On the boundaries of strict pseudospectra *

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Abstract

The boundary of the ordinary ε-pseudospectrum of a square matrix is contained in the boundary of the strict ε-pseudospectrum. This content relation may be strict in some cases.

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1 Introduction

The boundary of the ordinary pseudospectrum of a square matrix A of level ε, denoted by $\partial\Lambda_\varepsilon(A)$, is contained in the boundary of the strict pseudospectrum of the same level $\partial\Lambda'_\varepsilon(A)$ (see Remark 3.2, p. 280 in [1]). In this paper we will prove that in general these boundaries are not equal.

An equivalent problem is to determine whether the function of $z \mapsto \sigma_n(zI_n - A)$ can have local maxima. Thus, we will show that a complex number $z_0 \in \partial\Lambda'_\varepsilon(A) \setminus \partial\Lambda_\varepsilon(A)$ if and only if the function $z \mapsto \sigma_n(zI_n - A)$ reaches a local maximum at z_0. As a result, we will prove that the function $z \mapsto \sigma_n(zI_n - A)$ can have local maxima.

On the other hand, both the ordinary pseudospectrum of a matrix A of level ε and its boundary are semialgebraic sets [6]. We will prove that this property is also true for the strict pseudospectrum. This fact will allow us to prove that the set $\partial\Lambda'_\varepsilon(A) \setminus \partial\Lambda_\varepsilon(A)$ can be: empty, finite, or formed by the union of a finite set and a real analytic submanifold of dimension 1 with a finite number of connected components.

2 Previous notation and main results

For the inclusion relation between two sets X and Y we will use the notations $X \subset Y$ and $X \subsetneq Y$ to mean "X is contained in or equal to Y" and "X is strictly contained in Y", respectively. Let $\mathbb{C}^{n\times n}$ denote the space of $n \times n$ complex matrices. For any matrix $M \in \mathbb{C}^{n\times n}$ let

$$\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$$

denote its singular values in decreasing order. Let $\Lambda(A)$ denote the spectrum of the matrix $A \in \mathbb{C}^{n\times n}$. Given $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$ the *ordinary pseudospectrum of level* ε is the set

$$\Lambda_\varepsilon(A) := \bigcup_{\|\Delta\|\leq\varepsilon} \Lambda(A + \Delta),$$

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where $\|\cdot\|$ denotes the spectral norm. Analogously, the *strict pseudospectrum of level* ε is the set

$$\Lambda'_\varepsilon(A) := \bigcup_{\|\Delta\|<\varepsilon} \Lambda(A+\Delta).$$

Let g denote the function

$$g(z) := \sigma_n(zI_n - A), \quad z \in \mathbb{C}; \tag{1}$$

using this function a characterization of the pseudospectra is given by

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : g(z) \leq \varepsilon\}, \tag{2}$$

and

$$\Lambda'_\varepsilon(A) = \{z \in \mathbb{C} : g(z) < \varepsilon\}. \tag{3}$$

(See page 82 in [3]). For a subset S of $\mathbb{C}$ we will denote the boundary of S by ∂S and let S^{c} denote the complementary set of S with respect to $\mathbb{C}$. Moreover, $\overline{S}$ and $\mathring{S}$ will denote the closure and the interior of S, respectively.

Identifying $z = x + y\mathrm{i} \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$, notation (1) can be translated as follows. We define the function

$$(x, y) \mapsto \sigma_n((x + y\mathrm{i})I_n - A), \tag{4}$$

from $\mathbb{R}^2$ to $\mathbb{R}$, which we also denote by g.

It is known that a set $\mathcal{S} \subset \mathbb{R}^n$ is *semialgebraic* ([2],[6]) if there exist polynomials

$$p_j, q_l^j \in \mathbb{R}[W_1, W_2, \ldots, W_n], \quad l = 1, 2, \ldots, r_j, \quad j = 1, 2, \ldots, k,$$

such that

$$\mathcal{S} = \bigcup_{j=1}^{k} \{w \in \mathbb{R}^n : p_j(w) = 0, \, q_l^j(w) > 0, \, l = 1, 2, \ldots, r_j\}. \tag{5}$$

Remark 1. This notion can be extended to subsets of $\mathbb{C}^n$, identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Thus, given a set $\mathcal{S} \subset \mathbb{C}^n$, we identify each element of $s \in \mathcal{S}$ with the pair (w, l) where $w, l \in \mathbb{R}^n$ and $s = w + l\mathrm{i}$.Therefore, we can suppose that $\mathcal{S}$ is a subset of $\mathbb{R}^{2n}$. So, we conclude that the subset $\mathcal{S}$ of $\mathbb{C}^n$ (or $\mathbb{R}^{2n}$) is semialgebraic if there exist polynomials of $\mathbb{R}[W_1, W_2, \ldots, W_n, L_1, L_2, \ldots, L_n]$ such that

$$\mathcal{S} = \bigcup_{j=1}^{k} \{(w, l) \in \mathbb{R}^{n+n} : p_j(w, l) = 0, \, q_l^j(w, l) > 0, \, l = 1, 2, \ldots, r_j\}.$$

With this notation, the main results of this paper are the following ones.

Theorem 2. *For each $\varepsilon > 0$, the sets $\Lambda'_\varepsilon(A)$ and $\partial\Lambda'_\varepsilon(A)$ are semialgebraic.*

Theorem 3. *Given a finite set of points $\mathcal{P}$ of $\mathbb{C}$, there exist a matrix $A_0 \in \mathbb{C}^{n\times n}$ and an $\varepsilon_0 > 0$ such that:*

(1)

$$\mathcal{P} \cap \partial\Lambda_{\varepsilon_0}(A_0) = \emptyset, \quad \mathcal{P} \subset \partial\Lambda'_{\varepsilon_0}(A_0)$$

and each $z_0 \in \mathcal{P}$ is an isolated point of the set $\partial\Lambda'_{\varepsilon_0}(A_0)$.

(2) *For each $z_0 \in \mathcal{P}$, the function $z \mapsto \sigma_n(zI_n - A_0)$ has a strict local maximum at z_0.*

Remark 4. Given that $\overline{\Lambda'_\varepsilon(A)} = \Lambda_\varepsilon(A)$ (Corollary 4.3 of [3]), then $\partial\Lambda_\varepsilon(A) \subsetneq \partial\Lambda'_\varepsilon(A)$ if and only if $\Lambda'_\varepsilon(A) \subsetneq \overset{\circ}{\widehat{\Lambda_\varepsilon(A)}}$. As a consequence, by Theorem 3, for a finite set of points $\mathcal{P}$ of $\mathbb{C}$, there exist an $\varepsilon_0 > 0$ and a square matrix A_0 such that $\mathcal{P} \subset \overset{\circ}{\widehat{\Lambda_{\varepsilon_0}(A_0)}} \setminus \Lambda'_{\varepsilon_0}(A_0)$.

3 Auxiliary results

We begin this section by introducing the notion of semialgebraic function ([2],[6]). Given two nonempty semialgebraic sets $\mathcal{S}_i \subset \mathbb{R}^{n_i}$, $i = 1, 2$, a function $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ is said to be semialgebraic if its graph is a semialgebraic subset of $\mathbb{R}^{n_1+n_2}$. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. Let $\mathcal{S}$ be a semialgebraic subset of $\mathbb{R}^n$. It is said that a function $f\colon \mathcal{S} \to \overline{\mathbb{R}}$ is semialgebraic if the sets $f^{-1}(-\infty)$ and $f^{-1}(\infty)$ are semialgebraic and the restriction $f\Big|_{\mathcal{S}\setminus f^{-1}(\{-\infty,\infty\})}$ is a semialgebraic function.

Lemma 5. *With the above notation, we have:*

(a) *The composition of semialgebraic functions is a semialgebraic function.*

(b) *Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a semialgebraic function. Suppose that $\mathcal{B} \subset \mathcal{S}_2$ is a semialgebraic set. Then $f^{-1}(\mathcal{B})$ is a semialgebraic set.*

(c) *The function $\sigma_n\colon \mathbb{C}^{n\times n} \to [0, \infty)$ is semialgebraic, and for each $\varepsilon \geq 0$ the set $\Lambda_\varepsilon(A)$ is semialgebraic.*

(d) *The boundary of a semialgebraic set is semialgebraic.*

(e) *If $\mathcal{S} \subset \mathbb{R}^2$ is a nonempty semialgebraic set, then its boundary $\partial\mathcal{S}$ can be written as the disjoint union*

$$\partial\mathcal{S} = \mathcal{M}\dot{\cup}\mathcal{Q}$$

where $\mathcal{M}$ is the empty set or a real analytic submanifold of $\mathbb{R}^2$ of dimension 1 formed by a finite number of connected components and $\mathcal{Q}$ is a finite set or the empty set.

(f) *Let $\mathcal{S}_1, \mathcal{S}_2$ be semialgebraic sets. If $f : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathbb{R}$ is a semialgebraic function, then the functions*

$$f_1, f_2\colon \mathcal{S}_1 \to \overline{\mathbb{R}}, \quad f_1(x) := \inf_{y\in\mathcal{S}_2} f(x, y) \quad \text{and} \quad f_2(x) := \sup_{y\in\mathcal{S}_2} f(x, y)$$

are semialgebraic.

The statements in (a) and (b) can be found in Propositions 2.2.6 and 2.2.7 of [2]. The properties in (c), (d), (e) and (f) are in Corollary 3.1.22, Proposition 3.1.8, Proposition 3.1.9, Corollary 3.1.10 and Corollary 3.1.15 of [6].

Let us denote by $\mathcal{D}(z_0, \rho)$ the closed disk of $\mathbb{C}$ centered at $z_0 \in \mathbb{C}$ with radius $\rho > 0$. Some properties of the pseudospectra and the function $g\colon \mathbb{C} \to \mathbb{R}$, defined in (1), are as follows.

Proposition 6. *Let $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$. We have:*

(a) *The only local minima of g are the eigenvalues of the matrix A.*

(b) $\partial\Lambda_\varepsilon(A) \subset \partial\Lambda'_\varepsilon(A) = \{z \in \mathbb{C} : g(z) = \varepsilon\}$.

(c) *For $\alpha, \gamma \in \mathbb{C}$, with $|\gamma| > 0$, we have*

$$\Lambda_\varepsilon(\gamma A + \alpha I_n) = \alpha + \gamma\Lambda_{\varepsilon/|\gamma|}(A).$$

The same property holds if we replace Λ with Λ', $\partial\Lambda$ or $\partial\Lambda'$.

(d) *If $A = \operatorname{diag}(A_1, A_2)$, with A_1, A_2 square matrices, then $\Lambda_\varepsilon(A) = \Lambda_\varepsilon(A_1) \cup \Lambda_\varepsilon(A_2)$. The same property holds for the strict pseudospectrum.*

(e) *Suppose that A is nilpotent. Then*

$$\Lambda_\varepsilon(A) \subset \mathcal{D}(0, \varepsilon + \|A\|).$$

The statement in (a) is Theorem 4.2 of [3]. The assertion in (b) can be seen in Remark 3.2 of [1] and it is proved in Proposition 5.2.19 of [4]. Let us introduce another proof of (b), which is based on elementary properties of pseudospectra.

Let $z_0 \in \partial\Lambda'_\varepsilon(A)$. If we use (2) and (3) and the fact that $\overline{\Lambda'_\varepsilon(A)} = \Lambda_\varepsilon(A)$, we have

$$\left.\begin{array}{l} z_0 \in \overline{\Lambda'_\varepsilon(A)} \subset \Lambda_\varepsilon(A) \Rightarrow g(z_0) \leq \varepsilon, \\ z_0 \in \overline{(\Lambda'_\varepsilon(A))^c} \Rightarrow g(z_0) \geq \varepsilon \end{array}\right\} \Rightarrow g(z_0) = \varepsilon.$$

Conversely, let z_0 be such that $g(z_0) = \varepsilon$. If $z_0 \notin \partial\Lambda'_\varepsilon(A)$, then $z_0 \in (\Lambda'_\varepsilon(A))^c$. So, there exists a $\delta > 0$ such that $\mathcal{D}(z_0, \delta) \cap \Lambda'_\varepsilon(A) = \emptyset$ and, therefore, $\mathcal{D}(z_0, \delta) \subset (\Lambda'_\varepsilon(A))^c$. Under these conditions, $g(z) \geq \varepsilon$ holds for every $z \in \mathcal{D}(z_0, \delta)$. Hence z_0 is a local minimum of g and by (a) z_0 is an eigenvalue of A, which is impossible.

The properties in (c) and (d) are in [5], 23-2(3-a), and [1], Proposition 2.3, respectively. Finally, (e) is deduced from [6] Corollary 5.3.5 and Proposition 5.3.7, page 73.

The next result is the key to prove Theorem 3.

Theorem 7. *Given $\varepsilon_0 > 0$, let $z_0 \in \partial\Lambda'_{\varepsilon_0}(A)$. Then:*

(a) *z_0 is an isolated point of $\partial\Lambda'_{\varepsilon_0}(A)$ if and only if g attains a strict local maximum at z_0.*

(b) *If z_0 is an isolated point of $\partial\Lambda'_{\varepsilon_0}(A)$, then $z_0 \notin \partial\Lambda_{\varepsilon_0}(A)$.*

(c) *$z_0 \in \partial\Lambda'_{\varepsilon_0}(A) \setminus \partial\Lambda_{\varepsilon_0}(A)$ if and only if g attains a local maximum at z_0.*

(d) *$z_0 \in \partial\Lambda'_{\varepsilon_0}(A) \setminus \partial\Lambda_{\varepsilon_0}(A)$ if and only if z_0 is an interior point of $\Lambda_{\varepsilon_0}(A)$ and $g(z_0) = \varepsilon_0$.*

Proof.

First, we are going to prove (a). Let us suppose that z_0 were an isolated point of $\partial\Lambda'_{\varepsilon_0}(A)$. Then there would exist an open ball $\mathcal{B}$ centered at z_0 with radius $\delta > 0$, such that $\forall z \in \mathcal{B}' = \mathcal{B} - \{z_0\}$, $g(z) \neq \varepsilon_0$ would be fulfilled.

Let us see that either $g(z) > \varepsilon_0$, $\forall z \in \mathcal{B}'$; or $g(z) < \varepsilon_0$, $\forall z \in \mathcal{B}'$. Let us suppose the contrary. Let $z_1, z_2 \in \mathcal{B}'$ be such that $g(z_1) < \varepsilon_0 < g(z_2)$. Let $t \mapsto \gamma(t)$ be a continuous curve on $[0,1]$, with $\gamma(0) = z_1$, $\gamma(1) = z_2$, such that $\gamma(t) \in \mathcal{B}'$, $\forall t \in [0,1]$. Then, as $t \mapsto g(\gamma(t))$ would be continuous on $[0,1]$, by the Bolzano's Theorem there would exist a $z_3 \in \mathcal{B}'$ such that $g(z_3) = \varepsilon_0$. This is a contradiction.

Now, let us see that $g(z) < \varepsilon_0$ for each $z \in \mathcal{B}'$. Otherwise, z_0 would be a local minimum of g. Therefore by virtue of Proposition 6 (a) z_0 would be an eigenvalue of A, i.e. $\varepsilon_0 = 0$, which is a contradiction. This proves that g has a strict local maximum at z_0. The converse is immediate, so that (a) is proved.

To prove (b) we see that z_0 being an isolated point of $\partial\Lambda'_{\varepsilon_0}(A)$, by (a) it follows that $\forall z \in \mathcal{B}$, $g(z) \leq \varepsilon_0$. Then $\mathcal{B} \cap \Lambda_{\varepsilon_0}(A)^c = \emptyset$. Therefore $z_0 \notin \partial\Lambda_{\varepsilon_0}(A)$.

To prove (c) let us note that

$$z_0 \in \partial\Lambda'_{\varepsilon_0}(A) \setminus \partial\Lambda_{\varepsilon_0}(A) \text{ if and only if } \left\{\begin{array}{l} g(z_0) = \varepsilon_0, \\ \forall z \in \mathcal{B},\ g(z) \leq \varepsilon_0. \end{array}\right.$$

Finally, to prove (d) let us observe that for any subset S of $\mathbb{C}$ the equality $\overline{S} = \mathring{S} \,\dot\cup\, \partial S$ holds, where $\dot\cup$ means disjoint union. As $\Lambda_{\varepsilon_0}(A)$ is a closed set, we have

$$\Lambda_{\varepsilon_0}(A) = \overset{\circ}{\widehat{\Lambda_{\varepsilon_0}(A)}} \,\dot\cup\, \partial\Lambda_{\varepsilon_0}(A). \tag{6}$$

If $z_0 \in \partial\Lambda'_{\varepsilon_0}(A) \setminus \partial\Lambda_{\varepsilon_0}(A)$, then by Proposition 6(b) $g(z_0) = \varepsilon_0$; so, $z_0 \in \Lambda_{\varepsilon_0}(A)$. Thus, by (6),

$$z_0 \in \overset{\circ}{\widehat{\Lambda_{\varepsilon_0}(A)}}.$$

Conversely, if z_0 is an interior point of $\Lambda_{\varepsilon_0}(A)$ and $g(z_0) = \varepsilon_0$, by Proposition 6(b) we deduce that $z_0 \in \partial\Lambda'_{\varepsilon_0}(A)$. Since

$$\overset{\circ}{\widehat{\Lambda_{\varepsilon_0}(A)}} \quad \text{and} \quad \partial\Lambda_{\varepsilon_0}(A)$$

are disjoint sets, then $z_0 \notin \partial\Lambda_{\varepsilon_0}(A)$. So,

$$z_0 \in \partial\Lambda'_{\varepsilon_0}(A) \setminus \partial\Lambda_{\varepsilon_0}(A).$$

□

Remark 8. *As a consequence of Theorem 7(b), if $\partial\Lambda'_{\varepsilon_0}(A)$ has isolated points, then $\partial\Lambda_{\varepsilon_0}(A) \subsetneq \partial\Lambda'_{\varepsilon_0}(A)$.*

Before we state another result that we will use, let us recall the following standard notation. If $\mathcal{R}$ is a commutative ring, $\mathcal{R}[x,y]$ denotes the set of polynomials in the variables x, y with coefficients in $\mathcal{R}$, and $\mathcal{R}[\varepsilon]$ stands for the set of polynomials in the variable ε with coefficients in $\mathcal{R}$.

Lemma 9. *Given $A \in \mathbb{C}^{n\times n}$, for $(x, y, \varepsilon) \in \mathbb{R}^3$ consider*

$$q(x, y, \varepsilon) := \det \begin{pmatrix} \varepsilon I_n & (x + y\mathrm{i})I_n - A \\ (x - y\mathrm{i})I_n - A^* & \varepsilon I_n \end{pmatrix} \in \mathbb{R}[\varepsilon][x, y].$$

Let $(x_0, y_0) \in \mathbb{R}^2$ be such that $g(x_0, y_0) > 0$. Suppose that (x_0, y_0) is an isolated point of the curve

$$q(x, y, g(x_0, y_0)) = 0.$$

Then (x_0, y_0) is an isolated point of

$$\partial\Lambda'_{g(x_0,y_0)}(A).$$

Proof. First, by the Wielandt's lemma, ε is a singular value of $(x + y\mathrm{i})I_n - A$ if and only if $q(x, y, \varepsilon) = 0$. Set $\varepsilon_0 := g(x_0, y_0)$. Then, by Proposition 6(b), we have

$$(x_0, y_0) \in \partial\Lambda'_{\varepsilon_0}(A) \subset \{(x, y) \in \mathbb{R}^2 : q(x, y, \varepsilon_0) = 0\}.$$

Thus, if (x_0, y_0) is an isolated point of the curve $q(x, y, \varepsilon_0) = 0$, so is it of $\partial\Lambda'_{\varepsilon_0}(A)$. □

To conclude this section, we present a characterization of the isolated points of a curve $f(x, y) = 0$ in $\mathbb{R}^2$ defined by a real function f of class C^2. Its proof can be seen in [9], pages 254–255.

Theorem 10. *Let us consider a curve in $\mathbb{R}^2$ defined by $f(x, y) = 0$, with f being a real function of class C^2 in a neighborhood of the point $(x_0, y_0) \in \mathbb{R}^2$. If (x_0, y_0) satisfy the conditions*

$$\begin{array}{l} f(x_0, y_0) = 0, \\ f'_x(x_0, y_0) = 0, \\ f'_y(x_0, y_0) = 0, \\ f''_{xy}(x_0, y_0)^2 - f''_{xx}(x_0, y_0) f''_{yy}(x_0, y_0) < 0, \end{array}$$

then (x_0, y_0) is an isolated point of the curve.

4 Proof of Theorem 2.

The main objective of this section is the proof of Theorem 2.

Proof of Theorem 2.

Let us consider the following diagram

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{h} & \mathbb{C}^{n\times n} & \xrightarrow{\sigma_n} & [0, \infty) \\ (x, y) & \mapsto & (x + \mathrm{i}y)I_n - A & \mapsto & \sigma_n((x + \mathrm{i}y)I_n - A) = g(x, y). \end{array}$$

Identifying $\mathbb{C}^{n\times n}$ with $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ and $A \in \mathbb{C}^{n\times n}$ with $(\mathrm{re}(A), \mathrm{im}(A)) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, we have

$$\mathrm{graph}(h) = \{(x, y, xI - \mathrm{re}(A), yI - \mathrm{im}(A)) \colon (x, y) \in \mathbb{R}^2\},$$

where $\mathrm{re}(A)$ and $\mathrm{im}(A)$ denote the real and imaginary part of A, respectively. Hence using the polynomial

$$P(x, y, X, Y) := \|X - xI + \mathrm{re}(A)\|_F^2 + \|Y - yI + \mathrm{im}(A)\|_F^2$$

it follows that $\mathrm{graph}(h) = \{(x, y, X, Y) : P(x, y, X, Y) = 0\}$, where $x, y \in \mathbb{R}$ and $X, Y \in \mathbb{R}^{n\times n}$. Here $\|\cdot\|_F$ denotes the Frobenius norm. Therefore, the set $\mathrm{graph}(h)$ is semialgebraic. As a consequence, the function h is semialgebraic. Now, by (a) and (c) in Lemma 5, we conclude that the function

$$\begin{array}{rccl} g: & \mathbb{R}^2 & \longrightarrow & [0, \infty) \\ & (x, y) & \mapsto & g(x, y) = \sigma_n((x + \mathrm{i}y)I_n - A) \end{array}$$

is semialgebraic. As the set $[0, \varepsilon)$ is semialgebraic, by Lemma 5(b) we deduce that $g^{-1}[0, \varepsilon)$ is semialgebraic. But

$$g^{-1}[0, \varepsilon) = \{(x, y) : \sigma_n((x + y\mathrm{i})I_n - A) < \varepsilon\} = \{(x, y) : g(x, y) < \varepsilon\} = \Lambda'_\varepsilon(A).$$

For this reason, $\Lambda'_\varepsilon(A)$ is a semialgebraic set. Finally, from Lemma 5(d), we infer that the set $\partial\Lambda'_\varepsilon(A)$ is semialgebraic. □

5 Proof of Theorem 3.

Before proving Theorem 3 we will demonstrate a previous proposition.

Proposition 11. *For the matrix*

$$B := \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}, \tag{7}$$

it is true that $-4/15 + 0\mathrm{i}$ *is an isolated point of* $\partial\Lambda'_{1/15}(B)$.

Proof. First, considering $(x, y) \in \mathbb{R}^2$ and using the notation of Lemma 9 where A is replaced by the matrix B given in (7), we obtain a family of real algebraic curves $q(x, y, \varepsilon) = 0$, indexed by ε. Next, we try to find a value $\varepsilon_0 > 0$ of ε such that the curve $q(x, y, \varepsilon_0) = 0$ has isolated points. A little calculation shows that

$$q(x, y, \varepsilon) = \varepsilon^6 - 3(x^2 + y^2 + 6)\varepsilon^4 + (3(x^2 + y^2)^2 + 18(x^2 + y^2) + 8x + 1)\varepsilon^2 - (x^2 + y^2)^3.$$

Using Theorem 10, we seek $(x, y, \varepsilon) \in \mathbb{R}^3$ satisfying

$$\begin{array}{l} q(x, y, \varepsilon) = q'_x(x, y, \varepsilon) = q'_y(x, y, \varepsilon) = 0 \\ q''_{xy}(x, y, \varepsilon)^2 - q''_{xx}(x, y, \varepsilon)q''_{yy}(x, y, \varepsilon) < 0. \end{array}$$

One can verify that

$$(x, y, \varepsilon) = (-4/15, 0, 1/15)$$

satisfies

$$q(-4/15, 0, 1/15) = q'_x(-4/15, 0, 1/15) = q'_y(-4/15, 0, 1/15) = 0$$

and

$$q''_{xy}(-4/15, 0, 1/15)^2 - q''_{xx}(-4/15, 0, 1/15)q''_{yy}(-4/15, 0, 1/15) = -44/16875 < 0.$$

As a consequence, by Theorem 10, the point $(-4/15, 0)$ is isolated for the curve $q(x, y, 1/15) = 0$. Thus, by Lemma 9, to demonstrate the proposition it suffices to prove that $1/15$ is the smallest singular value of the matrix $(-4/15)I_3 - B$. But its singular values are the positive square roots of the roots of the polynomial

$$p(\lambda) = \det[\lambda I_3 - ((-4/15)I_3 - B)^*((-4/15)I_3 - B)] = \lambda^3 - \frac{1366}{75}\lambda^2 + \frac{2731}{16875}\lambda - \frac{4096}{11390625},$$

which are

$$\left(\frac{64}{15}\right)^2, \quad \left(\frac{1}{15}\right)^2, \quad \left(\frac{1}{15}\right)^2.$$

Hence $g(-4/15, 0) = \sigma_3((-4/15)I_3 - B) = 1/15$. Thus, by Lemma 9, we deduce that $-4/15 + 0\mathrm{i}$ is an isolated point of $\partial\Lambda'_{1/15}(B)$. According to Theorem 7(a) the function g attains a strict local maximum at this point. See Figure 1. □

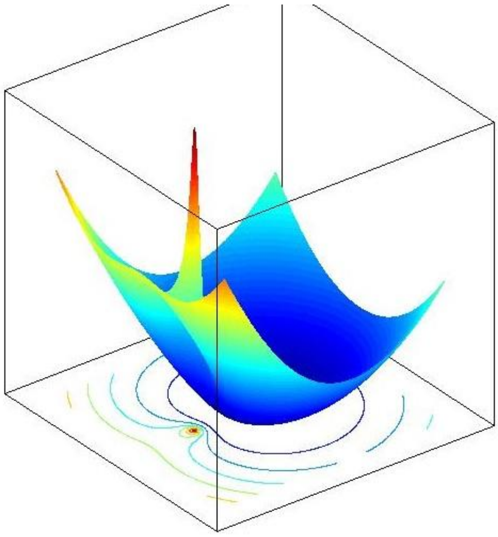

Figure 1: Strict local maximum of $g(x, y) = \sigma_3\big((x + y\mathrm{i})I_3 - B\big)$ at the point $(-4/15, 0)$.

Remark 12. There are some examples of matrices whose boundaries of the strict ε-pseudo-spectrum have an isolated point for a given ε.

Let $B_n := \mathrm{diag}(z_1, z_2, \ldots, z_n)$, where $z_k = \exp(2\pi\mathrm{i}k/n)$, $k = 1, \ldots, n$. Then $(0, 0)$ is an isolated point of $\partial\Lambda'_1(B_n)$. In fact, for each $z \in \mathbb{C}$ let

$$\mathrm{d}(z, \Lambda(B_n)) := \min_{k=1,\ldots,n} |z - z_k|$$

be the distance from z to $\Lambda(B_n)$; then $\sigma_n(zI_n - B_n) = \mathrm{d}(z, \Lambda(B_n))$. So, for each (x, y) in the deleted open ball centered at $(0, 0)$ and radius $1/2$, we have $\mathrm{d}\big(x + y\mathrm{i}, \Lambda(B_n)\big) < 1$; therefore, the function $(x, y) \mapsto \sigma_n\big((x + y\mathrm{i})I_n - B_n\big)$ has a strict local maximum at $(0, 0)$. This is a generalization of $B_4 := \mathrm{diag}(\mathrm{i}, -1, -\mathrm{i}, 1)$ (Example 2.6 of [8]). See Figure 2.

We are now ready to prove Theorem 3.

Proof of Theorem 3 To simplify the exposition, we will denote by

$$w_0 := \frac{-4}{15}, \quad \varepsilon_1 := \frac{1}{15}. \tag{8}$$

Let $\mathcal{P} := \{z_1, z_2, \ldots, z_k\} \subset \mathbb{C}$ and

$$\delta := \frac{1}{4} \min_{1 \le p < q \le k} |z_p - z_q|, \qquad \gamma := \frac{\delta}{\varepsilon_1 + \|B\|}, \qquad \varepsilon_0 = \gamma\varepsilon_1 = \gamma/15, \tag{9}$$

where B is the matrix in (7).

For $p = 1, 2, \ldots, k$, Proposition 6(c) implies

$$\Lambda'_{\varepsilon_0}(\gamma B + (z_p - \gamma w_0)I_3) = z_p - \gamma w_0 + \gamma\Lambda'_{\varepsilon_0/|\gamma|}(B) = z_p - \gamma w_0 + \gamma\Lambda'_{\varepsilon_1}(B). \tag{10}$$

Hence, from Proposition 11, we conclude that z_p is an isolated point of $\partial\Lambda'_{\varepsilon_0}(\gamma B + (z_p - \gamma w_0)I_3)$.

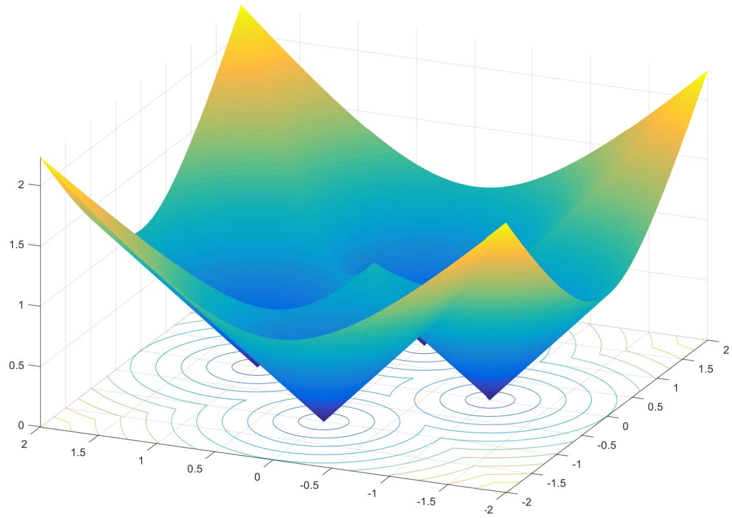

Figure 2: The function $(x, y) \mapsto \sigma_4\big((x + y\mathrm{i})I_4 - B_4\big)$ attains a strict local maximum at (0,0).

On the other hand, as the matrix B is nilpotent, from (10), Proposition 6(e) and using the notation introduced in (9), we obtain

$$\Lambda'_{\varepsilon_0}(\gamma B+(z_p-\gamma w_0)I_3) = z_p-\gamma w_0+\gamma\Lambda'_{\varepsilon_1}(B) \subset z_p-\gamma w_0+\gamma\mathcal{D}(0, \varepsilon_1+\|B\|) = \mathcal{D}(z_p-\gamma w_0, \delta). \quad (11)$$

Finally, taking

$$A_0 := \operatorname{diag}(\gamma B + (z_p - \gamma w_0)I_3)_{p=1}^k,$$

by Proposition 6(d) and (11), we get

$$\Lambda'_{\varepsilon_0}(A_0) = \bigcup_{p=1}^{k} \Lambda'_{\varepsilon_0}(\gamma B + (z_p - \gamma w_0)I_3) \subset \bigcup_{p=1}^{k} \mathcal{D}(z_p - \gamma w_0, \delta).$$

Note that $z_p \in \mathcal{D}(z_p - \gamma w_0, \delta)$. Besides, these disks are pairwise disjoint. As a consequence z_p is isolated for $\partial\Lambda'_{\varepsilon_0}(A_0)$. This proves the statements in (1) of the theorem. The assertion in (2) is deduced immediately from Theorem 7(a). □

Remark 13. A question arises: Fix a connected component of the strict ε-pseudospectrum that contains only one eigenvalue of a matrix A. Can it have two isolated points on its boundary? Michael Karow asked us this question in Barcelona in July of 2014 [7]. The answer is affirmative. It suffices to choose the matrix

$$\begin{pmatrix} 0 & 1 & 4 & 20 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the same way as in the proof of Proposition 11 it is found that the points $-4/19 \pm \mathrm{i}4\sqrt{5}/95$ are isolated for the boundary of the strict pseudospectrum of level $\varepsilon = \sqrt{5}/95$. Hence, by Theorem 7(a), the function g attains two strict local maxima at these points. See Figure 3.

Remark 14. By Proposition 6(c) we have $\partial\Lambda_\varepsilon(A) \subset \partial\Lambda'_\varepsilon(A)$. Moreover, according to Theorem 3, the sets $\partial\Lambda'_\varepsilon(A)$ and $\partial\Lambda_\varepsilon(A)$ can differ in a finite set of points. Hence, the question arises: Can both sets differ in an infinite number of points?

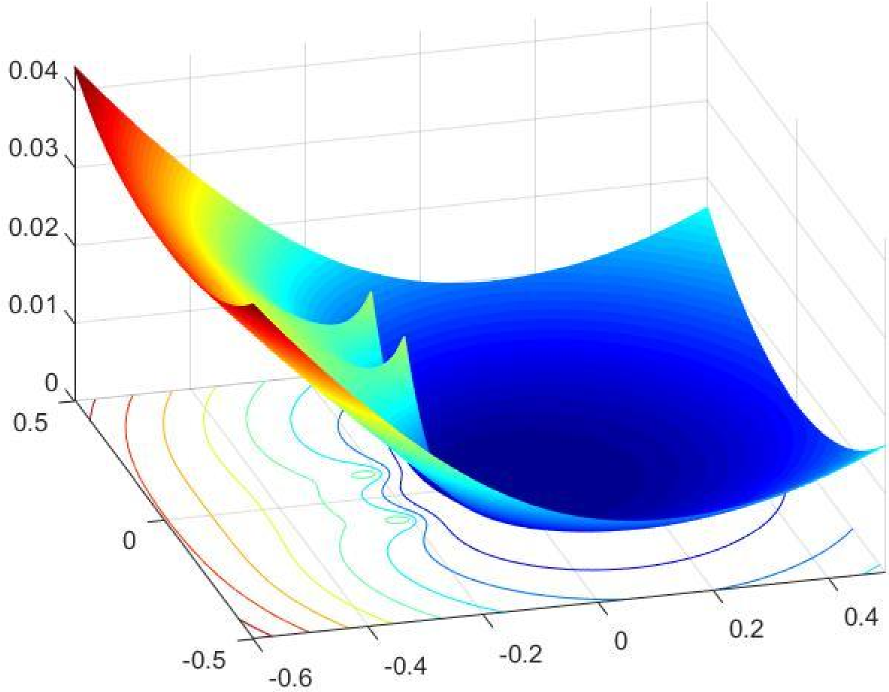

Figure 3: Strict local maxima of g at the points $(-\frac{4}{19}, \pm\frac{4\sqrt{5}}{95})$.

Let us suppose that the set $\partial\Lambda'_\varepsilon(A)\backslash\partial\Lambda_\varepsilon(A)$ had infinite points. Then a connected component of $\partial\Lambda'_\varepsilon(A)$, which we denote by $\mathcal{C}$, would contain a set of infinitely many points which would not be in $\partial\Lambda_\varepsilon(A)$. Let $\mathcal{X} := \mathcal{C} \cap (\partial\Lambda_\varepsilon(A))^{\mathrm{c}}$ denote this set. If $\mathcal{X}$ had infinite isolated points (with respect to $\mathcal{X}$), then $\mathcal{C} \cap \partial\Lambda_\varepsilon(A)$ would have infinitely many connected components; therefore, $\partial\Lambda_\varepsilon(A)$ would have infinitely many connected components, which is impossible by Lemma 5 (e). Hence, $\mathcal{X}$ can not have infinite isolated points; that is, $\mathcal{X}$ is a real analytic submanifold of dimension 1 of $\mathbb{R}^2$ with a finite number of connected components. Therefore, the sets $\partial\Lambda'_\varepsilon(A)$ and $\partial\Lambda_\varepsilon(A)$ differ at most in a finite set and in a real analytic submanifold of dimension 1 of $\mathbb{R}^2$ with a finite number of connected components.

An equivalent problem which arises from Theorem 7(c) is the following: Can the function g have local maxima that are not strict? If this were possible, then the function g would take the constant value ε on a real analytic submanifold $\mathcal{M}$ of dimension 1 of $\mathbb{R}^2$ and would attain a nonstrict local maximum at each point of $\mathcal{M}$. We think that this case can not occur.

5.1 Acknowledgements

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