stable letter of an HNN extension must be one of the two generators, the vertex group must be trivial and H is a free pro-p group. In all of the other cases, the vertex group has either rank one or two.

Let us now consider the case when Δ has a vertex w with a single loop. We are left to consider the case when the vertex group has either rank one or 2.

Suppose the vertex group has rank one. Since we can assume the edge group not to be trivial, it has to be also abelian of rank 1 and since edge groups are retractions, it follows that the edge group coincides with the vertex group. Since pro-$\mathcal{C}$ RAAGs are torsion-free, it follows that H splits as an HNN extension

$HNN(\mathcal{H}(w), \mathcal{H}(w), id)$, which is a free abelian pro-p group of rank two.

The last case to consider is when $\mathcal{H}(w)$ is free abelian of rank 2. As the group is two-generated and one generator must be the stable letter t of the HNN extension, the only possibility is that $\mathcal{H}(w) = \langle x, x^t \rangle$ for some $x, t \in G$, t power of the stable letter. Consider the retraction $f : G_\Gamma \to G_{\Gamma \setminus \{v\}}$. Since $\langle x, x^t \rangle$ is a free abelian group of rank 2, we have that $x = f(x), x^{f(t)}$ also generate an abelian group of rank 2 and so, in particular, $f(t) \in G_{\Gamma \setminus \{v\}}$ is nontrivial. Now, by induction hypothesis, the 2-generated subgroup $\langle x, f(t) \rangle < G_{\Gamma \setminus \{v\}}$ is either free or free abelian. The latter case implies that $x = x^{f(t)}$ contradicting the fact that $\langle x, x^{f(t)} \rangle$ is of rank 2. If x and $f(t)$ generate a free group, then $[x, x^f(t)] \neq 1$ contradicting that the fact that $\langle x, x^{f(t)} \rangle$ is abelian. This proves that this case cannot hold.

Since all the alternatives have been considered, the result follows. $\square$

6
Abelian splittings of RAAGs

In this section we will study how a pro-$\mathcal{C}$ RAAG can split as an amalgamated product or HNN extension over an abelian subgroup.

In [35], Hull and Groves proved that an abstract RAAG splits over an abelian subgroup if and only if the underlying graph either has a separating complete graph or it is disconnected. This result extends a previous theorem of Clay [21], who proved it in the case of cyclic splittings. In the first section we prove that the same conditions are necessary and sufficient in order to have abelian splittings of pro-$\mathcal{C}$ RAAGs. We also point out that, if the underlying graph is connected, a conjugate of a standard subgroup is always contained in the abelian amalgamated subgroup.

Describing all the abelian splittings of a group is in general difficult as some of them are not compatible with each other. In any case, there is a construction, called the JSJ decomposition, that encodes all the "universal" splittings of a group over a chosen class of subgroups. In the second section we give a description of JSJ decompositions in profinite groups, which is obtained following the approach of Guirardel and Levitt in [37]. In particular, we can define $\mathcal{A}$-JSJ decompositions, meaning that we describe all splittings of a group when the amalgamated subgroups are in the class of groups $\mathcal{A}$, and then relative $(\mathcal{A}, \mathcal{H})$-JSJ decompositions, in the sense that every subgroup in the class $\mathcal{H}$ is elliptic in the decomposition.

In the third section we obtain a $(\mathcal{A}, \mathcal{H})$-JSJ decomposition of pro-$\mathcal{C}$ RAAGs in the case that $\mathcal{A}$ is the class of abelian groups and $\mathcal{H}$ is the class of procyclic subgroups generated by a canonical generator. The proof is constructive, in the sense that it inheritely provides an algorithm to obtain the aforementioned decomposition.

In the last section we refine the relative decomposition in order to obtain a general $\mathcal{A}$-JSJ decomposition. We conclude with an explicit example showing the algorithm beneath the construction of these decompositions.

6.1 Abelian splittings of profinite RAAGs

The main goal of this section is to describe when and how a pro-$\mathcal{C}$ RAAGs splits over a pro-$\mathcal{C}$ abelian group. We begin with two auxiliary lemmas.

Lemma 6.1. *Let $G = G_\Gamma$ be a pro-$\mathcal{C}$ RAAG associated with a connected graph Γ. Suppose that G acts on a pro-$\mathcal{C}$ tree T without a global fixed point, and that all canonical generators are elliptic. Then there exist two canonical generators $v, w \in V(\Gamma)$ such that $(v, w) \notin E(\Gamma)$ and $\langle v, w\rangle$ does not stabilize any vertex of T.*

Proof. Let T^v be the subtree of fixed points of a canonical generator v. If, by contradiction, $T^v \cap T^w \neq \varnothing$ for each couple of canonical generators $v, w \in V(\Gamma)$, by Lemma 5.3 there is a point contained in $\bigcap_{v\in V(\Gamma)} T^v$ fixed by all the generators and so fixed by G, contradicting the hypothesis. This implies that there are at least two vertices $v, w \in V(\Gamma)$ such that $\langle v, w\rangle$ does not stabilize any vertex of T. Notice that such vertices cannot be adjacent by Lemma 5.5 (2). □

Lemma 6.2. *Let G_Γ be a pro-$\mathcal{C}$ RAAG over a connected graph Γ acting on a pro-$\mathcal{C}$ tree T with abelian edge stabilisers. Suppose that a canonical generator $v \in G_\Gamma$ is hyperbolic, then:*

1. $\mathrm{Star}(v)$ *is a complete graph;*
2. *either* $V(\Gamma) = \mathrm{Star}(v)$ *or the set*

$$S := \{u \in \mathrm{Link}(v) \mid \mathrm{Star}(u) \textit{ is not a complete graph}\}$$

   separates $\mathrm{Star}(v) \setminus S$ *and* $\Gamma \setminus \mathrm{Star}(v)$*;*

3. *the standard subgroup generated by S stabilizes an edge.*

Proof. 1. If there exists a single vertex adjacent to v, then the result holds. Suppose then that there exist two distinct vertices $w_1, w_2 \in \mathrm{Link}(v)$.

For each canonical generator w commuting with v we can restrict to the minimal subtree $T_{\langle v,w\rangle}$ on which the abelian subgroup $\langle v, w\rangle$ acts. By Theorems 5.7 and 5.8, the group $\langle v, w\rangle$ is a procyclic extension of the kernel of this action and since $\langle v, w\rangle$ is abelian of rank 2, there exists an element $g = ab$ with $a \in \langle v\rangle \leq G$ and $b \in \langle w\rangle \leq G$ with $b \neq 1$ (as v is hyperbolic) in the kernel of the action, i.e. g fixes pointwise the minimal subtree $T_{\langle v,w\rangle}$. Pick now two elements $g_i = a_i b_i$ with $i \in \{1, 2\}$ for $a_1, a_2 \in \overline{\langle v\rangle}$, $b_1 \in \overline{\langle w_1\rangle}$, $b_2 \in \overline{\langle w_2\rangle}$ such that b_1, b_2 are not trivial and such that g_1, g_2 stabilize pointwise T_v. By hypothesis g_1, g_2 are contained in the abelian stabilisers of the edges of T_v. Let $K = \langle g_1, g_2\rangle$ and let f be the retraction of G onto the standard subgroup generated by w_1, w_2. The image $f(K) \leq G_{\{w_1,w_2\}}$ is an abelian subgroup that contains b_1 and b_2. The element b_1 is in the centraliser of b_2 and they are both with minimal support among their conjugates, so applying Lemma 5.20 this can happen only if $w_1 \in \mathrm{Link}(b_2) = \mathrm{Link}(w_2)$, so w_1, w_2 are adjacent and $\mathrm{Star}(v)$ is a complete graph.

2. Suppose $V(\Gamma) \neq \mathrm{Star}(v)$, as Γ is connected we have that

$$S = \{u \in \mathrm{Link}(v) \mid \mathrm{Star}(u) \text{ is not a complete graph}\}$$

is non-empty. It is immediate to see that S separates the subgraphs generated by $\mathrm{Star}(v) \setminus S$ and $\Gamma \setminus \mathrm{Star}(v)$ because, as $\mathrm{Link}(v)$ is a complete graph by (1), each vertex in $\mathrm{Star}(v) \setminus S$ is connected only to vertices in $\mathrm{Star}(v)$.

3. By Lemma 5.5(1), each vertex of S fixes the subtree T_v, which contains at least an edge because v is hyperbolic. By (1), the action on T of any element of S is elliptic, and hence T_v is fixed pointwise by S.

□

We are now ready to prove the main theorem of this section.

Theorem 6.3. *Let $G = G_\Gamma$ be a pro-$\mathcal{C}$ RAAG associated with a connected graph Γ. Then G acts on a pro-$\mathcal{C}$ tree with abelian edge stabilisers without a global fixed point if and only if either Γ is a complete graph or Γ has a disconnecting complete graph.*
In the second case, there exists a disconnecting complete graph whose standard subgroup is contained in one edge stabiliser of T.

Proof. The case when Γ is a complete graph is clear: indeed, denoting by $\pi = \pi(\mathcal{C})$, the pro-$\mathcal{C}$ RAAG G_Γ is isomorphic to $\mathbb{Z}_\pi^n$, that splits as an HNN-extension

$HNN(\mathbb{Z}_\pi^{n-1}, \mathbb{Z}_\pi^{n-1}, id)$. Similarly, if there is a complete graph K that disconnects Γ, i.e. $\Gamma \setminus K = \Gamma_1 \cup \Gamma_2$ with Γ_1, Γ_2 disjoint subgraphs, then G splits as

$$G_\Gamma = G_{\Gamma_1 \cup K} \amalg_{G_K} G_{\Gamma_2 \cup K}$$

and so G acts on the standard pro-$\mathcal{C}$ tree associated with this splitting.
Suppose now that G acts on a pro-$\mathcal{C}$ tree T with abelian edge stabilisers. If there exists a hyperbolic canonical generator v of G, by Lemma 6.2 we know that either Γ is complete or the set $S := \{u \in \mathrm{Link}(v) \mid \mathrm{Star}(u) \text{ is not a complete graph}\}$ is a disconnecting complete graph contained in an edge stabiliser. We are left to the case when each canonical generator of G is elliptic.

By Lemma 6.1 there exist two vertices $v, w \in V(\Gamma)$ such that no vertex of T is stabilized by both v and w. As each canonical generator acts elliptically, let t_v, t_w be two vertices of T stabilized by v and w respectively. Let $S = [t_v, t_w]$ be the geodesic between these two vertices in T.
By Lemma 6.1 S contains at least one edge, and moreover there exists at least one edge of S that is not stabilised either by v or by w, as by collapsing the subtrees $S \cap T_v$ and $S \cap T_w$ to a point (noticing that we chose v, w such that $T_v \cap T_w = \varnothing$), we would otherwise have S to be disconnected.Define K as a maximal (by the number of vertices contained) complete subgraph of Γ contained in an edge group of S that is not stabilized by either v or w, say $e \in E(T)$. If K is empty, let e be any edge of S, not stabilized by v or w. It is important to notice that even if S might contain infinitely many edges satisfying the properties, Γ is finite hence K is well defined. We claim that K is a complete graph of Γ that disconnects the vertices v and w.
Suppose by contradiction that it is not, then we could find a finite path $p = (v, u_1, \ldots, u_k, w)$ in Γ, such that no vertex of p is contained in K. By Lemma 5.9 there exist some vertices $t_1, \ldots, t_{k+2}$ such that v stabilizes t_1, u_i stabilizes the geodesic $S_i = [t_i, t_{i+1}]$ for $i = 1, \ldots, k$, and w stabilizes $[t_{k+1}, t_{k+2}]$. Set $t_0 = t_v$ and $T_{k+3} = t_w$. In this setting, v stabilizes $S_0 = [t_0, t_1]$ and w stabilizes $S_{k+1} = [t_{k+1}, t_w]$. Furthermore, the union $S' = \bigcup_{i \in \{0, \ldots, k+1\}} S_i$ of the S_i is a pro-p tree that contains t_v and t_w, hence it contains the whole $[t_v, t_w]$. In particular, S' contains e, so $e \in [t_j, t_{j+1}]$ for some $j \in \{0, \ldots, k+3\}$. By the choice of e, it cannot be stabilized by v or w, so there exists a vertex u_j such that $u_j \in G_e$. Now G_e is an abelian pro-$\mathcal{C}$ subgroup of G that contains u_j and each vertex of K, but by Theorem 5.23 this is only possible if u_j is adjacent to every vertex of K. By maximality of K, $u_j \in K$, but this contradicts the fact that no element of the path p is contained in K. If we assumed K to be empty, we have anyway proved that there is a vertex u_j contained in an edge stabiliser of T contradicting that

$K = \varnothing$.
This proves that the graph K is a disconnecting complete graph contained in an edge stabiliser, as required. □

6.2 JSJ decompositions

Prerequisites

In the previous section, we have characterised when a pro-$\mathcal{C}$ RAAGs admits a splitting over an abelian subgroup. Our next goal is to describe all the splittings of these groups over abelian subgroups. In the abstract case, the abelian splittings of a finitely generated group are encoded in a construction called the JSJ decomposition of a group. We develop this theory following the approach of Guirardel and Levitt in [37]. We show that it can be naturally extended to the pro-$\mathcal{C}$ world; for additional results and alternative definitions on the theory of JSJ decompositions see the references in [37].

Definition 6.4 ($\mathcal{A}$-trees)**.** For each class of pro-$\mathcal{C}$ groups $\mathcal{A}$ closed for subgroups and conjugation, we define an $\mathcal{A}$-*tree* (T, G) as a pro-$\mathcal{C}$ tree T with an action of a pro-$\mathcal{C}$ group G such that each edge stabiliser is a group in the class $\mathcal{A}$.

We often denote the $\mathcal{A}$-tree as T rather than (T, G) whenever the pro-$\mathcal{C}$ group G acting on it is clear by the context and we will say that an $\mathcal{A}$-tree (T, G) is trivial if T consists of a single vertex stabilized by the whole G.

We say that a subgroup H of a pro-$\mathcal{C}$ group G is *universally elliptic* (for actions over $\mathcal{A}$-trees) if the action of H is elliptic over any $\mathcal{A}$-tree (T, G) on which G acts.

Definition 6.5 (JSJ decompositions)**.**

- An $\mathcal{A}$-tree (T, G) is *universally elliptic* if its edge stabilisers $G_e \leq G$ are universally elliptic for actions on $\mathcal{A}$-trees.
- An $\mathcal{A}$-tree (T, G) *dominates* another $\mathcal{A}$-tree (T', G) if the same group G acts on both of them and the action of vertex stabilisers G_v, $v \in T$ is elliptic on T' too.
- Two $\mathcal{A}$-trees (T, G) and (T', G) are *equivalent* if the same pro-$\mathcal{C}$ group G acts on both of them and they dominate each other. An equivalence class of $\mathcal{A}$-trees for this relation is said to be a *deformation space*.
- The deformation space of the $\mathcal{A}$-trees that are universally elliptic and that dominate any other universally elliptic $\mathcal{A}$-tree on which G acts is the *JSJ deformation space* and its elements are called the *JSJ tree decompositions*.

Notice that the deformation space is unique, but there might be many non-isomorphic tree decompositions of a pro-$\mathcal{C}$ group G.

Definition 6.6 (Rigid and flexible vertices)**.** A vertex v of a JSJ-tree is said to be *rigid* if it is universally elliptic for the action on any $\mathcal{A}$-tree (even if the tree is not universally elliptic) and *flexible* otherwise.

Notice that if all vertex groups of an $\mathcal{A}$-tree are rigid, then the $\mathcal{A}$-tree is a JSJ tree, but the converse is not true, as the following example shows.

Example 6.7. If $G \cong \mathbb{Z}_\rho^n$ for $n \geq 2$, ρ an arbitrary set of primes, the abelian JSJ decomposition is trivial.
We claim that for each element $g \in G$ we can produce an $\mathcal{A}$-tree (T, G) such that the action of g on T is hyperbolic. Consider a maximal procyclic subgroup C containing $g \in G$. Any generator of a maximal procyclic group can be part of a basis of $\mathbb{Z}_\rho^n$, so we can pick a complement $B \cong \mathbb{Z}_\rho^{n-1}$ of C in G and write $G = HNN(B, B, id)$ with a generator of C as the stable letter. The standard pro-$\mathcal{C}$ tree associated with this pro-$\mathcal{C}$ HNN extension is a vertex with a single loop and g is hyperbolic by construction. This proves that no edge group can be universally elliptic, hence there exists a single universally elliptic $\mathcal{A}$-tree (T, G) on which G acts, which is a tree T with a single point. This is the JSJ decomposition of G, which has a single flexible vertex.

Sometimes is convenient to study *relative* JSJ decompositions, which are defined as follows.

Definition 6.8 (Relative JSJ Decompositions)**.** Let $\mathcal{H}$ be an arbitrary family of subgroups of a pro-$\mathcal{C}$ group G. An $\mathcal{A}$-tree (T, G) is an $(\mathcal{A}, \mathcal{H})$-tree if all the subgroups in the class $\mathcal{H}$ are elliptic. An $(\mathcal{A}, \mathcal{H})$-tree is an $(\mathcal{A}, \mathcal{H})$-*JSJ decomposition* if it is universally elliptic for actions on $(\mathcal{A}, \mathcal{H})$-trees and it dominates every other universally elliptic $(\mathcal{A}, \mathcal{H})$-tree.

We now turn our attention to the study of the JSJ-decomposition of a pro-$\mathcal{C}$ RAAG over abelian groups. Let $G = G_\Gamma$ be a pro-$\mathcal{C}$ RAAG over a finite connected graph Γ. From here on, we assume $\mathcal{A}$ to be the class of abelian pro-$\mathcal{C}$ subgroups of G and $\mathcal{H}$ to be the class of procyclic groups generated by canonical generators of G.

We first construct by induction a decomposition of G over abelian subgroups relative to $\mathcal{H}$, and prove that it is actually an $(\mathcal{A}, \mathcal{H})$-JSJ decomposition. We then refine this decomposition in order to obtain the $\mathcal{A}$-JSJ decomposition of G.

As we are interested in splittings over standard subgroups of disconnecting complete graphs, we first need some basic properties of splittings of this type.

Lemma 6.9. *Let G_Γ be a pro-$\mathcal{C}$ RAAG over a finite connected graph Γ and $K \leq \Gamma$ be a complete subgraph of Γ.*

1. *If all cyclic subgroups generated by canonical generators in K are universally elliptic for their action on $\mathcal{A}$-trees, then the whole standard subgroup G_K is universally elliptic for its action on $\mathcal{A}$-trees.*
2. *If K is a minimal disconnecting complete graph (in the sense that no proper subset of K is a disconnecting complete graph), then the standard subgroup G_K is universally elliptic for its action on $\mathcal{A}$-trees.*
3. *If $\mathrm{Star}(v)$ is a complete graph for $v \in V(\Gamma)$, then there exists an $\mathcal{A}$-tree on which the action of v is hyperbolic.*

Proof.

1. Since by assumption Γ is connected, this follows as a consequence of Lemma 5.5 (2).
2. Assume that G acts on an $\mathcal{A}$-tree (T, G) and suppose that there exists at least one hyperbolic canonical generator $v \in V(K)$. By Lemma 6.2(1), we have that $\mathrm{Star}(v)$ is a complete graph. Since a complete graph does not have any disconnecting subgraphs, it follows that $\Gamma \neq \mathrm{Star}(v)$. From the minimality of the disconnecting complete graph K, we have that the full subgraph Γ' generated by $(V(\Gamma) \smallsetminus V(K)) \cup \{v\}$ is connected and v is a disconnecting vertex of Γ'. In particular, there are two vertices $w_1, w_2 \in V(\Gamma')$ that are adjacent to v but lie in different connected components of $\Gamma \smallsetminus K$. This contradicts the fact that $\mathrm{Star}(v)$ is a complete graph. Hence each canonical generator of K must be elliptic and by (1) the whole K is elliptic.
3. It suffices to notice that the standard pro-$\mathcal{C}$ tree associated with the splitting (5.2) has abelian edge stabilisers because $\mathrm{Link}(v)$ is a complete graph.

$\square$

We record the following graph theoretical observation.

Lemma 6.10 (Disconnecting graphs of components)**.** *Let Γ be a finite connected simplicial graph. Let K be a disconnecting complete subgraph of Γ and let $\{\Gamma^i \mid i \in \{1, \dots, m\}\}$ be the connected components of $\Gamma \smallsetminus K$.*

If K' is a disconnecting subgraph of $\Gamma^j \cup K$ for some $j \in \{1, \dots, m\}$, then K' is also a disconnecting subgraph of Γ.

Proof. Suppose on the contrary that $\Gamma \smallsetminus K'$ is connected. Since K is by assumption a disconnecting subgraph of Γ, it follows that K is not contained in K' and so $K \setminus K'$ is nonempty. Since $\Gamma \smallsetminus K'$ is connected and K is disconnecting, for each vertex v in $(\Gamma^j \cup K) \smallsetminus K'$ there is a vertex $w(v)$ in $K \smallsetminus K'$ such that v and $w(v)$ are connected by a path inside $(\Gamma^j \cup K) \smallsetminus K'$. As K is complete, there is an edge between any two vertices in K. It follows that any pair of vertices $v, v' \in (\Gamma^j \cup K) \smallsetminus K'$ are connected by the path which is the composition of the paths from v to $w(v)$, the edge $(w(v), w(v'))$ and the path from $w(v')$ to v'. Since this path is in $(\Gamma^j \cup K) \smallsetminus K'$, we have that $(\Gamma^j \cup K) \smallsetminus K'$ is connected, deriving a contradiction. □

6.3 $(\mathcal{A}, \mathcal{H})$-JSJ decomposition of pro-$\mathcal{C}$ RAAGs

We first construct the (relative) abelian JSJ decomposition of pro-$\mathcal{C}$ RAAGs under the assumption that all the subgroups in the class $\mathcal{H} = \{\langle v \rangle \mid v \in V(\Gamma)\}$ of procyclic subgroups generated by canonical generators are elliptic.

Theorem 6.11. *Let $G = G_\Gamma$ be a pro-$\mathcal{C}$ RAAG associated with a connected abstract finite graph Γ.*

There is a (possibly trivial) decomposition of G as a fundamental pro-$\mathcal{C}$ group of a reduced finite tree of pro-$\mathcal{C}$ groups $(\mathcal{G}_\Delta, \Delta)$ with the following properties:

- *vertex groups of $(\mathcal{G}_\Delta, \Delta)$ are standard subgroups which are either abelian or their underlying graph does not contain any disconnecting complete subgraph;*
- *each edge group of $(\mathcal{G}_\Delta, \Delta)$ is a standard subgroup associated with a disconnecting complete subgraph K_e of Γ and, moreover, K_e is a minimal (with respect to inclusion) disconnecting complete graph of a subgraph Γ' of Γ.*

Furthermore, the standard pro-$\mathcal{C}$ tree associated with this decomposition is an $(\mathcal{A}, \mathcal{H})$-JSJ tree decomposition (T_Δ, G) of G.

Proof. We prove the statements by induction on the number of generators of the pro-$\mathcal{C}$ RAAG.

Assume first that Γ has one vertex, i.e. $G = \mathbb{Z}_{\pi(\mathcal{C})}$. In this case, we consider the decomposition as a fundamental group of a graph of groups to be trivial, so Δ is a point and the associated group is $\mathbb{Z}_{\pi(\mathcal{C})}$. This decomposition satisfies the required conditions. Furthermore, since G is a standard subgroup, by assumption it is elliptic and so the $(\mathcal{A}, \mathcal{H})$-JSJ decomposition of G is trivial and agrees with the decomposition as a fundamental group of a graph of groups.

Assume that we have already established the decomposition of every pro-$\mathcal{C}$ RAAG whose underlying graph has at most $n-1$ vertices as a fundamental group of a graph of groups and that we have proved that the $(\mathcal{A},\mathcal{H})$-JSJ decomposition of G is determined by the group decomposition as a fundamental group of a graph of pro-$\mathcal{C}$ groups satisfying the properties of the theorem.

Let now Γ be a connected graph with n vertices, $n \geq 2$. Suppose first that Γ does not have any disconnecting complete subgraph. In this case, we consider the decomposition as a fundamental group of a graph of groups to be trivial and so Δ has one vertex with corresponding group G. This decomposition satisfies the requirements. If Γ is a complete graph, then $G \simeq \mathbb{Z}^n_{\pi(\mathcal{C})}$. Since by assumption, each canonical generator is elliptic, then by Lemma 6.9, the group G stabilizes a point, and hence the $(\mathcal{A},\mathcal{H})$-JSJ decomposition is trivial and coincides with the decomposition of G as the fundamental group of a graph of groups. If Γ is not complete and does not have any disconnecting complete subgraph, then by Theorem 6.3 G cannot act non-trivially on an $\mathcal{A}$-tree, so the $(\mathcal{A},\mathcal{H})$-JSJ decomposition is again trivial.

Suppose now that Γ has a disconnecting complete graph. Let K be a disconnecting complete graph such that $|V(K)|$ is minimal among disconnecting complete graphs.

We first construct a splitting of G as an amalgamated free product over the standard subgroup G_K. Assume that $\Gamma \smallsetminus K$ has $m \geq 2$ nontrivial connected components Γ^i, for $i \in \{1,\dots,m\}$. In this case, we consider the splitting of G as a pro-$\mathcal{C}$ amalgamated product of the form

$$G = \coprod_{i=1}^{m}{}_{G_K} G_{K\cup\Gamma^i}.$$

By Theorem 6.5.2 of [72], this decomposition corresponds to the pro-$\mathcal{C}$ fundamental group of a tree of groups $(\overline{\mathcal{G}_\Delta},\overline{\Delta})$ with m vertices $V(\overline{\Delta}) = \{x_1,\dots,x_m\}$, whose vertex groups $\{G_{K\cup\Gamma^i} \mid i \in \{1,\dots,m\}\}$ respectively, and with all edges of $E(\overline{\Delta})$ stabilised by G_K. Since K is a complete graph, G_K is a pro-$\mathcal{C}$ abelian subgroup and hence this decomposition is an $\mathcal{A}$-decomposition of G. Notice that if $m > 2$, the underlying tree $\overline{\Delta}$ is not unique. Indeed any tree with m vertices provides the same fundamental group G since all edge groups coincide. Without loss of generality, we choose the underlying graph $\overline{\Delta}$ to be a path consisting of m points and $m-1$ edges, vertex groups $G_{K\cup\Gamma^i}$ and edge groups G_K (with the natural embeddings). By construction, the graph of pro-$\mathcal{C}$ groups $(\overline{\mathcal{G}_\Delta},\overline{\Delta})$ has G as its pro-$\mathcal{C}$ fundamental group.

By the induction hypothesis, for each $i \in \{1, \ldots, m\}$ each vertex group $G_{K \cup \Gamma^i}$ has a decomposition as a fundamental group of a tree of pro-$\mathcal{C}$ groups $(\mathcal{G}_{\Delta_i}, \Delta_i)$ as in the statement and this decomposition determines an $(\mathcal{A}, \mathcal{H})$-JSJ tree decomposition.

For each $i \in \{1, \ldots, m\}$, by Lemma 6.9 the action of the group G_K is elliptic on any $(\mathcal{A}, \mathcal{H})$-tree $(T_{K \cup \Gamma^i}, G_{K \cup \Gamma^i})$ and so G_K is contained in a vertex stabiliser of $T_{K \cup \Gamma^i}$. Hence, a conjugate of G_K is contained in a vertex group of the graph of groups $(\mathcal{G}_{\Delta_i}, \Delta_i)$, namely $v_i \in \Delta_i$. By Lemma 5.15, if a conjugate of a canonical generator is contained in a standard subgroup, then the standard subgroup contains the generator. As each vertex group of $(\mathcal{G}_{\Delta_i}, \Delta_i)$ is a standard subgroup by induction, G_K itself is contained in the vertex group $\mathcal{G}_{\Delta_i}(v_i)$.

We construct a tree of groups $(\mathcal{G}_\Delta, \Delta)$ in the following way. Define $V(\Delta) = \bigcup_{i=1}^m V(\Delta_i)$ and

$$E(\Delta) = \{E(\Delta_i), (v_j, v_\ell) \mid i \in \{1, \ldots, m\}, (v_j, v_\ell) \in E(\overline{\Delta})\}.$$

For each $w \in V(\Delta)$ there is $i \in \{1, \ldots, m\}$ such that $w \in \Delta_i$ and we define the group $\mathcal{G}_\Delta(w)$ of $\mathcal{G}_\Delta$ to be $\mathcal{G}_\Delta(w) = \mathcal{G}_{\Delta_i}(w)$. Similarly, if e is an edge of Γ such that $e \in E(\Delta_i)$, then the corresponding group (and vertex embeddings) are induced from Δ_i. If $e = (v_j, v_\ell)$, we define $\mathcal{G}_\Delta(\delta) = G_K$ (with the natural embeddings). This graph of groups is well-defined as each edge group embeds in the adjacent vertex groups and its pro-$\mathcal{C}$ fundamental group is exactly G by construction (as the fundamental group of the graphs $(\mathcal{G}_{\Delta_i}, \Delta_i)$ are the standard subgroups $G_{K \cup \Gamma_i}$).

This graph of groups is reduced. Indeed by induction, edge groups of $(\mathcal{G}_{\Delta_i}, \Delta_i)$ do not coincide with the adjacent vertex groups. We next show that G_K cannot coincide with any vertex group in $\mathcal{G}_\Delta(v_i)$. Indeed, by definition, Γ_i is a nontrivial connected component of $\Gamma \setminus K$ and so $G_{K \cup \Gamma^i} \neq G_K$ and, in particular, if the decomposition of $G_{K \cup \Gamma^i}$ is trivial, then the unique vertex group does not coincide with G_K. Assume next that $G_{K \cup \Gamma^i}$ has a nontrivial decomposition satisfying the conditions of the statement and suppose by contradiction that there is a vertex group of $(\mathcal{G}_{\Delta_i}, \Delta_i)$ equal to G_K. In particular, since the graph $K \cup \Gamma^i$ is connected and edge groups are standard subgroups of complete disconnecting subgraphs of $K \cup \Gamma^i$, there would be disconnecting subgraph K' of $K \cup \Gamma^i$ contained in K. By Lemma 6.10, K' would also be a disconnecting complete subgraph of Γ, contradicting the minimality of K. Hence, we have shown that the graph of groups is reduced.

We next show that the decomposition as a fundamental group of a graph of groups satisfies the required properties. Indeed, by the inductive hypothesis on Δ_i, we have that each vertex group of $\mathcal{G}_\Delta$ is a standard subgroup which is either abelian or the underlying graph does not have disconnecting complete subgraphs; and the edge groups of $\mathcal{G}_\Delta$ are either G_K or, by induction, they are standard subgroups associated with disconnecting complete subgraphs of a certain $K \cup \Gamma^i$, which are also disconnecting subgraphs for Γ by Lemma 6.10. In the former case, by our choice G_K is the standard subgroup of a minimal complete disconnecting subgraph of Γ. In the latter case, the induction hypothesis assures that the associated disconnecting complete subgraph is minimal for a subgraph of $K \cup \Gamma^i$, which is also a subgraph of Γ.

Finally, we are left to check that the standard tree (T_Δ, G) associated with the decomposition as a fundamental group of a graph of groups given for G is an $(\mathcal{A}, \mathcal{H})$-JSJ tree. The tree (T_Δ, G) is universally elliptic since each edge stabiliser is either universally elliptic by the induction hypothesis or it is a conjugate of G_K and since G_K is abelian and generated by universally elliptic elements, by Lemma 6.9 (2), G_K acts universally elliptic on any $\mathcal{A}, \mathcal{H}$-tree.

In order to prove that (T_Δ, G) dominates any other $(\mathcal{A}, \mathcal{H})$-tree (T', G), consider a vertex stabiliser $H \leq G$ given by the decomposition of G as a fundamental group of graphs of groups. By construction, H is either a standard subgroup associated with a complete graph, with an elliptic action on any $(\mathcal{A}, \mathcal{H})$-tree by Lemma 6.9 (2), or it is a standard subgroup associated with a graph without disconnecting complete subgraphs, which is also elliptic for the action on any $(\mathcal{A}, \mathcal{H})$-tree by Theorem 6.3.

Therefore, the $(\mathcal{A}, \mathcal{H})$-tree is a JSJ-tree decomposition of G. □

6.4 $\mathcal{A}$-JSJ decomposition of pro-$\mathcal{C}$ RAAGs

In order to obtain the general $\mathcal{A}$-JSJ decomposition, we must further refine the $(\mathcal{A}, \mathcal{H})$-JSJ decomposition described in Theorem 6.11.

Definition 6.12 (Hanging vertex)**.** We say that a vertex v of Γ is a *hanging vertex* if $\mathrm{Star}(v)$ is a complete graph and for each $w \in \mathrm{Link}(v)$, $\mathrm{Star}(w)$ is not a complete graph.

Theorem 6.13. *Let $G = G_\Gamma$ be a pro-$\mathcal{C}$ RAAG associated with a connected abstract finite graph Γ.*

There is a (possibly trivial) decomposition of G as a fundamental pro-$\mathcal{C}$ group of a reduced finite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}_\Theta, \Theta)$ with the following properties:

- *the underlying graph Θ is either a tree or a tree with loops;*

- *vertex groups of $(\mathcal{G}_\Theta, \Theta)$ are standard subgroups which are either abelian or their underlying graph does not contain any disconnecting complete graph;*
- *each edge group of $(\mathcal{G}_\Theta, \Theta)$ is a standard subgroup associated with a disconnecting complete subgraph of Γ;*
- *hanging vertices do not belong to any vertex group.*

Furthermore, the standard pro-$\mathcal{C}$ tree associated with this decomposition is an $\mathcal{A}$-JSJ tree decomposition (T_Θ, G) of G.

Proof. Suppose first that $|V(\Gamma)| = 1$, so $G = \mathbb{Z}_{\pi(\mathcal{C})}$. In this case define a graph Θ consisting of a single vertex with a loop, so that $V(\Theta) = \{v_\Theta\}$, $E(\Theta) = \{e_\Theta\}$ with $d_0(e_\Theta) = d_1(e_\Theta) = v_\Theta$. Define the corresponding vertex and edge group as $\mathcal{G}_\Theta(e_\Theta) = \mathcal{G}_\Theta(v_\Theta) = 1$ (with the natural embedding). The graph of groups $(\mathcal{G}_\Theta, \Theta)$ satisfies the required properties and the associated pro-$\mathcal{C}$ tree is the $\mathcal{A}$-JSJ decomposition of G because the trivial element is always elliptic.

Assume now that $|V(\Gamma)| \geq 2$. If Γ does not have disconnecting complete graphs, then the trivial decomposition, with Θ consisting of a single vertex with corresponding group G, satisfies the requirements. If Γ is a complete graph, then the associated tree decomposition (T_Θ, G), which is trivial, is the $\mathcal{A}$-JSJ decomposition, see discussion in Example 6.7. Similarly, if Γ is not complete and it has no disconnecting complete subgraph, then the $\mathcal{A}$-JSJ decomposition is trivial by Theorem 6.3.

In the case when Γ has disconnecting complete subgraphs, we first consider the graph of pro-$\mathcal{C}$ groups $(\mathcal{G}_\Delta, \Delta)$ as described in Theorem 6.11.

Let $HV(\Gamma) \subset V(\Gamma)$ be the set of hanging vertices of Γ. We claim that, for each $v \in HV(\Gamma)$, the standard subgroup $G_{\mathrm{Star}(v)}$ coincides with an abelian vertex group of $(\mathcal{G}_\Delta, \Delta)$. Since by definition $\mathrm{Star}(v)$ is complete, $G_{\mathrm{Star}(v)}$ is abelian and so by Lemma 6.9 the action of this subgroup on T_Δ is elliptic and therefore a conjugate of this subgroup is contained in at least one vertex group of $(\mathcal{G}_\Delta, \Delta)$. As each of these vertex groups is a standard subgroup, by Lemma 5.15 $G_{\mathrm{Star}(v)}$ itself is contained in them. Notice that if $\mathrm{Star}(v)$ disconnects a graph, then so does $\mathrm{Link}(v)$, and similarly, if $\mathrm{Star}(v)$ is contained in a complete disconnecting subgraph K, then $K \setminus \{v\}$ is also a disconnecting subgraph. Since edge groups are minimal complete disconnecting subgraphs of a subgraph of Γ, see Theorem 6.11, from the latter observation we have that $G_{\mathrm{Star}(v)}$ cannot be contained in any edge group of the graph of groups $(\mathcal{G}_\Delta, \Delta)$ and so $G_{\mathrm{Star}(v)}$ is contained in a unique vertex group, namely the vertex group $G_{\Gamma'}$, which by Theorem 6.11 is a standard subgroup associated with some subgraph $\Gamma' < \Gamma$. If $G_{\mathrm{Star}(v)} \lneq G_{\Gamma'}$, then there is no edge between vertices in $\Gamma' \smallsetminus \mathrm{Star}(v) \neq \varnothing$ and v, and so $G_{\Gamma'}$ is not abelian and $\mathrm{Link}(v)$ is a disconnecting subgraph of Γ', contradicting the vertex group description of Theorem 6.11. Therefore, we have that $G_{\mathrm{Star}(v)}$ is precisely the vertex group.

Similarly, v cannot be contained in any edge group of $(\mathcal{G}_\Delta, \Delta)$ because such groups are minimal disconnecting complete graphs K, and $K \smallsetminus v$ would also be a disconnecting complete graph. For this reason, for each $v \in HV(\Gamma)$, there exists only a single vertex $d_v \in V(\Delta)$ such that $\langle v \rangle \leq \mathcal{G}_\Delta(d_v)$. Notice that, as $\mathcal{G}_\Delta(d_v)$ is abelian, it is immediate from the definition of hanging vertex that v is the only hanging vertex contained in $\mathcal{G}_\Delta(d_v)$, so $d_{v_1} \neq d_{v_2}$ for each distinct $v_1, v_2 \in HV(\Gamma)$.

We define a graph of groups $(\mathcal{G}_{\Delta_0}, \Delta_0)$ in the following way. We define $V(\Delta_0) = V(\Delta)$ and $E(\Delta_0) = E(\Delta) \cup \{e_v \mid v \in HV(\Gamma)\}$, where $d_0(e_v) = d_1(e_v) = d_v$. For $w \in V(\Delta_0)$, if $w = d_v$ for some $v \in HV(\Gamma)$, then we set $\mathcal{G}_{\Delta_0}(d_v) = \mathcal{G}_{\Delta_0}(e_v) = G_{\mathrm{Link}(v)}$ (the embeddings from the edge groups to the vertex groups are the identity) and otherwise, we set $\mathcal{G}_{\Delta_0}(w) = \mathcal{G}_\Delta(w)$ for $w \in V(\Delta_0), w \neq d_v$, v a handing vertex and $\mathcal{G}_{\Delta_0}(e) = \mathcal{G}_\Delta(e)$ for $e \in E(\Delta)$. As we observed above, since hanging vertices do not belong to any edge group, the embeddings from edges groups to vertex groups in $(\mathcal{G}_\Delta, \Delta)$ also define embeddings in $(\mathcal{G}_{\Delta_0}, \Delta_0)$.

The graph of pro-$\mathcal{C}$ groups $(\mathcal{G}_{\Delta_0}, \Delta_0)$ may not be reduced, so we define $(\mathcal{G}_\Theta, \Theta)$ as the reduced graph of groups obtained from $(\mathcal{G}_{\Delta_0}, \Delta_0)$. By construction, the graph of groups $(\mathcal{G}_\Theta, \Theta)$ is reduced. The underlying graph Θ is obtained from Δ_0 by collapsing some edges and in turn, the graph Δ_0 is obtained by adding loops to the tree Δ and therefore Θ is a tree with loops. Vertex and edge groups are either equal to $G_{\mathrm{Link}(v)}$ for some $v \in HV(\Gamma)$ or they inherit the structure of vertex and edge groups of $(\mathcal{G}_\Delta, \Delta)$. No hanging vertex can be contained in any vertex group by construction. As the pro-$\mathcal{C}$ fundamental group of each vertex with loop $(\mathcal{G}_{\Delta_0}, \{d_v, e_v\})$ is exactly the pro-$\mathcal{C}$ fundamental group of $(\mathcal{G}_\Delta, \{d_v\})$, the pro-$\mathcal{C}$ fundamental group of $(\mathcal{G}_\Theta, \Theta)$ is also G. Therefore, the decomposition of G as a fundamental group of graph of groups satisfies the requirements of the statement.

In order to conclude, we have to check that the standard tree (T_Θ, G) associated with the decomposition given for G as a fundamental group of a graph of groups is an $\mathcal{A}$-JSJ tree. Edge stabilisers are either conjugates of a standard subgroup $G_{\mathrm{Link}(v)}$ for some $v \in HV(\Gamma)$, and in this case they act universally elliptic on any $\mathcal{A}$-trees by Lemma 6.2, or they are conjugates of standard subgroups associated with disconnecting complete graphs of Γ, as in the $(\mathcal{A}, \mathcal{H})$-JSJ decomposition. In this case, there exist some subgraphs Γ' of Γ such that our disconnecting subgraphs are minimal among complete subgraphs that disconnect Γ'. By Lemma 6.9 (2), edge stabilisers of T_Θ act universally elliptic on each $\mathcal{A}$-tree on which $G_{\Gamma'}$ acts, and in particular over any $\mathcal{A}$-tree on which G acts. This shows that (T_Θ, G) is universally elliptic.

In order to prove that (T_Θ, G) dominates any other $(\mathcal{A})$-tree (T', G), we need to prove that the action on T' of a vertex stabiliser H of T_Θ is elliptic for each (T', G) universally elliptic $\mathcal{A}$-tree. Up to conjugation, we can assume that H is a standard subgroup that corresponds to a vertex group of $(\mathcal{G}_\Theta, \Theta)$. If H is non-abelian, then it is a standard subgroup associated with a subgraph without disconnecting complete graphs and its action on T' is elliptic by Theorem 6.3. Assume now that H is abelian and suppose that there exists a canonical generator v in H such that its action on T' is hyperbolic. We next show that v is a hanging vertex. By Lemma 6.2, $\mathrm{Star}(v)$ must be a complete graph. Let $w \in \mathrm{Link}(v)$ such that w is an element of H. Since H is abelian and $\langle v \rangle \cong \widehat{\mathbb{Z}}_{\mathcal{C}}$, H can not be virtually procyclic and so by Theorem 7.1.7 of [72] there must be a nontrivial element $g \in \langle v, w \rangle$ contained in an edge stabiliser of T'. As (T', G) is a universally elliptic $\mathcal{A}$-tree, g must be a universally elliptic element and $w \in \alpha(g)$. By Remark 5.18, the action of g on the standard pro-$\mathcal{C}$ tree of the pro-$\mathcal{C}$ HNN extension

$$G = HNN(G_{\Gamma \smallsetminus \{w\}}, G_{\mathrm{Link}(w)}, id)$$

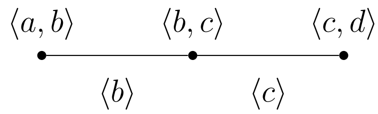
is hyperbolic, and since g is universally elliptic on $\mathcal{A}$-trees, this implies that $G_{\mathrm{Link}(w)}$ is not abelian and so $\mathrm{Star}(w)$ is not abelian either. As this is true for each $w \in \mathrm{Link}(v)$, we conclude that v is a hanging vertex. However, by the construction of the decomposition, hanging vertices are not contained in any vertex stabilisers of T_Θ and so we arrived at a contradiction. This proves that the action on T' of each canonical generator in H is elliptic and, applying Lemma 6.9 (1), we conclude that H is elliptic for its action on T', as desired.

This proves that (T_Θ, G) is an $\mathcal{A}$-JSJ decomposition of G. □

We provide an example of an $(\mathcal{A}, \mathcal{H})$-JSJ decomposition and an $\mathcal{A}$-JSJ decomposition of a pro-$\mathcal{C}$ RAAG.

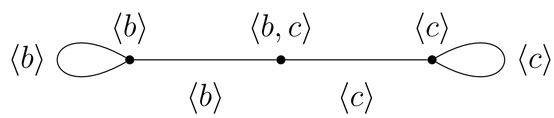
Example 6.14. Consider the pro-$\mathcal{C}$ RAAG associated with the graph P_4, which is

As the $(\mathcal{A}, \mathcal{H})$ and $\mathcal{A}$-JSJ decompositions are uniquely determined by the associated graph of groups, we describe only this graph of groups, writing edge and vertex groups next to the corresponding edge and vertex. A minimal disconnecting complete graph in P_4 is b. The subgroup $\langle a, b \rangle$ is abelian, whereas c is a disconnecting complete subgraph of the graph generated by b, c, d. The graph of groups decomposition of G_{P_4}

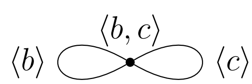


satisfies the conditions of Theorem 6.11 and so the corresponding tree defines an $(\mathcal{A}, \mathcal{H})$-JSJ decomposition of P_4.

The vertex a and d are hanging vertices, because $\mathrm{Star}(a)$, $\mathrm{Star}(d)$ are complete graphs and $\mathrm{Star}(b)$, $\mathrm{Star}(c)$ are not complete. For this reason we substitute each of the vertices corresponding to $\langle a, b\rangle$ and $\langle c, d\rangle$ with a vertex and a loop, both with associated group $G_{\mathrm{Link}(a)} = \langle b\rangle$ and $G_{\mathrm{Link}(d)} = \langle c\rangle$ respectively.



This graph of groups is not reduced. After reducing it,



we obtain a graph of groups decomposition of P_4 satisfying the conditions of Theorem 6.13 and so the associated tree is a $\mathcal{A}$-JSJ decomposition of G_{P_4}.

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