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Three Shapley rankings of poverty measures according to their sensitivity to the three components of poverty

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Abstract

Most poverty measures in the literature depend on the three components of poverty, incidence, intensity and inequality among the poor. However, the implication of each component on the final poverty measure is something that has not been studied. It has been shown that families of parameter-dependent poverty indices can be more or less sensitive to transfers among the poor, depending on the value of the parameter taken. This value will affect the sensitivity of the family of measures to the inequality among the poor. In this paper we intend to carry out an analogous study for the three components of poverty. We provide three poverty rankings of some poverty measures according to their sensitivity to incidence, intensity and inequality. In fact, we offer orderings for all the rank-dependent poverty measures and two families of rank-independent poverty measures.

Keywords: Incidence; Intensity; Inequality among the poor; Decomposable poverty measures

1. Introduction

Since the seminal work of [18] it has been widely accepted that a poverty measure should be sensitive to the incidence of poverty, the intensity of poverty and the inequality among the poor. Hence, several poverty measures have been decomposed in these three terms. Some of these decompositions can be seen in [13] and [1].

However, we could ask to what extent each component contributes to the final poverty value, and, we could also ask what difference it makes choosing one poverty index or another.

With respect to the inequality component of poverty, in the literature it is widely accepted that poverty measures should take into account distribution-sensitivity, that is, the inequality among the poor, see [7] and [24]. In fact, the degree of distribution-sensitivity of some transfers has always been regarded as an important factor. Nevertheless, there are few formal definitions of distribution-sensitivity comparisons of these measures, see [25], [6] and [2].

Zheng [25] provides a theoretical base to compare distribution-sensitivity for a class of additively separable poverty indices. However, the criterion introduced by Zheng does not allow comparisons among rank-dependent poverty mea-

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sures. Therefore, Bosmans [6] introduces a condition for rank-dependent poverty measures, based on the dominance of the indices' vector of weights, which allows such comparisons in terms of their distribution-sensitivity to some specific transfers. Aristondo and Ciommi [2] also propose a criterion to sort the rank-dependent poverty measures in terms of a mathematical value called orness, which can be interpreted as a distribution-sensitivity classification.

However, so far no study has been carried out comparing poverty measures in terms of their sensitivity to the other two components of poverty, that is, incidence and intensity. In fact, it would be very intuitive to think that if one measure is more sensitive than another with respect to the inequality among the poor, perhaps it is less sensitive to either one or both of the other two components. For this reason, in this paper we will focus on the sensitivity of decomposable poverty measures to their three poverty components. In addition, we will provide three rankings of the poverty measures according to their three components; incidence, intensity and inequality of the poor.

For this purpose, we use the Shapley decomposition method which has also been used in dynamic poverty studies such as [14], [4], and so on. Kakwani [14] decomposes poverty changes in growth and inequality changes, and proposes a way to apply the Shapley method. In fact, he provides a new method to eliminate each contributory factor by assigning the arithmetic mean of all the possible combinations that can be made with the value in time 1 and value in time 0. In addition, Aristondo and Onaindia [4] decompose poverty changes in incidence, intensity and inequality changes, and they follow Kakwani's procedure to eliminate the determinants in order to compute the marginal contributions of the changes in the three poverty components to the global poverty change.

All these dynamic methods are very interesting and could be applied in turn with the method proposed in this paper since they are complementary. In fact, in any poverty study, we could use first the method proposed in this paper to choose the most appropriate decomposable poverty index, and then we could apply any dynamic method to analyse changes in poverty and their different types of components, such as growth and distribution, or incidence, intensity and inequality, see also [16], [10].

Summarizing, we offer a classification for almost all the decomposable poverty measures in terms of their sensitivity to incidence, intensity and inequality among the poor. Firstly, we present the decompositions of the existing poverty measures in the literature in terms of these three components. Then, we compute the relative marginal contributions of the three components to the total poverty value using the Shapley decomposition methodology, see [19]. Finally, we provide three classifications for the poverty measures, depending on the three components' relative marginal contributions to the total poverty.

We want to note that these rankings will allow us to choose the most appropriate poverty measure for each empirical study according to which of the three components we want to give more strength or importance to. Moreover, in static poverty studies, this work will allow us to know the absolute and relative contributions to the final poverty value of each of the components for all the decomposable indices.

We wish to add that the rankings in terms of the sensitivity to inequality among the poor proposed in this paper coincide with those proposed by [6] and [2].

Finally, we will see that incidence and intensity's relative marginal contributions to poverty are exactly the same for all the measures, and all of them are delimited between one third and one half. On the other hand, inequality's relative marginal contributions are always lower than those of incidence and intensity for all the measures, and the values are higher than 0 and lower than one third.

The paper is classified as follows; section 2 is devoted to notations, definitions and the presentation of the poverty measures and their decompositions. In section 3 we show the Shapley decomposition in the three terms of the measures, and section 4 offers the three rankings of the measures in terms of their three contribution values. Finally, we can see the concluding remarks and the appendix with the proofs in section 5 and Appendix A, respectively.

2. Notations and definitions

In what follows, we introduce some basic notations and definitions. We consider a population consisting of $n \geq 3$ individuals. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be the income vector distribution, $x_i \in \mathbb{R}_{++}$ represent the income of the $i-$th individual and $D = \bigcup_{n\geq 3} \mathbb{R}^n_{++}$ be the set of all distributions.[1] For a given $\boldsymbol{x} \in D$, let us denote by $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$ the non-decreasing and non-increasing rearrangement of the coordinates of $\boldsymbol{x}$, respectively. Whereas,

[1] $\mathbb{R}_{++}$ is the set of real values higher than 0.

Table 1
Rank-dependent poverty measures.

Measure	w_i
PGR	$\frac{1}{n}$
S	$\frac{2(q+0.5-i)}{qn}$
SST	$\frac{2(n+0.5-i)}{n^2}$
T	$\frac{2(n+1-i)}{(n+1)n}$
T_τ	$\frac{\tau n+1-2i}{(\tau-1)n^2}, \tau \geq 2$

the arithmetic mean is denoted as $\mu(\boldsymbol{x}) = (x_1 + \cdots + x_n)/n$. Let us denote by $z \in \mathbb{R}_{++}$ the poverty line, such that, an individual $i \in \{1, \ldots, n\}$ is defined as *poor* if $x_i < z$ and as *non poor* if $x_i \geq z$.

So doing, $q = q(\boldsymbol{x}, z)$ denotes the number of poor people, for a given distribution $\boldsymbol{x}$, and the poor distribution and its mean is defined as $\boldsymbol{x}_p = (x_{(1)}, \ldots, x_{(q)})$ and $\mu_p = \mu(\boldsymbol{x}_p) = (x_{(1)} + \cdots + x_{(q)})/q$, respectively. Without loss of generality we establish that $n > q \geq 2$.

2.1. Rank-dependent poverty measures

A poverty measure is a non-constant function $P : D \times \mathbb{R}_{++} \to \mathbb{R}$ whose value $P(\boldsymbol{x}, z)$ denotes the poverty level associated with an income distribution $\boldsymbol{x}$ and the poverty line z. The normalized gaps for incomes below the poverty line are defined as the relative distance between the income value and the poverty line, and for incomes above, it is defined as zero. Formally: $g_i = \max\left\{\frac{z-x_i}{z}, 0\right\}$. Finally, $\boldsymbol{g} = (g_1, \ldots, g_n)$ denotes the normalized gap vector and $\boldsymbol{g}_p = (g_{[1]}, \ldots, g_{[q]})$ the normalized gap vector of the poor. The arithmetic mean of the normalized gaps of the poor is denoted as $\mu_{g_p} = \mu(\boldsymbol{g}_p) = (g_{[1]} + \cdots + g_{[q]})/q$. In poverty measurement, a rank-dependent poverty measure is a one whose individual's weight depends only on its place in the distribution with respect to the others. Formally:

Definition 1. A poverty measure $P : D \times \mathbb{R}_{++} \to \mathbb{R}$ is rank-dependent if for each income distribution $\boldsymbol{x} \in D$ and any fixed poverty line $z \in \mathbb{R}_{++}$, it takes the following expression

$$P(\boldsymbol{x}, z) = \sum_{i=1}^{q} w_i \frac{z - x_{(i)}}{z} = \sum_{i=1}^{q} w_i g_{[i]}, \tag{1}$$

where $w_1 \geq w_2 \geq \cdots \geq w_q$. In addition, if the weights decrease strictly then the *transfer axiom*[2] is satisfied.

Table 1 collects the rank-dependent poverty measures proposed in the literature, namely the Poverty Gap Ratio, PGR, [18] index, S, the [20] index, SST, the [23] index, T, and the Thon class, T_τ, [12].[3] We want to note that the Thon class on indices for $\tau = 2$ is exactly the SST index and for $\tau = 2 + \frac{1}{n}$ the T index is obtained.

The [22] index, namely T_a, is also a rank-dependent poverty measure. However it is defined in terms of the censored income, $\widetilde{x} = (x_{(1)}, \cdots, x_{(q)}, z \cdots, z)$, as follows, $T_a = \frac{2}{\mu(\widetilde{x})n^2} \sum_{i=1}^{n} (\mu(\widetilde{x}) - x_{(i)})$.

All these poverty measures can be decomposed in terms of the Headcount ratio, H, the Income gap ratio, M, and the Gini index of the gap of the poor, G. The Headcount ratio is defined as $H = H(x) = \frac{q}{n}$. The Income gap ratio

2 The *transfer axiom* states that, given other things, a pure transfer of income from a poor individual to any other individual that is richer must increase the poverty measure.

3 See [6] and [2] for detailed definition of the above listed rank-dependent poverty measures.

Table 2
Rank-dependent decompositions.

Measure	Decomposition
PGR	HM
S	$HM(1+G)$
SST	$HM(2+H(G-1))$
T	$\dfrac{HM(2n+1+nH(G-1))}{n+1}$
T_τ	$\dfrac{HM(\tau-H+HG)}{\tau-1}, \tau \geq 2$
T_a	$\dfrac{HM(1-H+HG)}{1-HM}$

as $M = M(x) = \frac{1}{q}\sum_{i=1}^{q}\frac{z-x_{(i)}}{z}$, and the Gini index of the gap of the poor has the following form; $G = G(g_p) = \frac{1}{2q^2\mu_{g_p}^2}\sum_{i=1}^{q}\sum_{j=1}^{q}\left|g_i - g_j\right|$.

Then the above rank-dependent poverty measures can be decomposed as in Table 2, see [1].

In the following section we will concentrate on rank-independent poverty measures.

2.2. Rank-independent poverty measures

The rank-independent poverty measures are those indices defined in terms of the income of each individual relative to the poverty line. That is, they do not depend on the place individuals take in the distribution. The family of poverty measures introduced by [9], henceforth FGT family, takes the form:

$$FGT_\alpha = FGT_\alpha(x) = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{z-x_{(i)}}{z}\right)^{\alpha} \qquad \alpha \geq 0, \tag{2}$$

where α is the poverty aversion. Note that for $\alpha = 0$ and $\alpha = 1$ we have the H and PGR measures, respectively. For $\alpha \geq 2$ the measure is averse to transfers of income from a poor person to a less poor one. That is, they will take into account the inequality among the poor. Hence, this family could be decomposed in terms of incidence, intensity and inequality among the gap of the poor for every $\alpha \geq 2$, see [3]. The inequality family used in this decomposition is the Generalized Entropy family measured for the gaps of the poor. It is defined as follows:

$$G_\alpha = GE_\alpha(g_p) = \frac{1}{q(\alpha^2-\alpha)}\sum_{i=1}^{q}\left(\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\alpha} - 1\right) \qquad for \quad \alpha \geq 2. \tag{3}$$

And the decomposition has the following form:

$$FGT_\alpha = HM^\alpha\left[1+(\alpha^2-\alpha)G_\alpha\right] \qquad for \quad \alpha \geq 2.$$

Finally, we introduce another family of poverty measures proposed by [8]. They notice that the Sen index can be written as a combination of incidence, intensity and inequality of the gap of the poor measured by the Gini index. Hence, they propose another poverty measure replacing the [11] inequality index with the [5] inequality measure as follows:

$$CHU_\alpha = CHU_\alpha(x) = \frac{q}{n}\left[\frac{1}{q}\sum_{i=1}^{q}\left(\frac{z-x_{(i)}}{z}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \qquad for \quad \alpha \geq 1. \tag{4}$$

Aristondo [1] proposes the decomposition of this family of poverty measures using the extended part of [5] family of inequality measures proposed by [15] for the gaps of the poor. It can be written as follows:

$$A_\alpha = A_\alpha(g_p) = 1 - \left(\frac{1}{q}\sum_{i=1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\alpha}\right)^{-\frac{1}{\alpha}} \qquad for \quad \alpha > 1. \tag{5}$$

And this family is decomposed as:

$$CHU_\alpha = \frac{HM}{1 - A_\alpha} \qquad for \quad \alpha > 1.$$

3. Shapley decomposition of the poverty measures

The [19] decomposition is a technique extended from game theory to applied economics by [21] and [17]. In this decomposition method, we need to assume an indicator, for example I, a function that depends on some determinants. In this paper, we will focus on three determinants, a, b and c. Hence, the indicator I will be a function of these three parameters, that is, $I(a, b, c)$. Now, we want to note that there are $3! = 6$ ways of ordering the three determinants;

$$(a, b, c), \quad (a, c, b), \quad (b, a, c), \quad (b, c, a), \quad (c, a, b) \ \text{ and } \ (c, b, a). \tag{6}$$

The Shapley decomposition will allow us to know the marginal contributions of each determinant to the overall value $I(a, b, c)$. The respective marginal contributions of the determinants a, b and c are defined as all the possible ways in which each of these determinants can be eliminated taking into account that these determinants can be eliminated first, second or third. For example, $C(a)$ is defined as the marginal contribution of a to the function $I(a, b, c)$. If a is eliminated first, its contribution to the overall value of function I will be $I(a, b, c) - I(0, b, c)$. The given weight to this possibility is $2/6$ since in equation (6) we can see that there are two cases in which a appears first. If a is eliminated second, it implies that another determinant has been eliminated first. Following equation (6) we can see that there exist two possibilities, eliminating first b or c. For (b, a, c), the contribution is $I(a, 0, c) - I(0, 0, c)$ and for (c, a, b), the contribution is $I(a, b, 0) - I(0, b, 0)$. The weights for these two cases are $1/6$, since it appears once. And finally, if a is eliminated third, the other two determinants have already been eliminated. In this case, we have two cases (b, c, a) and (c, b, a), which entails a weight of $2/6$, and the contribution of a is $I(a, 0, 0) - I(0, 0, 0) = I(a, 0, 0)$. Finally we may summarize the three cases in order to obtain the marginal contribution of determinant a, namely $C(a)$. The other two marginal contributions corresponding to determinants b and c, namely $C(b)$ and $C(c)$, can be similarly computed. And finally, it is proved that $I(a, b, c) = C(a) + C(b) + C(c)$ is satisfied.

In this paper we will apply this methodology to decomposable poverty measures in order to obtain the marginal contributions of the three components of poverty to the total poverty value.

As mentioned before, all the poverty measures presented in the paper can be decomposed in the three terms of poverty; incidence, intensity and inequality. Therefore, all the poverty measures can be written as an aggregative function of incidence, H, intensity, M, and inequality among the poor, I, as follows:

$$P = f_P(H, M, I). \tag{7}$$

Then, the Shapley decomposition can be applied to the aggregation function f_P and the determinants H, M and I. The marginal contribution of incidence, intensity and inequality among the poor to the total poverty value P are denoted as, $CH_P = C_P(H)$, $CM_P = C_P(M)$ and $CI_P = C_P(I)$, respectively. Note that we have added the suffix P to each marginal contribution in order to differentiate the marginal contributions associated to different poverty measures.

As mentioned before, there are $3! = 6$ ways of ordering the three determinants. And the marginal contributions of the three determinants are all the possible ways of eliminating them first, second and third.

Following the methodology presented, the three marginal contributions are written as follows:

$$\begin{aligned} CH_P = C_P(H) = &\frac{2}{6}\ \Big(f_P(H, M, I) - f_P(0, M, I)\Big) + \frac{1}{6}\ \Big(f_P(H, 0, I) - f_P(0, 0, I)\Big) + \\ &\frac{1}{6}\ \Big(f_P(H, M, 0) - f_P(0, M, 0)\Big) + \frac{2}{6}\ \Big(f_P(H, 0, 0)\Big), \end{aligned} \tag{8}$$

$$\begin{aligned} CM_P = C_P(M) = &\frac{2}{6}\ \Big(f_P(H, M, I) - f_P(H, 0, I)\Big) + \frac{1}{6}\ \Big(f_P(0, M, I) - f_P(0, 0, I)\Big) + \\ &\frac{1}{6}\ \Big(f_P(H, M, 0) - f_P(H, 0, 0)\Big) + \frac{2}{6}\ \Big(f_P(0, M, 0)\Big), \end{aligned} \tag{9}$$

Table 3
Poverty Contributions of the Rank-dependent poverty measures.

P	RCH$_P$	RCM$_P$	RCI$_P$
PGR	$\frac{1}{2}$	$\frac{1}{2}$	0
S	$\frac{2G+3}{6(G+1)}$	$\frac{2G+3}{6(G+1)}$	$\frac{G}{3(G+1)}$
SST	$\frac{H(2G-3)+6}{6H(G-1)+12}$	$\frac{H(2G-3)+6}{6H(G-1)+12}$	$\frac{GH}{3H(G-1)+6}$
T	$\frac{3+6n-3nH+2nHG}{6+12n+6nH(G-1)}$	$\frac{3+6n-3nH+2nHG}{6+12n+6nH(G-1)}$	$\frac{nHG}{3+6n+3nH(G-1)}$
T_τ	$\frac{H(2G-3)+3\tau}{6H(G-1)+6\tau}$	$\frac{H(2G-3)+3\tau}{6H(G-1)+6\tau}$	$\frac{HG}{3H(G-1)+3\tau}$
T_a	$\frac{H(2G-3)+3}{6H(G-1)+6}$	$\frac{H(2G-3)+3}{6H(G-1)+6}$	$\frac{HG}{3H(G-1)+3}$

$$CI_P = C_P(I) = \frac{2}{6}\ \Big(f_P(H,M,I) - f_P(H,M,0)\Big) + \frac{1}{6}\ \Big(f_P(0,M,I) - f_P(0,M,0)\Big) + \frac{1}{6}\ \Big(f_P(H,0,I) - f_P(H,0,0)\Big) + \frac{2}{6}\ \Big(f_P(0,0,I)\Big). \tag{10}$$

Then, from the Shapley definition, we have that;

$$f_P(H,M,I) = CH_P + CM_P + CI_P. \tag{11}$$

In this paper we will compute the marginal contributions of the three components for different poverty measures. However, in order to be able to make comparisons between indices and to provide rankings for different poverty measures according to their marginal contributions, we will use the relative marginal contributions for each poverty measure.

Therefore, the three relative marginal contributions of the determinants are defined by dividing the marginal contributions by the global function value as follows:

$$RCH_P = \frac{CH_P}{f_P(H,M,I)} \quad ; \quad RCM_P = \frac{CM_P}{f_P(H,M,I)} \quad ; \quad RCI_P = \frac{CI_P}{f_P(H,M,I)}. \tag{12}$$

Then, it is straightforward to see that we have $RCH_P + RCM_P + RCI_P = 1$.

Note that these three values will always be between 0 and 1. Therefore, multiplying by 100, these values could be interpreted as the contribution percentage to the global poverty value assigned to each poverty component. The vector of the relative marginal contributions for every poverty measure P can be written as (RCH_P, RCM_P, RCI_P).

In the following subsection we compute the relative marginal contributions of the three poverty components for all the poverty measures defined in section 2.

3.1. Rank-dependent poverty measures

All the above presented rank-dependent poverty measures can be expressed as an aggregative function of H, M and $I = G$, where G is the Gini index of the gap of the poor individuals as $f_P(H,M,G)$.

Proposition 1. *The relative marginal contributions associated to each component for all the rank-dependent poverty measures presented in the paper can be seen in Table 3.*

Proof. See in the appendix.

It is very interesting to note that if we focus on the relative contributions of the three components, those of incidence and intensity are exactly the same for all the rank-dependent poverty measures.

Table 4
RCH, RCM and RCI for FGT_α family.

P	RCH$_P$	RCM$_P$	RCI$_P$
FGT_α	$\frac{2\alpha^2 G_\alpha - 2\alpha G_\alpha + 3}{6\alpha^2 G_\alpha - 6\alpha G_\alpha + 6}$	$\frac{2\alpha^2 G_\alpha - 2\alpha G_\alpha + 3}{6\alpha^2 G_\alpha - 6\alpha G_\alpha + 6}$	$\frac{(\alpha - 1)\alpha G_\alpha}{3\alpha^2 G_\alpha - 3\alpha G_\alpha + 3}$

Table 5
RCH, RCM and RCI for CHU_α family.

P	RCH$_P$	RCM$_P$	RCI$_P$
CHU_α	$\frac{3 - A_\alpha}{6}$	$\frac{3 - A_\alpha}{6}$	$\frac{A_\alpha}{3}$

3.2. Rank-independent poverty measures

This section is devoted to computing the relative marginal contributions of two families of decomposable rank-independent poverty measures, FGT_α and CHU_α.

We have seen in the previous section that the FGT_α family of poverty measures can be rewritten as an aggregate measure of H, M, and $I = G_\alpha$, that is $f_{FGT_\alpha}(H, M, G_\alpha)$, where G_α is the Generalized Entropy family measured for the gaps of the poor.

Proposition 2. *The relative marginal contributions associated to the FGT_α family are shown in Table 4.*

Proof. See in the appendix.

With respect to the CHU_α family, it can also be rewritten as an aggregative function $f_{CHU_\alpha}(H, M, I = A_\alpha)$ where the inequality measure of the poor individuals is measured by the extended Atkinson family of inequality index.

Proposition 3. *The corresponding relative marginal contributions associated to the CHU_α family are shown in Table 5.*

Proof. See in the appendix.

We will now try to give a ranking of the poverty measures depending on these three aspects.

4. Poverty measures rankings in terms of incidence, intensity and inequality

In this section we will rank the poverty measures in terms of their components' relative marginal contributions. In fact, these rankings will allow us to choose the most appropriate measures depending on the study we want to carry out. In fact, we will be able to know to what extent the choice of one index or another will influence the results.

For this purpose we introduce some definitions.

Definition 2. A poverty measure P will have a higher (not lower) *Headcount ratio* contribution than a poverty measure P' if $RCH_P > (\geq) RCH_{P'}$ and will be denoted as $P \succ_H (\succeq_H) P'$.

Definition 3. A poverty measure P will have a lower (not higher) *Headcount ratio* contribution than a poverty measure P' if $RCH_P < (\leq) RCH_{P'}$ and will be denoted as $P \prec_H (\preceq_H) P'$.

Analogously, the definitions for the income gap ratio and the inequality among the poor.

Definition 4. A poverty measure P will have a higher (not lower) *Income gap ratio* contribution than a poverty measure P' if $RCM_P > (\geq) RCM_{P'}$ and will be denoted as $P \succ_M (\succeq_M) P'$.

Definition 5. A poverty measure P will have a lower (not higher) *Income gap ratio* contribution than a poverty measure P' if $RCM_P < (\leq) RCM_{P'}$ and will be denoted as $P \prec_M (\preceq_M) P'$.

Definition 6. A poverty measure P will have a higher (not lower) *Inequality* contribution, measured by the inequality index I, than a poverty measure P' if $RCI_P > (\geq) RCI_{P'}$ and will be denoted as $P \succ_I (\succeq_I) P'$.

Definition 7. A poverty measure P will have a lower (not higher) *Inequality* contribution, measured by the inequality index I, than a poverty measure P' if $RCI_P < (\leq) RCI_{P'}$ and will be denoted as $P \prec_I (\preceq_I) P'$.

4.1. Rank-dependent poverty measures

Once the contributions of the components to all the poverty indices have been computed, we will provide three rankings for the rank-dependent poverty measures according to the incidence, intensity and inequality contributions to the measure.

The following proposition shows the rank-dependent poverty measures previously presented ranked in terms of the *Headcount Ratio* contribution.

Proposition 4. *The poverty measures of S, SST, T and Ta can be ordered in terms of the Headcount Ratio's relative marginal contribution to the final poverty value as follows:*

$$\begin{aligned} &S \preceq_H Ta \preceq_H SST = T_2 \preceq_H T \preceq_H PGR\ , \quad if \quad H < \frac{1}{2}, \\ &Ta \preceq_H S \preceq_H SST = T_2 \preceq_H T \preceq_H PGR\ , \quad if \quad H > \frac{1}{2}, \\ &Ta \equiv_H S \preceq_H SST = T_2 \preceq_H T \preceq_H PGR\ , \quad if \quad H = \frac{1}{2}. \end{aligned} \tag{13}$$

Proof. See in the appendix.

The next proposition gives us a ranking for the family of T_τ indices:

Proposition 5. *Taking into account that $T_2 = SST$ and $T_{2+\frac{1}{n}} = T$ are satisfied, the T_τ family can be ranked in terms of the Headcount Ratio's relative marginal contribution for different values of τ as follows;*

$$SST = T_2 \preceq_H T_t \preceq_H T = T_{2+\frac{1}{n}} \preceq_H T_r \preceq_H T_s\ , \quad 2 < t < 2 + \frac{1}{n} \quad and \quad 2 + \frac{1}{n} < r < s. \tag{14}$$

Proof. See in the appendix.

Secondly, we will rank the poverty measures in terms of their sensitivity to the intensity of poverty. As mentioned, these orderings will be exactly the same as the ones for incidence.

Proposition 6. *The poverty measures of S, SST, T and Ta can be ordered in terms of the Income Gap Ratio's relative marginal contribution to the final poverty value as follows:*

$$\begin{aligned} &S \preceq_M Ta \preceq_M SST = T_2 \preceq_M T \preceq_M PGR\ , \quad if \quad H < \frac{1}{2}, \\ &Ta \preceq_M S \preceq_M SST = T_2 \preceq_M T \preceq_M PGR\ , \quad if \quad H > \frac{1}{2}, \\ &Ta \equiv_M S \preceq_M SST = T_2 \preceq_M T \preceq_M PGR\ , \quad if \quad H = \frac{1}{2}. \end{aligned} \tag{15}$$

Proof. It is analogous to Proposition 4. □

The next proposition gives us a ranking for the family of T_τ indices:

Proposition 7. *Taking into account that* $T_2 = SST$ *and* $T_{2+\frac{1}{n}} = T$ *are satisfied, the* T_τ *family can be ranked in terms of the Income Gap Ratio's relative marginal contribution for different values of* τ*:*

$$SST = T_2 \preceq_M T_t \preceq_M T = T_{2+\frac{1}{n}} \preceq_M T_r \preceq_M T_s\ , \quad 2 < t < 2 + \frac{1}{n} \quad and \quad 2 + \frac{1}{n} < r < s. \tag{16}$$

Proof. It is analogous to Proposition 2. □

Finally, we will be able to rank the poverty measures in terms of their sensitivity to the inequality among the poor individuals.

Proposition 8. *The poverty measures of* S*,* SST*,* T *and* Ta *can be ordered in terms of the inequality's relative marginal contribution to the final poverty value as follows:*

$$\begin{aligned} &PGR \preceq_G T \preceq_G SST = T_2 \preceq_G S \preceq_G Ta\ , \quad if \quad H < \frac{1}{2}, \\ &PGR \preceq_G T \preceq_G SST = T_2 \preceq_G Ta \preceq_G S\ , \quad if \quad H > \frac{1}{2}, \\ &PGR \preceq_G T \preceq_G SST = T_2 \preceq_G S \equiv_G Ta\ , \quad if \quad H = \frac{1}{2}. \end{aligned} \tag{17}$$

Proof. See in the appendix.

The next proposition gives us a ranking for the family of T_τ indices:

Proposition 9. *Taking into account that* $T_2 = SST$ *and* $T_{2+\frac{1}{n}} = T$ *are satisfied, the* T_τ *family can be ranked in terms of the inequality's relative marginal contribution for different values of* τ*:*

$$T_s \preceq_G T_r \preceq_G T = T_{2+1/n} \preceq_G T_t \preceq_G SST = T_2\ , \quad 2 < t < 2 + \frac{1}{n} \quad and \quad 2 + \frac{1}{n} < r < s. \tag{18}$$

Proof. See in the appendix.

4.2. Rank-independent poverty measures

The following propositions show the rank-independent poverty measures ranked in terms of their three contributions.

Proposition 10. *The family of poverty measures* FGT_α *can be ordered in terms of the contributions of their three components, for* $\forall \alpha, \beta \in \mathbb{R}$ *and* $2 \leq \alpha < \beta$*:*

$$\begin{aligned} &FGT_\beta \prec_H FGT_\alpha, \\ &FGT_\beta \prec_M FGT_\alpha, \\ &FGT_\beta \succ_{G_\alpha} FGT_\alpha. \end{aligned} \tag{19}$$

Proof. See the appendix.

Proposition 11. *The family of poverty measures* CHU_α *can be ordered in terms of the contributions of their three components, for* $\forall \alpha, \beta \in \mathbb{R}$ *and* $1 \leq \alpha < \beta$*:*

$$\begin{aligned} &CHU_\beta \prec_H CHU_\alpha, \\ &CHU_\beta \prec_M CHU_\alpha, \\ &CHU_\beta \succ_{A_\alpha} CHU_\alpha. \end{aligned} \tag{20}$$

Proof. See in the appendix.

Finally, we will also be able to delimit the three contributions associated to every poverty measure. The following proposition establishes the bounds for the contributions concluding that the contributions of incidence and intensity are equal, and always higher than that of inequality.

Proposition 12. *Given a poverty measure P, the vector of contributions (RCH_P, RCM_P, RCI_P) is always delimited as follows:*

$$(RCH_P, RCM_P, RCI_P) \in \left[\frac{1}{3}, \frac{1}{2}\right] \times \left[\frac{1}{3}, \frac{1}{2}\right] \times \left[0, \frac{1}{3}\right]. \tag{21}$$

Proof. See in the appendix.

5. Concluding remarks

We have computed the relative marginal contributions of the three components of poverty; incidence, intensity and inequality, to the total poverty value.

Indeed, we believe that knowing these contributions will help us to choose the most appropriate poverty measure when measuring poverty. In the literature there are numerous poverty families that depend on a parameter whose value varies according to the sensitivity of the measures to transfers among the poorest, i.e. the inequality of the poor.

With these rankings, we manage to rank families according to inequality's relative marginal contribution to poverty, and also to rank them according to the relative marginal contributions of incidence and intensity.

We would like to note that interestingly, the marginal contributions of incidence and intensity are identical for all measures. Moreover, not only have we ranked several families, but we have also managed to rank all the rank-dependent measures.

Declaration of competing interest

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Appendix A

Proof of Proposition 1. We need to compute equations (8), (9) and (10) for all the indices.

- For $PGR = HM$:
  $CH_{PGR} = \frac{2}{6}(HM - 0) + \frac{1}{6}(0 - 0) + \frac{1}{6}(HM - 0) + \frac{2}{6}(0) = \frac{1}{2}HM,$
  $CM_{PGR} = \frac{2}{6}(HM - 0) + \frac{1}{6}(0 - 0) + \frac{1}{6}(HM - 0) + \frac{2}{6}(0) = \frac{1}{2}HM,$
  $CI_{PGR} = \frac{2}{6}(HM - HM) + \frac{1}{6}(0 - 0) + \frac{1}{6}(0 - 0) + \frac{2}{6}(0) = 0.$
  Hence, the relative ones are $(RCH_{PGR}, RCM_{PGR}, RCI_{PGR}) = (1/2, 1/2, 0)$.

- For $S = HM(1+G)$:

$CH_S = \frac{2}{6}(HM(1+G)-0) + \frac{1}{6}(0-0) + \frac{1}{6}(HM-0) + \frac{2}{6}(0) = \frac{1}{6}HM(3+2G)$,

$CM_S = \frac{2}{6}(HM(1+G)-0) + \frac{1}{6}(0-0) + \frac{1}{6}(HM-0) + \frac{2}{6}(0) = \frac{1}{6}HM(3+2G)$,

$CI_S = \frac{2}{6}(HM(1+G)-HM) + \frac{1}{6}(0-0) + \frac{1}{6}(0-0) + \frac{2}{6}(0) = \frac{2}{6}HMG$.

Hence, dividing the three values by S and operating, we obtain the relative ones.

$$(RCH_S, RCM_S, RCI_S) = \left(\frac{3+2G}{6(1+G)}, \frac{3+2G}{6(1+G)}, \frac{G}{3(1+G)}\right).$$

- For $SST = HM(H(G-1)+2)$:

$CH_{SST} = \frac{2}{6}(HM(H(G-1)+2)-0) + \frac{1}{6}(0-0) + \frac{1}{6}(HM(2-H)-0) + \frac{2}{6}(0) = \frac{HM}{6}(H(2G-3)+6)$,

$CM_{SST} = \frac{2}{6}(HM(H(G-1)+2)-0) + \frac{1}{6}(0-0) + \frac{1}{6}(HM(2-H)-0) + \frac{2}{6}(0) = \frac{HM}{6}(H(2G-3)+6)$,

$CI_{SST} = \frac{2}{6}(HM(H(G-1)+2)-HM(2-H)) + \frac{1}{6}(0-0) + \frac{1}{6}(0-0) + \frac{2}{6}(0) = \frac{2}{6}HMG$.

Hence, dividing the three values by SST and operating, we obtain the relative ones.

$$(RCH_{SST}, RCM_{SST}, RCI_{SST}) = \left(\frac{H(2G-3)+6}{6H(G-1)+12}, \frac{H(2G-3)+6}{6H(G-1)+12}, \frac{G}{3H(G-1)+6}\right).$$

- For $T = \frac{HM(2n+1+nH(G-1))}{n+1}$:

$$CH_T = \frac{2}{6}\left(\frac{HM(2n+1+nH(G-1))}{n+1} - 0\right) + \frac{1}{6}(0-0) + \frac{1}{6}\left(\frac{HM(2n+1-nH)}{n+1} - 0\right) + \frac{2}{6}(0) = \frac{HM(3+6n-3nH+2nHG)}{6(n+1)},$$

$$CM_T = \frac{2}{6}\left(\frac{HM(2n+1+nH(G-1))}{n+1} - 0\right) + \frac{1}{6}(0-0) + \frac{1}{6}\left(\frac{HM(2n+1-nH)}{n+1} - 0\right) + \frac{2}{6}(0) = \frac{HM(3+6n-3nH+2nHG)}{6(n+1)},$$

$$CI_T = \frac{2}{6}\left(\frac{HM(2n+1+nH(G-1))}{n+1} - \frac{HM(2n+1-nH)}{n+1}\right) + \frac{1}{6}(0-0) + \frac{1}{6}(0-0) + \frac{2}{6}(0) = \frac{nH^2MG}{3(n+1)}.$$

Hence, dividing the three values by T and operating, we obtain the relative ones.

$$(RCH_T, RCM_T, RCI_T) = \left(\frac{3+6n-3nH+2nHG}{6+12n+6nH(G-1)}, \frac{3+6n-3nH+2nHG}{6+12n+6nH(G-1)}, \frac{nHG}{3+6n+3nH(G-1)}\right).$$

- For $T\tau = \frac{HM(\tau-H+HG)}{\tau-1}$:

$$CH_{T\tau} = \frac{2}{6}\left(\frac{HM(\tau-H+HG)}{\tau-1} - 0\right) + \frac{1}{6}(0-0) + \frac{1}{6}\left(\frac{HM(\tau-H)}{\tau-1} - 0\right) + \frac{2}{6}(0) = \frac{HM(3(\tau-H)+2HG)}{6(\tau-1)},$$

$$CM_{T\tau} = \frac{2}{6}\left(\frac{HM(\tau-H+HG)}{\tau-1} - 0\right) + \frac{1}{6}(0-0) + \frac{1}{6}\left(\frac{HM(\tau-H)}{\tau-1} - 0\right) + \frac{2}{6}(0) = \frac{HM(3(\tau-H)+2HG)}{6(\tau-1)},$$

$$CI_{T\tau} = \frac{2}{6}\left(\frac{HM(\tau - H + HG)}{\tau - 1} - \frac{HM(\tau - H)}{\tau - 1}\right) + \frac{1}{6}(0-0) + \frac{1}{6}(0-0) + \frac{2}{6}(0) = \frac{H^2MG}{3(\tau - 1)}.$$

Hence, dividing the three values by $T\tau$ and operating, we obtain the relative ones.

$$(RCH_{T\tau}, RCM_{T\tau}, RCI_{T\tau}) = \left(\frac{H(2G-3)+3\tau}{6H(G-1)+6\tau}, \frac{H(2G-3)+3\tau}{6H(G-1)+6\tau}, \frac{HG}{3(G-1)H+3\tau}\right).$$

- For $Ta = \frac{HM(1-H+HG)}{1-HM}$:

$$CH_{Ta} = \frac{2}{6}\left(\frac{HM(1-H+HG)}{1-HM} - 0\right) + \frac{1}{6}(0-0) + \frac{1}{6}\left(\frac{HM(1-H)}{1-HM} - 0\right) + \frac{2}{6}(0) =$$
$$= \frac{HM(3(1-H)+2HG)}{6(1-HM)},$$
$$CM_{Ta} = \frac{2}{6}\left(\frac{HM(1-H+HG)}{1-HM} - 0\right) + \frac{1}{6}(0-0) + \frac{1}{6}\left(\frac{HM(1-H)}{1-HM} - 0\right) + \frac{2}{6}(0) =$$
$$= \frac{HM(3(1-H)+2HG)}{6(1-HM)},$$
$$CI_{Ta} = \frac{2}{6}\left(\frac{HM(1-H+HG)}{1-HM} - \frac{HM(1-H)}{1-HM}\right) + \frac{1}{6}(0-0) + \frac{1}{6}(0-0) + \frac{2}{6}(0) = \frac{H^2MG}{3(1-HM)}.$$

Hence, dividing the three values by Ta and operating, we obtain the relative ones.

$$(RCH_{Ta}, RCM_{Ta}, RCI_{Ta}) = \left(\frac{H(2G-3)+3}{6H(G-1)+6}, \frac{H(2G-3)+3}{6H(G-1)+6}, \frac{HG}{3H(G-1)+3}\right). \quad \square$$

Proof of Proposition 2. We need to compute equations (8), (9) and (10).

We have that, $FGT_\alpha = HM^\alpha(1+(\alpha^2-\alpha)G_\alpha)$:

$$CH_{FGT_\alpha} = \frac{2}{6}\Big(HM^\alpha(1+(\alpha^2-\alpha)G_\alpha) - 0\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{1}{6}\Big(HM^\alpha - 0\Big) + \frac{2}{6}\Big(0\Big) = \frac{HM^\alpha(3+2(\alpha^2-\alpha)G_\alpha)}{6},$$
$$CM_{FGT_\alpha} = \frac{2}{6}\Big(HM^\alpha(1+(\alpha^2-\alpha)G_\alpha) - 0\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{1}{6}\Big(HM^\alpha - 0\Big) + \frac{2}{6}\Big(0\Big) = \frac{HM^\alpha(3+2(\alpha^2-\alpha)G_\alpha)}{6},$$
$$CI_{FGT_\alpha} = \frac{2}{6}\Big(HM^\alpha(1+(\alpha^2-\alpha)G_\alpha) - HM^\alpha\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{2}{6}\Big(0\Big) = \frac{(\alpha^2-\alpha)HM^\alpha G_\alpha}{3}.$$

Hence, dividing the three values by FGT_α and operating, we obtain the relative ones.

$$(RCH_{FGT\alpha}, RCM_{FGT\alpha}, RCI_{FGT\alpha}) = \left(\frac{2(\alpha^2-\alpha)G_\alpha+3}{6(\alpha^2-\alpha)G_\alpha+6}, \frac{2(\alpha^2-\alpha)G_\alpha+3}{6(\alpha^2-\alpha)G_\alpha+6}, \frac{(\alpha^2-\alpha)G_\alpha}{3(\alpha^2-\alpha)G_\alpha+3}\right). \quad \square$$

Proof of Proposition 3. We need to compute equations (8), (9) and (10).

We have that, $CHU_\alpha = \frac{HM}{1-A_\alpha}$:

$$CH_{CHU_\alpha} = \frac{2}{6}\Big(\frac{HM}{1-A_\alpha} - 0\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{1}{6}\Big(HM - 0\Big) + \frac{2}{6}\Big(0\Big) = \frac{HM(A_\alpha - 3)}{6(1-A_\alpha)},$$
$$CM_{CHU_\alpha} = \frac{2}{6}\Big(\frac{HM}{1-A_\alpha} - 0\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{1}{6}\Big(HM - 0\Big) + \frac{2}{6}\Big(0\Big) = \frac{HM(A_\alpha - 3)}{6(1-A_\alpha)},$$
$$CI_{CHU_\alpha} = \frac{2}{6}\Big(\frac{HM}{1-A_\alpha} - HM\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{1}{6}\Big(0-0\Big) + \frac{2}{6}\Big(0\Big) = \frac{HMA_\alpha}{3(1-A_\alpha)}.$$

Hence, dividing the three values by CHU_α and operating, we obtain the relative ones.

$$(RCH_{CHU\alpha}, RCM_{CHU\alpha}, RCI_{CHU\alpha}) = \left(\frac{3-A_\alpha}{6}, \frac{3-A_\alpha}{6}, \frac{A_\alpha}{3}\right). \quad \square$$

Proof of Proposition 4. We need to prove some statements.

- $S \preceq_H Ta$ for $H < \frac{1}{2}$, $Ta \preceq_H S$ for $H > \frac{1}{2}$ and $Ta \equiv_H S$ for $H = \frac{1}{2}$.
We need to compute,

$$RCH_S - RCH_{T_a} = \frac{2G+3}{6G+6} - \frac{(2G-3)H+3}{6H(G-1)+6}.$$

Equivalently and operating,

$$RCH_S - RCH_{T_a} = \frac{(6G(2H-1))}{(6G+6)(6H(G-1)+6)}.$$

Since $0 \leq H \leq 1$ and $0 \leq G \leq 1$, then $6G+6 \geq 0$, $6H(G-1)+6 \geq 0$ and then $6G(2H-1) \geq 0$ if $H \geq \frac{1}{2}$ and $6G(2H-1) \leq 0$ if $H \leq \frac{1}{2}$.

- $S \preceq_H SST$
We need to prove that,

$$RCH_S - RCH_{SST} = \frac{2G+3}{6G+6} - \frac{2GH-3H+6}{6H(G-1)+12} \leq 0.$$

Equivalently and operating,

$$RCH_S - RCH_{SST} = \frac{12G(H-1)}{(6G+6)(6H(G-1)+12)} \leq 0.$$

Since $0 \leq H \leq 1$ and $0 \leq G \leq 1$ are satisfied, then $6G+6 \geq 0$, $6H(G-1)+12 \geq 0$ and $12G(H-1) \leq 0$. Hence, it is proved.

- $Ta \preceq_H SST$
We need to prove that,

$$RCH_{Ta} - RCH_{SST} = \frac{(2G-3)H+3}{6(G-1)H+6} - \frac{2GH-3H+6}{6H(G-1)+12} \leq 0.$$

Equivalently and operating,

$$RCH_{Ta} - RCH_{SST} = \frac{-6GH}{(6(G-1)H+6)(6H(G-1)+12)} \leq 0.$$

Since $0 \leq H \leq 1$ and $0 \leq G \leq 1$ are satisfied, then $6(G-1)H+6 \geq 0$, $6H(G-1)+12 \geq 0$ and $-6GH \leq 0$. Hence, it is proved.

- $SST \preceq_H T$
We need to prove that,

$$RCH_{SST} - RCH_T = \frac{H(2G-3)+6}{6(G-1)H+12)} - \frac{nH(2G-3)+6n+3}{6(Hn(G-1)+2n+1)}.$$

Equivalently and operating,

$$RCH_{SST} - RCH_T = -\frac{HG}{6((G-1)H+2)((G-1)nH+2n+1)} \leq 0.$$

Since $n > 1$, $0 \leq H \leq 1$ and $0 \leq G \leq 1$ are satisfied, then $-HG \leq 0$, $(G-1)H+2 \geq 0$ and $(G-1)nH+2n+1 \geq 0$. Hence, it is proved.

- $T \preceq_H PGR$
We need to prove that,

$$RCH_T - RCH_{PGR} = \frac{nH(2G-3)+6n+3}{6(Hn(G-1)+2n+1)} - \frac{1}{2}.$$

Equivalently and operating,

$$RCH_T - RCH_{PGR} = -\frac{HGn}{6((G-1)Hn+2n+1)} \leq 0.$$

Since $n > 1$, $0 \leq H \leq 1$ and $0 \leq G \leq 1$ are satisfied, then $-HGn \leq 0$ and $(G-1)Hn + 2n + 1 \leq 0$. Hence, it is proved. □

Proof of Proposition 5. We need to prove $T_r \preceq_H T_s$, for $2 < r < s$. Hence, we may prove that

$$RCH_{T_r} - RCH_{T_s} = \frac{(2G-3)H+3r}{6((G-1)H+r)} - \frac{(2G-3)H+3s}{6((G-1)H+s)} \leq 0 \quad \text{, for } 2 < r < s.$$

Equivalently and operating

$$RCH_{T_r} - RCH_{T_s} = \frac{GH(r-s)}{6^2((G-1)H+r)((G-1)H+s)} \leq 0 \quad \text{, for } 2 < r < s.$$

Since $0 \leq H \leq 1$ and $0 \leq G \leq 1$ are satisfied, then $(G-1)H + r, (G-1)H + s \geq 0$ and we have prof the statement. □

Proof of Proposition 8. We need to prove some statements.

- $PGR \preceq_G T$
  It is true since $RCI_T > 0$ and $RCI_{PGR} = 0$.
- $T \preceq_G SST$
  We need to prove,

$$RCI_T - RCI_{SST} = \frac{nHG}{3+6n+3nH(G-1)} - \frac{HG}{3(H(G-1)+2)} \leq 0.$$

  Equivalently and operating,

$$RCI_T - RCI_{SST} = -\frac{HG}{(3+6n+3nH(G-1))(H(G-1)+2)} \leq 0.$$

  That is true since $q \geq 2$, $0 \leq H \leq 1$ and $0 \leq G \leq 1$.
- $SST \preceq_G S$
  We need to prove,

$$RCI_{SST} - RCI_S = \frac{GH}{3(G-1)H+6} - \frac{G}{3G+3} \leq 0.$$

  Equivalently and operating,

$$RCI_{SST} - RCI_S = \frac{2G(H-1)}{3(G+1)(GH-H+2)} \leq 0.$$

  That is true since $0 \leq H \leq 1$ and $0 \leq G \leq 1$.
- $SST \preceq_G Ta$
  We compute,

$$RCI_{SST} - RCI_{Ta} = \frac{GH}{3(G-1)H+6} - \frac{GH}{3(G-1)H+3} \leq 0.$$

  Equivalently and operating,

$$RCI_{SST} - RCI_{Ta} = -\frac{GH}{3(GH-H+1)(GH-H+2)} \leq 0.$$

  That is true since $0 \leq H \leq 1$ and $0 \leq G \leq 1$.

- $S \preceq_G Ta$ if $H \geq \frac{1}{2}$ and $Ta \preceq_G S$ if $H \leq \frac{1}{2}$.
  We compute,
  $$RCI_S - RCI_{Ta} = \frac{G}{3G+3} - \frac{GH}{3(G-1)H+3}.$$
  Equivalently and operating,
  $$RCI_S - RCI_{Ta} = -\frac{G(2H-1)}{3(G+1)(GH-H+1)}.$$
  Since $0 \leq H \leq 1$ and $0 \leq G \leq 1$ it is proved. □

Proof of Proposition 9. We need to prove $T_s \preceq_H T_r$, for $2 < r < s$. Hence, we may prove that,
$$RCH_{T_s} - RCH_{T_r} = \frac{HM((s-H)+HG)}{s-1} - \frac{HM((r-H)+HG)}{r-1} \leq 0 \quad \text{, for } \ 2 < r < s.$$
Equivalently and operating,
$$RCH_{T_s} - RCH_{T_r} = \frac{(r-s)(1-H+HG)}{(s-1)(r-1)} \leq 0 \quad \text{, for } \ 2 < r < s.$$
Since $0 \leq H \leq 1$ and $0 \leq G \leq 1$ are satisfied, then $1 - H + HG \geq 0$. Hence, we have prof the statement. □

Proof of Proposition 10. For $\forall \alpha, \beta \in \mathbb{R}$ and $2 \leq \alpha < \beta$, we have,
$RCH_{FGT_\alpha} = RCM_{FGT_\alpha} = \frac{2\alpha^2 G_\alpha - 2\alpha G_\alpha + 3}{6\alpha^2 G_\alpha - 6\alpha G_\alpha + 6}$ and $RCI_{FGT_\alpha} = \frac{(\alpha-1)\alpha G_\alpha}{3\alpha^2 G_\alpha - 3\alpha G_\alpha + 3}$.
We need to see that,
$$RCH_{FGT_\beta} - RCH_{FGT_\alpha} < 0 \Big(\text{analogously } RCM_{FGT_\beta} - RCM_{FGT_\alpha} < 0 \Big).$$
Operating we have,
$$\begin{aligned} RCH_{FGT_\beta} - RCH_{FGT_\alpha} &= \frac{2(\beta^2-\beta)G_\beta + 3}{6(\beta^2-\beta)G_\beta + 6} - \frac{2(\alpha^2-\alpha)G_\alpha + 3}{6(\alpha^2-\alpha)G_\alpha + 6} \\ &= \frac{(\alpha^2-\alpha)G_\alpha - (\beta^2-\beta)G_\beta}{6((\alpha^2-\alpha)G_\alpha + 1)((\beta^2-\beta)G_\beta + 1)}. \end{aligned}$$
Equivalently, we need to prove that,
$$(\alpha^2-\alpha)G_\alpha - (\beta^2-\beta)G_\beta = \sum_{i=1}^{q} \left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\alpha} - \sum_{i=1}^{q} \left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\beta} < 0.$$
With respect to the inequality relative marginal contributions:
$$RCI_{FGT_\beta} - RCI_{FGT_\alpha} = \frac{(\beta^2-\beta)G_\beta}{3(\beta^2-\beta)G_\beta + 3} - \frac{(\alpha^2-\alpha)G_\alpha}{3(\alpha^2-\alpha)G_\alpha + 3} = -\frac{(\alpha^2-\alpha)G_\alpha - (\beta^2-\beta)G_\beta}{3((\alpha^2-\alpha)G_\alpha + 1)((\beta^2-\beta)G_\beta + 1)}.$$
Hence, for $RCI_{FGT_\beta} - RCI_{FGT_\alpha} > 0$ we need to prove the same statement
$$(\alpha^2-\alpha)G_\alpha - (\beta^2-\beta)G_\beta = \sum_{i=1}^{q} \left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\alpha} - \sum_{i=1}^{q} \left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\beta} < 0.$$
Therefore, we only need to prove that $\sum_{i=1}^{q} \left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\beta} > \sum_{i=1}^{q} \left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\alpha}$ for all $\forall \alpha, \beta \in \mathbb{R}$ for $2 < \alpha < \beta$.

For this purpose, we will see that $f(a) = \sum_{i=1}^{q} \left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{a}$ is an increasing function in $\forall a \in \mathbb{R}$ and $a \geq 1$.

The derivative of the function with respect to a is:

$$f'(a)=\sum_{i=1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{a}\ln\left(\frac{g_{[i]}}{\mu_{g_p}}\right).$$

It will be proved by induction:
For $a=1$, we have,

$$f'(1)=\sum_{i=1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)\ln\left(\frac{g_{[i]}}{\mu_{g_p}}\right).$$

We have obtained the Theil index which is always higher than 0.
Now, suppose that it is true for a:

$$\sum_{i=1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{a}\ln\left(\frac{g_{[i]}}{\mu_{g_p}}\right)>0.$$

Then, we will see that is true for $a+1$:
Suppose that $g_{[1]}\ge g_{[2]}\ge\cdots\ge g_{[s]}\ge\mu_{g_p}\ge g_{[s+1]}\ge\cdots\ge g_{[q]}$. Then, we have

$$\sum_{i=1}^{s}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{a}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)\ln\left(\frac{g_{[i]}}{\mu_{g_p}}\right)+\sum_{i=s+1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{a}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)\ln\left(\frac{g_{[i]}}{\mu_{g_p}}\right).$$

For $g_i\le\mu_{g_p}$, we have

$$\left(\frac{g_i}{\mu_{g_p}}\right)^{a}\left(\frac{g_i}{\mu_{g_p}}\right)\ln\left(\frac{g_i}{\mu_{g_p}}\right)\ge\left(\frac{g_i}{\mu_{g_p}}\right)^{a}\ln\left(\frac{g_i}{\mu_{g_p}}\right),\ \text{since}\ \frac{g_i}{\mu_{g_p}}\le 1\ \text{and}\ \ln\left(\frac{g_i}{\mu_{g_p}}\right)\le 0.$$

And for $g_i\ge\mu_{g_p}$, we have

$$\left(\frac{g_i}{\mu_{g_p}}\right)^{a}\left(\frac{g_i}{\mu_{g_p}}\right)\ln\left(\frac{g_i}{\mu_{g_p}}\right)\ge\left(\frac{g_i}{\mu_{g_p}}\right)^{a}\ln\left(\frac{g_i}{\mu_{g_p}}\right),\ \text{since}\ \frac{g_i}{\mu_{g_p}}\ge 1\ \text{and}\ \ln\left(\frac{g_i}{\mu_{g_p}}\right)\ge 0.$$

Consequently;

$$\sum_{i=1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{a+1}\ln\left(\frac{g_{[i]}}{\mu_{g_p}}\right)>\sum_{i=1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{a}\ln\left(\frac{g_{[i]}}{\mu_{g_p}}\right)>0.\quad\square$$

Proof of Proposition 11. This proposition is satisfied if $A_\beta>A_\alpha$ for $\forall\alpha,\beta\in\mathbb{R}$ and $\beta>\alpha\ge 1$.
We will see that A_α is an increasing function on α.
Suppose we have,

$$f(\alpha)=A_\alpha=A_\alpha(g)=1-\left(\frac{1}{n}\sum_{i=1}^{q}\left(\frac{g_{[i]}}{\mu_{g_p}}\right)^{\alpha}\right)^{-\frac{1}{\alpha}}.$$

The derivative with respect to α:

$$f'(\alpha)=\frac{1}{\alpha}\left(\frac{1}{n}\sum_{i=1}^{q}\left(\frac{g_i}{\mu_{g_p}}\right)^{\alpha}\right)^{-\frac{1}{\alpha}-1}\cdot\left(\frac{1}{n}\sum_{i=1}^{q}\left(\frac{g_i}{\mu_{g_p}}\right)^{\alpha}\ln\left(\frac{g_i}{\mu_{g_p}}\right)\right).$$

The first part is positive and the second one is also positive for $\forall\alpha\in\mathbb{R}$ and $\alpha\ge 1$, since it has been proved in Proposition 10. □

Proof of Proposition 12. For RCH_P; we have that $RCH_P\le RCH_{PGR}=\frac{1}{2}$ for every P presented in the paper. On the other hand, we need to see that $RCH_S\ge\frac{1}{3}$ and $RCH_{T_a}\ge\frac{1}{3}$. Note that $RCH_S=\frac{2G+3}{6(G+1)}>\frac{G}{6(G+1)}=RCI_S$

and $RCH_{T_a}\frac{H(2G-3)+3}{6H(G-1)+6} > \frac{HG}{3H(G-1)+3} = RCI_{Ta}$. Hence, since $RCH_S = RCM_S$ and $RCH_{Ta} = RCM_{Ta}$ and $RCH_P + RCM_P + RCH_P = 1$ is satisfied for every measure, then $RCH_S, RCH_{Ta} \geq \frac{1}{3}$. And then, $\frac{1}{3} \leq RCH_P$ for every rank-dependent poverty measure P. Analogously, we have $\frac{1}{3} \leq RCM_P \leq \frac{1}{2}$ and consequently $0 \leq RCI_P \leq \frac{1}{3}$ for the rank-dependent poverty measures.

Now, we need to prove the same for the FGT_α and CHU_α families. It is very easy to see that $RCH_P = RCM_P > RCI_p$ for the two families. And they are also delimited by the PGR index $RCH_P \leq RCH_{PGR}$. Hence, the proposition is proved. □

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