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Perforated strict pseudospectra of Demmel's matrices *

Gorka Armentia†, Juan-Miguel Gracia†, Francisco-Enrique Velasco†

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Abstract

Regarding the pseudospectra of matrices it is easy to derive some conclusions from computer made graphics. However, those conclusions might be false without further analysis. Indeed, it happened when discussing the presence of a local maximum at the origin for the least singular value of the characteristic matrix of the $n \times n$ real Toeplitz matrix whose first row and first column are $(-1, -b, -b^2, \dots, -b^{n-1})$ and $(-1, 0, 0, \dots, 0)^{\mathrm{T}}$, respectively, which was first taken into account by Demmel.

In this paper, we determine the value of a local maximum for the last singular value of the characteristic matrix of such Toeplitz matrices in the cases $n = 3$ and $n = 5$. We also specify the point where this local maximum is attained at.

Keywords: singular values, local maxima, algebraic curves, isolated points.
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1 Notation

Let $\mathbb{C}^{n\times n}$ denote the space of $n \times n$ complex matrices. For any matrix $M \in \mathbb{C}^{n\times n}$, let

$$\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$$

be its ordered singular values. From now on, we will denote $\sigma_n(M)$ by $\sigma_{\min}(M)$. In this paper we will use the spectral norm of matrices; that is, $\|M\| = \sigma_1(M)$. Let us denote the spectrum of M by $\Lambda(M)$, and the transpose conjugate of M by M^*. It is said that a pair of vectors of unit length $u_i, v_i \in \mathbb{C}^{n\times 1}$ are left and right singular vectors of M associated with the singular value $\sigma_i(M)$ if $Mv_i = \sigma_i(M)u_i$ and $M^*u_i = \sigma_i(M)v_i$.

Let $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$ be given. Then the *ε-pseudospectrum* of A is the set $\{z \in \mathbb{C}\colon \sigma_{\min}(zI_n - A) \leq \varepsilon\}$. Analogously, the *strict ε-pseudospectrum* of A is the set $\{z \in \mathbb{C}\colon \sigma_{\min}(zI_n - A) < \varepsilon\}$. Identifying $z = x + y\mathrm{i} \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$, we can define the function $(x, y) \mapsto \sigma_{\min}((x + y\mathrm{i})I_n - A)$, from $\mathbb{R}^2$ to $\mathbb{R}$. Let $H \in \mathbb{C}^{n\times n}$ be Hermitian, it is known that all of its eigenvalues are real and, therefore, they can be written in decreasing order as

$$\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_n(H).$$

2 Introduction. Main theorems

The paper is about a very specific question. Namely, on the localization of the holes of a strict *ε-pseudospectrum* of $n \times n$ matrices of type

$$D_b := \begin{pmatrix} -1 & -b & -b^2 & \cdots & -b^{n-1} \\ 0 & -1 & -b & \cdots & -b^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

for $n = 3$ and $n = 5$, where b is a real number large enough. The searched value of ε is equal to $\frac{b}{b^2-1}$ and coincides with a strict local maximum of the function $(x, y) \mapsto \sigma_{\min}\big((x + y\mathrm{i})I_n - D_b\big)$, which is attained at $\big(\frac{1}{b^2-1}, 0\big)$.

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†Department of Applied Mathematics and Statistics, University of the Basque Country UPV/EHU, Faculty of Pharmacy, 7 Paseo de la Universidad, 01006 Vitoria-Gasteiz, Spain, gorka.armentia@ehu.eus, juanmiguel.gracia@ehu.eus, franciscoenrique.velasco@ehu.eus

In [1], we proved the existence of isolated points of the boundary of the strict ε-pseudospectrum of a matrix for some ε. A referee of that paper drew our attention to a Demmel's paper, see [4]. Although the aim of the paper was other, some figures of the pseudospectra of A_{100} were introduced [4, Fig. 3, p. 342], where

$$A_b := \begin{pmatrix} -1 & -b & -b^2 \\ 0 & -1 & -b \\ 0 & 0 & -1 \end{pmatrix}; \tag{1}$$

we recreate those graphs in Figure 1. Despite the lack of terminology on pseudospectra in that paper, what

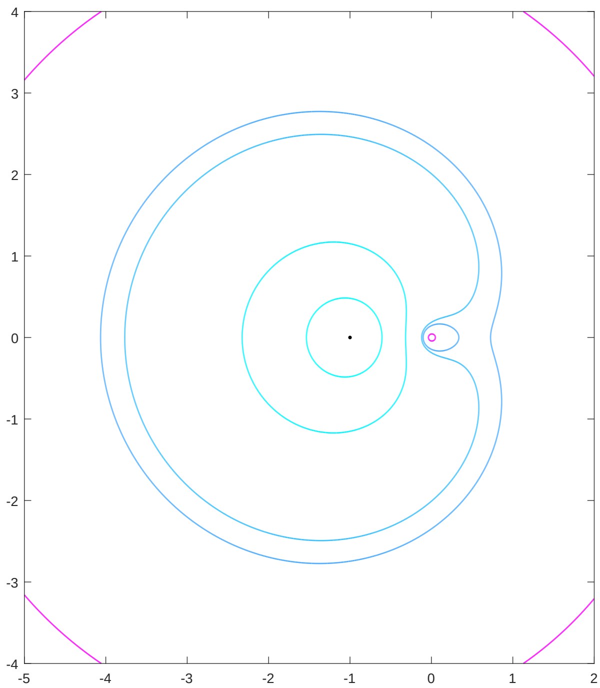

Figure 1: Pseudospectra of A_{100} for $\varepsilon = 10^{-5}$, 10^{-4}, $10^{-3.25}$, $10^{-3.15}$, $10^{-2.65}$.

Demmel pointed out is equivalent to saying some pseudospectra of A_{100} are not simply connected. Based on his graphs, Demmel stated that the function $(x, y) \mapsto \sigma_{\min}\big((x + y\mathrm{i})I_3 - A_{100}\big)$ reaches a local maximum at origin. In [3, p. 318], it is repeated the same assertion. We will prove that for every real b no local maximum of the function

$$(x, y) \mapsto \sigma_{\min}\big((x + y\mathrm{i})I_3 - A_b\big)$$

is attained at $(0, 0)$. This fact was remarked on in [7, p. 1060]. Moreover, we will show that a local maximum is reached at $\left(\frac{1}{b^2-1}, 0\right)$ and its value is $\frac{b}{b^2-1}$, for b large enough. In [6, Sec. 3.4], A_{100} was used to illustrate the differences between the unstructured and real structured pseudospectra.

For each $b, x, y \in \mathbb{R}$, let us define

$$F_b(x, y) := (x + y\mathrm{i})I_3 - A_b; \quad G_b(x) := xI_3 - A_b. \tag{2}$$

We want to prove the following theorems.

Theorem 1. *For no $b \in \mathbb{R}$, does the function $x \mapsto \sigma_{\min}\big(G_b(x)\big)$ from $\mathbb{R}$ to $\mathbb{R}$ have a local maximum at $x = 0$.*

As a consequence of the previous theorem we can deduce the following corollary.

Corollary 2. *For no $b \in \mathbb{R}$, does the function $(x, y) \mapsto \sigma_{\min}\big(F_b(x, y)\big)$ from $\mathbb{R}^2$ to $\mathbb{R}$ have a local maximum at $(0, 0)$.*

Theorem 3. *For each $b > 2 + \sqrt{3}$, the function $(x, y) \mapsto \sigma_{\min}\big(F_b(x, y)\big)$ from $\mathbb{R}^2$ to $\mathbb{R}$ has a local maximum at the point $\left(\frac{1}{b^2-1}, 0\right)$ and its value is $\frac{b}{b^2-1}$.*

In [7, Example 4.10, third example, p. 1060], for

$$T_b := \begin{pmatrix} -1 & -b & -b^2 & -b^3 & -b^4 \\ 0 & -1 & -b & -b^2 & -b^3 \\ 0 & 0 & -1 & -b & -b^2 \\ 0 & 0 & 0 & -1 & -b \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{3}$$

the authors asserted that for $b = 5$ the function $(x, y) \mapsto \sigma_{\min}\big((x + y\mathrm{i})I_5 - T_5\big)$ reaches a local maximum close to $(0, 0)$ but not exactly at $(0, 0)$. In this paper we prove the following result.

Theorem 4. *For $b \geq 4$, the function*

$$(x, y) \mapsto \sigma_{\min}\big((x + y\mathrm{i})I_5 - T_b\big)$$

attains a local maximum at $(1/(b^2 - 1), 0)$, and the value of this maximum is $b/(b^2 - 1)$.

Remark. For the analysis of the $n = 3$ case we use some elementary techniques (see [1]) since the singular value $b/(b^2 - 1)$ is double. However, for the $n = 5$ case this singular value is quadruple; which has led us to use results on Taylor's expansions of eigenvalues (see Lemma 10 and Corollary 11).

3 Proof of Theorem 1

In order to prove Theorem 1 we need three previous results.

Lemma 5. *For each $b \in \mathbb{R}, b \neq 0$, it is satisfied that*

(a) *None of the singular values of A_b is equal to 1.*

(b) *Each singular value of A_b is simple.*

Proof.
(a) The singular values of A_b are the nonnegative square roots of the zeros of the following polynomial

$$p_b(\lambda) := \det(\lambda I_3 - A_b A_b^*) = \lambda^3 - (b^4 + 2b^2 + 3)\lambda^2 + (2b^2 + 3)\lambda - 1, \tag{4}$$

Since $p_b(1) = -b^4$, we have $p_b(1) = 0$ if and only if $b = 0$.

(b) The discriminant of $p_b(\lambda)$ is equal to

$$4b^{10} + 13b^8 + 32b^6.$$

Since $b \neq 0$, it follows that the singular values of A_b are simple.

□

The next result we need can be seen in [9, Theorem 1.1].

Lemma 6. *Let $B \in \mathbb{R}^{n\times n}$. Let $\sigma_0 > 0$ be a simple singular value of B. Let $x \mapsto \mathcal{B}(x)$ be a real analytic function from some neighbourhood of 0 in $\mathbb{R}$ to $\mathbb{R}^{n\times n}$ such that $\mathcal{B}(0) = B$. Then there exists a real analytic function $x \mapsto \sigma(x)$ defined in a neighbourhood $\mathcal{N}_0$ of 0 in $\mathbb{R}$ such that, for every $x \in \mathcal{N}_0$, $\sigma(x)$ is a singular value of $\mathcal{B}(x)$, with $\sigma(0) = \sigma_0$.*

The proof of the following lemma can be seen in [5, p. 450, Theorem 7.3.3].

Lemma 7 (Wielandt's Lemma)**.** *Let $M \in \mathbb{C}^{n\times n}$. Let $s_1 \geq \cdots \geq s_n \geq 0$. Then, $s_1 \geq \cdots \geq s_n \geq -s_n \geq \cdots \geq -s_1$ are the ordered eigenvalues of the Hermitian matrix*

$$H := \begin{bmatrix} O & M \\ M^* & O \end{bmatrix}$$

if and only if $s_1 \geq \cdots \geq s_n$ are the ordered singular values of M.

Let $A \in \mathbb{C}^{n\times n}$. Let $x, y, s \in \mathbb{R}$, and set

$$q(x,y,s) := \det \begin{pmatrix} sI_n & (x+y\mathrm{i})I_n - A \\ (x-y\mathrm{i})I_n - A^* & sI_n \end{pmatrix} \tag{5}$$

By Lemma 7, $s \geq 0$ is a singular value of $(x+y\mathrm{i})I_n - A$ if and only if $q(x,y,s) = 0$. Moreover, by using Schur complement, we have

$$q(x,y,s) = \det\big(s^2 I_n - \big((x-y\mathrm{i})I_n - A^*\big)\big((x+y\mathrm{i})I_n - A\big)\big).$$

Proof of Theorem 1

First, let us assume that $b = 0$. By (1) and (2) it is obvious that $G_0(x) = (x+1)I_3$ and, therefore, $\sigma_{\min}(G_0(x)) = |x+1|$, which contradicts the local maximality of $x \mapsto \sigma_{\min}(G_0(x))$ at $x = 0$.

Hence, let us assume $b \neq 0$. By Lemma 5(b) the singular values of A_b are simple; from that fact and Lemma 6, it follows that the function $x \mapsto \sigma_b(x) := \sigma_{\min}(G_b(x))$ is real analytic in a neighbourhood of 0. Since $y = 0$ in this case, $q(x,y,s)$ is reduced to a polynomial which depends on s and x

$$\begin{aligned} q_b(x,s) :&= \det\big(s^2 I_3 - G_b(x)^* G_b(x)\big) = s^6 - \big(b^4 + 2b^2 + 3(x+1)^2\big)s^4 \\ &+ \big(b^4 x^2 + 2b^2(x+1)^2 + 3(x+1)^4\big)s^2 - (x+1)^6. \end{aligned} \tag{6}$$

We prove the theorem by contradiction. Assume that σ_b reaches a local maximum at the origin for some $b \in \mathbb{R}, b \neq 0$. Thus the following equalities hold.

$$q_b(0, \sigma_b(0)) = 0, \quad \sigma_b'(0) = 0. \tag{7}$$

From Lemma 5(b), we infer that $(q_b)'_s(0, \sigma_b(0)) \neq 0$. Therefore, since

$$\sigma_b'(0) = -\frac{(q_b)'_x(0, \sigma_b(0))}{(q_b)'_s(0, \sigma_b(0))},$$

we have $(q_b)'_x(0, \sigma_b(0)) = 0$. As a consequence of that, equalities (7) are equivalent to

$$q_b(0, \sigma_b(0)) = 0, \quad (q_b)'_x(0, \sigma_b(0)) = 0. \tag{8}$$

Setting $\sigma_0 := \sigma_b(0)$, we can write equalities (8) as

$$\begin{cases} -b^4\sigma_0^4 + 2b^2\sigma_0^2(1-\sigma_0^2) + (\sigma_0^2 - 1)^3 = 0, \\ 2\big(2b^2\sigma_0^2 - 3(\sigma_0^2-1)^2\big) = 0. \end{cases} \tag{9}$$

By multiplying on the left the invertible matrix

$$\begin{pmatrix} 4 & \dfrac{2b^2\sigma_0^2 + (\sigma_0^2 - 1)(3\sigma_0^2+1)}{2} \\ 0 & 1/2 \end{pmatrix}$$

by the column vector

$$\begin{pmatrix} -b^4\sigma_0^4 + 2b^2\sigma_0^2(1-\sigma_0^2) + (\sigma_0^2-1)^3 \\ 2\big(2b^2\sigma_0^2 - 3(\sigma_0^2-1)^2\big) \end{pmatrix},$$

we deduce that system (9), whose unknowns are σ_0, b, is equivalent to the system

$$\begin{cases} (1-\sigma_0^2)^3(9\sigma_0^2 - 1) = 0, \\ 2b^2\sigma_0^2 - 3(\sigma_0^2-1)^2 = 0. \end{cases}$$

By solving the first equation, we have $\sigma_0 = 1,\ 1/3$. However, by Lemma 5(a), the only feasible singular value of σ_0 is $1/3$. By substituting that value in the second equation, we obtain $b = \pm 4\sqrt{6}/3 =: \beta$. A little calculation shows that the singular values of A_β, for the two values of β, are

$$6 + \sqrt{33} \approx 11.7446, \quad 1/3 \approx 0.3333, \quad 6 - \sqrt{33} \approx 0.2554.$$

Therefore, $1/3$ is not the least singular value of A_β. This contradicts our assumption, namely, $x \mapsto \sigma_{\min}(G_b(x))$ reaches a local maximum at $x = 0$ for some $b \neq 0$.

□

4 Proof of Theorem 3

To prove Theorem 3, we need a previous result. By (5), for $(x, y, s) \in \mathbb{R}^3$ we have

$$q(x, y, s) := \det \begin{pmatrix} sI_3 & F_b(x, y) \\ F_b^*(x, y) & sI_3 \end{pmatrix} \in \mathbb{R}[x, y][s]. \tag{10}$$

So, combining Theorems 7 and 10 in [1] yields the following Lemma.

Lemma 8. *Let us assume that* $(x_0, y_0, \sigma_0) \in \mathbb{R}^3$ *satisfies the conditions*

$$\begin{cases} q(x_0, y_0, \sigma_0) = 0, \\ q'_x(x_0, y_0, \sigma_0) = 0, \\ q'_y(x_0, y_0, \sigma_0) = 0, \\ q''_{xy}(x_0, y_0, \sigma_0)^2 - q''_{xx}(x_0, y_0, \sigma_0) q''_{yy}(x_0, y_0, \sigma_0) < 0, \end{cases}$$

and $\sigma_0 = \sigma_{\min}(F_b(x_0, y_0))$*. Then the function*

$$(x, y) \mapsto \sigma_{\min}(F_b(x, y))$$

has a local maximum at (x_0, y_0)*.*

Proof of Theorem 3

From (1), (2) and (10), setting $x_0 := 1/(b^2 - 1)$, $y_0 := 0$ and $\sigma_0 := b/(b^2 - 1)$, we infer that

$$q(x_0, y_0, \sigma_0) = q'_x(x_0, y_0, \sigma_0) = q'_y(x_0, y_0, \sigma_0) = 0,$$

and that

$$q''_{xy}(x_0, y_0, \sigma_0)^2 - q''_{xx}(x_0, y_0, \sigma_0) q''_{yy}(x_0, y_0, \sigma_0) = -\frac{4b^8}{(b^2 - 1)^4}(b^4 - 14b^2 + 1).$$

Since the roots of $b^4 - 14b^2 + 1 = 0$ are $2 \pm \sqrt{3}$ and $-2 \pm \sqrt{3}$, it is easy to check that for $b > 2 + \sqrt{3}$ the inequality $q''_{xy}(x_0, y_0, \sigma_0)^2 - q''_{xx}(x_0, y_0, \sigma_0) q''_{yy}(x_0, y_0, \sigma_0) < 0$ holds. To complete the proof, let us observe that the positive roots of $q(x_0, y_0, s) = 0$ with respect to s are

$$\frac{b^4}{b^2 - 1}, \ \frac{b}{b^2 - 1}, \ \frac{b}{b^2 - 1};$$

therefore, $\sigma_0 = \sigma_{\min}\big(F_b(x_0, y_0)\big)$.

□

5 Proof of Theorem 4

For the proof of Theorem 4 we need some previous results. The following can be found in [2, Corollary III.2.6, p. 63, and Corollary III.1.5, p. 59].

Lemma 9. *Let* $H, G \in \mathbb{C}^{n \times n}$ *be Hermitian matrices and let* $H_k \in \mathbb{C}^{k \times k}$ *be a principal submatrix of* H*. Then*

(a) *Weyl's perturbation theorem:*

$$|\lambda_i(H + G) - \lambda_i(H)| \le \|G\|, \quad i = 1, \ldots, n.$$

(b) *Cauchy's interlacing theorem:*

$$\lambda_k(H_k) \ge \lambda_n(H).$$

By Lemma 9 (a) the function $Z \mapsto \lambda_i(Z)$ is continuous on the space of Hermitian matrices in $\mathbb{C}^{n \times n}$. Hence, the function

$$t \mapsto \lambda_i(H + tG)$$

from $\mathbb{R}$ to $\mathbb{R}$ is continuous, for $i = 1, \ldots, n$. The next lemma, on the behavior of the eigenvalues of Hermitian matrices of the form $H + tG$ when $t \to 0^+$, is adapted from Theorem A.4 of [8].

Lemma 10. *Let $H, G \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Let us suppose that the eigenvalue α of H has algebraic multiplicity k and*

$$\alpha = \lambda_{i_0+1}(H) = \cdots = \lambda_{i_0+k}(H).$$

Let us define

$$\delta_0 := \begin{cases} \min\limits_{j\notin\{i_0+1,\ldots,i_0+k\}} |\lambda_j(H)-\alpha| & \text{if } k<n \\ 1 & \text{if } k=n \end{cases} \qquad \delta := \frac{\|G\|}{\delta_0}, \tag{11}$$

$$r_0 := (2\delta + 4\delta^2)(\|H\|+\|G\|), \quad t_0 := \min\left\{\frac{1}{2}, \frac{1}{4\delta}\right\}. \tag{12}$$

Let $M \in \mathbb{C}^{n\times k}$ be a matrix whose columns are orthonormal eigenvectors of H corresponding to the eigenvalue $\lambda_{i_0+1}(H)$.

Then for each $j = 1, 2, \ldots, k$ the function

$$f_j(t,G) := \begin{cases} \dfrac{\lambda_{i_0+j}(H+tG) - \lambda_{i_0+1}(H) - t\lambda_j(M^*GM)}{t^2} & \text{if } t\in(0,t_0] \\ 0 & \text{if } t=0 \end{cases}$$

satisfies

$$|f_j(t,G)| \le r_0, \quad t\in[0,t_0].$$

PROOF

Case 1: $k < n$

For the sake of brevity we denote the eigenvalues of the Hermitian matrix M^*GM by $\mu_i := \lambda_i(M^*GM)$ for $i = 1, \ldots, k$. Let $M_1 \in \mathbb{C}^{n\times(n-k)}$ be a matrix whose columns are orthonormal eigenvectors of H corresponding to the eigenvalues $\lambda_i(H) \neq \alpha$, and let $R := (M, M_1) \in \mathbb{C}^{n\times n}$ be unitary. Let us define the matrices

$$\begin{aligned} &H_1 := R^*HR = \begin{pmatrix} \alpha I_k & O \\ O & \Sigma \end{pmatrix}, \\ &\Sigma := \operatorname{diag}(\lambda_1(H), \ldots, \lambda_{i_0}(H), \lambda_{i_0+k+1}(H), \ldots, \lambda_n(H)) \\ &G_1 := R^*GR = \begin{pmatrix} M^*GM & M^*GM_1 \\ M_1^*GM & M_1^*GM_1 \end{pmatrix}, \\ &G_1' := \begin{pmatrix} M^*GM & O \\ O & M_1^*GM_1 \end{pmatrix}. \end{aligned} \tag{13}$$

Let us define

$$V := \begin{pmatrix} O & -M^*GM_1(\Sigma-\alpha I_{n-k})^{-1} \\ (\Sigma-\alpha I_{n-k})^{-1}M_1^*GM & O \end{pmatrix}. \tag{14}$$

A little calculation shows that

$$VH_1 - H_1V + G_1 = G_1'. \tag{15}$$

Moreover, by (11) and (14) we deduce that

$$\|V\| \le \delta. \tag{16}$$

Since V is skew Hermitian, for each real t the matrix e^{tV} is unitary, that is

$$\left(\mathrm{e}^{tV}\right)^* = \left(\mathrm{e}^{-tV}\right) = \left(\mathrm{e}^{tV}\right)^{-1}. \tag{17}$$

Consequently, $H + tG$ and $\mathrm{e}^{tV}(H_1 + tG_1)\mathrm{e}^{-tV}$ are unitarily similar. Using the power series expansion of the matrix exponential, we get

$$\begin{aligned} &\mathrm{e}^{tV}(H_1+tG_1)\mathrm{e}^{-tV} \\ &= \left(I_n + tV + \sum_{j=2}^{\infty}\frac{(tV)^j}{j!}\right)(H_1+tG_1)\left(I_n - tV + \sum_{j=2}^{\infty}\frac{(-tV)^j}{j!}\right) \\ &= H_1 + t(VH_1 - H_1V + G_1) + t^2(VG_1 - VH_1V - G_1V) - t^3VG_1V \\ &+ (I_n+tV)(H_1+tG_1)\sum_{j=2}^{\infty}\frac{(-tV)^j}{j!} + \sum_{j=2}^{\infty}\frac{(tV)^j}{j!}(H_1+tG_1)\mathrm{e}^{-tV}. \end{aligned}$$

By (15) and the previous equality, we deduce that

$$\begin{aligned}
&\mathrm{e}^{tV}(H_1+tG_1)\mathrm{e}^{-tV}\\
&\qquad = H_1+tG_1'+t^2(VG_1-VH_1V-G_1V)-t^3VG_1V\\
&+(I_n+tV)(H_1+tG_1)\sum_{j=2}^{\infty}\frac{(-tV)^j}{j!}+\sum_{j=2}^{\infty}\frac{(tV)^j}{j!}(H_1+tG_1)\mathrm{e}^{-tV}.
\end{aligned} \tag{18}$$

In the remainder of Case 1 we assume t to run over $[0,t_0]$. From (12) and (16), it follows that $\|tV\|\le 1/4$. Furthermore, we can infer that

$$\begin{aligned}
\left\|\sum_{j=2}^{\infty}\frac{(tV)^j}{j!}\right\| &\le \sum_{j=2}^{\infty}\frac{\|tV\|^j}{j!}\le \|tV\|^2\sum_{j=0}^{\infty}\frac{\|tV\|^j}{(j+2)!}\le t^2\delta^2\sum_{j=0}^{\infty}\frac{\|tV\|^j}{(j+2)!}\\
&\le t^2\delta^2\sum_{j=0}^{\infty}\frac{1}{(j+2)!}\le t^2\delta^2;
\end{aligned}$$

hence

$$\left\|\sum_{j=2}^{\infty}\frac{(tV)^j}{j!}\right\|\le t^2\delta^2,\quad \left\|\sum_{j=2}^{\infty}\frac{(-tV)^j}{j!}\right\|\le t^2\delta^2;$$

it is easy to check that

$$\left\|\sum_{j=2}^{\infty}\frac{t^{j-2}V^j}{j!}\right\|\le \delta^2,\quad \left\|\sum_{j=2}^{\infty}\frac{(-t)^{j-2}V^j}{j!}\right\|\le \delta^2. \tag{19}$$

Now, using (13) too, we have

$$\begin{aligned}
&\|VG_1-VH_1V-G_1V\|\le 2\delta\|G\|+\delta^2\|H\|,\\
&\|VG_1V\|\le\delta^2\|G\|,\\
&\|I_n+tV\|\le 2,\\
&\|H_1+tG_1\|\le\|H\|+\|G\|.
\end{aligned} \tag{20}$$

Setting

$$\begin{aligned}
F(t):=&\,(VG_1-VH_1V-G_1V)-tVG_1V\\
&+(I_n+tV)(H_1+tG_1)\sum_{j=2}^{\infty}\frac{(-t)^{j-2}V^j}{j!}+\sum_{j=2}^{\infty}\frac{t^{j-2}V^j}{j!}(H_1+tG_1)\mathrm{e}^{-tV},
\end{aligned} \tag{21}$$

we can rewrite (18) as

$$\mathrm{e}^{tV}(H_1+tG_1)\mathrm{e}^{-tV}=H_1+tG_1'+t^2F(t). \tag{22}$$

From (12), (17), (19) and (20) we obtain

$$\begin{aligned}
\|F(t)\| &\le 2\delta\|G\|+\delta^2\|H\|+t\delta^2\|G\|+2\delta^2(\|H\|+\|G\|)+\delta^2(\|H\|+\|G\|)\\
&\le 4\delta^2\|H\|+(2\delta+3\delta^2+t\delta^2)\|G\|\le(2\delta+4\delta^2)(\|H\|+\|G\|),
\end{aligned}$$

where $t_0\le 1/2$ is used in the last inequality. Hence, by (12) we conclude that

$$\|F(t)\|\le r_0. \tag{23}$$

Let us now define the matrix $Z:=Z_1\oplus I_{n-k}\in\mathbb{C}^{n\times n}$, where $Z_1\in\mathbb{C}^{k\times k}$ is a unitary matrix such that $Z_1^*M^*GMZ_1=\operatorname{diag}(\mu_1,\mu_2,\ldots,\mu_k)$. Thus, from (13) and (22), we infer that $H+tG$ is unitarily similar to

$$\begin{pmatrix}\alpha I_k+t\operatorname{diag}(\mu_1,\ldots,\mu_k) & O\\ O & \Sigma+tM_1^*GM_1\end{pmatrix}+t^2Z^*F(t)Z. \tag{24}$$

For $i=1,2,\ldots,n$, let us define

$$\varphi_i(t):=\lambda_i\begin{pmatrix}\alpha I_k+t\operatorname{diag}(\mu_1,\ldots,\mu_k) & O\\ O & \Sigma+tM_1^*GM_1\end{pmatrix}. \tag{25}$$

Hence, as $\|M_1\| = 1$ and $|\mu_i| \le \|G\|$, applying Lemma 9 (a) we deduce that

$$\lambda_i \begin{pmatrix} \alpha I_k & O \\ O & \Sigma \end{pmatrix} - \|tG\| \le \varphi_i(t) \le \lambda_i \begin{pmatrix} \alpha I_k & O \\ O & \Sigma \end{pmatrix} + \|tG\|.$$

From (11) and (12) we have $t_0 \le 1/(4\delta) = \delta_0/(4\|G\|)$; we see that for each $i = 1, \ldots, n$

$$\lambda_i(H) - \delta_0/4 \le \varphi_i(t) \le \lambda_i(H) + \delta_0/4.$$

Consequently,

$$\begin{cases} \{\varphi_1(t), \varphi_2(t), \ldots, \varphi_{i_0}(t)\} \subset \left[\lambda_{i_0}(H) - \dfrac{\delta_0}{4}, \lambda_1(H) + \dfrac{\delta_0}{4}\right], \\ \{\varphi_{i_0+1}(t), \varphi_{i_0+2}(t), \ldots, \varphi_{i_0+k}(t)\} \subset \left[\alpha - \dfrac{\delta_0}{4}, \alpha + \dfrac{\delta_0}{4}\right], \\ \{\varphi_{i_0+k+1}(t), \varphi_{i_0+k+2}(t), \ldots, \varphi_n(t)\} \subset \left[\lambda_n(H) - \dfrac{\delta_0}{4}, \lambda_{i_0+k+1}(H) + \dfrac{\delta_0}{4}\right]. \end{cases} \tag{26}$$

Using again Lemma 9 (a), we deduce that

$$\Lambda(\Sigma + tM_1^* G M_1) \subset [\lambda_{i_0}(H) - \delta_0/4, \lambda_1(H) + \delta_0/4] \cup [\lambda_n(H) - \delta_0/4, \lambda_{i_0+k+1}(H) + \delta_0/4].$$

It is obvious that

$$\varphi_i(0) = \lambda_i(H) \in \begin{cases} [\lambda_{i_0}(H) - \delta_0/4, \lambda_1(H) + \delta_0/4], \text{ if } i \le i_0, \\ [\alpha - \delta_0/4, \alpha + \delta_0/4], \text{ if } i_0 + 1 \le i \le i_0 + k, \\ [\lambda_n(H) - \delta_0/4, \lambda_{i_0+k+1}(H) + \delta_0/4], \text{ if } i_0 + k + 1 \le i. \end{cases}$$

Let us recall that the image from an interval by a continuous function from $\mathbb{R}$ to $\mathbb{R}$ is an interval. Therefore, $\varphi_1([0, t_0])$,…,$\varphi_n([0, t_0])$ are intervals. This implies that for each $j = 1, \ldots, k$

$$\varphi_{i_0+j}([0, t_0]) \subset [\alpha - \delta_0/4, \alpha + \delta_0/4].$$

Since

$$\begin{aligned} &\Lambda \begin{pmatrix} \alpha I_k + t\,\mathrm{diag}(\mu_1, \ldots, \mu_k) & O \\ O & \Sigma + tM_1^* G M_1 \end{pmatrix} \\ &= \Lambda\big(\alpha I_k + t\,\mathrm{diag}(\mu_1, \ldots, \mu_k)\big) \mathbin{\dot\cup} \Lambda(\Sigma + tM_1^* G M_1), \end{aligned}$$

we conclude that $\varphi_{i_0+1}(t), \ldots, \varphi_{i_0+k}(t)$ are the eigenvalues of

$$\alpha I_k + t\,\mathrm{diag}(\mu_1, \ldots, \mu_k).$$

As $t \ge 0$ and $\mu_1 \ge \cdots \ge \mu_k$, it follows that

$$\varphi_{i_0+j}(t) = \alpha + t\mu_j, \qquad j = 1, \ldots, k.$$

From this and using Lemma 9 (a) in (24), we find that for $j = 1, \ldots, k$,

$$\alpha + t\mu_j - t^2\|F(t)\| \le \lambda_{i_0+j}(H + tG) \le \alpha + t\mu_j + t^2\|F(t)\|.$$

It suffices to use (23) to complete the proof in this case.

Case 2: $k = n$

If the Hermitian matrix $H \in \mathbb{C}^{n\times n}$ has an eigenvalue α of multiplicity n, then for $i = 1, \ldots, n$ and $t \ge 0$ we have

$$\lambda_i(H + tG) = \alpha + t\lambda_i(G).$$

□

As a consequence of this lemma, we have the following corollary on the behavior of the singular values of $A + tB$ when $t \to 0^+$. See [8, Lemma A.5].

Corollary 11. *Let $A, B \in \mathbb{C}^{n\times n}$. Let A have a positive singular value α of multiplicity k such that*

$$\alpha = \sigma_{i_0+1}(A) = \cdots = \sigma_{i_0+k}(A).$$

Let us define

$$\gamma_0 := \begin{cases} \min\limits_{j\notin\{i_0+1,\ldots,i_0+k\}} |\sigma_j(A)-\alpha| & \text{if } k<n \\ 1 & \text{if } k=n \end{cases} \qquad \gamma := \frac{\|B\|}{\gamma_0},$$

$$s_0 := (2\gamma+4\gamma^2)\big(\|A\|+\|B\|\big), \qquad t_0 := \min\left\{\frac{1}{2}, \frac{1}{4\gamma}\right\}.$$

Let $U = [u_{i_0+1},\ldots,u_{i_0+k}], V = [v_{i_0+1},\ldots,v_{i_0+k}] \in \mathbb{C}^{n\times k}$ be matrices of orthonormal columns and such that (u_{i_0+j}, v_{i_0+j}) is a pair of (left, right) singular vectors of A corresponding to $\sigma_{i_0+1}(A)$ for $j=1,\ldots,k$.

Then for each $j=1,2,\ldots,k$ the function

$$f_j(t,B) := \begin{cases} \dfrac{\sigma_{i_0+j}(A+tB) - \sigma_{i_0+1}(A) - t\lambda_j\big(\frac{U^*BV+V^*B^*U}{2}\big)}{t^2} & \text{if } t\in(0,t_0] \\ 0 & \text{if } t=0 \end{cases}$$

satisfies

$$|f_j\big(t,B\big)| \le s_0, \quad t\in[0,t_0]. \tag{27}$$

PROOF

Case 1: $k<n$.

Let $P := (U,U_1), Q := (V,V_1) \in \mathbb{C}^{n\times n}$ be unitary matrices such that

$$P^*AQ = S := \begin{pmatrix} \alpha I_k & O \\ O & \operatorname{diag}(\sigma_1(A),\ldots,\sigma_{i_0}(A),\sigma_{i_0+k+1}(A),\ldots,\sigma_n(A)) \end{pmatrix}.$$

Let us define

$$B_1 := P^*BQ = \begin{pmatrix} U^*BV & U^*BV_1 \\ U_1^*BV & U_1^*BV_1 \end{pmatrix}.$$

Note that for every $t\in[0,\infty)$ the singular values of $A+tB$ are those of $S+tB_1$, or equivalently, by Lemma 7, the nonnegative eigenvalues of

$$\begin{pmatrix} O & S+tB_1 \\ (S+tB_1)^* & O \end{pmatrix},$$

which is unitarily similar to

$$\begin{pmatrix} S & O \\ O & -S \end{pmatrix} + t\begin{pmatrix} \frac{1}{2}(B_1+B_1^*) & \frac{1}{2}(B_1^*-B_1) \\ \frac{1}{2}(B_1-B_1^*) & -\frac{1}{2}(B_1+B_1^*) \end{pmatrix},$$

where the unitary matrix of passage is $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} I_n & I_n \\ I_n & -I_n \end{smallmatrix}\right)$. We end the proof by applying Lemma 10 to the pair of Hermitian matrices

$$H := \begin{pmatrix} S & O \\ O & -S \end{pmatrix} \text{ and } G := \begin{pmatrix} \frac{1}{2}(B_1+B_1^*) & \frac{1}{2}(B_1^*-B_1) \\ \frac{1}{2}(B_1-B_1^*) & -\frac{1}{2}(B_1+B_1^*) \end{pmatrix}.$$

It is obvious that for each $j=1,\ldots,k$

$$\lambda_{i_0+j}(H) = \alpha.$$

Let us denote by $O_{k\times k}$ and $O_{(n-k)\times k}$ the zero matrices of sizes $k\times k$ and $(n-k)\times k$, respectively. Following from the definition of S the columns of

$$M := \begin{bmatrix} I_k \\ O_{(n-k)\times k} \\ O_{k\times k} \\ O_{(n-k)\times k} \end{bmatrix} \in \mathbb{C}^{2n\times k}$$

are orthonormal eigenvectors of H associated with the eigenvalue α. Besides, following from the definition of G and B_1, we infer that

$$G = \begin{matrix} & k & n-k & k & n-k \\ k \\ n-k \\ k \\ n-k \end{matrix} \begin{pmatrix} (V^*B^*U + U^*BV)/2 & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix};$$

hence,

$$M^*GM = \frac{1}{2}(U^*BV + V^*B^*U).$$

Therefore, for $t \in (0, s_0]$ and $j = 1, \ldots, k$ we see that

$$\left| \frac{\lambda_{i_0+j}(H + tG) - \lambda_{i_0+1}(H) - t\lambda_j\left(\frac{U^*BV+V^*B^*U}{2}\right)}{t^2} \right| \le s_0.$$

But $\lambda_{i_0+j}(H + tG) = \sigma_{i_0+j}(A + tB)$, which completes the proof.

Case 2: $k = n$**.** We have

$$\alpha = \sigma_1(A) = \cdots = \sigma_n(A).$$

The proof is trivial.

□

Remark 12. In Corollary 11 the value of $\lambda_j\left(\frac{U^*BV+V^*B^*U}{2}\right)$ depends only on B, and is independent of the matrices U and V chosen. In fact, from (27) we can deduce that $\lambda_j\left(\frac{U^*BV+V^*B^*U}{2}\right)$ is the value of the right derivative of the function

$$t \mapsto \sigma_{i_0+j}(A + tB)$$

at $t = 0$.

We are in a position to prove Theorem 4.

PROOF of Theorem 4

Let $C := \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ denote the unit circumference. From notation (3), for $(x, y) \in \mathbb{R}^2$ we write

$$\Phi(x, y) := (x + y\mathrm{i})I_5 - T_b \tag{28}$$

for short. We apply Corollary 11 to the matrices $A := \Phi(1/(b^2 - 1), 0)$ and $B := (x + y\mathrm{i})I_5$, where $(x, y) \in C$.

From (3) and (28), we have the matrix

$$\Phi(1/(b^2-1), 0) = \begin{pmatrix} \frac{b^2}{b^2-1} & b & b^2 & b^3 & b^4 \\ 0 & \frac{b^2}{b^2-1} & b & b^2 & b^3 \\ 0 & 0 & \frac{b^2}{b^2-1} & b & b^2 \\ 0 & 0 & 0 & \frac{b^2}{b^2-1} & b \\ 0 & 0 & 0 & 0 & \frac{b^2}{b^2-1} \end{pmatrix},$$

whose singular values are $b^6/(b^2-1)$ (simple) and $b/(b^2-1)$ (of multiplicity 4). Observe that $\|B\| = \|(x+y\mathrm{i})I_5\| = 1$, $i_0 = 1$ and $k = 4$. From notations in Corollary 11, we have

$$\gamma_0 = \frac{b^6 - b}{b^2 - 1}, \quad \gamma = \frac{b^2 - 1}{b^6 - b}.$$

Since $b \ge 4$, we have

$$\gamma = \frac{b^2 - 1}{b^6 - b} \le \frac{b^2}{b^6 - b} \le \frac{b^2}{b^6/2} = \frac{2}{b^4}.$$

An easy computation shows that $\|A\| = \|\Phi(1/(b^2-1),0)\| = b^6/(b^2-1)$. Therefore,

$$\begin{aligned}
&(2\gamma+4\gamma^2)(\|A\|+\|B\|) \le \frac{4(b^4+4)(b^6+b^2-1)}{b^8(b^2-1)} \le \frac{4(b^4+b^4)(b^6+b^2)}{b^8 b^2/2} = \frac{16(b^4+1)}{b^4} \le 17,\\
&\min\left\{\frac{1}{2},\frac{1}{4\gamma}\right\} = \min\left\{\frac{1}{2},\frac{b^4}{8}\right\} = \frac{1}{2}.
\end{aligned}$$

So, we can choose

$$s_0 := 17, \quad t_0 := \frac{1}{2}. \tag{29}$$

Let $U = [u_1, u_2, u_3, u_4]$, $V = [v_1, v_2, v_3, v_4] \in \mathbb{C}^{5\times 4}$ be matrices with orthonormal columns such that (u_i, v_i) is a pair of (left, right) singular vectors of $\Phi(1/(b^2-1),0)$ associated with the singular value $b/(b^2-1)$ for $i = 1,2,3,4$. For every $(x,y) \in C$, let $H(x,y)$ be the Hermitian matrix defined by

$$H(x,y) := \frac{1}{2}\big((x+y\mathrm{i})U^*V + (x-y\mathrm{i})V^*U\big). \tag{30}$$

According to Remark 12, we have $\lambda_4\big(H(x,y)\big)$ depends only on (x,y) and not on U and V. By Corollary 11 there exist $t_0 > 0$ and $s_0 > 0$, given in (29), such that the function $f\colon [0,t_0]\times C \to \mathbb{R}$ defined by

$$f(t,(x,y)) := \begin{cases} \dfrac{\sigma_{\min}\big(\Phi(1/(b^2-1)+tx,ty)\big) - \sigma_{\min}\big(\Phi(1/(b^2-1),0)\big) - t\lambda_4\big(H(x,y)\big)}{t^2} & \text{if } t>0\\ 0 & \text{if } t = 0\end{cases}$$

satisfies the inequality

$$|f(t,(x,y))| \le s_0 = 17, \qquad (t,(x,y)) \in [0,t_0]\times C. \tag{31}$$

In order to prove that the function

$$(x,y) \mapsto \sigma_{\min}\big(\Phi(x,y)\big)$$

from $\mathbb{R}^2$ in $\mathbb{R}$ has a local maximum at $(1/(b^2-1),0)$, we need to find a *global upper bound* a of $\lambda_4\big(H(x,y)\big)$ when (x,y) runs over C and such that a satisfies $at + s_0t^2 \le 0$ for $t>0$ small enough. To this end, computations made with the `Derive` and `Maple` programs lead us to choose the matrices U and V as follows: first, let us define the matrices

$$U_1 := \begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & \dfrac{b^2+1}{\sqrt{b^6+2b^4+2b^2+1}}\\ \dfrac{1}{\sqrt{b^2+1}} & \dfrac{-b^3}{\sqrt{b^6+2b^4+2b^2+1}}\\ \dfrac{-b}{\sqrt{b^2+1}} & \dfrac{-b^2}{\sqrt{b^6+2b^4+2b^2+1}}\end{pmatrix}, \quad V_1 := \begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & \dfrac{b(b^2+1)}{\sqrt{b^6+2b^4+2b^2+1}}\\ \dfrac{b}{\sqrt{b^2+1}} & \dfrac{-1}{\sqrt{b^6+2b^4+2b^2+1}}\\ \dfrac{-1}{\sqrt{b^2+1}} & \dfrac{-b}{\sqrt{b^6+2b^4+2b^2+1}}\end{pmatrix}.$$

Write $U_1 = [u_1, u_2]$,$V_1 = [v_1, v_2]$. Note that u_1, u_2 (resp. v_1, v_2) are orthonormal vectors; moreover, (u_i, v_i), $i = 1,2$, is a pair of (left, right) singular vectors of $\Phi(1/(b^2-1),0)$ associated with the singular value $b/(b^2-1)$. Second, let us choose pairs (u_i, v_i), $i = 3,4$, of singular vectors of $\Phi(1/(b^2-1),0)$ associated with $b/(b^2-1)$, such that $U := [u_1,u_2,u_3,u_4]$ and $V := [v_1,v_2,v_3,v_4]$ satisfy $U^*U = I_4$ and $V^*V = I_4$. Substituting *these matrices* into (30) we have $H(x,y)$. It is easy to see that

$$H_1(x,y) := \frac{1}{2}\big((x+y\mathrm{i})U_1^*V_1 + (x-y\mathrm{i})V_1^*U_1\big) \in \mathbb{C}^{2\times 2}$$

is a principal submatrix of $H(x,y)$. Since $x^2+y^2 = 1$, the characteristic polynomial $\det(\lambda I_2 - H_1(x,y))$ of $H_1(x,y)$ is given by

$$\lambda^2 - \frac{3bx(b^2+1)}{b^4+b^2+1}\lambda - \frac{b^4-2b^2(6x^2+1)+1}{4(b^4+b^2+1)}.$$

Computing the roots of $\det(\lambda I_2 - H_1(x,y)) = 0$ with respect to λ, we find that the least root is

$$\frac{3bx(b^2+1) - (b^2-1)\sqrt{b^4+b^2+1-3b^2x^2}}{2(b^4+b^2+1)}.$$

This expression, as a function of x, is increasing on $[-1,1]$. Thus its maximum is attained at $x = 1$ and has a value of

$$-\frac{b^2-4b+1}{2(b^2-b+1)} \le -\frac{1}{26} \text{ for all } b \ge 4.$$

Therefore, using Lemma 9 (b), we conclude that

$$\lambda_4\big(H(x,y)\big) \le \lambda_2\big(H_1(x,y)\big) \le -1/26 \tag{32}$$

for every $(x,y) \in C$. So, $a = -1/26$.

Note that for every $(t,(x,y)) \in [0,t_0] \times C$

$$\sigma_{\min}\big(\Phi(1/(b^2-1)+tx,ty)\big) - \sigma_{\min}\big(\Phi(1/(b^2-1),0)\big) = t\lambda_4\big(H(x,y)\big) + t^2 f(t,(x,y)) \le -(1/26)t + 17t^2.$$

The roots of $-(1/26)t + 17t^2 = 0$ are 0 and 1/442. Since $t_0 = 1/2 > 1/442$, we must restrain the interval of variation of t to ensure that

$$\sigma_{\min}\big(\Phi(1/(b^2-1)+tx,ty)\big) - \sigma_{\min}\big(\Phi(1/(b^2-1),0)\big) \le 0.$$

Therefore, let us choose for example $t_1 := 1/500$. In this way, for every $(X,Y) \in D\big((1/(b^2-1),0),t_1\big)$, open disk of centre $(1/(b^2-1),0)$ and radius t_1, we have

$$\sigma_{\min}\big(\Phi(X,Y)\big) \le \sigma_{\min}\big(\Phi(1/(b^2-1),0)\big).$$

Hence, the function

$$(x,y) \mapsto \sigma_{\min}\big(\Phi(x,y)\big)$$

has a strict local maximum at $(1/(b^2-1),0)$ and its value is $b/(b^2-1)$. □

6 Conclusions

We have determined both the ε-level and the coordinates of the holes of a strict ε-*pseudospectrum* of $n \times n$ matrices of type

$$D_b := \begin{pmatrix} -1 & -b & -b^2 & \cdots & -b^{n-1} \\ 0 & -1 & -b & \cdots & -b^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

where $b > 2+\sqrt{3}$ for $n = 3$ and $b \ge 4$ for $n = 5$. The determined value of ε is equal to $\frac{b}{b^2-1}$ and coincides with a strict local maximum of the function $(x,y) \mapsto \sigma_{\min}\big((x+y\mathrm{i})I_n - D_b\big)$, attained at $\big(\frac{1}{b^2-1},0\big)$.

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