Identical pseudospectra of any geometric multiplicity*

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Dedicated to Professor José António Dias da Silva

Abstract

If A, B are $n \times n$ complex matrices such that the singular values of $zI_n - A$ are the same as those of $zI_n - B$ for each $z \in \mathbb{C}$, then A and B are similar.

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1 Introduction

Let $M \in \mathbb{C}^{n\times n}$. Let $\Lambda(M)$ denote the spectrum of M and let $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$ denote the singular values of M arranged in decreasing order. We write $\|\cdot\|_2$ for the Euclidean norm on $\mathbb{C}^n$, defined by $\|x\|_2 := (\sum_{i=1}^n |x_i|^2)^{1/2}$, and $\|\cdot\|$ for the associated operator norm on $\mathbb{C}^{n\times n}$, defined by $\|M\| := \sup\{\|Mx\|_2 \colon \|x\|_2 = 1\}$. We write $\mathrm{GL}_n(\mathbb{C})$ for the group of invertible matrices of $\mathbb{C}^{n\times n}$. Given $\varepsilon \geq 0$, the ordinary ε-pseudospectrum of M can be defined as the set $\Lambda_\varepsilon(M) := \{z \in \mathbb{C} \colon \sigma_n(zI_n - M) \leq \varepsilon\}$.

In the Ph.D. Thesis of M. Karow [4] the relationship was shown between the condition numbers of eigenvalues of a matrix $M \in \mathbb{C}^{n\times n}$, whose spectrum is $\{\lambda_1, \ldots, \lambda_p\}$, and its Jordan decomposition

$$M = \sum_{i=1}^{p} (\lambda_i P_i + N_i),$$

where P_i is the Riesz projector corresponding to λ_i and N_i is the eigennilpotent matrix associated with λ_i. In particular, the index $\nu(\lambda_i)$ of each eigenvalue λ_i plays a major role. Moreover, in the same dissertation, the condition number of the eigenvalue λ_i is related to the connected component of the pseudospectrum

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$\Lambda_\varepsilon(M)$ containing λ_i. These facts led us to think that there should be a closer relationship between the Jordan canonical form of A and its pseudospectra.

Let k be an integer, $1 \le k \le n$. For $\varepsilon \ge 0$, the geometric ε-pseudospectrum of M of order k can be defined as the set

$$\Lambda_{\varepsilon,k}^{(\mathrm{g})}(M) = \{z \in \mathbb{C} : \sigma_{n-k+1}(zI_n - M) \le \varepsilon\}.$$

In this paper we are going to establish that the geometric pseudospectra $\Lambda_{\varepsilon,k}^{(\mathrm{g})}(A)$ for small enough ε determine the Jordan canonical form of A, or equivalently, determine its invariant factors. This is the content of the main theorem in this paper, which is the following.

Theorem 1 (Sufficient condition for similarity)**.** *Let $A, B \in \mathbb{C}^{n\times n}$. Let us assume that for each $z \in \mathbb{C}$ the singular values of $zI_n - A$ are the same as those of $zI_n - B$. Then A and B are similar matrices.*

This Theorem will be proved in Section 3.

Remark 1. Notice that if A and B are similar and both matrices are normal, then for each $z \in \mathbb{C}$ the singular values of $zI_n - A$ are the same as those of $zI_n - B$. This is no longer true if A and B are not assumed normal.

Theorem 1 was also inspired by Fact 5(b), page 16-2 in Chapter 16 on Pseudospectra written by M. Embree in the Handbook of Linear Algebra, edited by L. Hogben [3]. This Fact says that if A and B are $n \times n$ complex matrices that have the same *ordinary* ε-pseudospectrum for every $\varepsilon > 0$, then A and B have the same minimal polynomial. We remark that $\Lambda_\varepsilon(M) = \Lambda_{\varepsilon,1}^{(\mathrm{g})}(M)$.

Once we had proven our theorems, we read the paper by M. Fortier Bourque and T. Ransford [1], which came to confirm our hunch. Two matrices $A, B \in \mathbb{C}^{n\times n}$ are said to be unitarily similar if there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $B = U^*AU$, where * stands for the conjugate transpose. M. F. Bourque and T. Ransford say that the complex $n \times n$ matrices A and B have *super-identical pseudospectra* if, for each $z \in \mathbb{C}$, the singular values of $zI_n - A$ are the same as those of $zI_n - B$. In [1] it was also proved that this condition is excessive, and it is sufficient to require these equalities for a certain finite set F of $\mathbb{C}$; namely,

Theorem 2. *Let $F := \{r_p e^{i\theta_q} : p, q = 0, \dots, n\}$, where $0 < \theta_0 < \cdots < \theta_n < \pi$ and $0 < r_0 < \cdots < r_n$. Suppose that $A, B \in \mathbb{C}^{n\times n}$ satisfy*

$$\sigma_k(zI_n - A) = \sigma_k(zI_n - B) \quad (z \in F, k = 1, \dots, n).$$

Then A and B have super-identical pseudospectra.

Also they showed that: (a) $A, B \in \mathbb{C}^{2\times 2}$ have super-identical pseudospectra if and only if A is unitarily similar to B; (b) $A, B \in \mathbb{C}^{3\times 3}$ have super-identical pseudospectra if and only if A is unitarily similar to B or to its transpose; (c) there exist $A, B \in \mathbb{C}^{4\times 4}$ with super-identical pseudospectra such that $\|A^2\| \ne \|B^2\|$, this implies that A is not unitarily similar either to B or to its transpose.

We would note that there are problems in pure mathematics and control theory where the simultaneous consideration of all the singular values leads to more satisfactory solutions, like the problem of studying the approximation of a bounded matrix function on the unit circle by bounded analytic matrix functions on the unit disc [5].

The organization of this paper is as follows: Given $M \in \mathbb{C}^{n\times n}$ and z_0 an eigenvalue of M, we will analyze the asymptotic behavior of the singular values of the characteristic matrix $zI_n - M$ when $z \to z_0$ in Section 2. We will prove Theorem 1 in Section 3. In Section 4 we will give an extension of Theorem 1, and we will frame these results in the theory of pseudospectra.

2 Orders of the singular values of a characteristic matrix as infinitesimals

Let a matrix $M \in \mathbb{C}^{n\times n}$ and z_0 an eigenvalue of M. In this section we will study the asymptotic behavior of the singular values of the characteristic matrix $zI_n - M$, when $z \to z_0$. To that end, we need the following notations. Let $V'(z_0)$ be a punctured neighborhood of z_0 in $\mathbb{C}$, we consider the set $\mathcal{F}$ of real functions defined on $V'(z_0)$. Then, we have the following definition.

Definition 1. Let $f, g \in \mathcal{F}$. If there are constants $\delta, \Delta, d > 0$ such that for every $z \in B'(z_0, d)$ (open punctured disk centered at z_0 and radius d)

$$f(z) > 0, g(z) > 0 \text{ and } \delta \leq \frac{f(z)}{g(z)} \leq \Delta,$$

we write (with Hardy's notation [2])

$$f(z) \asymp g(z) \qquad (\text{when } z \to z_0).$$

We say that a function $f \in \mathcal{F}$ is an *infinitesimal* as $z \to z_0$ if $\lim_{z\to z_0} f(z) = 0$. If $f(z) \asymp |z - z_0|^k$ (with k integer ≥ 1) we say that $f(z)$ is an *infinitesimal of order* k as $z \to z_0$. The relation $\asymp$ is an equivalence relation.

Remark 2. If j, k are integers ≥ 0 and

$$|z - z_0|^j \asymp |z - z_0|^k \qquad (z \to z_0),$$

then $j = k$.

Remark 3. Recall that for positive functions $f, g \in \mathcal{F}$ the relation $f(z) \sim g(z)$ as $z \to z_0$ means

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = 1.$$

It is obvious that $f(z) \sim g(z)$ as $z \to z_0$ implies $f(z) \asymp g(z)$ as $z \to z_0$.

The main result of this section is the following lemma.

Lemma 3. *If $J_k(z_0)$ is the $k \times k$ Jordan block with eigenvalue z_0, then, as $z \to z_0$,*

$$\sigma_j\Big(zI_k - J_k(z_0)\Big) \sim \begin{cases} 1, & j = 1, \ldots, k-1, \\ |z - z_0|^k, & j = k. \end{cases}$$

Proof. Without loss of generality, we may suppose that $z_0 = 0$ and write simply $J_k := J_k(0)$. Since $J_k^* J_k = \mathrm{diag}(0, 1, \ldots, 1)$, it follows that the singular values of J_k are $1, \ldots, 1, 0$. Hence $\sigma_j(zI_k - J_k) \to 1$ as $z \to 0$ for $j = 1, 2, \ldots, k-1$. Also

$$\prod_{j=1}^{k} \sigma_j(zI_k - J_k)^2 = \det((zI_k - J_k)^*(zI_k - J_k)) = |\det(zI_k - J_k)|^2 = |z|^{2k},$$

whence it follows that $\sigma_k(zI_k - J_k) \sim |z|^k$ as $z \to 0$.

□

For the proof of Lemma 7, we need some preliminary results. The first one can be seen in [6].

Lemma 4. *Let $M_1, M_2, M_3 \in \mathbb{C}^{n\times n}$. Then, for $k = 1, 2, \ldots, n$,*

$$\sigma_n(M_1)\sigma_k(M_2)\sigma_n(M_3) \le \sigma_k(M_1 M_2 M_3) \le \|M_1\|\|M_3\|\sigma_k(M_2).$$

With this result we can prove the following.

Lemma 5. *Let $M \in \mathbb{C}^{n\times n}, P \in \mathrm{GL}_n(\mathbb{C})$ and $z_0 \in \mathbb{C}$. Then, for $j = 1, 2, \ldots, n$,*

$$\sigma_j(zI_n - P^{-1}MP) \asymp \sigma_j(zI_n - M) \qquad (z \to z_0).$$

Lemma 6. *Let $L \in \mathbb{C}^{q\times q}$ and z_0 be a complex number such that $z_0 \notin \Lambda(L)$. Then, for $j = 1, 2, \ldots, q$,*

$$\sigma_j(zI_q - L) \asymp 1 \qquad (z \to z_0).$$

Proof. For $j = 1, 2, \ldots, q$, the limit

$$\lim_{z\to z_0} \sigma_j(zI_q - L) = \sigma_j(z_0 I_q - L)$$

is nonzero and finite.

□

Lemma 7. *Let J be the Jordan form of a matrix $M \in \mathbb{C}^{n\times n}$. Let $z_0 \in \mathbb{C}$ and $k \in \{1, \ldots, n\}$. Then the number of $k \times k$ Jordan blocks in J with eigenvalue z_0 is equal to the number of $j \in \{1, \ldots, n\}$ such that $\sigma_j(zI_n - M) \asymp |z - z_0|^k$ as $z \to z_0$.*

Proof. By Lemma 6 if $z_0 \notin \Lambda(M)$, then for $j \in \{1, \ldots, n\}$,

$$\sigma_j(zI_n - M) \asymp 1 \quad (z \to z_0);$$

so, in this case, there is no j such that $\sigma_j(zI_n - M) \asymp |z - z_0|^k$ as $z \to z_0$.

If $z_0 \in \Lambda(M)$, by Lemma 5, for $j = 1, \ldots, n$,

$$\sigma_j(zI_n - M) \asymp \sigma_j(zI_n - J) \quad (z \to z_0).$$

Let

$$J = J_0 \oplus J_1,$$

where $J_0 \in \mathbb{C}^{n_0 \times n_0}$ is the direct sum of the t Jordan blocks associated with z_0, and $z_0 \notin \Lambda(J_1)$. When z is sufficiently close to z_0, the last t singular values of $zI_n - J$ are just the infinitesimal singular values of $zI_{n_0} - J_0$ as $z \to z_0$. Thus,

$$\lim_{z \to z_0} \sigma_j(zI_n - J) = 0, \qquad \text{for } j = n-t+1, \ldots, n-1, n.$$

The number of $j \in \{n-t+1, \ldots, n-1, n\}$ such that the order of the infinitesimal $\sigma_j(zI_n - J)$ as $z \to z_0$ is k, is equal to the number of $k \times k$ Jordan blocks in J_0 associated with z_0. For $j \in \{1, \ldots, n-t\}$, we have $\sigma_j(zI_n - J) \asymp 1$ as $z \to z_0$.

□

3 Proof of the main result

In this section, we will prove the main result of this paper.

Proof of Theorem 1.

Let $M \in \mathbb{C}^{n \times n}$ and $z_0 \in \mathbb{C}$. Then $z_0 \in \Lambda(M)$ if and only if $\sigma_n(z_0 I_n - M) = 0$. Since for each $z \in \mathbb{C}$, $\sigma_n(zI_n - A) = \sigma_n(zI_n - B)$, the eigenvalues of A and B are the same,

$$\Lambda(A) = \Lambda(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}.$$

As $\sigma_j(zI_n - A) = \sigma_j(zI_n - B)$ for $z \in \mathbb{C}$ and $j \in \{1, \ldots, n\}$, then for each $k \in \{1, \ldots, n\}$ and $\lambda_i \in \Lambda(A)$, the number of $j \in \{1, \ldots, n\}$ such that

$$\sigma_j(zI_n - A) \asymp |z - \lambda_i|^k \qquad \text{as } z \to \lambda_i$$

is equal to the number of $j \in \{1, \ldots, n\}$ such that

$$\sigma_j(zI_n - B) \asymp |z - \lambda_i|^k \qquad \text{as } z \to \lambda_i.$$

Thus, by Lemma 7, the number of $k \times k$ Jordan blocks associated with λ_i in the Jordan forms of A and B is the same. Given that this holds for every $\lambda_i \in \Lambda(A) = \Lambda(B)$, we infer that A and B are similar.

□

4 Remarks

Following a line of reasoning similar to that of Theorem 1, we can establish the following theorem.

Theorem 8. *Let $A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{m \times m}$. Let us suppose that $n \geq m$ and let*

$$g_i(\lambda)|g_{i+1}(\lambda)|\cdots|g_{m-1}(\lambda)|g_m(\lambda)$$

be the nontrivial invariant factors of B. Let us assume that for each $z \in \mathbb{C}$ and $k = 1, 2, \ldots, m-i+1$,

$$\sigma_{n-k+1}(zI_n - A) = \sigma_{m-k+1}(zI_m - B) \tag{1}$$

Then the last $m-i+1$ invariant factors of A,

$$f_{n-m+i}(\lambda)|f_{n-m+i+1}(\lambda)|\cdots|f_{n-1}(\lambda)|f_n(\lambda),$$

are nontrivial, and

$$f_n(\lambda) = g_m(\lambda),\ f_{n-1}(\lambda) = g_{m-1}(\lambda),\ \ldots, f_{n-m+i}(\lambda) = g_i(\lambda).$$

Let $M \in \mathbb{C}^{n\times n}$. For every real number $\varepsilon \geq 0$, another equivalent definition of the ordinary ε-pseudospectrum of M is

$$\Lambda_\varepsilon(M) := \bigcup_{\substack{X\in\mathbb{C}^{n\times n}\\ \|X-M\|\leq\varepsilon}} \Lambda(X).$$

For $z \in \mathbb{C}$ we denote by $\mathrm{gm}(z, M)$ the geometric multiplicity of z as eigenvalue of M. If $z \notin \Lambda(M)$, we agree that $\mathrm{gm}(z, M) = 0$. Let k be an integer, $1 \leq k \leq n$, and let $\Lambda_k^{(\mathrm{g})}(M)$ denote the set of $z \in \Lambda(M)$ such that $\mathrm{gm}(z, M) \geq k$. For $\varepsilon \geq 0$, the geometric ε-pseudospectrum of M of order k can be defined, alternatively, by

$$\Lambda_{\varepsilon,k}^{(\mathrm{g})}(M) := \bigcup_{\substack{X\in\mathbb{C}^{n\times n}\\ \|X-M\|\leq\varepsilon}} \Lambda_k^{(\mathrm{g})}(X).$$

5 Conclusions

Let A, B be $n \times n$ complex matrices such that the singular values of $zI_n - A$ are the same as those of $zI_n - B$ for each $z \in \mathbb{C}$. Then A and B are similar. A more general result for square matrices A and B of distinct sizes has been stated.

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