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# The Convergence Approach to Benardete's Paradox

**Jon Pérez Laraudogoitia**[1]

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## Abstract
The paper analyses Benardete's paradox of the gods from a more general perspective (the convergence approach) than several of the most important proposals made to date, but in close relation (and sharp contrast) with them. The new theory, based on the notion of limit, is systematically applicable in different possible scenarios involving a denumerable infinity of objects. In particular, it reveals in what way ω-consistency can be compromised in an otherwise consistent description of such "infinitary" situations.

**Keywords** Benardete's Paradox · Infinity · Limit · Convergence · ω-Consistency

Benardete's (1964) paradox of the gods has been the source of growing discussion in the literature. In its original formulation it reads as follows:

"A man decides to walk one mile from A to B. A god waits in readiness to throw up a wall blocking the man's further progress when the man has travelled ½ mile. A second god (unknown to the first) waits in readiness to throw up a wall of his own blocking the man's further progress when the man has travelled ¼ mile. A third god … &c. ad infinitum. It is clear that this infinite sequence of mere *intentions* (assuming the contrary-to-fact conditional that each god would succeed in executing his intention if given the opportunity) logically entails the consequence that the man will be arrested at point A; he will not be able to pass beyond it, even though not a single wall will in fact be thrown down in his path. The before-effect here will be described by the man as a strange field of force blocking his passage forward." (pp. 259–260).

The situation involves an infinity of objects (walls) that are capable of interacting with one another (the man). The paradox is that this interaction (which has nothing to do with gravitation) is not by contact. And one of the ways of dealing with the

✉ Jon Pérez Laraudogoitia
jon.perez@ehu.es

1 Departamento de Lógica Y Filosofía de La Ciencia. Facultad de Letras, Universidad del País Vasco, C/ Paseo de La Universidad 5, 01006 Vitoria-Gasteiz, Spain

problem consists of directly analysing this actual infinity, extracting consequences from it. This is what Hawthorne (2000) does, for example, thereby attempting to explain what the solution is. He argues that the paradox shows the falsity of what he calls the "Change Principle":

> "If x is the fusion of y's and y's are individually capable only of producing effect e by undergoing change, then x cannot, (without the addition of some non-supervening causal power), produce effect e without undergoing change." (p. 630).

Although he gives no explanation of what "change" refers to, it is clear from the context that he means change in the physical state. The fusion of the gods is thus supposed to be able to prevent the man's movement (indeed, the man cannot reach any of points $x = 1/2$, $x = 1/4$, …, $x = 1/2^n$, …) without change, which no god alone can do. This solution to the problem is essentially the same as the one Benardete seems to advocate: the fusion of the gods' causal power creates the strange force field to which the latter refers. I shall call it the mysterious fusion solution (MFS). An alternative approach is to begin from experience we have of finite sets of interacting objects and generalise some feature of the former deemed essential to the infinite case. There are two significant versions of this approach in the literature, both of which are widely supported: the causal finitism and the unsatisfiable pair diagnosis. It is not my intention here to expound these two positions (the reader's knowledge of which is presupposed), but simply to mention the parts of them that are relevant to the problem at hand (Benardete's paradox). According to causal finitism, nothing can be affected by infinitely many causes and, in addition (according to one of its refinements), excludes "cases where an infinite number of things each has the power to contribute to some effect,[ …] even if they do not actually do so" (Pruss, 2018, p. 144). This directly blocks the infinite system in which the paradox of the gods is formulated. The unsatisfiable pair diagnosis (Shackel, 2005) blocks in a different way. It considers that the fusion of the gods cannot have causal effects (nor the fusion of the walls, contrary to Laraudogoitia, 2003) because the man can only interact (collide) with specific and well-determined walls, as is necessarily the case in systems with a finite number of objects. Consequently, if no god-m ($m > n$) stops the man at respective point $x_m = 1/2^m$ controlled by the former, it does not follow that the latter has not passed point $x = 0$ rather that, on the contrary, he has done so (since only a wall could stop him) and is now approaching point $x_n = 1/2^n$ (where god-n stops him).[1] Symbolically:

$$\forall n(\neg\exists m(m > n)\text{the man is stopped at } x_m \rightarrow \text{the man is stopped at } x_n) \quad (1)$$

Yet we knew from the very outset, given the conditions of the situation, that:

$$\forall n(\text{the man is stopped at } x_n \rightarrow \neg\exists m(m > n)\text{the man is stopped at } x_m) \quad (2)$$

[1] This is basically the condition that Caie (2018) calls "Only Walls", and is also taken for granted in his analysis of Benardete's paradox.

Conjunction of the two gives:

$$\forall n(\text{the man is stopped at } x_n \leftrightarrow \neg\exists m(m > n)\text{the man is stopped at } x_m) \quad (3)$$

Finally, as has been stipulated:

$$\text{the set of points on line } \{x_1, x_2, ..., x_n, ...\} \text{ have no first member} \quad (4)$$

(3) and (4) form an unsatisfiable pair of conditions. The conclusion follows that it is impossible for the gods to fulfil their plan, and the paradox disappears.

## 1 The Convergence Approach

Against the background of the scheme of options discussed above, the proposal herein to analyse the paradox of the gods can be better understood. It shares with causal finitism (CF) and the unsatisfiable pair diagnosis (UPD) the intuition that it is the experience we have of finite sets of interacting objects which should provide us with the key to generalising to the infinite case. Nonetheless, it differs from them in what this key is. The key for CF and UPD is ontological; it is linked to a certain conception of causal action that is considered generalisable. The key for the convergence approach (CA) is purely methodological (or, if preferred, purely formal): the infinite system's behaviour (time evolution) must be "as similar as possible" (in a formal sense to be specified below) to that of a finite system. In more suggestive language, an infinite system's time evolution should be understood as the limit of a finite system's time evolution when the latter's number of components tends towards infinity. The most obvious understanding of a system's time evolution is the idea of its spatial configuration's evolution: the history of each of its parts' spatial location in time. Sometimes, however, it is not the spatial configuration that is of interest (for example, since it can be determined that it does not change over time), in which case, rather than the location of the parts, some other physical quantity related to them may be of interest. Nonetheless, the convergence approach can be based on the following:

Special Limit Principle (SLP): Consider the isolated system of infinite components $O = \{O_1, O_2, \ldots\}$, where the state of $O_i$ at instant t is $S_t(O_i)$, and $O_{(n)} = \{O_1, O_2, \ldots O_n\}$ is an n-component subsystem of O in corresponding states $S_t(O_i)$, considered an isolated system itself. The value of physical quantity M involving object $O_i$ of infinite system O at any instant t*: $M(O, O_i, t^*)$ is $\lim_{n\to\infty} M(O_{(n)}, O_i, t^*)$ if said limit exists.[2]

[2] Circumstances can easily be imagined where such a limit does not exist. Admittedly, this is a limitation of SLP, though not necessarily of the convergence approach. This is because far more general concepts of convergence are possible (e.g. see Dolecki & Mynard 2016) in which the appropriate handling of such cases is possible. However, such technicalities are of no relevance here and, in any event, SLP provides a sufficiently precise idea of what a more far-reaching convergence approach would look like. Also note that the reference to states $S_t(O_i)$ is essential in SLP. Indeed, it allows us to specify exactly and at any given moment in time how sub-systems $O_{(n)} = \{O_1, O_2, \ldots O_n\}$ are obtained from system $O = \{O_1, O_2, \ldots\}$ prior to going to the limit. And the specification is simple: the state of $O_i$ ($1 \leq i \leq n$) is the same in $O_{(n)}$ and in O.

Reference to physical quantities in SLP is justified by the fact that they (typically, but not exclusively, positions in space) have obvious and natural metric characteristics, a consequence of the usual metric associated with the set of real numbers. This makes the going to the limit referred to in SLP well defined. On the contrary, the limit of a sequence of nonphysical characteristics is not defined in principle (at least insofar as the latter cannot be quantified in a natural manner).

SLP is very circumspect about its commitments. It does not postulate anything specific for infinite systems, and any laws or physical intuitions apply exclusively to finite systems. Infinite systems as such are only known to us because of the going-to-the-limit process. However, SLP does not specify any criterion for establishing what a component of a system is and, therefore, for distinguishing finite from infinite systems. It simply presupposes some such criterion (which need not be unique, but which may depend on the context and the purpose of analysis, as seems reasonable). A simple application of SLP can also highlight the difference between the convergence approach (CA), the mysterious fusion solution (MFS) and causal finitism (CF) in a particularly interesting way. Consider a homogeneous cubic block B of unit edge and weight. B is at rest on a surface S and exerts unit force upon it: F(B,S)↓ = 1 (obviously also F(S,B)↑ = 1; in both cases the arrow is not strictly necessary, because I could write for example F(B,S) = − 1 and F(S,B) = 1; however, it clarifies the force direction, so I will use it on more occasions). Suppose now that the homogeneous cubic object of unit edge and weight is not a block but a stack of slabs B*. The lower half of the stack, $B^*_1$ (the first slab, in contact with S) has weight 1/2 and height 1/2. The remaining lower half of the stack, $B^*_2$ (the second slab, in contact with $B^*_1$) has weight $1/2^2$ and height $1/2^2$, and so on. Determining F(B*,S)↓ is an immediate application of the Special Limit Principle (SLP).[3] If a stack $B^*_{(n)}$ formed solely by slabs $B^*_1$ to $B^*_n$ is considered, it is clear that $F(B^*_{(n)}, S)\downarrow = 1/2 + 1/2^2 + \ldots + 1/2^n = 1 - 1/2^n$. So, according to CA, the value of physical quantity involving surface S, and which represents the force exerted on it by the stack of slabs on top of it, is $F(B^*,S)\downarrow = \lim_{n\to\infty} F(B^*_{(n)}, S)\downarrow = \lim_{n\to\infty} (1 - 1/2^n) = 1 = F(B,S)\downarrow$. This is eminently reasonable and in stark contrast to causal finitism, for which F(B,S)↓ = 1, while it is unacceptable to assert that F(B*,S)↓ = 1. Since SLP enables stack B*'s behaviour to be determined in a very simple manner (and that of many other infinite systems) in a multitude of particular circumstances, CF's extreme demand to consider them all metaphysically illegitimate entities is incomprehensible. The contrast with the mysterious fusion solution (MFS) is far more interesting because it is (unlike the contrast with the causal finitism) rather unexpected. Let us explain the reason for this in detail, which ultimately lies in the absence in MFS of any kind of hypothesis or postulate that allows the properties of infinite systems to be related to those of finite systems (as CA does by means

[3] It can be trivially determined by SLP that the stack of slabs' configuration does not change over time. This is why the focus is on the application of this principle to the forces present; not to the locations. Note also that although stack B* is not an isolated system, joint system B* + S indeed is (B*'s weight is due to S's gravitational field). SLP is applied to this.

of SLP). As seen, according to CA, $F(B^*,S)\downarrow = F(B^*_1,S)\downarrow = 1$[4] and, as one might expect, $F(B^*_2,B^*_1)\downarrow = 1/2$, $F(B^*_3,B^*_2)\downarrow = 1/2^2$, … can immediately be verified. In general, $F(B^*_{n+1},B^*_n)\downarrow = 1/2^n$. As $B^*_n$'s weight is $1/2^n$, it seems that, according to MFS, it should also be $F(B^*_1,S)\downarrow = 1$ and $F(B^*_{n+1},B^*_n)\downarrow = 1/2^n$. However, this is not true. Note that in MFS the actual infinity of the stack of slabs is directly considered, and in doing so an indeterminacy is discovered in the force that it exerts on surface S. Even though the stack weighs one unit, all that MFS permits us to conclude is that $F(B^*_1,S)\downarrow = 1 + \alpha$ (with α any non-negative real number). Where does the "excess" force on S of magnitude α come from? From the excess force that $B^*_2$ exerts on $B^*_1$, since $F(B^*_2,B^*_1)\downarrow$ is not 1/2 but $F(B^*_2,B^*_1)\downarrow = 1/2 + \alpha$. The excess force on $B^*_1$ comes in turn from the excess force that $B^*_3$ exerts on $B^*_2$, since $F(B^*_3,B^*_2)\downarrow = 1/2^2 + \alpha$ … and so on. Addressing the interaction with infinite systems directly (without going to the limit), as MFS does, leads to wild indeterminacy in the case of the stack of slabs (indeterminacy brought about, one might say, by the fusion of the slabs). This undesirable consequence (which, as such, I consider to be a compelling argument against MFS) could be used by a causal finitist to defend CF, thereby rejecting the role of fusions. However, CA shows that such an extreme move is unjustified: the Special Limit Postulate is, as seen, sufficient to cope with the situation without falling into indeterminacy. In the case of the paradox of the gods, MFS does not lead to indeterminism but to a mysterious force (brought about this time by the fusion of the gods, or of the walls). The role of the Special Limit Principle in such a case will now be discussed.

## 2 Benardete's Paradox of the Gods and the Convergence Approach

If the number of gods (and walls) were finite, termed n, then the man would evolve freely from $x = 0$ until he is stopped at $x = 1/2^n$ by the force exerted by $wall_n$. He exerts an equal and opposite force on $wall_n$ and zero force on every $wall_m$ $(m < n)$. At limit $n \to \infty$ the man cannot begin his journey. He is stopped at $x = 0$ by a force[5] which is not exerted by any wall (there is no $wall_\infty$), and at the same time he exerts no force on any wall. There would therefore seem to be the presence of a mysterious force stopping the man's progress and, consequently, there is no significant improvement on this point with regard to the verdict of the mysterious fusion solution. However, this is not the case. This misleading guise stems from ambiguity in the expression "mysterious force". If what is meant by this term is a quite different type of force in nature from other forces present in a Newtonian possible world, then CA indeed leads, as does MFS, to mysterious forces. However, if what is meant

[4] Note, however, that $F(B^*_{(1)},S)\downarrow = F(B^*_{(1)1},S)\downarrow = 1/2$. $B^*_{(1)1}$ is the first (and only) slab in $B^*_{(1)}$, while $B^*_1$ is the first slab in B*.

[5] Consider that in the finite case of n walls, a force to the left $\overleftarrow{F}$ acts on the man at $x = 1/2^n$ which is numerically equal to the force that friction with the ground exerts on him to the right. Reasonably, $\overleftarrow{F}$ does not rely on the specific value of n, so at limit $n \to \infty$ the force stopping the man from moving is precisely $\overleftarrow{F}$.

by "mysterious force" is a force whose presence and effects are almost completely beyond our capability to predict and calculate, then the force preventing the man from moving is mysterious in the context of MFS, yet not in the context of CA. This is because the Special Limit Principle enables us to predict when such a force will appear, and even its precise effects in a wide variety of circumstances.[6] In this respect, it resembles the typical force of universal gravitation. Universal gravitation remains mysterious in Newtonian worlds in that it is quite different to usual contact forces. Yet it is not mysterious at all insofar as its presence and effects can be predicted with remarkable accuracy. The force that holds the man back in the paradox of the gods can then be described as a natural manifestation of the sets of material objects at the infinite limit.[7] In this sense, the actual physical infinity opens up new perspectives as did the actual mathematical infinity. So, just as the Cantorian theory turned a paradox of infinity (equinumerosity with a proper part) into an immediate consequence of a natural definition of infinity that enables the systematic study of infinite sets, CA does something similar by turning paradoxes such as Benardete's into the immediate consequence of a natural procedure (SLP) for the systematic study of infinite physical systems. Some examples of the Special Limit Principle's predictive capacity (logical fertility) will be illuminating.

a) Suppose that the man is able (under normal conditions) to travel at unit speed. Suppose also that, although no wall can stop him, passing $\text{wall}_n$ requires such expenditure of energy that to go from $x = 1/2^n$ to $x = 1/2^{n-1}$ (where he will encounter the following wall or, if $n = 1$, final point B of his journey) will not take the usual time $1/2^n$ (distance $1/2^n$ travelled at unit speed) but rather the longer time $2/n^2$. In the case of a finite number of n gods, the man is able go from A to B. He takes time $1/2^n$ to go from A to the barrier located at $x = 1/2^n$ and time $2(1/n^2 + 1/(n+1)^2 + \ldots + 1/2^2 + 1/1^2)$ to go from there to point B. Total time: $1/2^n + 2(1/n^2 + 1/(n+1)^2 + \ldots + 1/2^2 + 1/1^2)$. According to SLP, in the case of infinite gods, this time is simply $2\sum_{n=1}^{n=\infty}\left(\frac{1}{n^2}\right) = \pi^2/3$.

b) Let us now change just one thing: the energy expenditure required to pass $\text{wall}_n$ means that the time taken to go from $x = 1/2^n$ to $x = 1/2^{n-1}$ is not $2/n^2$ but slightly longer: $2/n$. Under these conditions, as it is immediate, SLP implies that the total time to go from A to B is now $2\sum_{n=1}^{n=\infty}\left(\frac{1}{n}\right)$, an infinite quantity given the series' divergence. That is, the man cannot make the journey. He cannot even advance beyond A. The force preventing him from doing so is the result of the limit when $n \to \infty$ of the forces act upon him in the finite case (forces that slow him down, causing the

[6] Note (2), regarding limit concepts linked to more general notions of convergence, is relevant here.

[7] The way in which taking this infinite limit then throws some light on the nature of such a force can thus readily be seen. Why can wall w stop the man? Because, if it does not, he will interpenetrate with it in his progress; and the interpenetration of material objects is physically impossible. This is exactly why the infinite walls (the infinite gods) can stop the man's progress in the paradox of the gods: if they do not do so, he will interpenetrate with infinitely many of them. So the force in Benardete's paradox is less mysterious than it might seem at first sight because (like usual interaction via contact) it is based on the impenetrability of matter. It is nonetheless (concurring with that stated earlier in the main text) more mysterious than the usual force via contact because it takes place at a distance.

abovementioned energy expenditure). What is interesting about case b) is that (unlike a)) the gods are able to stop the man's movement even though none are individually able to do so. This shows that the convergence approach can systematically and rigorously address subtle forms of causal efficacy that go unnoticed from other points of view. In particular, it directly refutes the intuition voiced by Uzquiano (2012):

> "[If] *x* is capable of producing effect *e* by means of an open-ended series of hypothetical events only if *x* is capable of producing effect *e* by means of each event." (p. 263).

## 3 Possibilities, Impossibilities and the ω-Consistency

Benardete's paradox can be further formulated from (3) and (4) as follows:

$$\forall n[\text{god}-n \text{ places a barrier at } x_n = 1/2^n \leftrightarrow (\text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n) \text{ god}-m \text{ places a barrier at } x_m = 1/2^m)] \tag{5}$$

If the man progresses from A (x = 0), it is easy to see that (5) leads to a contradiction. Does this prove that such progress is impossible? No, says Yablo (2000). What it proves (according to UPD) is that the gods cannot fulfil the plan (5): "Logic stops them" (p. 150). This goes against CA. According to CA, (5) is fulfilled and no god will place any barrier (because the man does not go beyond point x = 0). To test the plausibility of UPD against CA, the story can be modified as follows: rather than placing a barrier at $x_n = 1/2^n$, what god-n will do is wink at $x_n = 1/2^n$. The new version of the paradox is described as follows:

$$\forall n[\text{god}-n \text{ winks at } x_n = 1/2^n \leftrightarrow (\text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n) \text{ god}-m \text{ winks at } x_m = 1/2^m)] \tag{6}$$

To Yablo (and UPD), (5) and (6) are similar. In both cases it follows that the man is able to advance, making it impossible for the gods to fulfil their plan. CA, on the other hand, explains why it is illusory to attempt to hold the man back by winking (as in (6)), but not by placing barriers (as in (5)). As there are infinite systems in both cases, SLP is applied on the basis of the corresponding finite systems. With a finite number of gods (termed N gods), (6) has no causal effect on the man, who can advance unhindered. (5), however, does have an effect: the man will be stopped at $x = 1/2^N$. Taking limit $N \rightarrow \infty$, CA thus explains why the gods in the infinite case are able to stop the man's progress if they are willing to place barriers, but are unable to do so if they are only willing to wink. I consider that the ability to address this difference constitutes a clear advantage of CA over UPD: (5) presupposes elementary causal links which are not present in (6), and this should be reflected in a suitable theory of infinity. According to CA, the ensemble formed by the man, the barriers and the gods is a physical system (which can be assumed to be isolated) with infinite components $O = \{O_1, O_2, \ldots\}$, and as such is addressed in SLP. This even enables something interesting to be said about plan (6) besides the trivial assertion that it is impossible for the gods to fulfil it. In particular, it reveals that there are not many ways in which the gods can fail to fulfil it—there is just one! This follows from SLP in that it considers the gods for what they are: physical objects. Returning to the finite case of N gods, we now know that god-N will wink (which will have no effect on the man's motion), and no other god will do so. Therefore, at limit $N \to \infty$ (remember that there is no god-∞) no god will wink. This is the only way in which (6) can be violated according to SLP and the convergence approach CA. Finally, the idea can be applied to Benardete's original paradox by considering the man as a physical system. In the case of N gods, the man's inner state corresponds to the experience (among other observations) that some god is preventing him from advancing (it happens to be the N-th god), and that neither god-1, nor god-2, …, nor god-N-1 are stopping him from doing so. At the infinite limit, $N \to \infty$, the man's experience (according to SLP) is that some god is preventing him from advancing even though neither god-1, nor god-2, … neither god-N, nor god-(N + 1), … are doing so. That is, Benardete's man will interpret the situation by stating that the description of reality produces ω-inconsistency. This is as close as can be reached to Priest's (1999) conclusion that "motion produces contradictions".

## 4 On Winking. The Role of SLP. Response to One First Criticism

It has been seen that if the gods decide to follow the prescription (6) to wink, the man will not therefore be halted by them. Conversely, it is they who in this case will be unable to fulfil their plan, which obviously amounts to:

$$\forall n[\text{god} - n \text{ winks at } x_n = 1/2^n \leftrightarrow \neg\exists m(m > n) \text{ god} - m \text{ winks at } x_m = 1/2^m] \tag{6-1}$$

Moreover, we know that (according to SLP) no god will fulfil (6) because no god will wink. We also know that if the number of gods is finite (N), they will all therefore fulfil (6). The following question immediately arises in view of this: isn't SLP being violated by acknowledging that the finite case (where all the gods fulfil (6)) is radically opposed to the infinite case (where no god fulfils (6)). No, it is not being violated, and there are two main reasons for this:

I) Special Limit Principle only refers to limit values of PHYSICAL QUANTITIES defined for physical objects $O_i$ (such as gods or man) that form part of a system of infinite physical objects $\{O_1, O_2, \ldots\}$ (such as the system formed by the gods and man). Note that SLP states that $M(O, O_i, t^*) = \lim_{n\to\infty} M(O_{(n)}, O_i, t^*)$, where M is a PHYSICAL MAGNITUDE involving $O_i$, and that at this limit $O_{(n)}$ is variable but $O_i$ is not: SLP specifically refers to $O_i$, albeit in the context of O. PHYSICAL CHARACTERISTICS (such as the physical configuration corresponding to the fact of winking or not winking, referred to here as the ocular configuration) of god-i, $M(O, O_i, t^*)$, can be determined, at least in principle, at the limit of infinite gods by calculating $\lim_{n\to\infty} M(O_{(n)}, O_i, t^*)$. However, SLP does not authorise the determination of NON-PHYSICAL CHARACTERISTICS (such as the fulfilment or non-fulfilment of plan (6)) of god-i, $M(O, O_i, t^*)$, at the limit of infinite gods by calculating $\lim_{n\to\infty} M(O_{(n)},$

$O_i$, t*). SLP does not therefore authorise the conclusion that god-i will fulfil plan (6) in the case of infinite gods on the basis that it will fulfil in the finite case (and does not authorise it because fulfilling this plan or not IS NOT A PHYSICAL CHARACTERISTIC involving god-i $O_i$, unlike, for example, its ocular configuration, which is).

II) Furthermore, SLP does not authorise the determination of characteristics for the SET OF GODS, M(O, t*), at the limit of infinite gods by calculating these characteristics for finite sets of gods, $M(O_{(n)}, t^*)$, and by then taking limit $\lim_{n\to\infty} M(O_{(n)}, t^*)$. On the contrary, in SLP the limit is taken in relation to a given characteristic M (more specifically $M(O_{(n)}, O_i, t^*)$) of A WELL-DEFINED OBJECT $O_i$ in infinite system O, an object whose identity remains unchanged in the limit-taking process. In other words, $\lim_{n\to\infty} M(O_{(n)}, O_i, t^*)$ is calculated, where $M(O_{(n)}, O_i, t^*)$ is a property of $O_i$, and not $\lim_{n\to\infty} M(O_{(n)}, t^*)$, where $M(O_{(n)}, t^*)$ is a property of $O_{(n)}$. SLP does not therefore authorise the conclusion that infinite gods will fulfil the plan (6) on the basis that, in the case of N gods, the number of gods fulfilling the plan (6) is N and by then taking limit $N \to \infty$. This explains why the case involving winking rather than placing barriers fails to constitute a counterexample to SLP.

## 5 Further Comments on Winking and Placing Barriers. The Role of SLP. Response to a Second Criticism

It was seen in Benardete's paradox that the man is stopped at $x = 0$ by a force. The force has its origin in the disposition of the gods to act in accordance with (5)[8] and SLP (applied to this case) is a formal expression of such disposition. We also know that if the gods intend to act in accordance with (6) none will do so, for none will wink. Analogously, this strange conclusion that no god will wink, this singular unanimity among them all, follows from the disposition of the gods to act in accordance with (6), and SLP (applied to this case) is again a formal expression of such disposition. The surprising consequence that none of the infinite gods will wink could thus be interpreted as an argument against the plausibility of SLP, but that is not so. Imagine the most natural case, where each god-n proposes to stop the man when (and only when) he passes point $x_n = 1/2^n$ by simply winking at $x_n = 1/2^n$. Each god-n thus acts according to the conjunction of

$$\forall n(\text{the man reaches } x_n = 1/2^n \rightarrow \text{god}-n \text{ winks at } x_n = 1/2^n) \quad (7)$$

and

$$\forall n(\text{god}-n \text{ winks at } x_n = 1/2^n \rightarrow \text{the man reaches } x_n = 1/2^n) \quad (8)$$

From (7) it logically follows that

[8] There is of course nothing mysterious about moving an object by exerting a force originating from our desire or disposition to do so.

$$\forall n[(\text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n)\ \text{god}-m \text{ winks at } x_m = 1/2^m) \rightarrow \text{god}-n \text{ winks at } x_n = 1/2^n] \tag{9}$$

However, from (8) it does not logically follow that

$$\forall n[\text{god}-n \text{ winks at } x_n = 1/2^n \rightarrow (\text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n)\ \text{god}-m \text{ winks at } x_m = 1/2^m)] \tag{10}$$

(10) is in fact a rather extravagant rule of action (which exceeds the mere conjunction of (7) and (8)) as it conditions winking at $x_n = 1/2^n$ in such a way that no god-m ($m > n$) winks at the corresponding point, and this is neither explicit nor implicit in the conjunction of (7) and (8). So,

$$\forall n[\text{god}-n \text{ winks at } x_n = 1/2^n \leftrightarrow (\text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n)\text{god}-m \text{ winks at } x_m = 1/2^m)] \tag{11}$$

is the conjunction of (9) and (10). Thus, the inadequacy (extravagance) of (10) on the basis of the conjunction (7) & (8) translates into the inadequacy (extravagance) of (11) on the same basis. This explains the strange unanimity of the gods in not winking: they propose to act according to a plan, (11), which artificially exceeds the natural basis (7) & (8). Acting under (7) & (8), all the gods will wink, who are powerless to stop the man with such a ruse. This is a perfectly natural conclusion which is also justified by SLP.

It may be interesting to compare the above with a parallel analysis of Benardete's paradox. The analogues of (7), (8), (9), (10) and (11) are evidently:

$$\forall n(\text{the man reaches } x_n = 1/2^n \rightarrow \text{god}-n \text{ places a barrier at } x_n = 1/2^n) \tag{12}$$

$$\forall n(\text{god}-n \text{ places } a \text{ barrier at } x_n = 1/2^n \rightarrow \text{the man reaches } x_n = 1/2^n) \tag{13}$$

$$\forall n(\text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n)\text{god}-m \text{ places a barrier at } x_m = 1/2^m \rightarrow \text{god}-n \text{ places a barrier at } x_n = 1/2^n) \tag{14}$$

$$\forall n(\text{god}-n \text{ places } a \text{ barrier at } x_n = 1/2^n \rightarrow \text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n)\text{god}-m \text{ places } a \text{ barrier at } x_m = 1/2^m) \tag{15}$$

$$\forall n[\text{god}-n \text{ places } a \text{ barrier at } x_n = 1/2^n \leftrightarrow (\text{the man reaches } x_n = 1/2^n \& \neg\exists m(m > n)\text{god}-m \text{ places a barrier at } x_m = 1/2^m)] \tag{16}$$

The key to the difference between the two cases lies in the contrast between (10) and (15). In contrast to (10), (15) is not extravagant, rather it is implicit in the conditions of the situation presented by Benardete ((12)&(13)). The reason being that, contrary to what occurs by winking, the placing of a barrier has the causal power to stop the man. Hence, if god-n places a barrier at $x_n = 1/2^n$, then not only is it true (because of (13)) that the man reaches $x_n = 1/2^n$ but also that $\neg\exists m$ ($m > n$) god-m places a barrier at $x_m = 1/2^m$ (since, in the event that $\exists m$ ($m > n$) god-m places a barrier at $x_m = 1/2^m$, the causal power of the barrier to stop the man at $x_m = 1/2^m$, stops the man from reaching $x_n = 1/2^n$ and, because of (13), stops god-n from placing a barrier there). Consequently

(15) is justified on the basis of (12) and (13). Now (5) is the conjunction of (14) and (15). Thus, the adequacy of (15) on the basis of conjunction (12) & (13) translates into the adequacy of (5) on the same basis. Acting in accordance with plan (5) in no way exceeds the natural basis (12) & (13) given our elementary intuitions concerning the causal power of barriers (as opposed to the fictitious role of winking).

## 6 Sensors, Marks and the Role of SLP. Response to a Third Criticism

As a final application of SLP, two new and apparent counterexamples are considered. First, suppose that, in addition to gods with their familiar winking plan, there is a sensor S. S beeps every time a god winks. Therefore, whenever there is only a finite number of gods, the man walks without altering his motion and sensor S beeps. This is because there is a first god (termed god-n*) that the man encounters. Consequently, god-n* will wink and, immediately afterwards, S will beep. We shall now turn to the case of infinite gods. In such circumstances it has been seen that if SLP is true, no god will wink. However, since there is no wink, then sensor S will not beep either. It seems therefore that SLP is false because the sensor is triggered (i.e. there is a physical magnitude, the position of its control switch, which turns "on") when there is a finite number of gods, but not when there is an infinite number of gods.

The response to this criticism can be succinctly expressed as follows: it is not true that in the infinite case the sensor does not beep. In order to see this, SLP shall be first applied by taking sensor S (which is a component of the complete infinite system, and which also includes the gods and the man) as object $O_i$ and the control switch position of S (which allows two positions, "on" and "off") as physical quantity M. When there is a finite number of gods, S' control switch always turns "on". SLP therefore leads to the fact that at the limit of infinite gods this switch will also turn "on" (and thus beep). Even if it is accepted that in the finite case the sensor is triggered if and only if a god winks, such a thing cannot hold in the infinite case. The fact that the sensor is triggered ("turns on") under these conditions (the infinite case) even if none of the infinite gods winks is nothing more than another variant of the Benardete paradox, where the man cannot advance even if none of the infinite walls prevents him from doing so. Assuming that, under the conditions of this new situation, only the winking of a god can trigger the sensor is tantamount to assuming that, under the conditions of the Benardete paradox, only a wall can stop the man. Both assumptions are false. It is now clear how SLP proves to be true with no contradiction: the sensor always beeps when there are finitely many gods, and also when there are infinitely many of them.

Another apparent counterexample to SLP is obtained by suppressing sensor S and assuming instead that each god-n has a small particle $p_n$ at its disposal (alongside, at $x_n = 1/2^n$). Furthermore, it is also assumed that whenever a god-n winks (and it is known that this occurs if and only if the man passes by it), the god "marks" the man by attaching particle $p_n$ to his forehead. Therefore, whenever there is only a finite number of gods, the man walks without altering his motion and is "marked" by a particle attached to his forehead. This is because there is a first god (termed god-n*) that the man encounters. Consequently god-n* will wink and, immediately afterwards, attach $p_{n*}$ to the man's forehead. We shall now turn to the case of infinite gods. In such circumstances it has been seen that if SLP is true, no god will wink. So, as there is no wink, the man will not end up with a particle attached to his forehead either. It therefore seems that SLP is false because the man ends up with a particle attached to his forehead when there is a finite number of gods, but not when there is an infinite number of gods.

The response to this criticism can be succinctly expressed as follows: the argument for the falsity of SLP is incorrect. In order to see this, it should be noted that the system of infinite components $O = \{O_1, O_2, \ldots\}$ has each god-n, each $p_n$ particle and, finally, the man as its elements $O_i$. The man being $O_1$, god-n $O_{2n}$, and $p_n$ $O_{2n+1}$. For any physical magnitude M involving the man ($O_1$), SLP affirms that $M(O, O_1, t^*)$ is $\lim_{n\to\infty} M(O_{(n)}, O_1, t^*)$ (if said limit exists). t* can be taken as the instant when the man has completed his journey by arriving at B. If M(t) is the physical quantity that gives the number of particles attached to the man's forehead at instant t, then $M(t^*) = 1$ in the finite case but $M(t^*) = 0$ in the infinite case. This is true, but does not contradict SLP. The reason being that SLP does not refer to $M(t^*)$ but to $M(O, O_1, t^*)$ and requires that, in equality $M(O, O_1, t^*) = \lim_{n\to\infty} M(O_{(n)}, O_1, t^*)$, M is a physical quantity INVOLVING $O_1$. However, the number of particles attached to the man's forehead is a physical magnitude that does not exactly involve $O_1$. It actually involves system $\{O_1, O_3\}$ if there is only one god (god-1), system $\{O_1, O_3, O_5\}$ if there are only two gods (god-1 and god-2),…, system $\{O_1, O_3, O_5, \ldots, O_{2n+1}\}$ if there are only n gods (god-1, god-2, …, god-n), and so on. That is, it does not involve a well-defined object in infinite system O whose identity remains unchanged in the limit-taking process. Consequently, what is stated in II) on p. 14 regarding the response to one first criticism is not fulfilled. Therefore, SLP is not applicable to the physical quantity that gives the number of particles attached to the man's forehead. This does not however mean that SLP is not applicable to this type of situation. Indeed it is. In order to see how, consider any of the infinite number of $p_m$ particles that could in principle "mark" the man. Consider $M(O_{(n)}, p_m, t^*)$ as the physical quantity that gives the position of $p_m$ in an n-component subsystem of O. Even though $M(O_{(2m+1)}, p_m, t^*)$ places $p_m$ on the man's forehead (since the first god that he encounters is god-m), it is clear that, for $n > m$, $M(O_{(2n+1)}, p_m, t^*) = 1/2^m$ (this is the location of god-m, which neither winks nor moves its $p_m$ particle from its original position at $x_m = 1/2^m$). Thus, $\lim_{n\to\infty} M(O_{(2n+1)}, p_m, t^*) = 1/2^m$. So, in the case of infinite gods, $p_m$ remains with god-m. As this holds for any m, it follows that when there are infinite gods, every particle $p_n$ remains with god-n without being manipulated by it. Consequently, it can simply be deduced that, along with there being no winking, in the infinite case the man will not end up with a particle attached to his forehead. Thus, in both the finite and the infinite case, the man completes his journey with a particle attached to his forehead if and only if a god winks.

## 7 An Illuminating Comparison

In the first example given in the previous section (the example with sensor S), we saw that the equivalence:

$$\text{sensor S is triggered if and only if a god winks} \tag{17}$$

is not true in general (it is not true in the case of infinite gods). On the contrary, in the second example (the example with "marks"), the following equivalence was reached with no exceptions:

the man ends up with a particle attached to his forehead if and only if a god winks (18)

The difference between the two cases lies in one very clear fact: only the former is a manifestation of Benardete's paradox. Although not only a god's wink can trigger sensor S (or, under the conditions of Benardete's original formulation, not only a wall can stop the man), it is clear that only a god's wink can place a mark (an attached particle) on the man's forehead. This is because each god can only act (by winking) on a different mark (god-n on, and exclusively on, particle $p_n$) whereas all gods can in principle act on sensor S (triggering it by winking). That is, in the first case, case (18), each god is disposed to act on a different object (god-n on $p_n$) while in the second case, case (17), there is one object (sensor S) on which each god is disposed to act. The former could be called a case of distributive disposition (dispositions to act distributed over different objects, particles $p_1$, $p_2$, $p_3$, …) and the latter a case of collective disposition (dispositions to act on a single object S). As is known, precisely the collective dispositions in this sense are what lie at the basis of the original Benardete paradox and its variants. Hence, only the example involving the sensor (but not the example involving the marks) provides a model of the Benardete paradox in the context of the Special Limit Principle. Evidently, if in place of infinite particles $p_1$, $p_2$, $p_3$, … there is a single particle p (all else remaining equal), the original distributive disposition then becomes a collective disposition and, in such a case, the example involving the "marks" also constitutes a realisation of Benardete's original paradox.[9] Assuming that, under these conditions (with a single particle p, all else remaining equal) only a god can place a mark (i.e., attach p) on the man's forehead is tantamount to assuming that, under the conditions of the case involving sensor S, only a god's wink can trigger S (or that, under the conditions of Benardete's paradox, only a wall can stop the man).

## 8 Final Words on Barriers and Sensors

Benardete's paradox with barriers (BPB) involves the man, gods with causal power over barriers and barriers with causal power over the man. It has been explained how SLP leads to (I) the gods being able to fulfil their plan of action and (II) the man being stopped without the gods acting on any barriers. Bernardete's paradox with a sensor (BPS) involves the man, sensor S and gods with causal power over S (god-n exerts causal power over S by winking). In this case it was seen that SLP

[9] This realisation has a remarkable feature: the disposition of the gods to act on particle p (by means of winking) now has the effect of SETTING p IN MOTION (until it is attached to the man's forehead). And it does so with no god performing this operation. Note that in Benardete's original paradox the gods' disposition to act on the man (by placing a wall) has instead the effect of PREVENTING the man's MOTION. However, it also does so with no god performing such an operation.

leads to (I) the gods being unable to fulfil their plan of action, (II) the man not being stopped and (III) sensor S being triggered (beeping) with no god acting on it. Also, the gods in BPS are known to act collectively, not individually, on S and are what triggers it. Let us imagine that the sensor is not triggered under BPS conditions. This outcome could be called "static sensor". This is incompatible with SLP. Is the static sensor outcome plausible? As the gods are unable to fulfil their plan in BPS, it would seem that they must also be unable to act collectively and, consequently, that S would therefore not be triggered. Is this scenario involving a static sensor at least as plausible as SLP? The argument below shows that it is clearly less plausible than SLP. We shall now construct a new variant of Benardete's paradox; Benardete's paradox with a causal sensor (BPSC). The difference from BPS is that the sensor S has an additional causal power Ad*: each time S is triggered (AND ONLY IN THIS CASE), it acts on the advancing man by paralysing him, while S remains triggered from then on. It is obvious that the man will now be stopped at $x = 0$,[10] as in the case of BPB barriers. BPSC is indeed an interesting variation of BPB, but instead of infinite barriers capable of blocking the man, there is just one "barrier" capable of doing so (S now fulfils this function). The gods are known to have power over S and to trigger it (as in BPB, the gods act collectively on S, not individually). And now S also has power over the man (unlike BPS) and stops him. As mentioned, when S is triggered, it has causal power over the man and, once triggered (on), it remains so thereafter. The gods trigger S through their collective disposition to act on S. And S stops the man, this time by means of a conventional (non-infinitary) causal connection involving just two objects: S and the man. We shall now consider one final situation (BPSC⁻), which is identical to BPSC apart from the fact that S' previous causal power over the man is removed, leaving all else unchanged. BPSC⁻ thus describes the same situation as BPS. Compared with BPSC, only the causal action of triggered sensor S on the man has disappeared (since the only thing that changes when transitioning from BPSC to BPSC⁻ is that S lacks additional causal power Ad*). Consequently, the collective action of the gods on S is maintained in BPSC⁻ and, therefore, S is ultimately triggered (beeps). Since BPSC⁻ describes the same situation as BPS, this explains (and makes very plausible) the verdict of SLP on S in BPS: S is ultimately triggered. Opposed to this, the alternative scenario with a static sensor (which is incompatible with SLP) seems clearly indefensible.

The barriers version (BPB) and S-sensor version (BPS) are characteristic examples of Benardete's paradox where a single instantaneous action is performed: stopping the man in the first case, and triggering sensor S in the second. It is therefore inappropriate to label Benardete's paradox as a supertask, which is not uncommon in the literature. Laraudogoitia (2016) proposed the term subtask for this kind of Benardete-type situation. This seems justifiable because, even if a single task or action is performed (such as stopping the man or triggering sensor S), the collective disposition involved entails (as seen) that NONE

[10] Since from (6) and the sequence of trivial equivalences god-n winks at $x_n = 1/2^n$ if and only if god-n triggers S if and only if the man is stopped at $x_n = 1/2^n$ it follows that $\forall n$[the man is stopped at $x_n = 1/2^n \leftrightarrow$ (the man reaches $x_n = 1/2^n$ & $\neg\exists m$ $(m > n)$ the man is stopped at $x_m = 1/2.^m$)].

OF THE INFINITE ACTORS SUSTAINING SUCH A DISPOSITION EVER PERFORMS A TASK OR ACTION OF ANY KIND. In this sense the actors would execute a subtask, with a certain task (action) as an outcome (analogously to how the execution of a supertask also involves a certain task or action as an outcome). A supertask's outcome requires infinite actions—a subtask's outcome requires none.

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## Declarations

**Conflict of Interest** The authors declare that they have no conflict of interest.



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