Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 466 (2023) 108460

FUZZY sets and systems

www.elsevier.com/locate/fss

Short communication

A note on the orness classification of the rank-dependent welfare functions and rank-dependent poverty measures

Oihana Aristondo [a,*], Ainhoa Iñiguez [b]

[a] *BRIDGE Research Group, Departamento de Matemática Aplicada, Universidad del País Vasco, Avenida de Otaola 29, 20600 Eibar (Gipuzkoa), Spain*

[b] *Departamento de Matemática Aplicada, Universidad del País Vasco, Avenida de Otaola 29, 20600 Eibar (Gipuzkoa), Spain*

Received 26 May 2022; received in revised form 20 December 2022; accepted 23 December 2022
Available online 2 January 2023

Abstract

This note focuses on ordering two families of rank-dependent poverty measures in terms of their *distribution-sensitivity*. It has been proved that a real value, between $1/2$ and 1, called *orness*, which is assigned to every rank-dependent poverty measure, can be interpreted as a *distribution-sensitivity* indicator. Therefore, the rank-dependent poverty measures can be classified in terms of their *distribution-sensitivity* using the *orness* value assigned to them. This ranking has already been carried out for numerous poverty measures. However, two families of poverty measures, the Kakwani and the S-Gini families, which are defined for every real parameter larger than one, have only been ranked for natural values of their parameters. This note broadens the classification of these families for every real parameter larger than one, that is, for every member of these two families. It also provides a ranking between the two families for the same parameter. It concludes that for higher values of the parameter, the families will be more sensitive to the bottom part of the distribution. Furthermore, for the same value of the parameter, the Kakwani index will be more sensitive to poor incomes than the S-Gini index. In addition, we will see that the proposed ranking for the two families in terms of the *orness* value will be analogous to other *distribution-sensitivity* criteria existing in the literature.

Keywords: Aggregation functions; OWA operators; Orness; Rank-dependent poverty measure

1. Introduction

In the literature, it is widely accepted that an income increase of a poor individual should decrease poverty, namely the monotonicity axiom. In addition, Kakwani [19] argues that a poverty measure should be more sensitive to what happens among the bottom levels of the distribution and he proposes some sensitivity axioms related with income increments and income transfers. This means that poverty measures should be more sensitive to income increments the lower that income is. With respect to income transfers, it is widely accepted that an income transfer from a better-off poor individual to a worse-off poor one, namely the *transfer principle*, should decrease poverty, see Sen [23]

* Corresponding author.
E-mail addresses: oihana.aristondo@ehu.es (O. Aristondo), ainhoa.iniguez@ehu.eus (A. Iñiguez).

https://doi.org/10.1016/j.fss.2022.12.016
0165-0114/

and [24]. In this case, the degree of *distribution-sensitivity* imposed by the *transfer principle* is minimal, since the distributional improvement produced by the transfer does not involve the loss of the mean income, see Chakravarty [9] and Zheng [30]. Nevertheless, the poverty measures go beyond *transfer principle* and most of them are able to tolerate some sacrifices of the mean income in return for a distributional improvement. Moreover, we want to note that there exist poverty measures that are able to sacrifice higher mean income values than others, for a distributional improvement. That is, the satisfaction of this property will depend on the amount of the mean loss.

There are numerous papers in the literature that offer poverty measures' rankings according to their *distribution-sensitivity* to income increments/decrements, or to different income transfers, see Zheng [31], Bosmans [8], Aristondo and Ciommi [2]. More recent papers analyze the *distribution-sensitivity* of poverty measures using the Shapley method. Datt [12] studies the case of multidimensional poverty measures and Aristondo [1] offers the ranking of many poverty measures not only in terms of their *distribution-sensitivity* but also in terms of their incidence and intensity sensitivity.

In addition, Urrutia and Puerta [22] propose some new transfers that will be more sensitive to high incomes, that is, at the top of the distribution.

Zheng [31] was the first to offer a theoretical method and a ranking for the class of *subgroup-consistent* poverty measures in terms of their *distribution-sensitivity*. Bosmans [8] compares rank-dependent poverty measures in terms of their *distribution-sensitivity* to two transfers called *lossy transfers* and *lossy equalization transfers*, which involve the loss of the mean income as a consequence of distributional improvement. Aristondo and Ciommi [2] expand Bosmans' proposal to welfare functions and they also propose a new ranking criterion based on a mathematical value, called *orness*, assigned to every welfare and poverty measure.

The *orness* value is a numerical value assigned to every ordered weighted averaging, or OWA, operator. The OWA operators were introduced by Yager [26] as a new aggregation technique and in recent years they have received great attention, and have been applied in different fields, such as decision making under uncertainty, fuzzy system, welfare and so on (see Yager and Kreinovich [29], Fodor and Roubens [15], Yager [28], García-Lapresta et al. [16], Aristondo et al. [5] and [6] and Aristondo and Ciommi [3]). The *orness* of an OWA operator was also introduced with the intention of offering a ranking of the OWA operators. This ranking classifies the OWA operators with regard to their location between two extreme situations, the OR and the AND. The OR value is the maximum *orness* value, and it means full compensation among criteria and the last minimum one. The AND means that a higher degree of satisfaction of one of the criteria can compensate for a lower degree of satisfaction of another.

Aristondo and Ciommi [3] show that every rank-dependent poverty measure can be decomposed in terms of an OWA operator, and then, an *orness* value can be assigned. Therefore, they show that all the rank-dependent poverty measures can be classified in terms of their corresponding *orness* value. And following the *orness* definition, they show that the *orness* value will be greater for higher weights applied to smaller income values, that is, the sensitivity of the measures for low incomes would be higher for higher *orness* values. Consequently, Aristondo and Ciommi [3] prove that the *orness* value assigned to every welfare and poverty index can be interpreted as a *distribution-sensitivity* indicator. Additionally, they prove that for some specific welfare functions and poverty measures, those with linear weights, the *orness* classification and the classifications offered by Bosmans [8], in terms of *lossy transfers* and *lossy equalizations*, are equivalent.

Two of the poverty families classified according to Aristondo and Ciommi's [2] *orness* classification are the Kakwani family [19] and the S-Gini family [25]. These two families are defined in terms of real parameters larger than one. However, Aristondo and Ciommi [3] only provide an *orness* classification of these two families for natural values of the parameters. Therefore, the aim of this note is to extend the classification of these two families for all real parameters greater than one, which is precisely the set where these two families are defined. In addition, we also offer a ranking between the two families for the same value of the parameter.

With these orderings we will offer a classification of the two families according to the weights assigned to the bottom of the distribution, that is, the individuals most affected by poverty. This will enable us to choose the most appropriate measure for any empirical work.

The paper is organized as follows. Section 2 introduces aggregation functions, OWA operators and the *orness* value. Section 3 is devoted to poverty measures, and more precisely to rank-dependent poverty measures and the way they can be rewritten as OWA operators. In section 4 different *distribution-sensitivity* criteria are introduced and in section 5 the *orness* classification for the two families is provided. Finally, section 6 offers some concluding remarks.

2. Aggregation functions and OWA operators

In this subsection we begin with a brief summary of the basic notations about aggregation functions and OWA operators.

Consider the $[0,1]^n$ domain with $n \geq 2$. Vectors in $[0,1]^n$ are denoted by $\mathbf{x} = (x_1, \ldots, x_n)$, with $\boldsymbol{1} = (1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for $\forall i \in \{1, \ldots, n\}$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. For $\mathbf{x} \in [0,1]^n$, the non-decreasing and non-increasing form of the vector are denoted as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. And the arithmetic mean of $\boldsymbol{x} \in [0,1]^n$ is denoted by $\mu(\boldsymbol{x}) = (x_1 + \cdots + x_n)/n$.

Then, we define an aggregation function.

Definition 1. A function $A : [0,1]^n \longrightarrow [0,1]$ is called an *n-ary aggregation function* if it is monotonic[1] and $A(\boldsymbol{0}) = 0$, $A(\boldsymbol{1}) = 1$.[2]

An *ordered weighted averaging operator* is a particular case of an aggregation function, hereafter *OWA* operator, introduced by Yager [26].

Definition 2. Given a vector of weights $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ satisfying $\sum\limits_{i=1}^{n} w_i = 1$, the OWA operator associated with $\mathbf{w}$ is the aggregation function $A_w : [0,1]^n \longrightarrow [0,1]$ defined as follows,

$$A_w(\mathbf{x}) = \sum_{i=1}^{n} w_i \, x_{[i]} \ . \tag{1}$$

And every OWA operator has an assigned numerical value called *orness*.

Definition 3. Given an *OWA* operator A_w associated with a system of weights $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$ satisfying $\sum\limits_{i=1}^{n} w_i = 1$, the *orness* of an OWA operator is defined as follows,

$$orness(A_w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i \ . \tag{2}$$

The maximum *orness* value is obtained with the weights $w = (1, 0, \cdots, 0)$, that is $orness(w) = 1$, while the minimum *orness* value is obtained with weights $w = (0, 0, \cdots, 1)$ and gives $orness(w) = 0$. The average, $orness(w) = 1/2$, is obtained with weights $w = (1/n, 1/n, \cdots, 1/n)$. The OWA operators with monotonic weights are either or-like or and-like. Accurately, for non-increasing weights $w_1 \geq w_2 \geq \cdots \geq w_n$ we have or-like OWA operators, while for non-decreasing weights $w_1 \leq w_2 \leq \cdots \leq w_n$ we obtain and-like OWA operators.

3. Poverty measures and rank-dependent poverty measures

Firstly, we present some notations, basic definitions and axioms about poverty measures.

Consider a population of $n \geq 3$ individuals. An income vector distribution is defined as $\boldsymbol{x} = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{R}_{++}$ is the income of the i-th individual and $D = \bigcup_{n \geq 3} \mathbb{R}^n_{++}$ represents the set of all distributions. The poverty line is defined as $z \in \mathbb{R}_{++}$; and an individual $i \in \{1, \ldots, n\}$ is defined as *poor* if $x_i < z$ and as *non-poor* if $x_i \geq z$. We denote $Q = Q(\boldsymbol{x}; z) = \{i \in \{1, \cdots, n\} : x_i < z\}$, and $q = q(\boldsymbol{x}; z)$ the set and the number of poor individuals, respectively, where $n > q \geq 2$. The total distribution mean is defined as $\mu(\boldsymbol{x}) = (x_1 + \cdots + x_n)/n$. With the intention of analyzing the individual shortfall, normalized gaps are defined as $g_i = \max\left\{\frac{z - y_i}{z}, 0\right\}$ and the normalized gap vector is denoted by $\boldsymbol{g} = (g_1, \ldots, g_n)$ which is defined in $[0,1]^n$. Without loss of generality, any $\boldsymbol{x} \in D$ is ordered in

1 A is monotonic if $\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in [0;1]^n$. Given $\mathbf{x}, \mathbf{y} \in D$, by $\mathbf{x} \geq \mathbf{y}$ we mean $x_i \geq y_i$ $\forall i \in \{1, \cdots, n\}$.

2 In what follows, the n-arity is omitted whenever it is clear from the context.

a non-decreasing way; $x_1 \leq \cdots \leq x_n$. Consequently, the normalized gaps of the poor are defined in a non-increasing way; $g_1 \geq \cdots \geq g_q$.

Once the poor individuals are defined, we need to aggregate individual poverty values in order to obtain a global poverty value.

Definition 4. A poverty measure is defined as a non-constant function $P(\boldsymbol{x}; z) : D \times \mathbb{R}_{++} \to [0, 1]$ that measures the poverty level associated with the distribution $\boldsymbol{x}$ and the poverty line z.

A number of axioms are usually assumed for a poverty measure, see other papers [30], [4] and [17].

- *Focus axiom (F):* $P(\boldsymbol{y}; z) = P(\boldsymbol{x}; z)$ whenever $\boldsymbol{y} \in D$ is obtained from $\boldsymbol{x} \in D$ by a change to a non-poor individual that is also non-poor after the change.
- *Replication Invariance axiom (RI):* $P(\boldsymbol{y}; z) = P(\boldsymbol{x}; z)$ whenever $\boldsymbol{y} \in D$ is obtained from $\boldsymbol{x} \in D$ by a k replication, that is $\boldsymbol{y} = \overbrace{(\boldsymbol{x}, ..., \boldsymbol{x})}^{k}$ for some $k \in N$.
- *Symmetry axiom (S):* $P(\boldsymbol{y}; z) = P(\boldsymbol{x}; z)$ whenever $\boldsymbol{y} \in D$ is obtained from $\boldsymbol{x} \in D$ by a permutation.
- *Monotonicity axiom (M):* $P(\boldsymbol{y}; z) < P(\boldsymbol{x}; z)$ whenever $\boldsymbol{y} \in D$ is obtained from $\boldsymbol{x} \in D$ by a simple increment to a poor person.
- *Normalization (N):* $P(x; z) = 0$ iff no one lives in poverty.
- *Weak Transfer axiom (WT):* $P(\boldsymbol{y}; z) < P(\boldsymbol{x}; z)$ $(P(\boldsymbol{y}; z) > P(\boldsymbol{x}; z))$ whenever $\boldsymbol{y} \in D$ is obtained from $\boldsymbol{x} \in D$ by a progressive (regressive) transfer[3] with at least the recipient (donor) being poor with no one crossing the poverty line as a consequence of the transfer.[4]
- *Monotonicity Sensitivity axiom (MS):* $P(\boldsymbol{y}; z) - P(\boldsymbol{x}; z) > P(\boldsymbol{y}'; z) - P(\boldsymbol{x}; z)$ whenever $\boldsymbol{y}, \boldsymbol{y}' \in D$ are obtained from $\boldsymbol{x} \in D$ by the same amount of decrement to poor incomes x_i and x_j, respectively, where $x_i < x_j$.

The first poverty measure introduced in the literature is the *headcount-ratio*, denoted by $H = q/n$, which is the percentage of poor people. It captures exactly the incidence of poverty and satisfies *F*, *RI*, *S* and *N*. However, it violates *M*, *WT* and *MS* since it does not take into account the intensity and the differences between the poor.

If we compute the mean of the normalized gaps with respect to the population, we obtain another well-known measure of poverty, named the *poverty gap ratio*, and defined as

$$PGR = PGR(\boldsymbol{x}; z) = \frac{1}{n} \sum_{i=1}^{q} \frac{z - x_i}{z} = \frac{1}{n} \sum_{i=1}^{q} g_i \ . \tag{3}$$

It captures the incidence and the intensity of poverty and satisfies *F*, *RI*, *S*, *N* and *M*. However, it violates *WT* and *MS* since it does not take into account the inequality among the poor. However, in the literature there exist numerous poverty measures that satisfy the transfer *(WT)* axiom. In this paper we will focus on two families of rank-dependent poverty measures.

Rank-dependent poverty measures are those poverty indices for which individuals' weights depend only on their place in the distribution with respect to the others. The definition is introduced below.

Definition 5. A poverty measure $P(\boldsymbol{x}; z) : D \times \mathbb{R}_{++} \to [0, 1]$ is rank-dependent if for each income distribution $\boldsymbol{x} \in D$ and any fixed poverty line $z \in \mathbb{R}_{++}$, it takes the following expression

$$P(\boldsymbol{x}; z) = \sum_{i=1}^{q} w_i \frac{z - x_i}{z} = \sum_{i=1}^{q} w_i g_i \ , \tag{4}$$

where as mentioned, $g_1 \geq \cdots \geq g_q$ and $x_1 \leq \cdots \leq x_q$. In addition, a poverty measure needs to satisfy $w_1 \geq w_2 \geq \cdots \geq w_q$ and if the weights decrease strictly then the *transfer axiom* (*WT*) is satisfied.

---

3 Progressive (Regressive) transfer: $\boldsymbol{y} \in D$ is obtained from $\boldsymbol{x} \in D$ by a progressive (regressive) transfer if there exists i and j, $i < j$, such that $y_i - x_i = x_j - y_j > 0$ (< 0), $y_j > x_i$ and $y_k = x_k$ for all $k \neq i, j$.

4 There are numerous transfer axioms depending on whether they are poor or not before and after the transfer, see Zheng [30].

The two rank-dependent poverty measures we focus on are the Kakwani family of poverty measures [19], K_k, and the S-Gini class of poverty measures, G_σ; see Kakwani [19], Donaldson and Weymark [13] and Chakravarty's [10]. The two families are defined as follows:

$$K_k(\boldsymbol{x}; z) = \sum_{i=1}^{q} \left[\frac{q(q+1-i)^k}{n \sum_{i=1}^{q} i^k} \right] g_i \quad , \quad k \geq 1 \quad , \quad k \in \mathbb{R} \; . \tag{5}$$

$$G_\sigma(\boldsymbol{x}; z) = \sum_{i=1}^{q} \left[\left(\frac{n+1-i}{n} \right)^\sigma - \left(\frac{n-i}{n} \right)^\sigma \right] g_i \quad , \quad \sigma \geq 1 \quad , \quad \sigma \in \mathbb{R} \; . \tag{6}$$

These two families satisfy F, S, M, N and MS; and WT is satisfied for every *Kakwani* index and for every S-Gini index for $\alpha > 1$. In the literature, it is well known that the parameters k and σ are directly related with the measure's sensitivity to income transfers at different income positions. That is, for larger values of k and σ, the measures are more sensitive for transfers at the bottom of the distribution. In fact, in the literature, the two parameters are considered as the measures' poverty aversion indicators.

Now, if we pay attention to the previous section, we can see that the definition of OWA operators and the rank-dependent poverty measures are very close. In general, rank-dependent poverty measures are not *OWA* operators, since they do not fulfill $\sum_{i=1}^{q} w_i = 1$. However, every rank-dependent poverty measure can be normalized and rewritten as the product of a normalization factor, invariant to transfers, and its normalized poverty measure, which will be an OWA operator; see Aristondo and Ciommi [3]. In what follows, we add the prefix N to the name of each rank-dependent poverty index in order to refer to the normalized rank-dependent poverty measure.

Therefore, we rewrite the K_k and the G_σ measures as the product of a normalization factor and their normalized poverty index NK_k and NG_σ:

$$\begin{aligned} K_k(\boldsymbol{x}; z) &= H \cdot \sum_{i=1}^{q} \left[\frac{(q+1-i)^k}{\sum_{i=1}^{q} i^k} \right] g_i \\ &= H \cdot NK_k(\boldsymbol{x}; z) \quad , \quad 1 \leq k \in \mathbb{R} \; . \end{aligned} \tag{7}$$

$$\begin{aligned} G_\sigma(\boldsymbol{x}; z) &= \Big(1 - (1-H)^\sigma\Big) \cdot \sum_{i=1}^{q} \left[\frac{(n+1-i)^\sigma - (n-i)^\sigma}{n^\sigma - (n-q)^\sigma} \right] g_i \\ &= \Big(1 - (1-H)^\sigma\Big) \cdot NG_\sigma(\boldsymbol{x}; z) \quad , \quad 1 \leq \sigma \in \mathbb{R} \; , \end{aligned} \tag{8}$$

where $H = q/n$ is the headcount ratio. The proof of these two statements, (7) and (8), can be seen in Aristondo and Ciommi [3].

Now the *orness* values of the two families, the Kakwani and the S-Gini families, can be computed. For more information see Aristondo and Ciommi [3].

$$orness(NK_k) = \frac{1}{(q-1)\sum_{i=1}^{q} i^k} \sum_{i=1}^{q} \left(i^{k+1} - i^k \right) \quad , \quad 1 \leq k \in \mathbb{R} \; . \tag{9}$$

$$orness(NG_\sigma) = \frac{1}{(q-1)(n^\sigma - (n-q)^\sigma)} \sum_{i=1}^{q} \Big((n+1-i)^\sigma - (n-i)^\sigma \Big)(q-i) \quad , \quad 1 \leq \sigma \in \mathbb{R} \; . \tag{10}$$

From the definition of the rank-dependent poverty measures we know that the weights are ordered in a non-decreasing way. Consequently, the weights of the corresponding OWA operator will also be ordered in the same way. Following OWA literature, see Yager [27], the OWA operators with weights ordered in a non-decreasing way are named or-like operators and those with weights ordered in a non-increasing way, and-like. Liu and Lou [20] show that the *orness* value for the or-like operators are always between $1/2$ and 1, and between 0 and $1/2$ for the and-like ones.

Hence, for the non-dressiness of the rank-dependent poverty measures' weights, our orness values will be always between $1/2$ and 1.

4. Distribution-sensitivity criteria using the orness values

In this section we will concentrate on the poverty measures' classification in terms of their *distribution-sensitivity*. As mentioned before, the two poverty families presented in the paper satisfy the *monotonicity sensitivity axiom*. This axiom states that a poverty measure should be more sensitive to income decrements/increments in a poor person's income, the poorer the person is.

By *orness* definition, we know that the *orness* value is greater for higher weights at the top of the normalized gap distribution. That is, the greater the *orness* value is, the higher the weights applied to small incomes are. Hence, this sensitivity at the bottom of the income distribution can be interpreted as a *distribution-sensitivity* measure. Therefore, poverty measures could be classified in terms of their *orness* value. In fact, the concept of *orness* is defined as a measure of optimism that lies within the unit interval and between $1/2$ and 1 for rank-dependent poverty measures. This numerical value indicates how close the measure of poverty is to the maximum operator (OR) or the minimum operator (AND). The maximum *orness* value, or OR value, is obtained with the weights $w = (1, 0, \cdots, 0)$ which gets $orness(W) = 1$ and it is exactly the relative gap of the poorest individual, that is $W = g_1$. While the minimum *orness* value in the poverty field is $w' = (1/n \cdots, 1/n)$. For these weights we have $orness(W') = 1/2$ and the measures obtained is the poverty gap ratio (PGR). Note that the W will only be affected by transfers of increments/decrements to the poorest individual. On the other hand, $W' = PGR$ index is not affected by any transfers, and the increments/decrements in a poor person's income will not have a greater effect on the measure the poorer the person is.

In this paper we focus on classifying the rank-dependent poverty measures in term of their assigned *orness* value. Let us see the following definition.

Definition 6. Let P and Q be two rank-dependent poverty measures and NP and NQ their corresponding normalized measures. Then, if $orness(NP) < (\leq)\ orness(NQ)$ is satisfied we will denote $P \prec (\preceq)\ Q$.

In addition, we want to note that there exists a link between the rank-dependent poverty measures' classification through the *orness* value and the classification of these measures in terms of their sensitivity to *lossy* transfers. That is, transfers from a better-off poor individual to a worse-off poor individual that involve the loss of the mean income with the benefit of a distributional improvement.

A minimal transfer axiom considers an income transfer from a better-off poor individual to a worse-off poor individual where the amount given by the donor is exactly the amount received by the recipient. However, if the donor gives more than the recipient gets, then only those poverty measures which value sufficiently the redistribution will approve the transfer. Note that this kind of transfer will also depend on the amount of the mean loss.

Atkinson [7] and Okun [21] were the first to define this type of *lossy transfers* to measure the relative importance attributed to the distribution. Since then, *lossy transfers* have been used to explain the inequality aversion of many social welfare functions (see [18], [14] and [11]).

For this purpose, we need to define the following two transfers among the poor.

Definition 7. Let $\mathbf{x}$ and $\mathbf{y}$ be two income distributions in D. Then $\mathbf{y}$ is obtained from $\mathbf{x}$ by a lossy transfer among the poor if $n_x = n_y = n$, $q_x = q_y = q$ and $\mathbf{y} = (x_1, x_2, \ldots, x_i + \alpha, \ldots, x_j - \beta, \ldots, x_q, x_{q+1}, \ldots, x_n)$ where $0 < \alpha < \beta$ and $x_i < x_i + \alpha \leq x_j - \beta < x_j < z$.

Definition 8. Let $\mathbf{x}$ and $\mathbf{y}$ be two income distributions in D. Then $\mathbf{y}$ is obtained from $\mathbf{x}$ by a lossy equalization among the poor if $n_x = n_y = n$, $q_x = q_y = q$ and $\mathbf{y} = (\theta, \ldots, \theta, x_{q+1}, x_{q+2}, \ldots, x_n)$ where $q \cdot \theta < \sum_{i=1}^{q} x_i$.

Now, we say that a poverty measure P is at least as distribution-sensitive for *lossy transfers* among the poor or *lossy equalization* transfers among the poor as a poverty measure Q, if P registers a poverty increment for each *lossy* transfer or *lossy equalization transfer* among the poor for which Q does. The definitions are shown below:

Definition 9. Let $P(\cdot; z)$ and $Q(\cdot; z)$ be two poverty measures and suppose that **y** is obtained from **x** by a lossy transfer among the poor. Then P is at least as $distribution - sensitive^*$ for lossy transfers among the poor as Q if $Q(\mathbf{y}; z) \leq Q(\boldsymbol{x}; z)$ implies $P(\mathbf{y}; z) \leq P(\mathbf{x}; z)$.

Definition 10. Let $P(\cdot; z)$ and $Q(\cdot; z)$ be two poverty measures and suppose that **y** is obtained from **x** by a lossy equalization among the poor. P is at least as $distribution - sensitive^{**}$ for lossy equalization among the poor as Q if $Q(\mathbf{y}; z) \leq Q(\mathbf{x}; z)$ implies $P(\mathbf{y}; z) \leq P(\mathbf{x}; z)$.

Moreover, a poverty measure $P(\cdot; z)$ is *more distribution-sensitive* than $P(\cdot; z)$ for *lossy* (*lossy equalization*) transfers among the poor if $P(\cdot; z)$ is at least as *distribution-sensitive* as $P(\cdot; z)$ and $P(\cdot; z)$ is not at least as *distribution-sensitive* as $P(\cdot; z)$ for *lossy* (*lossy equalization*) transfers among the poor.

Aristondo and Ciommi [3] prove that if two poverty measures can be ranked in terms of *lossy transfers* or *lossy equalization transfers* criteria, they can also be classified in terms of the *orness* value. In addition, they also show that the classification for the first order rank-dependent poverty measures in terms of their *orness* value is equivalent to the classification in terms of their *distribution-sensitivity* to *lossy* transfers or *lossy equalization* transfers introduced by Bosmans [8].[5]

In addition, in this paper we will see that the Kakwani and S-Gini indices are ordered equivalently with respect to their parameter value for the three *distribution-sensitivity* rankings; *lossy* transfers, *lossy equalization* transfers and *orness* value.

5. Orness classification

As mentioned, the *orness* value can be interpreted as a *distribution sensitivity* indicator of the rank-dependent poverty measures and they can be ordered in terms of this value. Aristondo and Ciommi [3] classify most of the rank-dependent poverty measures in terms of their assigned *orness* value. However, they do not offer a classification for every member of the Kakwani and S-Gini families. The *orness* ranking of these two families has only been done for natural values of the two parameters k and σ. In fact, Aristondo and Ciommi [3] prove that $K_k \prec K_{k+1}$ and $G_\sigma \prec G_{\sigma+1}$ for $\forall \sigma, k, q, n \in \mathbb{N}$, $n > q \geq 2$, $k \geq 1$ and $\sigma \geq 1$.

Nevertheless, both the *Kakwani* and *S-Gini* families can be computed for any real value of the parameters, $k \in \mathbb{R}$ with $k \geq 1$ for K_k and $\sigma \in \mathbb{R}$ with $\sigma \geq 1$ for G_σ. Therefore, in this note we offer an *orness* classification for these two families for every real value of the parameters larger than one.

The following propositions show the *orness* classification for the family of Kakwani indices. Focusing on the *orness* value for the members of the Kakwani family we can classify them as follows:

Proposition 1. *The members of the Kakwani family of poverty indices,* $\{K_k\}_{k\geq 1}$, *can be classified with respect to their orness value as follows:*

$$K_k \prec K_m\,, \qquad 1 \leq k < m\,, \forall k, m \in \mathbb{R}\,. \tag{11}$$

Proof of Proposition 1. See Appendix.

This proposition shows that the larger the k value, the larger the *orness* value.

The next proposition offers the *orness* classification for every member of the S-Gini family, that is, every parameter $\sigma \in \mathbb{R}$.

Proposition 2. *The members of the Kakwani family of poverty indices,* $\{G_\sigma\}_{\sigma\geq 1}$, *can be classified with respect to their orness value as follows:*

$$G_\sigma \prec G_\beta\,, \qquad 1 \leq \sigma < \beta\,, \forall \sigma, \beta \in \mathbb{R}\,. \tag{12}$$

[5] First order rank-dependent poverty measures are those for which the weights are linear, that is, their form is $w_i = e + (i-1)d$, where e and d do not depend on i.

Proof of Proposition 2. See Appendix.

Therefore, we have ranked all the members of the K_k and the G_σ families. These results complement the ranking obtained by Aristondo and Ciommi [3], given that all the rank-dependent poverty measures are ordered.

Now, we will see that the maximum *distribution-sensitivity* value is obtained for K_k when k tends to infinity, and for G_σ when σ tends to infinity.

Proposition 3. *The orness value of the K_k and G_σ families tend to the maximum orness value 1 when parameters k and σ tend to infinity, respectively. That is,*

$$P \preceq K_\infty \quad \text{and} \quad P \preceq G_\infty \tag{13}$$

for every rank-dependent poverty measure P since $orness(NG_\infty) = orness(NK_\infty) = 1$.

Proof of Proposition 3. See Appendix.

Consequently, the *orness* value for the two limit rank-dependent poverty measures K_∞ and G_∞ is equal to one. In addition, we know that the maximum *orness* values are obtained for $w = (1, 0, \cdots, 0)$ weights. Hence, the corresponding normalized measures of the two poverty measures must be exactly the relative gap of the poorest individual.

$$NK_\infty = g_1 = NG_\infty \, .$$

It can be noted that the limit measures have the following form; $K_\infty = H \cdot g_1$ and $NG_\infty = g_1$.

The Kakwani index has a normalization factor, H, that is invariant to *lossy transfers* and *lossy equalizations*. Hence, their sensitivity to these kind of transfers will be the same. To conclude, the *distribution-sensitivity* of these two measures will only focus on transfers that affect the poorest individual.

Finally, we will provide an additional poverty ordering between the members of the two families presented in the paper for the same value of the parameter. Proposition 4 shows that the K_k poverty index is more *distribution-sensitive* than G_k for every $k \in \mathbb{N}$.

Proposition 4. *The families $\{K_k\}_{k\geq 1}$ and $\{G_k\}_{k\geq 1}$ can be ordered in terms of their orness values for the same parameter k as follows:*

$$G_k \preceq K_k \;\; \text{for} \;\; k \in \mathbb{R} \, . \tag{14}$$

Proof of Proposition 4. See Appendix.

Finally, we want to focus on the measures' classifications in terms of *lossy transfers* and *lossy equalization transfers*. Aristondo and Ciommi [3] prove that the *orness* classification is equivalent to the two classifications when weights are linear. The weights of the Kakwani and the S-Gini families are not linear and the equivalency can not be directly concluded. The measure rankings presented in this paper in terms of the *orness* value are exactly the same rankings as those in terms of the *distribution-sensitivity* of the measures to *lossy transfers* and *lossy equalization transfers*. In fact, Bosmans [8] proves that *distribution-sensitivity* for *lossy transfers* and *lossy equalization transfers* increases with k and σ for the Kakwani and the S-Gini families, respectively.

6. Concluding remarks

We provide an easy-to-check criterion which is able to order rank-dependent poverty measures in terms of their *distribution-sensitivity* using a real value between $1/2$ and 1, called *orness*. Most of these indices have been ranked in terms of this criterion. However, the classifications provided for the Kakwani and the S-Gini families of poverty indices are incomplete since only the rankings for natural values of the index parameters have been provided. In this note, we provide the *orness* classification for all the members of the two families in terms of their family parameter. We conclude that the Kakwani and the S-Gini families are more sensitive to the lower part of the distribution for higher values of the parameter. In addition, we have been able to rank the two families for the same value of the parameter, concluding that for this value the Kakwani index is more sensitive to increments or transfers the lower the income is. Given a fixed poverty line, this ranking will allow poverty results to be compared for different measures depending on their sensitivity to lower incomes values. Alternatively, it will also allow a choice between the appropriate poverty measures taking into account their distribution sensitivity.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

Oihana Aristondo thanks the Spanish Ministerio de Economía y Competitividad the Grant PID2019-107539GB-I00 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe” and the Basque Government (IT1697-22) for research support.

Appendix A

Proof of Proposition 1. We prove this proposition by mathematical induction on q.

For $q=2$, we find $orness(NK_k)=\frac{1}{\sum_{i=1}^{2} i^k}\sum_{i=1}^{2}\left(i^{k+1}-i^k\right)=\frac{2^k}{1+2^k}$. We need to prove $\frac{2^k}{1+2^k}<\frac{2^s}{1+2^s}$, or equivalently, $2^k\left(1+2^s\right)<2^s\left(1+2^k\right)$ which is true for every $1\le k<s$.

Let us assume that it is true for q:

$$\frac{1}{(q-1)\sum_{i=1}^{q} i^k}\sum_{i=1}^{q}\left(i^{k+1}-i^k\right)<\frac{1}{(q-1)\sum_{i=1}^{q} i^s}\sum_{i=1}^{q}\left(i^{s+1}-i^s\right).$$

Analogously,

$$\sum_{i=1}^{q} i^s\cdot\sum_{i=1}^{q}\left(i^{k+1}-i^k\right)<\sum_{i=1}^{q} i^k\cdot\sum_{i=1}^{q}\left(i^{s+1}-i^s\right).$$

We need to show that it is true for $q+1$. That is,

$$\sum_{i=1}^{q+1} i^s\cdot\sum_{i=1}^{q+1}\left(i^{k+1}-i^k\right)<\sum_{i=1}^{q+1} i^k\cdot\sum_{i=1}^{q+1}\left(i^{s+1}-i^s\right).$$

Again, operating we have,

$$\left(\sum_{i=1}^{q} i^s+(q+1)^s\right)\cdot\left(\sum_{i=1}^{q}(i^{k+1}-i^k)+q(q+1)^k\right)<\left(\sum_{i=1}^{q} i^k+(q+1)^k\right)\cdot\left(\sum_{i=1}^{q}(i^{s+1}-i^s)+q(q+1)^s\right);$$

which simplifies to,

$$\sum_{i=1}^{q} i^s\cdot\sum_{i=1}^{q}\left(i^{k+1}-i^k\right)+q(q+1)^k\sum_{i=1}^{q} i^s+(q+1)^s\sum_{i=1}^{q}\left(i^{k+1}-i^k\right)$$
$$-\sum_{i=1}^{q} i^k\cdot\sum_{i=1}^{q}\left(i^{s+1}-i^s\right)-q(q+1)^s\sum_{i=1}^{q} i^k-(q+1)^k\sum_{i=1}^{q}\left(i^{s+1}-i^s\right)<0.$$

Using the induction step, it reduces to show that

$$\sum_{i=1}^{q} i^s \cdot \sum_{i=1}^{q}\left(i^{k+1}-i^k\right) + q(q+1)^k \sum_{i=1}^{q} i^s + (q+1)^s \sum_{i=1}^{q}\left(i^{k+1}-i^k\right)$$
$$= (q+1)^k \sum_{i=1}^{q} i^k (i^{s-k}-(q+1)^{s-k})(q+1-i) < 0;$$

which is trivial to prove since $i^{s-k}-(q+1)^{s-k}<0$ and $q+1-i>0$ for $1<i<q+1$ and $1\le k<s$. □

Proof of Proposition 2. We define the following function

$$f(\sigma) = \frac{\sum_{i=1}^{q}\left((n+1-i)^\sigma-(n-i)^\sigma\right)(q-i)}{(q-1)(n^\sigma-(n-q)^\sigma)} \quad for \quad \sigma \ge 2\,.$$

We need to prove that it is an increasing function in σ for every $\sigma \ge 2$. Equivalently, we will see that the derivative of $f(\sigma)$ is positive for $\sigma \ge 2$. Once derived, we obtain,

$$f'(\sigma) = \Big((q-1)\big(n^\sigma \ln n - (n-q)^\sigma \ln(n-q)\big)\Big)\Big((q-1)(n^\sigma-(n-q)^\sigma)\Big)^{-2}$$
$$\times\left(\sum_{i=1}^{q}\Big((n+1-i)^\sigma-(n-i)^\sigma\Big)(q-i)\right)+$$
$$+\left(\sum_{i=1}^{q}\Big((n+1-i)^\sigma\ln(n+1-i)-(n-i)^\sigma\ln(n-i)\Big)(q-i)\right)\Big((q-1)(n^\sigma-(n-q)^\sigma)\Big)^{-1}.$$

Since $q,n\in\mathbb{N}$ and $n>q\ge 2$ then $(n-q)\ge 1$. Consequently we have that for any $\sigma\ge 2$ $n^\sigma-(n-q)^\sigma\ge 0$, $\big(n^\sigma\ln n-(n-q)^\sigma\ln(n-q)\big)\ge 0$, $(n+1-i)^\sigma\ln(n+1-i)-(n-i)^\sigma\ln(n-i)\ge 0$, $(n+1-i)^\sigma-(n-i)^\sigma\ge 0$ and $(q-i)\ge 0$ $\forall i=1,\cdots,q$.

Hence, $f'(\sigma)\ge 0$, and consequently $f(\sigma)$ is an increasing function in σ. □

Proof of Proposition 3. For the K_k family, we need to prove

$$\lim_{x\to\infty} orness(NK_k) = 1\,.$$

Substituting the *orness* value,

$$\lim_{x\to\infty} orness(NK_k) = \lim_{x\to\infty}\frac{\sum\limits_{i=1}^{q}\left(i^{k+1}-i^k\right)}{(q-1)\sum\limits_{i=1}^{q} i^k} = \lim_{x\to\infty}\frac{\sum\limits_{i=1}^{q}\left(\left(\frac{i}{q}\right)^{k+1}-\frac{1}{q}\left(\frac{i}{q}\right)^{k}\right)}{\frac{q-1}{q}\sum\limits_{i=1}^{q}\left(\frac{i}{q}\right)^k} = 1\,.$$

For the G_σ family, we also need to prove

$$\lim_{x\to\infty} orness(NG_\sigma) = 1\,.$$

Substituting the *orness* value,

$$\lim_{x\to\infty} orness(NG_\sigma) = \lim_{x\to\infty}\sum_{i=1}^{q}\frac{(n+1-i)^\sigma-(n-i)^\sigma}{(q-1)(n^\sigma-(n-q)^\sigma)}(q-i)\,.$$

Operating,

$$\lim_{x\to\infty} orness(NG_\sigma) = \lim_{x\to\infty}\sum_{i=1}^{q}\frac{\left(\frac{n+1-i}{n}\right)^\sigma-\left(\frac{n-i}{n}\right)^\sigma}{(q-1)\left(1-\left(\frac{n-q}{n}\right)^\sigma\right)}(q-i) = 1\,. \quad \square$$

Proof of Proposition 4. In order to prove Proposition 4, we need two auxiliary Lemmas.

Lemma 1. *The $orness(NG_k)$ is a decreasing function in n.*

Proof of Lemma 1. Let us define $f(n)$ a continuous function on $n \in R$:

$$f(n) = \frac{\sum_{i=1}^{q}\left((n+1-i)^k - (n-i)^k\right)(q-i)}{n^k - (n-q)^k}.$$

In order to show that $f(n)$ is a decreasing function in n, we will prove that $f'(n) < 0$:

$$f'(n) = \frac{\sum_{i=1}^{q}\left(k(n+1-i)^{k-1} - k(n-i)^{k-1}\right)(q-i)(n^k-(n-q)^k)}{(n^k-(n-q)^k)^2} - \frac{\sum_{i=1}^{q}\left((n+1-i)^k - (n-i)^k\right)(q-i)(kn^{k-1}-k(n-q)^{k-1})}{(n^k-(n-q)^k)^2}.$$

Hence, equivalently we need to prove that

$$\sum_{i=1}^{q}\left((n+1-i)^{k-1} - (n-i)^{k-1}\right)(q-i)(n^k-(n-q)^k) - \sum_{i=1}^{q}\left((n+1-i)^k - (n-i)^k\right)(q-i)(n^{k-1}-(n-q)^{k-1}) < 0.$$

Operating we have,

$$\left(n^k-(n-q)^k\right)\left(n^{k-1}-(n-q)^{k-1}\right)(q-1)\left(orness(NG_{k-1}) - orness(NG_k)\right) < 0,$$

which is true from Proposition 2. Hence, if it is a decreasing function for real values, it is also decreasing for natural values. □

Lemma 2. *For every $q \in \mathbb{N}$ and $k \in \mathbb{R}$ the following inequality is satisfied:*

$$(q+1)^k - \left(\frac{(q+1)^k - q^k}{q^k}\right)\sum_{i=1}^{q} i^k \geq 0.$$

Proof of Lemma 2. Let us define $f(q) = (q+1)^k - \left(\frac{(q+1)^k - q^k}{q^k}\right)\sum_{i=1}^{q} i^k$, for some $k \in \mathbb{R}$.

It will suffice to show $f(q+1) \geq f(q)$ and $f(1) \geq 0$, for any $k \in \mathbb{R}$.

For $q = 1$,

$$f(1) = 2^k - (2^k - 1) = 1 \geq 0.$$

Now, we will see that $f(q+1) \geq f(q)$ for any $q \in \mathbb{N}$ and $k \in \mathbb{R}$.

$$\begin{aligned} f(q+1) - f(q) &= (q+2)^k - \left(\frac{(q+2)^k - (q+1)^k}{(q+1)^k}\right)\sum_{i=1}^{q+1} i^k - (q+1)^k + \left(\frac{(q+1)^k - q^k}{q^k}\right)\sum_{i=1}^{q} i^k \\ &= \frac{\left((q+1)^2\right)^k - (q(q+2))^k}{q^k(q+1)^k}\sum_{i=1}^{q} i^k \geq 0, \end{aligned}$$

which is true since $q, (q+1) \geq 0$ and $(q+1)^2 \geq q(q+2)$ for any $q \in \mathbb{N}$, $k \in \mathbb{R}$. □

Hence we can now prove **Proposition 4**.

We need to prove that for $k \in \mathbb{R}$, $q, n \in \mathbb{N}$ and $n > q \geq 2$,

$$orness(NK_k) - orness(NG_k) \geq 0.$$

That is,

$$\frac{\sum_{i=1}^{q} i^k\,(i-1)}{\sum_{i=1}^{q} i^k} - \frac{\sum_{i=1}^{q}\left((n+1-i)^k-(n-i)^k\right)(q-i)}{n^k-(n-q)^k} \geq 0\,.$$

From Lemma 1, we know that $orness(NG_k)$ is a decreasing function in n. Hence, it will suffice to prove for $n=q$. That is, we need to prove the following:

$$\frac{\sum_{i=1}^{q} i^k\,(i-1)}{\sum_{i=1}^{q} i^k} - \frac{\sum_{i=1}^{q}\left((q+1-i)^k-(q-i)^k\right)(q-i)}{q^k} \geq 0\,.$$

Operating, we have that,

$$\frac{\sum_{i=1}^{q} i^{k+1}-i^k}{\sum_{i=1}^{q} i^k} - \frac{q^{k+1}-\sum_{i=1}^{q} i^k}{q^k} \geq 0\,.$$

Hence, we need to prove the following:

$$q^k\sum_{i=1}^{q} i^{k+1} + \left(\sum_{i=1}^{q} i^k\right)^2 - (q+1)q^k\sum_{i=1}^{q} i^k \geq 0\,.$$

We proceed by induction on q. Firstly, we will see that it is true for $q=2$:

$$2^k\left(1+2^{k+1}\right)+\left(1+2^k\right)^2-(3)2^k\left(1+2^k\right)=1\geq 0\,.$$

Now suppose that is true for q,

$$q^k\sum_{i=1}^{q} i^{k+1} + \left(\sum_{i=1}^{q} i^k\right)^2 - (q+1)q^k\sum_{i=1}^{q} i^k \geq 0\,.$$

Now computing it for $q+1$:

$$\begin{aligned}&(q+1)^k\sum_{i=1}^{q+1} i^{k+1} + \left(\sum_{i=1}^{q+1} i^k\right)^2 - (q+2)(q+1)^k\sum_{i=1}^{q+1} i^k\\ &=(q+1)^k\left(\sum_{i=1}^{q} i^{k+1}\right) + \left(\sum_{i=1}^{q} i^k\right)^2 + 2(q+1)^k\sum_{i=1}^{q} i^k - (q+2)(q+1)^k\sum_{i=1}^{q} i^k\,.\end{aligned}$$

Using the inductive step for q,

$$\begin{aligned}&(q+1)^k\left(\sum_{i=1}^{q} i^{k+1}\right) + \left(\sum_{i=1}^{q} i^k\right)^2 + 2(q+1)^k\sum_{i=1}^{q} i^k - (q+2)(q+1)^k\sum_{i=1}^{q} i^k\\ &\geq \frac{(q+1)^k}{q^k}\left((q+1)q^k\sum_{i=1}^{q} i^k - \left(\sum_{i=1}^{q} i^k\right)^2\right) + \left(\sum_{i=1}^{q} i^k\right)^2 + 2\left(\sum_{i=1}^{q} i^k\right)(q+1)^k - (q+2)(q+1)^k\sum_{i=1}^{q} i^k\,.\end{aligned}$$

And operating we have that

$$\begin{aligned}&(q+1)^k\left(\sum_{i=1}^{q} i^{k+1}\right) + \left(\sum_{i=1}^{q} i^k\right)^2 + 2(q+1)^k\sum_{i=1}^{q} i^k - (q+2)(q+1)^k\sum_{i=1}^{q} i^k\\ &\geq \sum_{i=1}^{q} i^k\left((q+1)^k - \left(\frac{(q+1)^k-q^k}{q^k}\right)\sum_{i=1}^{q} i^k\right)\geq 0,\end{aligned}$$

which holds from Lemma 2. □

References

[1] O. Aristondo, Three Shapley rankings of poverty measures according to their sensitivity to the three components of poverty, Fuzzy Sets Syst. (2022).
[2] O. Aristondo, M. Ciommi, The decompositions of rank-dependent poverty measures using ordered weighted averaging operators, Int. J. Approx. Reason. 76 (2016) 47–62.
[3] O. Aristondo, M. Ciommi, The orness value for rank-dependent welfare functions and rank-dependent poverty measures, Fuzzy Sets Syst. 325 (2017) 114–136.
[4] O. Aristondo, J.L. García-Lapresta, C.L. de la Vega, R.A.M. Pereira, Classical inequality indices, welfare and illfare functions, and the dual decomposition, Fuzzy Sets Syst. 228 (2013) 114–136.
[5] O. Aristondo, J.L. García-Lapresta, C. Lasso de la Vega, R.A.M. Pereira, The Gini index, the dual decomposition of aggregation functions, and the consistent measurement of inequality, Int. J. Hybrid Intell. Syst. 27 (2) (2012) 132–152.
[6] O. Aristondo, J.L. García-Lapresta, C. Lasso de la Vega, R.A.M. Pereira, Classical inequality indices, welfare and illfare functions, and the dual decomposition, Fuzzy Sets Syst. 228 (2013) 114–136.
[7] A.B. Atkinson, Wealth, Income and Inequality: Selected Readings, Penguin Education, 1973.
[8] K. Bosmans, Distribution-sensitivity of rank-dependent poverty measures, J. Math. Econ. 51 (2014) 69–76.
[9] S. Chakravarty, Ethical Social Index Numbers, Springer Verlag, New York, 1990.
[10] S.R. Chakravarty, A new index of poverty, Math. Soc. Sci. 6 (3) (1983) 307–313.
[11] F. Cowell, Measuring Inequality, Oxford University Press, 2011.
[12] G. Datt, Distribution-sensitive multidimensional poverty measures, World Bank Econ. Rev. 33 (3) (2019) 551–572.
[13] D. Donaldson, J.A. Weymark, A single-parameter generalization of the Gini indices of inequality, J. Econ. Theory 22 (1) (1980) 67–86.
[14] J.-Y. Duclos, Gini indices and the redistribution of income, Int. Tax Public Finance 7 (2) (2000) 141–162.
[15] J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, vol. 14, Springer Science & Business Media, 1994.
[16] J.L. García-Lapresta, R.A.M. Pereira, The self-dual core and the anti-self-dual remainder of an aggregation operator, Fuzzy Sets Syst. 159 (1) (2008) 47–62.
[17] J.L. García-Lapresta, R.A.M. Pereira, The dual decomposition of aggregation functions and its application in welfare economics, Fuzzy Sets Syst. 281 (2015) 188–197.
[18] S.P. Jenkins, P.J. Lambert, Three I's of poverty curves, with an analysis of UK poverty trends, Oxf. Econ. Pap. 49 (3) (1997) 317–327.
[19] N. Kakwani, On a class of poverty measures, Econometrica: Journal of the Econometric Society (1980) 437–446.
[20] X. Liu, H. Lou, Parameterized additive neat Owa operators with different orness levels, Int. J. Hybrid Intell. Syst. 21 (10) (2006) 1045–1072.
[21] A.M. Okun, Equality and Efficiency: The Big Tradeoff, The Brookings Institution, Washington, DC, 1975.
[22] C. Puerta, A. Urrutia, et al., Lower and upper tail concern and the rank dependent social evaluation functions, Econ. Bull. 32 (4) (2012) 3250–3259.
[23] A. Sen, Poverty: an ordinal approach to measurement, Econometrica: Journal of the Econometric Society (1976) 219–231.
[24] H.W. Watts, An Economic Definition of Poverty, Institute for Research on Poverty, 1968.
[25] J.A. Weymark, Generalized Gini inequality indices, Math. Soc. Sci. 1 (4) (1981) 409–430.
[26] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decisionmaking, IEEE Trans. Syst. Man Cybern. 18 (1) (1988) 183–190.
[27] R.R. Yager, Families of Owa operators, Fuzzy Sets Syst. 59 (2) (1993) 125–148.
[28] R.R. Yager, Aggregation operators and fuzzy systems modeling, Fuzzy Sets Syst. 67 (2) (1994) 129–145.
[29] R.R. Yager, V. Kreinovich, Decision Making Under Interval Probabilities, 1999.
[30] B. Zheng, Aggregate poverty measures, J. Econ. Surv. 11 (2) (1997) 123–162.
[31] B. Zheng, Minimum distribution-sensitivity, poverty aversion, and poverty orderings, J. Econ. Theory 95 (1) (2000) 116–137.