Stability of reducing subspaces*

Gorka Armentia†, Juan-Miguel Gracia‡, Francisco-Enrique Velasco‡

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Dedicated to Professor Leiba Rodman on the occasion of his 65th birthday

Abstract

We characterize the stability of reducing subspaces of a rectangular matrix pencil of complex matrices $\lambda B - A$, except for the special case in which the pencil has no eigenvalues and only has one row and one column minimal indices and both are different from zero.

Keywords: rectangular matrix pencils, Kronecker canonical form, reducing subspaces, deflating subspaces, stability, perturbation.

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1 Introduction

Given two matrices $A, B \in \mathbb{C}^{m\times n}$, we call *matrix pencil* the first order matrix polynomial $\lambda B - A$. For simplicity, we will denote the set of matrix pencils of the form $\lambda B - A$, with $A, B \in \mathbb{C}^{m\times n}$, by $\mathcal{P}[\lambda]^{m\times n}$. We define the *normal rank* of a pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$, and we denote it by $\operatorname{nrank}(\lambda B - A)$, to be the greatest order of the minors of $\lambda B - A$ that are different from the zero polynomial. If $m = n$ and $\operatorname{nrank}(\lambda B - A) = n$, the pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ is said to be *regular*. Otherwise, the pencil is said to be *singular*.

Note by $\mathbb{C}(\lambda)$ the field of rational fractions in λ. If we consider $\lambda B - A$ as a linear map from the vector space $\mathbb{C}(\lambda)^n$ into $\mathbb{C}(\lambda)^m$, both over the field $\mathbb{C}(\lambda)$, we have

$$\operatorname{nrank}(\lambda B - A) = \dim_{\mathbb{C}(\lambda)} \operatorname{Im}(\lambda B - A),$$

(see [4]). We define the nullity of $\lambda B - A$ by $\nu(\lambda B - A) := \dim_{\mathbb{C}(\lambda)} \operatorname{Ker}(\lambda B - A)$. From

$$n = \dim_{\mathbb{C}(\lambda)} \operatorname{Ker}(\lambda B - A) + \dim_{\mathbb{C}(\lambda)} \operatorname{Im}(\lambda B - A),$$

$$\nu(\lambda B - A) = n - \operatorname{nrank}(\lambda B - A).$$

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†Department of Mathematical Engineering and Computer Science, The Public University of Navarre, Campus de Arrosadía, 31006 Pamplona, Spain. gorka.armentia@unavarra.es

‡Department of Applied Mathematics and Statistics, University of the Basque Country UPV/EHU, Faculty of Pharmacy, 7 Paseo de la Universidad, 01006 Vitoria-Gasteiz, Spain, juanmiguel.gracia@ehu.es, franciscoenrique.velasco@ehu.es

As usual we identify a matrix $M \in \mathbb{C}^{m\times n}$ with the linear map $x \mapsto Mx$ from $\mathbb{C}^n \equiv \mathbb{C}^{n\times 1}$ into $\mathbb{C}^m \equiv \mathbb{C}^{m\times 1}$. Let $\mathcal{N}$ be a subspace of $\mathbb{C}^n$, we define $M(\mathcal{N})$ as the subspace of $\mathbb{C}^m$ formed by all matrix products Mx with $x \in \mathcal{N}$. Van Dooren proved that

$$\dim(A(\mathcal{N}) + B(\mathcal{N})) \geq \dim \mathcal{N} - \nu(\lambda B - A), \tag{1}$$

where $A(\mathcal{N}) + B(\mathcal{N})$ is the sum of these subspaces of $\mathbb{C}^m$. See [17, Equation (2.16) on page 63 and in the line following (2.25 a) and (2.25 b) on page 65]. In the case the equality holds in (1), the subspace $\mathcal{N}$ is called a *reducing subspace* for the pencil (see [17]) or, equivalently, that $\mathcal{N}$ is a $(\lambda B - A)$-*reducing subspace*. Observe that if the pencil is regular then $\nu(\lambda B - A) = 0$. So, in this case $\mathcal{N}$ is a reducing subspace if and only if $\dim(A(\mathcal{N}) + B(\mathcal{N})) = \dim \mathcal{N}$. These reducing subspaces are also called *deflating subspaces* for regular pencils (see [16]). In order to simplify here on, we will denote by $(\lambda B - A)(\mathcal{N})$ the subspace $A(\mathcal{N}) + B(\mathcal{N})$.

We use the operator norm induced by the Euclidean norms on $\mathbb{C}^m$ and $\mathbb{C}^n$, also called the spectral norm,

$$\|M\| := \max_{\substack{x\in\mathbb{C}^n \\ \|x\|_2=1}} \|Mx\|_2.$$

The *gap* between subspaces $\mathcal{M}$ and $\mathcal{N}$ (in $\mathbb{C}^n$) is defined as

$$\theta(\mathcal{M}, \mathcal{N}) := \|P_{\mathcal{M}} - P_{\mathcal{N}}\|$$

where $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are the orthogonal projectors on $\mathcal{M}$ and $\mathcal{N}$, respectively.

Let $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ a matrix pencil. A reducing subspace $\mathcal{N}$ of $\mathbb{C}^n$ for $(\lambda B - A)$ is said to be *stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every matrix pencil $\lambda B' - A' \in \mathcal{P}[\lambda]^{m\times n}$ that satisfies

$$\|A' - A\| + \|B' - B\| < \delta$$

has a $(\lambda B' - A')$-reducing subspace $\mathcal{N}'$ for which the inequality $\theta(\mathcal{N}', \mathcal{N}) < \varepsilon$ holds. In the same way, we will say that the subspace $\mathcal{N}$ of $\mathbb{C}^n$ is *Lipschitz stable* if there exist $K, \varepsilon > 0$ such that every matrix pencil $\lambda B' - A' \in \mathcal{P}[\lambda]^{m\times n}$ that satisfies $\|A' - A\| + \|B' - B\| < \varepsilon$ has a $(\lambda B' - A')$-reducing subspace $\mathcal{N}'$ for which the inequality

$$\theta(\mathcal{N}', \mathcal{N}) \leq K(\|A' - A\| + \|B' - B\|)$$

holds. To simplify, we will often say that a subspace $\mathcal{N}$ is $(\lambda B - A)$-stable to mean that $\mathcal{N}$ is a $(\lambda B - A)$-reducing subspace and is stable to perturbations in the matrices B and A. For a clear motivation see Chapters 13 and 15 of [5].

A previous paper on this topic was published by the second and third authors [7]. A characterization of the stability or Lipschitz stability of deflating subspaces of a *regular matrix pencil* was already given there. In the current paper, we address this stability problem for the case of reducing subspaces of singular pencils. Before stating the main result of this paper, recall some properties of the pencils of matrices.

Two matrix pencils $\lambda B - A, \lambda D - C \in \mathcal{P}[\lambda]^{m\times n}$ are said to be *strictly equivalent* if there exist invertible matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$ such that

$\lambda D - C = P(\lambda B - A)Q$. Remark that two strictly equivalent pencils have the same normal rank. Hence a subspace $\mathcal{N}$ is $(\lambda B - A)$-reducing if and only if the subspace $Q^{-1}(\mathcal{N})$ is $(\lambda D - C)$-reducing. Set $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We will use the notation $\infty B - A := B$. We will say that the element $\alpha \in \overline{\mathbb{C}}$ is an eigenvalue of the pencil $\lambda B - A$ if

$$\operatorname{rank}(\alpha B - A) < \operatorname{nrank}(\lambda B - A).$$

An eigenvalue α of $\lambda B - A$ is *finite* if $\alpha \in \mathbb{C}$, and it is *infinite* if $\alpha = \infty$. We call *spectrum* of the pencil $\lambda B - A$ and we denote it by $\Lambda(\lambda B - A)$, the set of its eigenvalues. It is a subset of $\overline{\mathbb{C}}$.

The well-known Kronecker canonical form for the strict equivalence of matrix pencils is given in the following result.

Lemma 1 (Kronecker canonical form [4])**.** *Given a matrix pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$, there always exist invertible matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, such that $P(\lambda B - A)Q$ has the form*

$$\begin{pmatrix} \lambda B_c - A_c & O & O \\ O & \lambda B_r - A_r & O \\ O & O & \lambda B_f - A_f \end{pmatrix}, \tag{2}$$

where

$$\lambda B_c - A_c = (O_{(n_1-t_0)\times(t_0-t_1)}, \operatorname{diag}(L_{r_1}, L_{r_2}, \dots, L_{r_{t_1}})) \in \mathcal{P}[\lambda]^{(n_1-t_0)\times n_1}; \tag{3}$$

$L_{r_i} := \lambda(I_{r_i}, O) - (O, I_{r_i}) \in \mathcal{P}[\lambda]^{r_i\times(r_i+1)}$; and the pencil

$$\lambda B_r - A_r := \begin{pmatrix} \lambda N - I & O \\ O & \lambda I - J \end{pmatrix} \in \mathcal{P}[\lambda]^{n_2\times n_2}, \tag{4}$$

with a nilpotent matrix N, is regular; and, lastly,

$$\lambda B_f - A_f := \begin{pmatrix} O_{(s_0-s_1)\times(m_3-s_0)} \\ \operatorname{diag}(L^T_{\ell_1}, L^T_{\ell_2}, \dots, L^T_{\ell_{s_1}}) \end{pmatrix} \in \mathcal{P}[\lambda]^{m_3\times(m_3-s_0)}. \tag{5}$$

Remark 2. The sequences

$$(r_1, r_2, \dots, r_{t_1}, \overbrace{0, \dots, 0}^{t_0-t_1}), \quad (\ell_1, \ell_2, \dots, \ell_{s_1}, \overbrace{0, \dots, 0}^{s_0-s_1})$$

are the *column minimal indices* and the *row minimal indices*, respectively. The elementary divisors of the matrices N and J, are called *infinite elementary divisors* and *finite elementary divisors*, respectively, of the matrix pencil. Recall that $\Lambda(\lambda B - A)$ denotes the *spectrum* of the pencil $\lambda B - A$, then $\Lambda(\lambda B - A) = \Lambda(\lambda B_r - A_r)$.

Thus, a *complete system of invariants* for the strict equivalence of two matrix pencils is formed by the finite sequences of row and column minimal indices and the system of elementary divisors (finite and infinite). For some particular pencils some of these invariant can be absent. Strictly speaking it is not the same the *set* of row minimal indices and the *sequence* of row minimal indices. But from now on we will loosely speak and —for example— we will say that a pencil has two row minimal indices to mean that the sequence of row minimal indices has two terms, which might be equals.

Remark 3. Since $t_0 = \nu(\lambda B - A)$, where t_0 is the number of column minimal indices (see [4]), from (1) we deduce that for each subspace $\mathcal{N}$ of $\mathbb{C}^n$,

$$\dim(A(\mathcal{N}) + B(\mathcal{N})) \geq \dim \mathcal{N} - t_0.$$

Let X be a basis matrix of $\mathcal{N}$. As $\dim(A(\mathcal{N}) + B(\mathcal{N})) = \operatorname{rank}(AX, BX)$ we have

$$\operatorname{rank}(AX, BX) \geq \operatorname{rank}(X) - t_0. \tag{6}$$

As a consequence, the subspace $\mathcal{N}$ is $(\lambda B - A)$-reducing if and only if

$$\operatorname{rank}(AX, BX) = \operatorname{rank}(X) - t_0. \tag{7}$$

Recall that a matrix pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m \times n}$ is said to be *right regular* if $\operatorname{nrank}(\lambda B - A) = n$, or equivalently $\nu(\lambda B - A) = 0$, or equivalently, it has no column minimal index. In an analogous way, we will say that a pencil $\lambda B - A$ is *left regular* if $\lambda B^T - A^T$ is right regular, that is, if $\lambda B - A$ has no row minimal index.

Another previous work on the topic of the stability of reducing subspaces was the one by Demmel in [2]. He studied the stability of some reducing subspaces for *singular matrix pencils*, but under additional conditions. We will explain it briefly. Let $\lambda B - A$ be a singular pencil and let $\mathcal{N}$ be a reducing subspace for this pencil. Then, according to our notations, there is no loss of generality if we suppose that $\lambda B - A$ and $\mathcal{N}$ have the form

$$\lambda B - A = \begin{pmatrix} \lambda B_c - A_c & O & O & O \\ O & \lambda B_r^1 - A_r^1 & \lambda B_r^2 - A_r^2 & O \\ O & O & \lambda B_r^3 - A_r^3 & O \\ O & O & O & \lambda B_f - A_f \end{pmatrix},$$

$$\mathcal{N} = \left\langle \begin{pmatrix} I_{n_1} & O \\ O & I_{n_2} \\ O & O \\ O & O \end{pmatrix} \right\rangle,$$

where the $(m_1 \times n_1)$-pencil $\lambda B_c - A_c$ only has column minimal indices, $\lambda B_f - A_f$ only has row minimal indices and the pencil

$$\begin{pmatrix} \lambda B_r^1 - A_r^1 & \lambda B_r^2 - A_r^2 \\ O & \lambda B_r^3 - A_r^3 \end{pmatrix}$$

is regular, where $\lambda B_r^1 - A_r^1$ is a pencil of size $n_2 \times n_2$. In [2] it is supposed that

$$\Lambda(\lambda B_r^1 - A_r^1) \cap \Lambda(\lambda B_r^3 - A_r^3) = \emptyset.$$

Under these hypotheses, in Theorem 6, page 26 of [2], *some results* are given on the stability of the subspace $\mathcal{N}$, but *assuming also that the perturbed pencils have reducing subspaces of the same dimension as* $\mathcal{N}$.

There is an ample literature on the use of reducing subspaces of matrix pencils as a tool for factorizing rational matrices and for solving Riccati equations. One can see many references in the book by Ionescu, Oară and Weiss [10]. See also [13].

With theses notations, the main result of the paper is the following.

Theorem 4. *Let $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ be a singular matrix pencil. The following assertions are true:*
(1) *If the pencil has row minimal indices, column minimal indices and eigenvalues, then no reducing subspace is stable.*
(2) *If the pencil has no row minimal index, then the unique stable and Lipschitz stable reducing subspace is $\mathbb{C}^n$.*
(3) *If the pencil has no column minimal index, then the unique stable and Lipschitz stable reducing subspace is $\{0\}$.*
(4) *If the pencil has column minimal indices and, at least, two row minimal indices, then no reducing subspace is stable.*
(5) *If the pencil only has one row minimal index which is equal to zero, and has no eigenvalues, then the only stable and Lipschitz stable reducing subspace is $\mathbb{C}^n$.*
(6) *If the pencil only has one row minimal index which is different than zero, has not eigenvalues and has at least two column minimal indices, then no reducing subspace is stable.*
(7) *If the pencil only has one row minimal index which is different than zero, and has one column minimal index, which is equal to zero, and has not eigenvalues, then the unique stable and Lipschitz stable reducing subspace is* Ker$A$.

The organization of this paper is the following. In Section 2 algebraic properties of the reducing subspaces of pencils of linear maps are established. In Section 3, these properties are translated into terms of matrix pencils. In Section 4 the problem of the stability of reducing subspaces is addressed by means of converging sequences of matrix pencils and basis matrices of subspaces. In Sections 5 to 9 the proof of Theorem 4 (Main Theorem) is developed. In Section 5 Assertions (1), (2) and (3) of the Theorem are proved. In Sections 6, 7, 8 and 9 Assertions (4), (5), (6) and (7), respectively, are proved.

2 Properties of the reducing subspaces of linear map pencils

In this section we give a characterization of the reducing subspaces for pencils of linear maps. Its proof will be made in the following section, translating these results to the matrix pencils.

First, remark that the concepts of normal rank and reducing subspace can be extended to the case of a pair of linear maps. Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over $\mathbb{C}$ and let $\boldsymbol{A}, \boldsymbol{B} : \mathcal{U} \to \mathcal{V}$ be linear maps. The normal rank of the *pencil of linear maps* $\lambda\boldsymbol{B} - \boldsymbol{A}$ is defined by

$$\mathrm{nrank}(\lambda\boldsymbol{B} - \boldsymbol{A}) := \max_{z\in\mathbb{C}} \mathrm{rank}(z\boldsymbol{B} - \boldsymbol{A}),$$

where $z\boldsymbol{B} - \boldsymbol{A} : \mathcal{U} \to \mathcal{V}$ is a linear map for each $z \in \mathbb{C}$. A pencil of linear maps $\lambda\boldsymbol{B}-\boldsymbol{A}$ is said to be *regular* if $\dim\mathcal{U} = \dim\mathcal{V}$ and the linear map $z\boldsymbol{B}-\boldsymbol{A} : \mathcal{U} \to \mathcal{V}$ is invertible for every $z \in \mathbb{C}$, except for at most a finite number of complex numbers. Otherwise, we will say that the pencil is *singular*. For each $x \in \mathcal{U}$ we define

$$(\lambda\boldsymbol{B} - \boldsymbol{A})(x) := \boldsymbol{B}(x) + \boldsymbol{A}(x).$$

From this definition it is deduced that for every subspace $\mathcal{N}$ of $\mathcal{U}$, $(\lambda\boldsymbol{B}-\boldsymbol{A})(\mathcal{N})=\boldsymbol{B}(\mathcal{N})+\boldsymbol{A}(\mathcal{N})$. Therefore, the subspace $\mathcal{N}$ of $\mathcal{U}$ is said to be $(\lambda\boldsymbol{B}-\boldsymbol{A})$-*reducing* if

$$\dim(\lambda\boldsymbol{B}-\boldsymbol{A})(\mathcal{N})=\dim\mathcal{N}-\min_{z\in\mathbb{C}}\dim_{\mathbb{C}}\operatorname{Ker}(z\boldsymbol{B}-\boldsymbol{A}).$$

To write the statements of the main theorems in this section, we need some previous definitions and notations. Let $\mathcal{U}^k$ denote the Cartesian product $\mathcal{U}\times\cdots\times\mathcal{U}$, $k-$times. Given a pair of linear maps $\boldsymbol{A},\boldsymbol{B}:\mathcal{U}\to\mathcal{V}$ and $\alpha\in\mathbb{C}$, for $k=1,2,\ldots$, consider the linear maps

$$\boldsymbol{T}^k_{\lambda\boldsymbol{B}-\boldsymbol{A}}:\mathcal{U}^k\to\mathcal{V}^{k+1},\quad \boldsymbol{P}^{k,\alpha}_{\lambda\boldsymbol{B}-\boldsymbol{A}},\boldsymbol{P}^{k,\infty}_{\lambda\boldsymbol{B}-\boldsymbol{A}}:\mathcal{U}^k\to\mathcal{V}^k$$

defined for $x=(x_1,x_2,\ldots,x_k)\in\mathcal{U}^k$ by means of

$$\boldsymbol{T}^k_{\lambda\boldsymbol{B}-\boldsymbol{A}}(x):=(\boldsymbol{B}(x_1),-\boldsymbol{A}(x_1)+\boldsymbol{B}(x_2),\ldots,-\boldsymbol{A}(x_{k-1})+\boldsymbol{B}(x_k),-\boldsymbol{A}(x_k))\,, \quad (8)$$

$$\boldsymbol{P}^{k,\alpha}_{\lambda\boldsymbol{B}-\boldsymbol{A}}(x):=((\alpha\boldsymbol{B}-\boldsymbol{A})(x_1),\boldsymbol{B}(x_1)+(\alpha\boldsymbol{B}-\boldsymbol{A})(x_2),\ldots,\boldsymbol{B}(x_{k-1})+(\alpha\boldsymbol{B}-\boldsymbol{A})(x_k))\,, \quad (9)$$

$$\boldsymbol{P}^{k,\infty}_{\lambda\boldsymbol{B}-\boldsymbol{A}}(x):=(\boldsymbol{B}(x_1),-\boldsymbol{A}(x_1)+\boldsymbol{B}(x_2),\ldots,-\boldsymbol{A}(x_{k-1})+\boldsymbol{B}(x_k))\,. \quad (10)$$

Given $x=(x_1,x_2,\ldots,x_k)\in\mathcal{U}^k$, for $i=1,2,\ldots,k$ we define the projections $\pi^k_i(x)=x_i$. Now, for every $\alpha\in\bar{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ and $k=1,2,\ldots$, we define the subspaces:

$$\mathcal{S}^k_{\lambda\boldsymbol{B}-\boldsymbol{A}}:=\sum_{i=1}^k\pi^k_i\left(\operatorname{Ker}(\boldsymbol{T}^k_{\lambda\boldsymbol{B}-\boldsymbol{A}})\right), \quad (11)$$

$$\mathcal{S}^{k,\alpha}_{\lambda\boldsymbol{B}-\boldsymbol{A}}:=\sum_{i=1}^k\pi^k_i\left(\operatorname{Ker}(\boldsymbol{P}^{k,\alpha}_{\lambda\boldsymbol{B}-\boldsymbol{A}})\right), \quad (12)$$

$$\mathcal{D}^k_{\lambda\boldsymbol{B}-\boldsymbol{A}}:=\mathcal{S}^k_{\lambda\boldsymbol{B}-\boldsymbol{A}}+\sum_{\alpha\in\Lambda(\lambda\boldsymbol{B}-\boldsymbol{A})}\mathcal{S}^{k,\alpha}_{\lambda\boldsymbol{B}-\boldsymbol{A}}. \quad (13)$$

With these notations we obtain the first result in this section.

Theorem 5. *Given two linear maps $\boldsymbol{A},\boldsymbol{B}:\mathcal{U}\to\mathcal{V}$, then* (a) *The subspaces $\mathcal{S}^n_{\lambda\boldsymbol{B}-\boldsymbol{A}}$ and $\mathcal{D}^n_{\lambda\boldsymbol{B}-\boldsymbol{A}}$ are $(\lambda\boldsymbol{B}-\boldsymbol{A})$-reducing.*
(b) *For every $(\lambda\boldsymbol{B}-\boldsymbol{A})$-reducing subspace $\mathcal{N}$ we have $\mathcal{S}^n_{\lambda\boldsymbol{B}-\boldsymbol{A}}\subset\mathcal{N}\subset\mathcal{D}^n_{\lambda\boldsymbol{B}-\boldsymbol{A}}$.*

Remark 6. Theorem 5 will be proven by means of matrix pencils in Theorems 12 and 20 in Section 3.

Proposition 7. *To prove Theorem 5 there is no loss of generality if, instead of the linear map pencil $\lambda\boldsymbol{B}-\boldsymbol{A}$, we consider the linear map pencil $\lambda\boldsymbol{D}-\boldsymbol{C}=\boldsymbol{P}\circ(\lambda\boldsymbol{B}-\boldsymbol{A})\circ\boldsymbol{Q}$ (with $\boldsymbol{P},\boldsymbol{Q}$ invertible transformations of $\mathcal{V}$ and $\mathcal{U}$, respectively).*

Proof. Consider the linear maps $\boldsymbol{Q}_1:\mathcal{U}^k\to\mathcal{U}^k$ and $\boldsymbol{P}_1:\mathcal{V}^{k+1}\to\mathcal{V}^{k+1}$ defined by

$$\boldsymbol{Q}_1(x_1,\ldots,x_k):=(\boldsymbol{Q}(x_1),\ldots,\boldsymbol{Q}(x_k))$$

and

$$\boldsymbol{P}_1(y_1,\ldots,y_{k+1}):=(\boldsymbol{P}(y_1),\ldots,\boldsymbol{P}(y_{k+1})).$$

From (8) we immediately deduce

$$\boldsymbol{T}^k_{\lambda \boldsymbol{D}-\boldsymbol{C}} = \boldsymbol{P}_1 \circ \boldsymbol{T}^k_{\lambda \boldsymbol{B}-\boldsymbol{A}} \circ \boldsymbol{Q}_1.$$

Therefore,

$$\boldsymbol{T}^k_{\lambda \boldsymbol{B}-\boldsymbol{A}}(x) = 0 \Leftrightarrow \boldsymbol{P}_1 \circ \boldsymbol{T}^k_{\lambda \boldsymbol{B}-\boldsymbol{A}} \circ \boldsymbol{Q}_1 \circ \boldsymbol{Q}_1^{-1}(x) = 0 \Leftrightarrow \boldsymbol{T}^k_{\lambda \boldsymbol{D}-\boldsymbol{C}} \circ \boldsymbol{Q}_1^{-1}(x) = 0.$$

That is, $\mathrm{Ker}(\boldsymbol{T}^k_{\lambda \boldsymbol{D}-\boldsymbol{C}}) = \boldsymbol{Q}_1^{-1}(\mathrm{Ker}(\boldsymbol{T}^k_{\lambda \boldsymbol{B}-\boldsymbol{A}}))$. , from (11) we infer that

$$\mathcal{S}^k_{\lambda \boldsymbol{D}-\boldsymbol{C}} = \boldsymbol{Q}^{-1}(\mathcal{S}^k_{\lambda \boldsymbol{B}-\boldsymbol{A}}). \tag{14}$$

Using the same arguments from (9), (10) and (12), we obtain $\mathcal{S}^{k,\alpha}_{\lambda \boldsymbol{D}-\boldsymbol{C}} = \boldsymbol{Q}^{-1}(\mathcal{S}^{k,\alpha}_{\lambda \boldsymbol{B}-\boldsymbol{A}})$, and substituing (14) in (13), we have

$$\mathcal{D}^k_{\lambda \boldsymbol{D}-\boldsymbol{C}} = \boldsymbol{Q}^{-1}(\mathcal{D}^k_{\lambda \boldsymbol{B}-\boldsymbol{A}}). \tag{15}$$

As $\mathcal{N}$ is a $(\lambda\boldsymbol{B} - \boldsymbol{A})$-reducing subspace if and only if $\boldsymbol{Q}^{-1}(\mathcal{N})$ is $(\lambda\boldsymbol{D} - \boldsymbol{C})$-reducing, from (14) and (15) we conclude that $\mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and $\mathcal{D}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ are $(\lambda\boldsymbol{B}-\boldsymbol{A})$-reducing if and only if $\mathcal{S}^n_{\lambda \boldsymbol{D}-\boldsymbol{C}}$ and $\mathcal{D}^n_{\lambda \boldsymbol{D}-\boldsymbol{C}}$ are $(\lambda\boldsymbol{D}-\boldsymbol{C})$-reducing. Moreover it is clear that $\mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}} \subset \mathcal{N} \subset \mathcal{D}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ if and only if $\mathcal{S}^n_{\lambda \boldsymbol{D}-\boldsymbol{C}} \subset \boldsymbol{Q}^{-1}(\mathcal{N}) \subset \mathcal{D}^n_{\lambda \boldsymbol{D}-\boldsymbol{C}}$.
□

For the second result we need some notations. Let $\mathcal{K}$ be a direct complement of $\mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ in $\mathcal{D}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and let $\pi_{\mathcal{K}} : \mathcal{D}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}} \to \mathcal{K}$ be the projection over $\mathcal{K}$ along $\mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$. That is, $\mathrm{Im}\,\pi_{\mathcal{K}} = \mathcal{K}$ and $\mathrm{Ker}\pi_{\mathcal{K}} = \mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$. Denote

$$\mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}} := (\lambda\boldsymbol{B} - \boldsymbol{A})(\mathcal{D}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}), \quad \mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}} := (\lambda\boldsymbol{B} - \boldsymbol{A})(\mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}). \tag{16}$$

Now, let $\mathcal{L}$ be a direct complement of $\mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ in $\mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and let $\pi_{\mathcal{L}} : \mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}} \to \mathcal{L}$ be the projection over $\mathcal{L}$ along $\mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$. With these notations, we have the following result.

Theorem 8. *Let $\boldsymbol{A}, \boldsymbol{B} : \mathcal{U} \to \mathcal{V}$ be linear maps. Then a subspace $\mathcal{N}$ of $\mathcal{U}$ is $(\lambda\boldsymbol{B} - \boldsymbol{A})$-reducing if and only if the subspace $\pi_{\mathcal{K}}(\mathcal{N})$ of $\mathcal{K}$ is deflating for the regular pencil*

$$\pi_{\mathcal{L}} \circ (\lambda\boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}} : \mathcal{K} \to \mathcal{L}.$$

Remark 9. Theorem 8 will be proved in Theorem 20 in Section 3.

Proposition 10. *The conclusions of Theorem 8 do not depend on the choice of the subspaces $\mathcal{K}$, $\mathcal{L}$.*

Proof. Let $\mathcal{K}_1$ be another direct complement of $\mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ in $\mathcal{D}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and let $\pi_{\mathcal{K}_1} : \mathcal{D}_{\lambda \boldsymbol{B}-\boldsymbol{A}} \to \mathcal{K}_1$ be the projection over $\mathcal{K}_1$ along $\mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$. In the same manner, let $\mathcal{L}_1$ be another direct complement of $\mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ in $\mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and let $\pi_{\mathcal{L}_1} : \mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}} \to \mathcal{L}_1$ be the projection over $\mathcal{L}_1$ along $\mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$. Then, (see [15, Remark 2, p. 402]), there exist invertible linear maps

$$\boldsymbol{Q} : \mathcal{K} \to \mathcal{K}_1, \qquad \boldsymbol{P} : \mathcal{L} \to \mathcal{L}_1$$

such that

$$\forall x \in \mathcal{K},\, x - \boldsymbol{Q}(x) \in \mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}, \quad \forall y \in \mathcal{L},\, y - \boldsymbol{P}(y) \in \mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}}, \tag{17}$$

and moreover,

$$\pi_{\mathcal{K}_1} = \boldsymbol{Q} \circ \pi_{\mathcal{K}}, \quad \pi_{\mathcal{L}_1} = \boldsymbol{P} \circ \pi_{\mathcal{L}}. \tag{18}$$

See first that the pencils $\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}}$ and $\pi_{\mathcal{L}_1} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}_1}$ are strictly equivalent. So, one is regular if and only if the other is. Observe that $\boldsymbol{Q}$ are $\boldsymbol{P}$ are invertible, it suffices to see that $\boldsymbol{P} \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}} = \pi_{\mathcal{L}_1} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}_1} \circ \boldsymbol{Q}$. Given that $\pi_{\mathcal{L}_1} = \boldsymbol{P} \circ \pi_{\mathcal{L}}$ by (18), it is sufficient to prove that

$$\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}} = \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}_1} \circ \boldsymbol{Q}. \tag{19}$$

Let $x \in \mathcal{K}$. Then, as $\boldsymbol{Q}(x) \in \mathcal{K}_1$, to prove (19) it suffices to see that $(\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(x - \boldsymbol{Q}(x)) = 0$. But given that, by (17) , $x - \boldsymbol{Q}(x) \in \mathcal{S}^n_{\lambda \boldsymbol{B} - \boldsymbol{A}}$, from the notations of (16) we see that $(\lambda \boldsymbol{B} - \boldsymbol{A})(x - \boldsymbol{Q}(x)) \in \mathcal{M}_{\lambda \boldsymbol{B} - \boldsymbol{A}}$. Therefore $(\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(x - \boldsymbol{Q}(x)) = 0$, which proves (19).

Now see that

$$(\pi_{\mathcal{L}_1} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}_1}(\mathcal{N})) = (\boldsymbol{P} \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}}(\mathcal{N})). \tag{20}$$

As by (18), we have

$$\pi_{\mathcal{K}_1}(\mathcal{N}) = \boldsymbol{Q}(\pi_{\mathcal{K}}(\mathcal{N})) = \pi_{\mathcal{K}}(\mathcal{N}) + (\boldsymbol{Q} - I)(\pi_{\mathcal{K}}(\mathcal{N})),$$

and $\pi_{\mathcal{L}_1} = \boldsymbol{P} \circ \pi_{\mathcal{L}}$, we deduce that

$$\begin{array}{rcl}(\pi_{\mathcal{L}_1} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}_1}(\mathcal{N})) & = & (\boldsymbol{P} \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}}(\mathcal{N})) \\ & + & (\boldsymbol{P} \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))((\boldsymbol{Q} - I)\pi_{\mathcal{K}}(\mathcal{N})).\end{array} \tag{21}$$

Now, as (17) implies $(\boldsymbol{Q} - I)(\pi_{\mathcal{K}}(\mathcal{N})) \subset (\boldsymbol{Q} - I)(\mathcal{K}) \subset \mathcal{S}^n_{\lambda \boldsymbol{B} - \boldsymbol{A}}$, from the notations of (16) we obtain

$$(\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))((\boldsymbol{Q} - I)\pi_{\mathcal{K}}(\mathcal{N})) \subset (\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\mathcal{S}^n_{\lambda \boldsymbol{B} - \boldsymbol{A}}) = \pi_{\mathcal{L}}(\mathcal{M}(\boldsymbol{B}, \boldsymbol{A})) = \{0\}.$$

This last expression together with (21) yields (20).

Finally, as $\pi_{\mathcal{K}_1}(\mathcal{N}) = \boldsymbol{Q}(\pi_{\mathcal{K}}(\mathcal{N}))$ with $\boldsymbol{Q}$ invertible, we see that $\dim(\pi_{\mathcal{K}_1}(\mathcal{N})) = \dim(\pi_{\mathcal{K}}(\mathcal{N}))$. This fact together with (20) implies

$$\begin{array}{c}\dim\left((\pi_{\mathcal{L}_1} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}_1}(\mathcal{N}))\right) = \dim(\pi_{\mathcal{K}_1}(\mathcal{N})) \\ \Updownarrow \\ \dim\left((\boldsymbol{P} \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}}(\mathcal{N}))\right) = \dim(\pi_{\mathcal{K}}(\mathcal{N})).\end{array}$$

Consequently, $\pi_{\mathcal{K}}(\mathcal{N})$ is $\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}}$-deflating if and only if $\pi_{\mathcal{K}_1}(\mathcal{N})$ is $\pi_{\mathcal{L}_1} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}_1}$-deflating. □

Proposition 11. *In the conclusions of Theorem* 8 *there is no loss of generality if we consider the strictly equivalent pencil* $\lambda \boldsymbol{D} - \boldsymbol{C} = \boldsymbol{P} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}) \circ \boldsymbol{Q}$*, with invertible transformations* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *of* $\mathcal{V}$ *and* $\mathcal{U}$*, respectively.*

Proof. Observe that from (14), (15) and (16) we obtain

$$\mathcal{H}_{\lambda \boldsymbol{D} - \boldsymbol{C}} = \boldsymbol{P}(\mathcal{H}_{\lambda \boldsymbol{B} - \boldsymbol{A}}), \quad \mathcal{M}_{\lambda \boldsymbol{D} - \boldsymbol{C}} = \boldsymbol{P}(\mathcal{M}_{\lambda \boldsymbol{B} - \boldsymbol{A}}). \tag{22}$$

Therefore, as $\mathcal{D}^n_{\lambda \boldsymbol{B} - \boldsymbol{A}} = \mathcal{S}^n_{\lambda \boldsymbol{B} - \boldsymbol{A}} \oplus \mathcal{K}$ and $\mathcal{H}_{\lambda \boldsymbol{B} - \boldsymbol{A}} = \mathcal{M}_{\lambda \boldsymbol{B} - \boldsymbol{A}} \oplus \mathcal{L}$, from (14), (15) and (22) we deduce that

$$\mathcal{D}^n_{\lambda \boldsymbol{D}-\boldsymbol{C}} = \mathcal{S}^n_{\lambda \boldsymbol{D}-\boldsymbol{C}} \oplus \boldsymbol{Q}^{-1}(\mathcal{K}), \quad \mathcal{H}_{\lambda \boldsymbol{D}-\boldsymbol{C}} = \mathcal{M}_{\lambda \boldsymbol{D}-\boldsymbol{C}} \oplus \boldsymbol{P}(\mathcal{L}). \tag{23}$$

To prove the Proposition, see first that

$$\pi_{\boldsymbol{Q}^{-1}(\mathcal{K})}(\boldsymbol{Q}^{-1}(\mathcal{N})) = (\boldsymbol{Q}^{-1} \circ \pi_{\mathcal{K}})(\mathcal{N}). \tag{24}$$

Let x be a vector of $\mathcal{N}$. Then, by 5(b), we see $x = y + z$ with $y \in \mathcal{S}^n_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and $z \in \mathcal{K}$. Hence $\pi_{\mathcal{K}}(x) = \pi_{\mathcal{K}}(z) = z$. On the other hand, as $\boldsymbol{Q}^{-1}(x) = \boldsymbol{Q}^{-1}(y) + \boldsymbol{Q}^{-1}(z)$, with $\boldsymbol{Q}^{-1}(y) \in \mathcal{S}^n_{\lambda \boldsymbol{D}-\boldsymbol{C}}$ and $\boldsymbol{Q}^{-1}(z) \in \boldsymbol{Q}^{-1}(\mathcal{K})$, we infer that

$$\pi_{\boldsymbol{Q}^{-1}(\mathcal{K})}(\boldsymbol{Q}^{-1}(x)) = \pi_{\boldsymbol{Q}^{-1}(\mathcal{K})}(\boldsymbol{Q}^{-1}(z)) = \boldsymbol{Q}^{-1}(z) = \boldsymbol{Q}^{-1}(\pi_{\mathcal{K}}(x)),$$

which proves (24).

Now consider the invertible linear maps

$$\boldsymbol{Q}_1^{-1} := \boldsymbol{Q}^{-1}\,|_{\mathcal{K}} : \mathcal{K} \to \boldsymbol{Q}^{-1}(\mathcal{K}), \quad \boldsymbol{P}_1 := \boldsymbol{P}\,|_{\mathcal{L}} : \mathcal{L} \to \boldsymbol{P}(\mathcal{L}).$$

With the notations of (16), see that

$$\pi_{\boldsymbol{P}(\mathcal{L})} \circ \boldsymbol{P}\,\big|_{\mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}}} = \boldsymbol{P}_1 \circ \pi_{\mathcal{L}}\,\big|_{\mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}}}. \tag{25}$$

Let $x = y + z$ be a vector of $\mathcal{H}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ with $y \in \mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and $z \in \mathcal{L}$. Then, as $\pi_{\mathcal{L}}(y) = 0$ and $\pi_{\mathcal{L}}(z) = z$, we have

$$\boldsymbol{P}_1(\pi_{\mathcal{L}}(x)) = \boldsymbol{P}_1(\pi_{\mathcal{L}}(y)) + \boldsymbol{P}_1(\pi_{\mathcal{L}}(z)) = \boldsymbol{P}_1(z) = \boldsymbol{P}(z). \tag{26}$$

On the other hand, as $y \in \mathcal{M}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$, by (22), we see that $\boldsymbol{P}(y) \in \mathcal{M}_{\lambda \boldsymbol{D}-\boldsymbol{C}}$. Therefore $\pi_{\boldsymbol{P}(\mathcal{L})}(\boldsymbol{P}(y)) = 0$. Moreover, since $z \in \mathcal{L}$, it follows $\boldsymbol{P}(z) \in \boldsymbol{P}(\mathcal{L})$. Hence $\pi_{\boldsymbol{P}(\mathcal{L})}(\boldsymbol{P}(z)) = \boldsymbol{P}(z)$. Thus $\pi_{\boldsymbol{P}(\mathcal{L})}(\boldsymbol{P}(x)) = \boldsymbol{P}(z)$. This equality together with (26) proves (25).

Now see that

$$\pi_{\boldsymbol{P}(\mathcal{L})} \circ (\lambda \boldsymbol{D} - \boldsymbol{C})\,\big|_{\boldsymbol{Q}^{-1}(\mathcal{K})} \circ \boldsymbol{Q}_1^{-1} = \boldsymbol{P}_1 \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}}. \tag{27}$$

Let $x \in \mathcal{K}$. Then, as $\boldsymbol{Q}_1^{-1}(x) = \boldsymbol{Q}^{-1}(x) \in \boldsymbol{Q}^{-1}(\mathcal{K})$, we have

$$(\pi_{\boldsymbol{P}(\mathcal{L})} \circ (\lambda \boldsymbol{D} - \boldsymbol{C}))(\boldsymbol{Q}_1^{-1}(x)) = (\pi_{\boldsymbol{P}(\mathcal{L})} \circ \boldsymbol{P} \circ \boldsymbol{P}^{-1} \circ (\lambda \boldsymbol{D} - \boldsymbol{C}) \circ \boldsymbol{Q}^{-1})(x) = (\pi_{\boldsymbol{P}(\mathcal{L})} \circ \boldsymbol{P} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(x).$$

Therefore, using (25), we conclude that

$$(\pi_{\boldsymbol{P}(\mathcal{L})} \circ (\lambda \boldsymbol{D} - \boldsymbol{C}))(\boldsymbol{Q}_1^{-1}(x)) = (\boldsymbol{P}_1 \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(x).$$

Hence, the pencils $\pi_{\boldsymbol{P}(\mathcal{L})} \circ (\lambda \boldsymbol{D} - \boldsymbol{C})\,\big|_{\boldsymbol{Q}^{-1}\mathcal{K}}$ and $\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}}$ are strictly equivalent. Therefore one is regular if and only if the other is too. Finally, from (27) we have

$$(\pi_{\boldsymbol{P}(\mathcal{L})} \circ (\lambda \boldsymbol{D} - \boldsymbol{C}))(\pi_{\boldsymbol{Q}^{-1}(\mathcal{K})}(\boldsymbol{Q}^{-1}(\mathcal{N})) = (\boldsymbol{P}_1 \circ \pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}}(\mathcal{N})),$$

given that $\boldsymbol{P}_1$ is invertible we deduce that the subspaces

$$(\pi_{\boldsymbol{P}(\mathcal{L})} \circ (\lambda \boldsymbol{D} - \boldsymbol{C}))(\pi_{\boldsymbol{Q}^{-1}(\mathcal{K})}(\boldsymbol{Q}^{-1}(\mathcal{N}))$$

and $(\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A}))(\pi_{\mathcal{K}}(\mathcal{N}))$ have the same dimension. Moreover, from (24) we deduce that $\pi_{\boldsymbol{Q}^{-1}(\mathcal{K})}(\boldsymbol{Q}^{-1}(\mathcal{N}))$ and $\pi_{\mathcal{K}}(\mathcal{N})$ have the same dimension, we conclude that $\pi_{\boldsymbol{Q}^{-1}(\mathcal{K})}(\boldsymbol{Q}^{-1}(\mathcal{N}))$ is $\pi_{\boldsymbol{P}(\mathcal{L})} \circ (\lambda \boldsymbol{D} - \boldsymbol{C})\,\big|_{\boldsymbol{Q}^{-1}(\mathcal{K})}$-deflating if and only if $\pi_{\mathcal{K}}(\mathcal{N})$ is $\pi_{\mathcal{L}} \circ (\lambda \boldsymbol{B} - \boldsymbol{A})\,|_{\mathcal{K}}$-deflating. □

3 Properties of the reducing subspaces for matrix pencils

In this section we formulate Theorems 5 and 8 in terms of matrix pencils. First observe that if $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ is a matrix representation of $\lambda \boldsymbol{B} - \boldsymbol{A}$, then $\boldsymbol{T}^k_{\lambda \boldsymbol{B}-\boldsymbol{A}}$, $\boldsymbol{P}^{k,\alpha}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$ and $\boldsymbol{P}^{k,\infty}_{\lambda \boldsymbol{B}-\boldsymbol{A}}$, defined in (8), (9) and (10) respectively, have the following matrix representations

$$T^k_{\lambda B-A} := \begin{pmatrix} B & O & O & \cdots & O & O \\ -A & B & O & \cdots & O & O \\ O & -A & B & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & -A & B \\ O & O & O & \cdots & O & -A \end{pmatrix} \in \mathbb{C}^{(k+1)m\times kn}, \tag{28}$$

for each $\alpha \in \mathbb{C}$,

$$P^{k,\alpha}_{\lambda B-A} := \begin{pmatrix} \alpha B - A & O & \cdots & O & O \\ B & \alpha B - A & \cdots & O & O \\ O & B & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & B & \alpha B - A \end{pmatrix} \in \mathbb{C}^{km\times kn}, \tag{29}$$

and finally

$$P^{k,\infty}_{\lambda B-A} := \begin{pmatrix} B & O & \cdots & O & O \\ -A & B & \cdots & O & O \\ O & -A & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & -A & B \end{pmatrix} \in \mathbb{C}^{km\times kn}. \tag{30}$$

In the same way, as in (11), (12) and (13), we define the subspaces

$$\mathcal{S}^k_{\lambda B-A} := \sum_{i=1}^{k} \pi^k_i \left(\operatorname{Ker}(T^k_{\lambda B-A})\right), \tag{31}$$

$$\mathcal{S}^{k,\alpha}_{\lambda B-A} := \sum_{i=1}^{k} \pi^k_i \left(\operatorname{Ker}(P^{k,\alpha}_{\lambda B-A})\right), \tag{32}$$

$$\mathcal{D}^k_{\lambda B-A} := \mathcal{S}^k_{\lambda B-A} + \sum_{\alpha\in\Lambda(\lambda B-A)} \mathcal{S}^{k,\alpha}_{\lambda B-A}. \tag{33}$$

With these notations, we have a first result.

Theorem 12. *Given the matrix pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ in the form (2), we have*

$$\mathcal{S}^n_{\lambda B-A} = \left\langle \begin{pmatrix} I_{n_1} \\ O \end{pmatrix} \right\rangle, \quad \mathcal{D}^n_{\lambda B-A} = \left\langle \begin{pmatrix} I_{n_1+n_2} \\ O \end{pmatrix} \right\rangle.$$

To prove this theorem we need the following result that can be seen in [9, Teorema 1.7, p. 99], [12, Corollary 3.2] and [11, Section 5].

Lemma 13. *Given the pencil $\lambda D - C \in \mathcal{P}[\lambda]^{m\times n}$, let c_j be the number of column minimal indices $\geq j$ for $j = 0, 1, 2, \ldots$. Then, for $k = 1, 2, \ldots,$ we have*

$$\dim \operatorname{Ker}(T^k_{\lambda D-C}) = kc_0 - \sum_{i=1}^{k} c_i. \tag{34}$$

Moreover, given $\alpha \in \overline{\mathbb{C}}$, if m_α is the multiplicity of α as an eigenvalue of $\lambda D - C$ and t_0 is the number of column minimal indices of the pencil, then for $k \geq n$ we have

$$\dim \operatorname{Ker}(P^{k,\alpha}_{\lambda D-C}) = m_\alpha + kt_0, \tag{35}$$

The proof of Theorem 12 is going to be structured in several lemmas. The first one is the following.

Lemma 14. *Assume that the pencil $\lambda D - C \in \mathcal{P}[\lambda]^{m\times n}$ is in the form*

$$\lambda D - C = \begin{pmatrix} \lambda D_1 - C_1 & O \\ O & \lambda D_2 - C_2 \end{pmatrix},$$

with $\lambda D_1 - C_1 \in \mathcal{P}[\lambda]^{p\times q}$ and $\lambda D_2 - C_2 \in \mathcal{P}[\lambda]^{r\times s}$. Then, for $k = 1, 2, \ldots,$ and $\alpha \in \overline{\mathbb{C}}$

$$\mathcal{S}^k_{\lambda D-C} = \begin{pmatrix} \mathcal{S}^k_{\lambda D_1-C_1} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^k_{\lambda D_2-C_2} \end{pmatrix}, \quad \mathcal{S}^{k,\alpha}_{\lambda D-C} = \begin{pmatrix} \mathcal{S}^{k\alpha}_{\lambda D_1-C_1} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^{k\alpha}_{\lambda D_2-C_2} \end{pmatrix}.$$

Remark 15. By the above lemma, it follows

$$\mathcal{D}^k_{\lambda D-C} = \begin{pmatrix} \mathcal{D}^k_{\lambda D_1-C_1} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{D}^k_{\lambda D_2-C_2} \end{pmatrix}$$

Proof. Let

$$x := (y_1^T, z_1^T, y_2^T, z_2^T, \ldots, y_k^T, z_k^T)^T \in \mathbb{C}^{kn\times 1},$$

where, for $i = 1, 2, \ldots, k$, $y_i \in \mathbb{C}^{q\times 1}$ and $z_i \in \mathbb{C}^{s\times 1}$. Denote

$$y := (y_1^T, y_2^T, \ldots, y_k^T)^T \in \mathbb{C}^{kq\times 1}, \quad z := (z_1^T, z_2^T, \ldots, z_k^T)^T \in \mathbb{C}^{ks\times 1}.$$

From (28), it follows that $T^k_{\lambda D-C} x = 0$ if and only if $T^k_{\lambda D_1-C_1} y = 0$ and $T^k_{\lambda D_2-C_2} z = 0$. Therefore

$$\operatorname{Ker}(T^k_{\lambda D-C}) = \{(y_1^T, 0, y_2^T, 0, \ldots, y_k^T, 0)^T, (0, z_1^T, 0, z_2^T, \ldots, 0, z_k^T)^T\},$$

where

$$(y_1^T, y_2^T, \ldots, y_k^T)^T \in \operatorname{Ker}(T^k_{\lambda D_1-C_1}) \text{ and } (z_1^T, z_2^T, \ldots, z_k^T)^T \in \operatorname{Ker}(T^k_{\lambda D_2-C_2}).$$

Hence, from (31), we obtain

$$\mathcal{S}^k_{\lambda D-C} = \begin{pmatrix} \mathcal{S}^k_{\lambda D_1-C_1} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^k_{\lambda D_2-C_2} \end{pmatrix}.$$

Using the same arguments as in (29), (30) and (32) the decomposition for $\mathcal{S}^{k,\alpha}_{\lambda D-C}$ can be proved. $\square$

Reasoning in an analogous way, we can prove the following result.

Lemma 16. *Suppose that $\lambda D - C = (O, \lambda D_1 - C_1) \in \mathcal{P}[\lambda]^{m\times n}$, where $O \in \mathbb{C}^{m\times q}$. Then, for $k = 1, 2, \dots,$ and $\alpha \in \overline{\mathbb{C}}$, we have*

$$\mathcal{S}^k_{\lambda D-C} = \left\langle \begin{pmatrix} I_q \\ O \end{pmatrix} \right\rangle \oplus \begin{pmatrix} O \\ \mathcal{S}^k_{\lambda D_1 - C_1} \end{pmatrix}, \quad \mathcal{S}^{k,\alpha}_{\lambda D-C} = \left\langle \begin{pmatrix} I_q \\ O \end{pmatrix} \right\rangle \oplus \begin{pmatrix} O \\ \mathcal{S}^{k,\alpha}_{\lambda D_1 - C_1} \end{pmatrix}.$$

Lemma 17. *Suppose that $\lambda D - C = \lambda[I_p, O] - [O, I_p] \in \mathcal{P}[\lambda]^{p\times(p+1)}$. Then*

$$\mathcal{S}^k_{\lambda D-C} = \begin{cases} \{0\} & \text{if } k < p+1 \\ \mathbb{C}^{p+1} & \text{if } k \geq p+1. \end{cases}$$

Proof. Let c_j be the number of column minimal indices $\geq j$. For $\lambda D - C$ we have $c_0 = c_1 = \cdots = c_p = 1$ and $c_{p+1} = 0$. Thus, by (34), we deduce that $\dim \operatorname{Ker}(T^k_{\lambda D-C}) = 0$ for $k = 1, 2 \dots, p$. Hence $\mathcal{S}^k_{\lambda D-C} = \{0\}$. Denoting by e_i the $i-$th vector of the canonical basis of $\mathbb{C}^{p+1}$, we see that

$$T^{p+1}_{\lambda D-C}(e^T_{p+1}, e^T_p, \dots, e^T_1)^T = 0.$$

Therefore

$$\mathbb{C}^{p+1} =< (e_1, e_2, \dots, e_{p+1}) >\subset \mathcal{S}^{p+1}_{\lambda D-C} \subset \mathbb{C}^{p+1},$$

that is $\mathcal{S}^{p+1}_{\lambda D-C} = \mathbb{C}^{p+1}$. To conclude the proof it suffices to remark that $\mathcal{S}^{p+1}_{\lambda D-C} \subset \mathcal{S}^{p+r}_{\lambda D-C} \subset \mathbb{C}^{p+1}$ for $r = 2, 3, \dots$. □

Lemma 18. *Let $\lambda D - C \in \mathcal{P}[\lambda]^{p\times p}$ be a regular pencil. Then, for $k = 1, 2, \dots,$ we have $\mathcal{S}^k_{\lambda D-C} = \{0\}$. Moreover, with the notations in (32), for $k \geq p$*

$$\sum_{\alpha \in \Lambda(\lambda D - C)} \mathcal{S}^{k,\alpha}_{\lambda D-C} = \mathbb{C}^p.$$

Proof. First, as the pencil $\lambda D - C$ has no column minimal indices, from (34) we infer that $\dim \operatorname{Ker}(T^k_{\lambda D-C}) = 0$. Hence $\mathcal{S}^k_{\lambda D-C} = \{0\}$.

On the other hand, because $\lambda D - C$ is regular, from Lemma 1, we can assume that

$$\lambda D - C = \operatorname{diag}(\lambda D_1 - C_1, \lambda D_2 - C_2, \dots, \lambda D_h - C_h),$$

$\lambda D_i - C_i \in \mathcal{P}[\lambda]^{p_i\times p_i}$ being a regular pencil with only one eigenvalue α_i. Therefore, applying Lemma 14, for each $\alpha \in \bar{\mathbb{C}}$ we have

$$\mathcal{S}^{k,\alpha}_{\lambda D-C} = \begin{pmatrix} \mathcal{S}^{k,\alpha}_{\lambda D_1 - C_1} \\ O \\ \vdots \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^{k,\alpha}_{\lambda D_2 - C_2} \\ \vdots \\ O \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} O \\ O \\ \vdots \\ \mathcal{S}^{k,\alpha}_{\lambda D_h - C_h} \end{pmatrix}. \tag{36}$$

Now, as the pencil $\lambda D_i - C_i \in \mathcal{P}[\lambda]^{p_i\times p_i}$ is regular and α_i is its only eigenvalue, then its multiplicity is p_i. Hence, from (35) we deduce that for $k \geq p_i$

$$\dim \operatorname{Ker}(P^{k,\alpha_i}_{\lambda D_i - C_i}) = p_i.$$

Consequently, there exists a basis matrix of $\operatorname{Ker}(P^{k,\alpha}_{\lambda D_i - C_i})$ of the form

$$X = (X^T_1, X^T_2, \dots, X^T_k)^T \in \mathbb{C}^{kp_i\times p_i}, \text{ where } X_i \in \mathbb{C}^{p_i\times p_i},$$

with $\operatorname{rank}(X) = p_i$. Therefore $\operatorname{rank}(X_1, X_2, \dots, X_k) = p_i$, and from (32) we see that $\mathcal{S}^{k,\alpha_i}_{\lambda D_i - C_i} = \mathbb{C}^{p_i}$. The proof is completed by taking these results into (36). □

Lemma 19. *Suppose that the pencil $\lambda D - C \in \mathcal{P}[\lambda]^{p\times q}$ only has row minimal indices. Then using the notations of* (33), *for $k = 1, 2, \ldots,$ we have*

$$\mathcal{D}^k_{\lambda D-C} = \{0\}.$$

Proof. It suffices to prove that $\mathcal{S}^k_{\lambda D-C} = \{0\}$ and $\mathcal{S}^{k,\alpha}_{\lambda D-C} = \{0\}$ for each $k = 1, 2\ldots,$ and $\alpha \in \overline{\mathbb{C}}$. First of all, as the pencil $\lambda D - C$ has no column minimal indices, from (34) we deduce that $\dim \operatorname{Ker}(T^k_{\lambda D-C}) = 0$. Hence $\mathcal{S}^k_{\lambda D-C} = \{0\}$. Analogously, as $\lambda D - C$ has no eigenvalues, from (35) we see that $\dim \operatorname{Ker}(P^{k,\alpha}_{\lambda D-C}) = 0$, that is $\mathcal{S}^{k,\alpha}_{\lambda D-C} = \{0\}$. □

We are now ready to prove Theorem 12.

Proof of Theorem 12.

First, as $\lambda B - A$ is in the form (2), applying Lemmas 14, 18 and 19 we have

$$\mathcal{S}^n_{\lambda B-A} = \begin{pmatrix} \mathcal{S}^n_{\lambda B_c-A_c} \\ O \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^n_{\lambda B_r-A_r} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ O \\ \mathcal{S}^n_{\lambda B_f-A_f} \end{pmatrix} = \begin{pmatrix} \mathcal{S}^n_{\lambda B_c-A_c} \\ O \\ O \end{pmatrix}.$$

Now, by Lemmas 16 and 17 we deduce that $\mathcal{S}^n_{\lambda B_c-A_c} = \mathbb{C}^{n_1}$.

On the other hand, denoting by $\lambda B_1 - A_1 := \operatorname{diag}(\lambda B_r - A_r, \lambda B_f - A_f)$, by Lemmas 14 and 19 we infer that

$$\mathcal{S}^{n,\alpha}_{\lambda B_1-A_1} = \begin{pmatrix} \mathcal{S}^{n,\alpha}_{\lambda B_r-A_r} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^{n,\alpha}_{\lambda B_f-A_f} \end{pmatrix} = \begin{pmatrix} \mathcal{S}^{n,\alpha}_{\lambda B_r-A_r} \\ O \end{pmatrix}.$$

Therefore, as $\Lambda(\lambda B - A) = \Lambda(\lambda B_r - A_r)$, by Lemma 18 we have

$$\sum_{\alpha\in\Lambda(\lambda B-A)} \mathcal{S}^{n,\alpha}_{\lambda B_r-A_r} = \sum_{\alpha\in\Lambda(\lambda B_r-A_r)} \mathcal{S}^{n,\alpha}_{\lambda B_r-A_r} = C^{n_2}.$$

This last expression together with $\mathcal{S}^n_{\lambda B_c-A_c} = \mathbb{C}^{n_1}$ prove the theorem. □

Taking into account Theorem 12, Propositions 7, 10 and 11, Theorems 5 and 8 can be reformulated in the following way.

Theorem 20. *Let $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ be a matrix pencil in the form* (2). *Let $\mathcal{N}$ be a subspace of $\mathbb{C}^n$. Then $\mathcal{N}$ is $(\lambda B - A)$-reducing if and only if there exists a matrix $X \in \mathbb{C}^{n_2\times q}$ of full column rank such that*

$$\mathcal{N} = \left\langle \begin{pmatrix} I_{n_1} & O \\ O & X \\ O & O \end{pmatrix} \right\rangle \tag{37}$$

and the subspace $\langle X \rangle$ is deflating for the pencil $\lambda B_r - A_r$.

To prove this theorem we need several previous results. The first one is a direct consequence of Theorems 2.3, 2.4 and Corollary 2.1 of [17].

Lemma 21. *Let $\lambda D - C \in \mathcal{P}[\lambda]^{p\times q}$ be a pencil. If this pencil only has column minimal indices, then its unique reducing subspace is C^q. If this pencil only has row minimal indices, then its unique reducing subspace is $\{0\}$.*

Lemma 22. *Let* $A_c, B_c \in \mathbb{C}^{(n_1-t_0)\times n_1}$ *be matrices in the form* (3) *and let* $A_f, B_f \in \mathbb{C}^{m_3\times(m_3-s_0)}$ *be matrices in the form* (5)*. Then there exist matrices* $Q \in \mathbb{C}^{n_1\times n_1}$ *and* $P \in \mathbb{C}^{m_3\times m_3}$ *such that*

$$B_c = A_c Q, \qquad B_f = P A_f.$$

Moreover, given a finite subset Λ_1 *of* $\mathbb{C}$*, we can assume that* P *and* Q *satisfy one of the following conditions:*
a) $\Lambda(P) \cap \Lambda(Q) = \emptyset$ *and* $\Lambda_1 \cap \big(\Lambda(P) \cup \Lambda(P^{-1}) \cup \Lambda(Q) \cup \Lambda(Q^{-1})\big) = \emptyset$.
b) $0 \in \Lambda(Q)$.
c) $\Lambda_1 \cap \Lambda(Q) \neq \emptyset$.
d) $0 \in \Lambda(P)$ *and* $\Lambda_1 \cap \Lambda(Q) \neq \emptyset$.

Proof. It suffices to observe that

$$(I_{r_i}, O) = (O, I_{r_i}) \begin{pmatrix} O & \alpha \\ I_{r_i} & O \end{pmatrix}, \text{ and } \begin{pmatrix} I_{l_j} \\ O \end{pmatrix} = \begin{pmatrix} O & I_{l_j} \\ \beta & O \end{pmatrix} \begin{pmatrix} O \\ I_{l_j} \end{pmatrix}$$

for any $\alpha, \beta \in \mathbb{C}$. □

We are already in conditions to prove Theorem 20.

Proof of Theorem 20

Let $\mathcal{N} \in \mathbb{C}^n$ be a subspace $(\lambda B - A)$-reducing of dimension p and let $Y \in \mathbb{C}^{n\times p}$ be a basis matrix of $\mathcal{N}$. Decomposing Y according to the decomposition of the pencil given in (2) we have

$$Y = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} Y_{11} & O & O \\ Y_{21} & Y_{22} & O \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \end{array}, \text{ where } \quad \operatorname{rank}(Y_{ii}) = p_i,\ i = 1, 2, 3. \qquad (38)$$

As the subspace $\mathcal{N}$ is $(\lambda B - A)$-reducing and t_0 is the number of column minimal indices of the pencil, from (7) we have $\operatorname{rank}(AY, BY) = p - t_0$. Therefore, operating according to (2) and (38) we get

$$\operatorname{rank} \begin{pmatrix} A_cY_{11} & O & O & B_cY_{11} & O & O \\ A_rY_{21} & A_rY_{22} & O & B_rY_{21} & B_rY_{22} & O \\ A_fY_{31} & A_fY_{32} & A_fY_{33} & B_fY_{31} & B_fY_{32} & B_fY_{33} \end{pmatrix} = p_1 + p_2 + p_3 - t_0. \qquad (39)$$

Now, as the pencil $\lambda B_c - A_c$ has the same column minimal indices as the pencil $\lambda B - A$ and $\lambda B_r - A_r$ and $\lambda B_f - A_f$ have no column minimal indices, from (6) we deduce that $\operatorname{rank}(A_cY_{11}, B_cY_{11}) \geq p_1 - t_0$, $\operatorname{rank}(A_rY_{22}, B_rY_{22}) \geq p_2$ and $\operatorname{rank}(A_fY_{33}, B_fY_{33}) \geq p_3$. Taking these three inequalities into (39) we see that

$$\begin{aligned} p_1 + p_2 + p_3 - t_0 = \;& \operatorname{rank}(AY, BY) \\ \geq \;& \operatorname{rank}(A_cY_{11}, B_cY_{11}) + \operatorname{rank}(A_rY_{22}, B_rY_{22}) + \operatorname{rank}(A_fY_{33}, B_fY_{33}) \\ \geq \;& p_1 + p_2 + p_3 - t_0. \end{aligned}$$

$$\operatorname{rank}(A_cY_{11}, B_cY_{11}) = p_1 - t_0,\ \operatorname{rank}(A_rY_{22}, B_rY_{22}) = p_2,\ \operatorname{rank}(A_fY_{33}, B_fY_{33}) = p_3,$$

that is, the subspaces $\langle Y_{11}\rangle$, $\langle Y_{22}\rangle$ and $\langle Y_{33}\rangle$ are reducing for the pencils $\lambda B_c - A_c$, $\lambda B_r - A_r$ and $\lambda B_f - A_f$, respectively. Therefore, applying Lemma 21 we deduce that $\langle Y_{11}\rangle = \mathbb{C}^{n_1}$ and $Y_{33} = O$. Hence we can assume that $Y_{11} = I_{n_1}$. Moreover, since $\lambda B_r - A_r$ is a regular pencil, denoting $X := Y_{22}$, it follows that the subspace $\langle X\rangle$ is $(\lambda B_r - A_r)$-deflating. Therefore (38) and (39) are transformed into

$$Y = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & p_2 \end{array} \\ \begin{pmatrix} I_{n_1} & O \\ Y_{21} & X \\ Y_{31} & Y_{32} \end{pmatrix} \end{array}, \tag{40}$$

with $\langle X\rangle$ a $(\lambda B_r - A_r)$-deflating subspace and

$$\operatorname{rank}\begin{pmatrix} A_c & O & B_c & O \\ A_rY_{21} & A_rX & B_rY_{21} & B_rX \\ A_fY_{31} & A_fY_{32} & B_fY_{31} & B_fY_{32} \end{pmatrix} = n_1 + p_2 - t_0. \tag{41}$$

Now observe that, since $\operatorname{rank}(A_c) = n_1 - t_0$, from (41) and (6) we infer that

$$\operatorname{rank}\begin{pmatrix} A_rX & B_rX \\ A_fY_{32} & B_fY_{32} \end{pmatrix} = p_2. \tag{42}$$

Moreover, as $\langle X\rangle$ is $(\lambda B_r - A_r)$-deflating, we can assume that

$$\lambda B_r - A_r = \begin{pmatrix} \lambda N_1 - I_{q_1} & \lambda N_3 & O & O \\ O & \lambda N_2 - I & O & O \\ O & O & \lambda I_{q_2} - J_1 & -J_3 \\ O & O & O & \lambda I - J_2 \end{pmatrix} \text{ and } X = \begin{pmatrix} I_{q_1} & O \\ O & O \\ O & I_{q_2} \\ O & O \end{pmatrix}, \tag{43}$$

with nilpotent matrices N_1, N_2. Hence, partitioning $Y_{32} = (Z, H)$ with $Z \in \mathbb{C}^{m_3\times q_1}$, from (42) we see that

$$\operatorname{rank}\begin{pmatrix} I_{q_1} & O & N_1 & O \\ O & J_1 & O & I_{q_2} \\ A_fZ & A_fH & B_fZ & B_fH \end{pmatrix} = p_2 = q_1 + q_2.$$

Now, from Lemma 22 there exists a matrix P such that $B_f = PA_f$, and thus,

$$\begin{aligned} q_1 + q_2 &= \operatorname{rank}\begin{pmatrix} I_{q_1} & O & N_1 & O \\ O & J_1 & O & I_{q_2} \\ A_fZ & P^{-1}B_fH & PA_fZ & B_fH \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} I_{q_1} & O & O & O \\ O & O & O & I_{q_2} \\ A_fZ & P^{-1}B_fH - B_fHJ_1 & PA_fZ - A_fZN_1 & B_fH \end{pmatrix}. \end{aligned}$$

Therefore $P^{-1}B_fH - B_fHJ_1 = O$ and $PA_fZ - A_fZN_1 = O$. Choosing P in such a way that $\Lambda(J_1) \cap \big(\Lambda(P) \cup \Lambda(P^{-1})\big) = \emptyset$, (Lema 22), we conclude that $B_fH = O$ and $A_fZ = O$, that is, H and Z are null matrices, and hence $Y_{32} = O$.

On the other hand, since $\operatorname{rank}(A_rX, B_rX) = p_2$, from (41) we deduce that

$$\operatorname{rank}\begin{pmatrix} A_c & B_c \\ A_fY_{31} & B_fY_{31} \end{pmatrix} = n_1 - t_0,$$

hence, by Lemma 22 there exist invertible matrices P and Q such that $B_c = A_cQ$, $B_f = PA_f$ and $\Lambda(P) \cap \Lambda(Q) = \emptyset$. As a consequence,

$$n_1 - t_0 = \operatorname{rank}\begin{pmatrix} A_c & A_cQ \\ A_fY_{31} & PA_fY_{31} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_c & O \\ A_fY_{31} & PA_fY_{31} - A_fY_{31}Q \end{pmatrix}.$$

Therefore, since $\operatorname{rank}(A_c) = n_1 - t_0$, we have $PA_fY_{31} - A_fY_{31}Q = O$. Now, as $\Lambda(P) \cap \Lambda(Q) = \emptyset$, we see that $A_fY_{31} = O$, and hence $Y_{31} = O$.

To conclude the proof it suffices to prove that we can choose $Y_{21} = O$. Partitioning Y_{21} according to (43),

$$Y_{21} = \begin{pmatrix} O \\ V \\ O \\ W \end{pmatrix},$$

as Y_{31} and Y_{32} are zero matrices, from (40), (41) and (43) we deduce that

$$\operatorname{rank}\begin{pmatrix} A_c & O & O & B_c & O & O \\ O & I_{q_1} & O & N_3V & N_1 & O \\ V & O & O & N_2V & O & O \\ J_3W & O & J_1 & O & O & I_{q_2} \\ J_2W & O & O & W & O & O \end{pmatrix} = n_1 + q_1 + q_2 - t_0.$$

Therefore, as by Lemma 22 we have $B_c = A_cQ$ for some matrix Q, then

$$n_1 - t_0 = \operatorname{rank}\begin{pmatrix} A_c & A_cQ \\ V & N_2V \\ J_2W & W \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_c & O \\ V & N_2V - VQ \\ J_2W & W - J_2WQ \end{pmatrix}.$$

Hence, as $\operatorname{rank}(A_c) = n_1 - t_0$ we have $N_2V - VQ = O$ and $W - J_2WQ = O$. Or equivalently $WQ^{-1} - J_2W = O$. Choosing Q in such a way that $\Lambda(J_2) \cap (\Lambda(Q) \cup \Lambda(Q^{-1})) = \emptyset$, we deduce that V and W are null matrices and therefore $Y_{21} = O$. The converse is immediate. □

4 Properties of the stability

In this section we give some auxiliary results about the stability of reducing subspaces. First, observe that from (7), if X is a basis matrix of the subspace $\mathcal{N}$, then $\mathcal{N}$ is $(\lambda B - A)$-reducing if and only if $\operatorname{rank}(AX, BX) = \operatorname{rank}(X) - t_0$. This definition enables us to make a reformulation of the concept of stability and Lipschitz stability of a reducing subspace in terms of limits of sequences of matrices. To do so, we will use the following result on the convergence of a sequence of subspaces that one deduces straightforwardly from ([1], Section 1.5, p. 29–31), ([5], Theorem 13.5.1) and ([3], Theorem I-2-6).

Proposition 23. *Let $\mathcal{N}$ be a p-dimensional subspace of $\mathbb{C}^n$ and let $\{\mathcal{N}_q\}_{q=1}^{\infty}$ be a sequence of subspaces of $\mathbb{C}^n$ that converges to $\mathcal{N}$ in the gap metric. Then, for each $X \in \mathbb{C}^{n\times p}$, basis matrix of $\mathcal{N}$, there exist a sequence of matrices $\{X_q\}_{q=1}^{\infty}$ converging to X, two positive constants K_1, K_2, and a positive integer q_0, such that for $q \geq q_0$, X_q is a basis matrix of $\mathcal{N}_q$, and*

$$K_1\|X_q - X\| \leq \theta(\mathcal{N}_q, \mathcal{N}) \leq K_2\|X_q - X\|.$$

From Proposition 23, we can reformulate the concept of stable and Lipschitz stable subspace in terms of the convergence of sequences of matrices. The result is the following.

Proposition 24. *Let $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ be a matrix pencil and let $\mathcal{N}$ be a $(\lambda B - A)$-reducing subspace such that $\dim \mathcal{N} = p$. Then $\mathcal{N}$ is $(\lambda B - A)$-stable if and only if for every basis matrix $X \in \mathbb{C}^{n\times p}$ of $\mathcal{N}$, and for every sequence of matrix pencils $\lambda B_q - A_q \to \lambda B - A$, there exist a sequence of matrices $X_q \to X$ and a positive integer q_0, such that for $q \geq q_0$: X_q is a matrix of rank p and the subspace $\langle X_q \rangle$ is $(\lambda B_q - A_q)$-reducing.*

Moreover, $\mathcal{N}$ is $(\lambda B - A)$-Lipschitz stable if and only if there exist a constant $K > 0$ and a positive integer q_0 such that for $q \geq q_0$

$$\|X_q - X\| \leq K(\|A_q - A\| + \|B_q - B\|).$$

In addition, if $X = \left(\begin{smallmatrix} I_p \\ 0 \end{smallmatrix}\right)$, then for $q \geq q_0$ we can choose $X_q = \left(\begin{smallmatrix} I_p \\ Y_q \end{smallmatrix}\right)$, where $Y_q \to 0$.

We will see some results that will simplify the statements of Theorem 4 and some proofs. The first is the following, which can be proved from Proposition 24 and using the techniques employed in the proof of Proposition 3.3 of [18].

Proposition 25. *Let $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ be a matrix pencil and let $\lambda D - C \in \mathcal{P}[\lambda]^{m\times n}$ be a pencil strictly equivalent to $\lambda B - A$; that is to say, $\lambda D - C = P(\lambda B - A)Q$ with $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ invertible matrices. Let $\mathcal{N}$ be a $(\lambda B - A)$-reducing subspace. Then, $\mathcal{N}$ is $(\lambda B - A)$-stable (or Lipschitz stable) if and only if $Q^{-1}\mathcal{N}$ is $(\lambda B - A)$-stable (or Lipschitz stable).*

Remark 26. As a consequence of this Proposition, when studying the stability (or Lipschitz stability) of a reducing subspace, no generality is lost if we consider another strictly equivalent pencil and the corresponding transformed subspace.

To prove the following result we need some previous notations. Given a matrix pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$, we denote by $\mathcal{CS}(\lambda B - A)$ the set of all sequences of matrix pencils that converge to $\lambda B - A$. Let $\tilde{\mathcal{CS}}(\lambda B - A)$ be a subset of $\mathcal{CS}(\lambda B - A)$. We will say that a set $\mathcal{G} \subset \tilde{\mathcal{CS}}(\lambda B - A)$ is a *Lipschitz generator subset of* $\tilde{\mathcal{CS}}(\lambda B - A)$ if for every sequence

$$\{(A_q, B_q)\}_{q=1}^{\infty} \in \tilde{\mathcal{CS}}(\lambda B - A),$$

there exist sequences

$$\{(\lambda \overline{B}_q - \overline{A}_q)\}_{q=1}^{\infty} \in \mathcal{G} \text{ and } \{(P_q, Q_q)\}_{q=1}^{\infty} \text{ converging to } (I_m, I_n),$$

and there exist a positive integer number q_0 and a constant $K > 0$, that depends on the preceding sequences, such that for $q \geq q_0$,

$$\begin{cases} \lambda B_q - A_q = P_q(\lambda \overline{B}_q - \overline{A}_q)Q_q, \\ \max\{\|P_q - I_m\|, \|Q_q - I_m\|\} \leq K(\|A_q - A\| + \|B_q - B\|). \end{cases}$$

With the preceding notation we have the following proposition, whose demonstration is similar to that one of Proposition 3.5 of [8] using Proposition 25.

Proposition 27. *Let $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ be a matrix pencil, and let $\mathcal{N}$ be a $(\lambda B - A)-$reducing subspace and X a basis matrix of $\mathcal{N}$. Let $\tilde{\mathcal{CS}}(\lambda B - A)$ be a subset of $\mathcal{CS}(\lambda B - A)$ and $\mathcal{G}$ a Lipschitz generator subset of $\tilde{\mathcal{CS}}(\lambda B - A)$. Then the assertions below are equivalent.*

(i) *For every sequence $\{(\lambda B_q - A_q)\}_{q=1}^{\infty} \in \tilde{\mathcal{CS}}(\lambda B - A)$, there exist a sequence of matrices $X_q \to X$, a constant $K_1 > 0$ and a positive integer q_1, such that for $q \geq q_1$, the subspace $\langle X_q \rangle$ is $(\lambda B_q - A_q)-$reducing, and*

$$\|X_q - X\| \leq K_1(\|A_q - A\| + \|B_q - B\|).$$

(ii) *For every sequence $\{(\lambda \overline{B}_q - \overline{A}_q)\}_{q=1}^{\infty} \in \mathcal{G}$, there exist a sequence of matrices $\overline{X}_q \to X$, a constant $K_2 > 0$ and a positive integer q_2, such that for $q \geq q_2$, the subspace $\langle \overline{X}_q \rangle$ is $(\lambda \overline{B}_q - \overline{A}_q)-$reducing, and*

$$\|\overline{X}_q - X\| \leq K_2(\|\overline{A}_q - A\| + \|\overline{B}_q - B\|).$$

In addition, if $\mathcal{G}_1$ is a Lipschitz generator subset of $\mathcal{G}$ then $\mathcal{G}_1$ is a Lipschitz generator subset of $\tilde{\mathcal{CS}}(\lambda B - A)$.

Remark 28. In the above results, the existence of a positive integer q_0 is required in such a way that the results are true for $q \geq q_0$. To simplify, without loss of generality, we will assume hereafter that $q_0 = 1$.

5 Proof of Theorem 4. Assertions (1), (2) and (3).

To prove Assertions (1), (2) and (3) of Theorem 4, we need some lemmas. Let $\lambda B - A$ be a pencil in the form (2) and let $\mathcal{N}$ be a subspace $(\lambda B - A)$-reducing, which, by Theorem 20 can be put in the form (37).

Lemma 29. *With the previous notations, if the subspace $\mathcal{N}$ is $(\lambda B - A)$-stable, then $\langle X \rangle$ is $(\lambda B_r - A_r)$-stable subspace.*

Proof. Consider an arbitrary sequence of matrix pencils $\lambda B_r^q - A_r^q$ converging to $\lambda B_r - A_r$ as $q \to \infty$. From now on we will summarize this with the notation $S_q \to L$ to mean that S_q is a sequence of mathematical objects converging to the limit L when $q \to \infty$. Then

$$\lambda B_q - A_q := \begin{pmatrix} \lambda B_c - A_c & 0 & 0 \\ 0 & \lambda B_r^q - A_r^q & 0 \\ 0 & 0 & \lambda B_f - A_f \end{pmatrix} \to \lambda B - A.$$

Now, as $\mathcal{N}$ is a subspace $(\lambda B - A)$-stable, there exists a sequence of subspaces $\mathcal{N}_q \to \mathcal{N}$ such that $\mathcal{N}_q$ is a $(\lambda B_q - A_q)$-reducing subspace for every q. By the form of $\lambda B_q - A_q$, from Theorem 20 we know that there exists a sequence of matrices $X_q \to X$ where

$$\mathcal{N}_q = \left\langle \begin{pmatrix} I_{n_1} & O \\ O & X_q \\ O & O \end{pmatrix} \right\rangle,$$

$\langle X_q \rangle$ being a $(\lambda B_r^q - A_r^q)$-deflating subspace. Hence the subspace $\langle X_q \rangle$ is $(\lambda B_r - A_r)$-stable. □

Remark 30. Observe that by [5, Theorem 14.3.1, p. 429] and [7], if $\langle X \rangle$ is $(\lambda B_r - A_r)$-stable, then $\langle X \rangle$ is isolated; that is, there exists a neighbourhood of the subspace $\langle X \rangle$ such that the unique $(\lambda B_r - A_r)$-deflating subspace that is in this neighbourhood is $\langle X \rangle$ itself.

Lemma 31. *Consider a pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ and a $(\lambda B - A)$-reducing subspace $\mathcal{N}$, both in the form*

$$\lambda B - A = \begin{matrix} \\ m_1 \\ m_2 \end{matrix} \begin{matrix} n_1 & n_2 \\ \begin{pmatrix} \lambda D - C & O \\ O & \lambda F - E \end{pmatrix} \end{matrix}, \quad \mathcal{N} = \left\langle \begin{pmatrix} X \\ O \end{pmatrix} \right\rangle, \tag{44}$$

with $X \in \mathbb{C}^{n_1 \times p}$ a matrix of rank p, the pencil $\lambda D - C$ is left regular and the pencil $\lambda F - E$ only has row minimal indices. Then the subspace $\langle X \rangle$ is $(\lambda D - C)$-reducing. Moreover, if the subspace $\mathcal{N}$ is $(\lambda B - A)$-stable, then $\langle X \rangle$ is $(\lambda D - C)$-stable.

Proof. As $\lambda D - C$ is left regular and $\lambda F - E$ only has row minimal indices, it follows that $\nu(\lambda B - A) = \nu(\lambda D - C)$. Hence, as $\mathcal{N}$ is $(\lambda B - A)$-reducing we have $\dim(A(\mathcal{N}) + B(\mathcal{N})) = p - \nu(\lambda D - C)$. Therefore, from (44) we deduce that $\operatorname{rank}(CX, DX) = \operatorname{rank}(X) - \nu(\lambda D - C)$, that is, $\langle X \rangle$ is $(\lambda D - C)$-reducing.

Consider now an arbitrary sequence $\lambda D_q - C_q \to \lambda D - C$. Then the sequence

$$\lambda B_q - A_q = \begin{pmatrix} \lambda D_q - C_q & O \\ O & \lambda F - E \end{pmatrix} \to \lambda B - A, \tag{45}$$

and moreover, as $\lambda F - E$ only has row minimal indices,

$$\nu(\lambda B_q - A_q) = \nu(\lambda D_q - C_q). \tag{46}$$

Now then, as $\mathcal{N}$ is $(\lambda B - A)$-stable, there exists a sequence of subspaces $\mathcal{N}_q \to \mathcal{N}$ such that $\mathcal{N}_q$ is $(\lambda B_q - A_q)$-reducing for every q. Due to the form of $\lambda B_q - A_q$, given in (45), from Theorem 20 we see that there exists a sequence of matrices $X_q \to X$ such that for every q

$$\mathcal{N}_q = \left\langle \begin{pmatrix} X_q \\ O \end{pmatrix} \right\rangle. \tag{47}$$

Therefore, because $\mathcal{N}_q$ is a $(\lambda B_q - A_q)$-reducing subspace, from (46) it follows that $\dim(A_q(\mathcal{N}_q) + B_q(\mathcal{N}_q)) = p - \nu(\lambda D_q - C_q)$. That is, from (45), (46) and (47), we infer that

$$\operatorname{rank}(X_q) - \nu(\lambda D_q - C_q) = p - \nu(\lambda D_q - C_q) = \operatorname{rank} \begin{pmatrix} C_q X_q & D_q X_q \\ O & O \end{pmatrix} = \operatorname{rank}(C_q X_q, D_q X_q).$$

Hence $\langle X_q \rangle$ is $(\lambda D_q - C_q)$-reducing for every q. Lastly, as $X_q \to X$ we have $\langle X \rangle$ is $(\lambda D - C)$-stable. □

Lemma 32. *Consider a pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ and a $(\lambda B - A)$-reducing subspace $\mathcal{N}$, both in the form*

$$\lambda B - A = \begin{matrix} \\ m_1 \\ m_2 \end{matrix} \begin{matrix} n_1 & n_2 \\ \begin{pmatrix} \lambda D - C & O \\ O & \lambda F - E \end{pmatrix} \end{matrix}, \quad \mathcal{N} = \left\langle \begin{pmatrix} I_{n_1} & O \\ O & X \end{pmatrix} \right\rangle, \tag{48}$$

with $X \in \mathbb{C}^{n_2 \times p}$ a matrix of rank p, the pencil $\lambda D - C$ is left regular and the pencil $\lambda F - E$ is right regular. Then the subspace $\langle X \rangle$ is $(\lambda F - E)$-reducing. Moreover, if the subspace $\mathcal{N}$ is $(\lambda B - A)$-stable we have $\langle X \rangle$ is $(\lambda F - E)$-stable.

Proof. Observe first that $\nu(\lambda B - A) = \nu(\lambda D - C) = n_1 - m_1$. Thus, as $\mathcal{N}$ is $(\lambda B - A)$-reducing, it follows that $\dim(A(\mathcal{N}) + B(\mathcal{N})) = p + m_1$, and hence, from (48), we see that

$$\operatorname{rank}\begin{pmatrix} C & O & D & O \\ O & EX & O & FX \end{pmatrix} = p + m_1.$$

Therefore, as $\operatorname{rank}(C, D) = m_1$ and $\nu(\lambda F - E) = 0$, we have $\operatorname{rank}(EX, FX) = p = \operatorname{rank}(X) - \nu(\lambda F - E)$, that is, $\langle X \rangle$ is $(\lambda F - E)$-reducing.

Consider now an arbitrary sequence $\lambda F_q - E_q \to \lambda F - E$. Then the sequence

$$\lambda B_q - A_q = \begin{pmatrix} \lambda D - C & O \\ O & \lambda F_q - E_q \end{pmatrix} \to \lambda B - A, \tag{49}$$

and moreover, as $\lambda F_q - E_q$ is right regular,

$$\nu(\lambda B_q - A_q) = \nu(\lambda D - C) = n_1 - m_1. \tag{50}$$

Now, given that $\mathcal{N}$ is $(\lambda B - A)$-stable, from (48) we deduce that there exist two sequences of matrices, $X_q \to X$ and $Y_q \to O$, such that for every q the subspace

$$\mathcal{N}_q = \left\langle \begin{pmatrix} I_{n_1} & O \\ Y_q & X_q \end{pmatrix} \right\rangle. \tag{51}$$

is $(\lambda B_q - A_q)$-reducing. Therefore, from (49) and (51) we infer that

$$p + m_1 = \operatorname{rank}\begin{pmatrix} C & O & D & O \\ E_q Y_q & E_q X_q & F_q Y_q & F_q X_q \end{pmatrix} \geq \operatorname{rank}(C, D) + \operatorname{rank}(E_q X_q, F_q X_q).$$

Now, as $\operatorname{rank}(C, D) = m_1$ and $\nu(\lambda F_q - E_q) = 0$ we have

$$\operatorname{rank}(E_q X_q, F_q X_q) = p = \operatorname{rank}(X_q) - \nu(\lambda F_q - E_q),$$

therfore $\langle X_q \rangle$ is $(\lambda F_q - E_q)$-reducing for every q. Finally, since $X_q \to X$, it follows that $\langle X \rangle$ is $(\lambda F - E)$-stable. □

Lemma 33. *Suppose that the matrix pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m \times n}$ has no row minimal indices. Then the unique $(\lambda B - A)$-stable subspace is $\mathbb{C}^n$.*

Proof. Let $\mathcal{N}$ be a $(\lambda B - A)$-stable subspace. Given that the pencil has no row minimal indices, from Remark 26, (2) and Theorem 20 we can assume that $\lambda B - A$ and $\mathcal{N}$ are in the form

$$\lambda B - A = \begin{matrix} \\ m_1 \\ n_2 \end{matrix} \begin{matrix} n_1 & n_2 \\ \left(\begin{matrix} \lambda B_c - A_c \\ O \end{matrix}\right. & \left.\begin{matrix} O \\ \lambda B_r - A_r \end{matrix}\right) \end{matrix}, \quad \mathcal{N} = \left\langle \begin{matrix} \\ n_1 \\ n_2 \end{matrix} \begin{matrix} n_1 & p \\ \left(\begin{matrix} I_{n_1} \\ O \end{matrix}\right. & \left.\begin{matrix} O \\ X \end{matrix}\right) \end{matrix} \right\rangle, \tag{52}$$

where $\lambda B_c - A_c$ is a pencil with only column minimal indices, $\lambda B_r - A_r$ is a regular pencil, X is a matrix of rank p and $\langle X \rangle$ is a $(\lambda B_r - A_r)$-deflating subspace. Thus by [7] we can assume that

$$\lambda B_r - A_r = \begin{pmatrix} \lambda N_1 - I_{p_1} & \lambda N_3 & O & O \\ O & \lambda N_2 - I_{r_1} & O & O \\ O & O & \lambda I_{p_2} - J_1 & -J_3 \\ O & O & O & \lambda I_{r_2} - J_2 \end{pmatrix}, \; X = \begin{pmatrix} I_{p_1} & O \\ O & O \\ O & I_{p_2} \\ O & O \end{pmatrix}, \tag{53}$$

with N_1, N_2 and N_3 nilpotent matrices and $p_1 + p_2 = p$.

Consider now two arbitrary sequences of matrices $E_q \to O \in \mathbb{C}^{q_2 \times n_1}$ and $F_q \to O \in \mathbb{C}^{q_1 \times n_1}$. As the sequence

$$\lambda B_q - A_q = \begin{pmatrix} \lambda B_c - A_c & O & O & O & O \\ O & \lambda N_1 - I_{p_1} & \lambda N_3 & O & O \\ \lambda F_q & O & \lambda N_2 - I_{r_1} & O & O \\ O & O & O & \lambda I_{p_2} - J_1 & -J_3 \\ E_q & O & O & O & \lambda I_{r_2} - J_2 \end{pmatrix} \tag{54}$$

converges to $\lambda B - A$ and the subspace $\mathcal{N}$ is $(\lambda B - A)$-stable, it follows that there exist sequences of matrices $Y_q, Z_q, U_q, V_q, W_q, H_q$ of adequate sizes, all of them converging to O, such that for every q the subspace

$$\mathcal{N}_q = \left\langle \begin{pmatrix} I_{n_1} & O & O \\ O & I_{p_1} & O \\ Y_q & U_q & W_q \\ O & O & I_{p_2} \\ Z_q & V_q & H_q \end{pmatrix} \right\rangle \tag{55}$$

is $(\lambda B_q - A_q)$-reducing, that is $\dim(A_q(\mathcal{N}_q) + B_q(\mathcal{N}_q)) = \dim(\mathcal{N}_q) - \nu(\lambda B_q - A_q)$. But as the pencil $\lambda B - A$ has no row minimal indices, we have $\nu(\lambda B_q - A_q) = \nu(\lambda B - A) = n_1 - m_1$. Therefore $\dim(A_q(\mathcal{N}_q) + B_q(\mathcal{N}_q)) = n_1 + p + (n_1 - m_1) = m_1 + p$. Thus from (54) and (55) we conclude that

$$\operatorname{rank} \begin{array}{c} \\ m_1 \\ p_1 \\ r_1 \\ p_2 \\ r_2 \end{array} \begin{array}{c} \begin{array}{cccccc} n_1 & p_1 & p_1 & n_1 & p_2 & p_2 \end{array} \\ \begin{pmatrix} A_c & O & O & B_c & O & O \\ O & I_{p_1} & O & N_3 Y_q & N_1 + N_3 U_q & N_3 W_q \\ Y_q & U_q & W_q & F_q + N_2 Y_q & N_2 U_q & N_2 W_q \\ J_3 Z_q & J_3 V_q & J_1 + J_3 H_q & O & O & I_{p_2} \\ E_q + J_2 Z_q & J_2 V_q & J_2 H_q & Z_q & V_q & H_q \end{pmatrix} \end{array} = m_1 + p. \tag{56}$$

Since $\operatorname{rank}(A_c) = \operatorname{rank}(B_c) = m_1$, from (56) we see that

$$\operatorname{rank} \begin{pmatrix} I_{p_1} & O & N_1 + N_3 U_q & N_3 W_q \\ U_q & W_q & N_2 U_q & N_2 W_q \\ J_3 V_q & J_1 + J_3 H_q & O & I_{p_2} \\ J_2 V_q & J_2 H_q & V_q & H_q \end{pmatrix} = p.$$

Consequently, from (53), the subspace $\mathcal{M}_q$ generated by the columns of the matrix

$$X_q = \begin{pmatrix} I_{p_1} & O \\ U_q & W_q \\ O & I_{p_2} \\ V_q & H_q \end{pmatrix}$$

is $(\lambda B_r - A_r)$-stable. Hence, by Remark 30 we infer that $\langle X_q \rangle = \langle X \rangle$ for every q, and therefore the matrices of the sequences U_q, V_q, W_q, H_q are all zero. So, from (56) we deduce that

$$\operatorname{rank} \begin{array}{c} \\ m_1 \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_1 \end{array} \\ \begin{pmatrix} A_c & B_c \\ Y_q & F_q + N_2 Y_q \\ E_q + J_2 Z_q & Z_q \end{pmatrix} \end{array} = m_1. \tag{57}$$

At this point, note that to prove the lemma it suffices to verify that $r_1 = r_2 = 0$. For the sake of contradiction, assume first that $r_1 > 0$, then as $\operatorname{rank}(A_c) = \operatorname{rank}(B_c) = m_1$, from (57) we deduce that

$$\operatorname{rank} \begin{pmatrix} A_c & B_c \\ Y_q & F_q + N_2 Y_q \end{pmatrix} = m_1. \tag{58}$$

Now, by Lemma 22 there exists a matrix $Q \in \mathbb{C}^{n_1 \times n_1}$ with $0 \in \Lambda(Q)$ such that $B_c = A_c Q$. Then from (58) we immediately obtain

$$m_1 = \operatorname{rank}(A_c) = \operatorname{rank} \begin{pmatrix} A_c & A_c Q \\ Y_q & F_q + N_2 Y_q \end{pmatrix} = \operatorname{rank} \begin{pmatrix} A_c & O \\ Y_q & F_q + N_2 Y_q - Y_q Q \end{pmatrix},$$

and hence $F_q + N_2 Y_q - Y_q Q = O$. In conclusion, if it were true that $r_1 > 0$, we would have proved that for every sequence of matrices $F_q \to O$ there exits a sequence of matrices $Y_q \to O$ such that for each q it satisfies $F_q + N_2 Y_q - Y_q Q = O$, with $\Lambda(N_2) \cap \Lambda(Q) \neq \emptyset$, which is impossible. Therefore, $r_1 = 0$.

If it were true that $r_2 > 0$, as by Lemma 22 there exists a matrix $Q \in \mathbb{C}^{n_1 \times n_1}$ with $\Lambda(J_2) \cap \Lambda(Q) \neq \emptyset$, so that $A_c = B_c Q$, applying the previous reasoning we would lead to a contradiction. Thus $r_2 = 0$. □

Lemma 34. *Given a matrix pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m \times n}$, suppose that it has no column minimal indices. Then the unique $(\lambda B - A)$-stable subspace is $\{0\}$.*

Proof. Let $\mathcal{N}$ be a $(\lambda B - A)$-stable subspace. We can now proceed analogously to the proof of the previous lemma. So, we can assume that $\lambda B - A$ and $\mathcal{N}$ are in the form

$$\lambda B - A = \begin{array}{c} \\ n_2 \\ m_3 \end{array} \begin{array}{c} \begin{array}{cc} n_2 & n_3 \end{array} \\ \begin{pmatrix} \lambda B_r - A_r & O \\ O & \lambda B_f - A_f \end{pmatrix} \end{array}, \quad \mathcal{N} = \left\langle \begin{array}{c} \\ n_2 \\ n_3 \end{array} \begin{array}{c} p \\ \begin{pmatrix} X \\ O \end{pmatrix} \end{array} \right\rangle, \tag{59}$$

where $\lambda B_r - A_r$ is a regular pencil and $\langle X \rangle$ is a $(\lambda B_r - A_r)$-deflating subspace, both in the form (53). Moreover, $\lambda B_f - A_f$ is a pencil that only has row minimal indices.

Now consider two arbitrary sequences of matrices $E_q \to O \in \mathbb{C}^{m_3 \times p_2}$ and $F_q \to O \in \mathbb{C}^{m_3 \times p_1}$. As the sequence

$$\lambda B_q - A_q = \begin{pmatrix} \lambda N_1 - I_{p_1} & \lambda N_3 & O & O & O \\ O & \lambda N_2 - I_{r_1} & O & O & O \\ O & O & \lambda I_{p_2} - J_1 & -J_3 & O \\ O & O & O & \lambda I_{r_2} - J_2 & O \\ \lambda F_q & O & -E_q & O & \lambda B_f - A_f \end{pmatrix}, \tag{60}$$

converges to $\lambda B-A$ and the subspace $\mathcal{N}$ is $(\lambda B-A)$-stable, there exist sequences of matrices $Y_q, Z_q, U_q, V_q, W_q, H_q$, of adequate sizes, that converge to O, such that for every q the subspace

$$\mathcal{N}_q = \left\langle \begin{pmatrix} I_{p_1} & O \\ U_q & W_q \\ O & I_{p_2} \\ V_q & H_q \\ Y_q & Z_q \end{pmatrix} \right\rangle \tag{61}$$

is $(\lambda B_q - A_q)$-reducing; that is $\dim(A_q(\mathcal{N}_q)+B_q(\mathcal{N}_q)) = \dim(\mathcal{N}_q)-\nu(\lambda B_q-A_q)$. But as the pencil $\lambda B - A$ has no column minimal indices, then $\nu(\lambda B_q - A_q) = \nu(\lambda B - A) = 0$. Therefore $\dim(A_q\mathcal{N}_q + B_q\mathcal{N}_q) = p$. Thus from (60) and (61) we see that

$$\operatorname{rank} \begin{array}{c} \\ p_1 \\ r_1 \\ p_2 \\ r_2 \\ m_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_1 & p_2 \end{array} \\ \begin{pmatrix} I_{p_1} & O & N_1 + N_3U_q & N_3W_q \\ U_q & W_q & N_2U_q & N_2W_q \\ J_3V_q & J_1 + J_3H_q & O & I_{p_2} \\ J_2V_q & J_2H_q & V_q & H_q \\ A_fY_q & E_q + A_fZ_q & F_q + B_fY_q & B_fZ_q \end{pmatrix} \end{array} = p. \tag{62}$$

Now, as the sequences $Y_q, Z_q, U_q, V_q, W_q, H_q$ converge to O, from (62)

$$\operatorname{rank} \begin{pmatrix} I_{p_1} & O & N_1 + N_3U_q & N_3W_q \\ U_q & W_q & N_2U_q & N_2W_q \\ J_3V_q & J_1 + J_3H_q & O & I_{p_2} \\ J_2V_q & J_2H_q & V_q & H_q \end{pmatrix} = p;$$

that implies for (53) that the subspace $\mathcal{M}_q$ generated by the matrix

$$X_q = \begin{pmatrix} I_{p_1} & O \\ U_q & W_q \\ O & I_{p_2} \\ V_q & H_q \end{pmatrix}$$

is $(\lambda B_r - A_r)$-deflating. Hence, like in the previous lemma, all the terms of the sequences of matrices U_q, V_q, W_q, H_q are O. Thus from (62) we infer that

$$\operatorname{rank} \begin{array}{c} \\ p_1 \\ p_2 \\ m_3 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_1 & p_2 \end{array} \\ \begin{pmatrix} I_{p_1} & O & N_1 & O \\ O & J_1 & O & I_{p_2} \\ A_fY_q & E_q + A_fZ_q & F_q + B_fY_q & B_fZ_q \end{pmatrix} \end{array} = p;$$

that is

$$F_q + B_fY_q - A_fY_qN_1 = O, \qquad E_q + A_fZ_q - B_fZ_qJ_1 = O. \tag{63}$$

Now, by Lemma 22 there exist matrices $P, Q \in \mathbb{C}^{m_3\times m_3}$ with $0 \in \Lambda(P)$ and $\Lambda(J_2) \cap \Lambda(Q) \neq \emptyset$ such that $B_f = PA_f$ and $A_f = QB_f$. Then we immediately see from (63) that for every pair of sequences of matrices $E_q, F_q \to O$ there are sequences $Y_q, Z_q \to O$ that satisfy

$$F_q + PA_fY_q - A_fY_qN_1 = O, \qquad E_q + QB_fZ_q - B_fZ_qJ_1 = O,$$

for every q. This contradicts to the choice of P and Q. Consequently $p_1 = p_2 = 0$ and therefore $\mathcal{N} = \{0\}$. $\square$

We are now ready to prove Assertions (1) (2) and (3) of Theorem 4.

Proof Theorem 4: Assertions (1), (2) and (3).

First, note that Assertions (2) and (3) follow straightforward from Lemmas 33 and 34, respectively.

Second, to prove Assertion (1), by Remark 28, we can assume that $\lambda B - A$ is in the form given in (2), with n_1, n_2, m_3 nonzero . Let $\mathcal{N}$ be a $(\lambda B - A)$-reducing subspace. Then, by Theorem 20, we can assume that

$$\mathcal{N} = \left\langle \begin{matrix} \\ n_1 \\ n_2 \\ m_3 \end{matrix} \overset{\begin{matrix} n_1 & p \end{matrix}}{\begin{pmatrix} I_{n_1} & O \\ O & X \\ O & O \end{pmatrix}} \right\rangle,$$

where $\operatorname{rank}(X) = p$. Suppose that $\mathcal{N}$ is $(\lambda B - A)$-stable. Hence, as the pencil $\operatorname{diag}(\lambda B_c - A_c, \lambda B_r - A_r)$ is left regular and the pencil $\lambda B_f - A_f$ only has row minimal indices, applying Lemma 31, we deduce that the subspace

$$\left\langle \begin{pmatrix} I_{n_1} & O \\ O & X \end{pmatrix} \right\rangle$$

is $\operatorname{diag}(\lambda B_c - A_c, \lambda B_r - A_r)$-stable. Therefore, by Lemma 33 we have $X = I_{n_2}$.

On the other hand, applying Lemma 32 to the pencils $\lambda B_c - A_c$ (left regular) and $\operatorname{diag}(\lambda B_r - A_r, \lambda B_f - A_f)$ (right regular), since $\mathcal{N}$ is $(\lambda B - A)$-reducing, we see that the subspace generated by the columns of the matrix $\left(\begin{smallmatrix} I_{n_2} \\ O \end{smallmatrix}\right)$ is $\operatorname{diag}(\lambda B_r - A_r, \lambda B_f - A_f)$-stable, which contradicts Lemma 34. $\square$

6 Proof of Theorem 4: Assertion (4).

In this section we prove Assertion (4) of Theorem 4. Therefore in all the section we will assume that the pencil $\lambda B - A$ has row minimal indices, at least two column minimal indices, and no eigenvalues. The following result will allow us to simplify the proofs.

Lemma 35. *Let $\lambda B - A \in \mathcal{P}[\lambda]^{m \times n}$ be a pencil without eigenvalues and let $\mathcal{N}$ be a $(\lambda B - A)-$reducing subspace, which are given by*

$$\lambda B - A = \begin{matrix} \\ m_1 \\ n_2 \end{matrix} \overset{\begin{matrix} n_1 & \quad & n_2 \end{matrix}}{\begin{pmatrix} \lambda D - C & 0 \\ 0 & \lambda F - E \end{pmatrix}}, \quad \mathcal{N} = \left\langle \begin{matrix} \\ n_1 \\ n_2 \end{matrix} \overset{\begin{matrix} p_1 & p_2 \end{matrix}}{\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}} \right\rangle,$$

where X, Y are matrices of full column rank. Let $\mathcal{M} := \langle X \rangle$. Then if $\mathcal{N}$ is $(\lambda B - A)-$stable it follows that $\mathcal{M}$ is $(\lambda D - C)-$stable.

Proof. Note that as the pencil $\lambda F - E$ has no eigenvalues, from (31) and (33) we see that $\mathcal{D}^k_{\lambda B - A} = \mathcal{S}^k_{\lambda B - A}$. Hence by Theorem 20 we infer that

$$\langle Y \rangle = \mathcal{D}^n_{\lambda F - E} = \mathcal{S}^n_{\lambda F - E}. \tag{64}$$

Consider now an arbitrary sequence $\lambda D_q - C_q \to \lambda D - C$. Then as the sequence

$$\lambda B_q - A_q = \begin{pmatrix} \lambda D_q - C_q & 0 \\ 0 & \lambda F - E \end{pmatrix}$$

converges to $\lambda B - A$ and $\mathcal{N}$ is $(\lambda B - A)-$stable, there exist sequences $X_q \to X$, $Y_q \to Y$, $Z_q \to 0$, $V_q \to 0$, such that for every q the subspace

$$\mathcal{N}_q := \left\langle \begin{pmatrix} X_q & Z_q \\ V_q & Y_q \end{pmatrix} \right\rangle \tag{65}$$

is $(\lambda B_q - A_q)-$reducing. Therefore, by Theorem 20, $\mathcal{N}_q \subset \mathcal{D}^n_{\lambda B_q - A_q}$. Now applying Lemma 14 and (64), we obtain

$$\mathcal{N}_q \subset \mathcal{D}^n_{\lambda B_q - A_q} = \begin{pmatrix} \mathcal{D}^n_{\lambda D_q - C_q} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{D}^n_{\lambda F - E} \end{pmatrix} \subset \begin{pmatrix} I_{n_1} \\ O \end{pmatrix} \oplus \left\langle \begin{pmatrix} O \\ Y \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} I_{n_1} & O \\ O & Y \end{pmatrix} \right\rangle.$$

For this reason from (65) there exist matrices of adequate sizes Q_i, $i = 1, 2, 3, 4$, such that

$$\begin{pmatrix} X_q & Z_q \\ V_q & Y_q \end{pmatrix} = \begin{pmatrix} I_{n_1} & O \\ O & Y \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}.$$

Observe that $Y_q = YQ_4$. Moreover, as $Y_q \to Y$ and Y is of full column rank, we deduce that Q_4 is invertible. Hence, as $V_q = YQ_3$, it follows that $V_q = Y_q Q_4^{-1} Q_3$. Thus, in (65), if we subtract to the first column the second one multiplied by $Q_4^{-1} Q_3$ we obtain

$$\mathcal{N}_q = \left\langle \begin{pmatrix} X_q - Z_q Q_4^{-1} Q_3 & Z_q \\ O & YQ_4 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} X_q - Z_q Q_4^{-1} Q_3 & Z_q Q_4^{-1} \\ O & Y \end{pmatrix} \right\rangle. \tag{66}$$

In the same way, from Theorem 20, Lemma 14 and (64) we obtain

$$\mathcal{N}_q \supset \mathcal{S}^n_{\lambda B_q - A_q} = \begin{pmatrix} \mathcal{S}^n_{\lambda D_q - C_q} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^n_{\lambda F - E} \end{pmatrix} = \begin{pmatrix} \mathcal{S}^n_{\lambda D_q - C_q} \\ O \end{pmatrix} \oplus \left\langle \begin{pmatrix} O \\ Y \end{pmatrix} \right\rangle \supset \left\langle \begin{pmatrix} O \\ Y \end{pmatrix} \right\rangle.$$

From (66) we deduce that there exist matrices of adequate sizes P_1, P_2 such that

$$\begin{pmatrix} O \\ Y \end{pmatrix} = \begin{pmatrix} X_q - Z_q Q_4^{-1} Q_3 & Z_q Q_4^{-1} \\ O & Y \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}.$$

Hence $P_2 = I_{p_2}$ and $Z_q Q_4^{-1} = -(X_q - Z_q Q_4^{-1} Q_3) P_1$. Therefore denoting $\tilde{X}_q := X_q - Z_q Q_4^{-1} Q_3 \to X$, from (66) we see that

$$\mathcal{N}_q = \left\langle \begin{pmatrix} \tilde{X}_q & O \\ O & Y \end{pmatrix} \right\rangle.$$

Finally, as $\mathcal{N}_q$ is $(\lambda B - A)-$reducing

$$\operatorname{rank} \begin{pmatrix} C_q \tilde{X}_q & O & D_q \tilde{X}_q & O \\ O & EY & O & FY \end{pmatrix} = p_1 + p_2 - \nu(\lambda D_q - C_q) - \nu(\lambda F - E). \tag{67}$$

But given that $\langle Y \rangle$ is a $(\lambda F - E)-$reducing subspace, it follows that $\operatorname{rank}(EY, FY) = p_2 - \nu(\lambda F - E)$. Thus from (67) we conclude that $\operatorname{rank}(C_q \tilde{X}_q, D_q \tilde{X}_q) = p_1 - \nu(\lambda D_q - C_q)$; that is the subspace $\left\langle \tilde{X}_q \right\rangle$ is $(\lambda D_q - C_q)-$reducing. Consequently $\mathcal{M}$ is $(\lambda D - C)-$stable. □

Remark 36. Assume that $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$ is a pencil without eigenvalues, and has the form

$$\lambda B - A = \begin{pmatrix} \lambda D - C & O \\ O & \lambda F - E \end{pmatrix},$$

where $\lambda D - C$ is a pencil with only two row minimal indices and one column minimal index. Then, by Theorem 20 and Lemma 14 the unique reducing subspace for $\lambda B - A$ is

$$\mathcal{N} = \begin{pmatrix} \mathcal{S}^n_{\lambda D - C} \\ O \end{pmatrix} \oplus \begin{pmatrix} O \\ \mathcal{S}^n_{\lambda F - E} \end{pmatrix}.$$

Now, if we had proved that the subspace $\mathcal{S}^n_{\lambda D - C}$ were not $(\lambda D - C)-$stable, then, by Lemma 35, we would have proved that $\mathcal{N}$ cannot be $(\lambda B - A)-$stable. Therefore, from here to the end of the section, we will assume that the pencil $\lambda B - A$ only has two row minimal indices and one column minimal index. Moreover, by Remark 26, we can assume that $\lambda B - A$ is in the canonical form (2).

We prove Assertion (4) of Theorem 4. For do that, consider three cases: (a) two row minimal indices equal to zero; (b) one row minimal index equal to zero and another row minimal index which is different than zero; (c) two row minimal indices which are different than zero.

6.1 Two row minimal indices which are equal to zero

Distinguish two subcases: one column minimal index which is equal to zero and one column minimal index which is different than zero.

One column minimal index which is equal to zero. In this case we have

$$\lambda B - A = \lambda \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathcal{P}[\lambda]^{2\times 1}, \quad \mathcal{N} = \mathbb{C}.$$

Take the sequence

$$\lambda B_q - A_q = \lambda \begin{pmatrix} 1/q \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/q \end{pmatrix} \to \lambda B - A.$$

As $\lambda B_q - A_q$ only has a row minimal index equal to 1, from Theorem 20 we infer that $\mathcal{N}_q = \{0\}$ is its unique reducing subspace. Hence, as $\mathcal{N}_q \to \{0\}$ we have that $\mathcal{N}$ is not $(\lambda B - A)-$stable.

One column minimal index which is different than zero. Denoting by $D := (I_k, 0), C := (0, I_k) \in \mathbb{C}^{k\times(k+1)}$, we have

$$\lambda B - A = \lambda \begin{pmatrix} D \\ O \\ O \end{pmatrix} - \begin{pmatrix} C \\ O \\ O \end{pmatrix} \in \mathcal{P}[\lambda]^{(k+2)\times(k+1)}, \quad \mathcal{N} = \mathbb{C}^{k+1}.$$

Consider the sequence

$$\lambda B_q - A_q = \lambda \begin{pmatrix} I_k & 0 \\ 0 & 1/q \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & I_k \\ 0 & 0 \\ 1/q & 0 \end{pmatrix} \to \lambda B - A.$$

Note that $\nu(\lambda B_q - A_q) = 0$. Hence $\mathcal{N}$ is $(\lambda B - A)-$stable if and only if for every q the subspace $\mathbb{C}^{k+1}$ is $(\lambda B_q - A_q)-$reducing; that is, $\operatorname{rank}(A_q, B_q) = k + 1$. Which is a contradiction, because $\operatorname{rank}(A_q, B_q) = k + 2$.

6.2 One row minimal index which is equal to zero and another which is different than zero

As in the previous subsection, we consider two subcases: one column minimal index which is equal to zero and one column minimal index which is different than zero.

One column minimal index which is equal to zero. Denoting by $D := (I_k, 0)^T, C := (0, I_k)^T \in \mathbb{C}^{(k+1)\times k}$, we have

$$\lambda B - A = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & D \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C \end{pmatrix} \in \mathcal{P}[\lambda]^{(k+2)\times(k+1)}, \quad \mathcal{N} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle.$$

Consider the sequence

$$\lambda B_q - A_q = \lambda \begin{pmatrix} 1/q & 0 \\ 0 & 0 \\ 0 & D \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1/q & 0 \\ 0 & C \end{pmatrix} \to \lambda B - A.$$

Note that $\nu(\lambda B_q - A_q) = 0$. Suppose that $\mathcal{N}$ is $(\lambda B - A)-$stable. Then, by Proposition 24, there exists a sequence $X_q \to 0 \in \mathbb{C}^{k\times 1}$ such that for every q the subspace generated by the columns of the matrix $\left(\begin{smallmatrix} 1 \\ X_q \end{smallmatrix}\right)$ is $(\lambda B_q - A_q)-$reducing; that is

$$\operatorname{rank} \begin{pmatrix} 0 & 1/q \\ 1/q & 0 \\ CX_q & DY_q \end{pmatrix} = 1,$$

which is a contradiction since the rank of this matrix is 2.

One column minimal index which is different than zero. Let $D := (I_k, 0), C := (0, I_k) \in \mathbb{C}^{k\times(k+1)}$, $F := (I_l, 0)^T, E := (0, I_l) \in \mathbb{C}^{(l+1)\times l}$. We have

$$\lambda B - A = \lambda \begin{pmatrix} D & 0 \\ 0 & 0 \\ 0 & F \end{pmatrix} - \begin{pmatrix} C & 0 \\ 0 & 0 \\ 0 & E \end{pmatrix} \in \mathcal{P}[\lambda]^{(k+l+2)\times(k+l+1)}, \quad \mathcal{N} = \left\langle \begin{pmatrix} I_{k+1} \\ 0 \end{pmatrix} \right\rangle.$$

Taking the sequences of matrices

$$a_q = (0, \cdots, 0, 1/q) \in \mathbb{C}^{1\times(k+1)},\ b_q = \begin{pmatrix} 1/q & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{(l+1)\times(k+1)},$$

we see that

$$\lambda B_q - A_q = \lambda \begin{pmatrix} D & 0 \\ a_q & 0 \\ 0 & F \end{pmatrix} - \begin{pmatrix} C & 0 \\ 0 & 0 \\ b_q & E \end{pmatrix} \to \lambda B - A.$$

Observe that $\nu(\lambda B_q - A_q) \leq \nu(B_q) = 0$. Suppose that $\mathcal{N}$ is $(\lambda B - A)-$stable. Then, by Proposition 24, there exists a sequence of matrices $X_q \to 0 \in \mathbb{C}^{(k+1)\times l}$

such that for every q the subspace generated by the columns of the matrix $\begin{pmatrix} I_{k+1} \\ X_q \end{pmatrix}$ is $(\lambda B_q - A_q)-$reducing. That is, for every q

$$k+1 = \operatorname{rank}\begin{pmatrix} C & D \\ 0 & a_q \\ b_q + EX_q & FX_q \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & I_k & I_k & 0 \\ 0 & 0 & 0 & 1/q \\ 1/q & 0 & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

$$\geq \operatorname{rank}\begin{pmatrix} 0 & I_k & 0 \\ 0 & 0 & 1/q \\ 1/q & 0 & \star \end{pmatrix} = k+2,$$

which is a contradiction. Therefore $\mathcal{N}$ is not $(\lambda B - A)-$stable.

6.3 Two row minimal indices which are different than zero

Analogously, we will distinguish two subcases: one column minimal index which is equal to zero and one column minimal index which is different than zero.

One column minimal index which is equal to zero. Define

$$D := \begin{pmatrix} I_k \\ 0 \end{pmatrix}, C := \begin{pmatrix} 0 \\ I_k \end{pmatrix} \in \mathbb{C}^{(k+1)\times k},\ F := \begin{pmatrix} I_l \\ 0 \end{pmatrix}, E := \begin{pmatrix} 0 \\ I_l \end{pmatrix} \in \mathbb{C}^{(l+1)\times l}.$$

Then in this subcase we have

$$\lambda B - A = \begin{pmatrix} 0 & \lambda D - C & 0 \\ 0 & 0 & \lambda F - E \end{pmatrix} \in \mathcal{P}[\lambda]^{(k+l)\times(k+l+3)}, \quad \mathcal{N} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle.$$

Taking the sequences of matrices

$$a_q = (1/q, 0, \cdots, 0)^T \in \mathbb{C}^{(k+1)\times 1},\ b_q = (0, \cdots, 0, 1/q)^T \in \mathbb{C}^{(l+1)\times 1},$$

we infer that

$$\lambda B_q - A_q = \begin{pmatrix} a_q & \lambda D - C & 0 \\ b_q & 0 & \lambda F - E \end{pmatrix} \to \lambda B - A.$$

Moreover, $\nu(\lambda B_q - A_q) = \nu(A_q) = 0$. Assume that $\mathcal{N}$ is $(\lambda B - A)-$stable. Then, by Proposition 24, there exist sequences of matrices $X_q \to 0 \in \mathbb{C}^{k\times q}$ and $Y_q \to 0 \in \mathbb{C}^{l\times 1}$ such that for every q the subspace

$$\left\langle \begin{pmatrix} 1 \\ X_q \\ Y_q \end{pmatrix} \right\rangle$$

is $(\lambda B_q - A_q)-$reducing. That is, for every q

$$\operatorname{rank}\begin{pmatrix} a_q + CX_q & DX_q \\ EY_q & b_q + FY_q \end{pmatrix} = 1. \tag{68}$$

Define

$$X_q := (x_1^q, x_2^q, \cdots, x_k^q)^T,\ Y_q := (y_1^q, y_2^q, \cdots, y_l^q)^T,$$

from (68) we conclude that

$$\operatorname{rank}\begin{pmatrix}1/q & x_1^q & \cdots & x_{k-1}^q & x_k^q & 0 & y_1^q & \cdots & y_{l-1}^q & y_l^q\\ x_1^q & x_2^q & \cdots & x_k^q & 0 & y_1^q & y_2^q & \cdots & y_l^q & 1/q\end{pmatrix}=1.$$

Now, it is immediate to see that this implies $x_i^q = y_j^q = 0$. Hence

$$\operatorname{rank}\begin{pmatrix}1/q & 0\\ 0 & 1/q\end{pmatrix}=1,$$

which is a contradiction.

One column minimal index which is different than zero

Denote $D := (I_n, 0), C := (0, I_n) \in \mathbb{C}^{n\times(n+1)}$ and

$$F := \begin{pmatrix}I_m\\ 0\end{pmatrix}, E := \begin{pmatrix}0\\ I_m\end{pmatrix} \in \mathbb{C}^{(m+1)\times m},\ H := \begin{pmatrix}I_p\\ 0\end{pmatrix}, G := \begin{pmatrix}0\\ I_p\end{pmatrix} \in \mathbb{C}^{(p+1)\times p}.$$

We have

$$\lambda B - A = \begin{pmatrix}\lambda D - C & 0 & 0\\ 0 & \lambda F - E & 0\\ 0 & 0 & \lambda H - G\end{pmatrix} \in \mathcal{P}[\lambda]^{(n+m+p+2)\times(n+m+p+1)},\ \mathcal{N} = \left\langle \begin{pmatrix}I_{n+1}\\ 0\\ 0\end{pmatrix}\right\rangle.$$

Define the sequences of matrices

$$a_q := \begin{pmatrix}0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0\\ 0 & \cdots & 0 & 1/q\end{pmatrix} \in \mathbb{C}^{(m+1)\times(n+1)},\ b_q := \begin{pmatrix}1/q & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix} \in \mathbb{C}^{(p+1)\times(n+1)}.$$

Then

$$\lambda B_q - A_q = \lambda\begin{pmatrix}D & 0 & 0\\ a_q & F & 0\\ 0 & 0 & H\end{pmatrix} - \begin{pmatrix}C & 0 & 0\\ 0 & E & 0\\ b_q & 0 & G\end{pmatrix} \to \lambda B - A.$$

Moreover, as $\nu(A_q) = 0$ we see that $\nu(\lambda B_q - A_q) = 0$. Hence if $\mathcal{N}$ is $(\lambda B - A)-$stable, by Proposition 24, there exist sequences of matrices $X_q \to 0 \in \mathbb{C}^{m\times(n+1)}$ and $Y_q \to 0 \in \mathbb{C}^{l\times(n+1)}$ such that for every q the subspace

$$\mathcal{N}_q := \left\langle \begin{pmatrix}I_{n+1}\\ X_q\\ Y_q\end{pmatrix}\right\rangle$$

is $(\lambda B_q - A_q)-$reducing. That is, for every q

$$\operatorname{rank}\begin{pmatrix}C & D\\ EX_q & a_q + FX_q\\ b_q + GY_q & HY_q\end{pmatrix} = n+1. \tag{69}$$

Now, as $\nu(A_q) = 0$, from (69) we deduce that

$$\operatorname{rank}\begin{pmatrix}C\\ EX_q\\ b_q + GY_q\end{pmatrix} = n+1. \tag{70}$$

From Lemma 22, it follows that

$$D = C\begin{pmatrix} 0 & 1 \\ I_n & 0 \end{pmatrix},$$

then from (69) and (70), we have

$$\begin{pmatrix} D \\ a_q + FX_q \\ HY_q \end{pmatrix} = \begin{pmatrix} C \\ EX_q \\ b_q + GY_q \end{pmatrix}\begin{pmatrix} 0 & 1 \\ I_n & 0 \end{pmatrix},$$

hence

$$a_q + FX_q = EX_q\begin{pmatrix} 0 & 1 \\ I_n & 0 \end{pmatrix}. \tag{71}$$

Let $X_q := (x_{ij})_{0\le i\le n,\, 0\le j\le n}$. With this notation, from (71) we infer that

$$\begin{pmatrix} x_{00} & x_{01} & \cdots & x_{0n-1} & x_{0n} \\ x_{10} & x_{11} & \cdots & x_{1n-1} & x_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m0} & x_{m1} & \cdots & x_{mn-1} & x_{mn} \\ 0 & 0 & \cdots & 0 & 1/q \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ x_{00} & x_{01} & \cdots & x_{0n-1} & x_{0n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1,0} & x_{m-1,1} & \cdots & x_{m-1,n-1} & x_{m-1,n} \\ x_{m0} & x_{m1} & \cdots & x_{mn-1} & x_{mn} \end{pmatrix}.$$

Now, it is immediate to see that $x_{ij} = 0$, which implies $1/q = 0$. A contradiction.

7 Proof of Theorem 4: Assertion (5)

In this section we prove Assertion (5) of Theorem 4. That is, if the pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m\times n}$, without eigenvalues, only has one row minimal index which is equal to zero, then the unique reducing subspace — by Theorem 20 it is $\mathbb{C}^n$— is $(\lambda B - A)$-stable. To prove this result we will consider two cases: in the first one we will assume that all column minimal indices are equal to zero; in the second one we will assume that there is at least a column minimal index is different than zero.

7.1 All column minimal indices are equal to zero.

In this case we have $\lambda B - A = \lambda 0 - 0 \in \mathcal{P}[\lambda]^{1\times n}$. Now consider a sequence $\lambda B_q - A_q \to \lambda B - A$. Then we have two possible subcases: $\nu(\lambda B_q - A_q) = n$ or $\nu(\lambda B_q - A_q) = n - 1$. For the first subcase, $\nu(\lambda B_q - A_q) = n$, we obtain $\lambda B_q - A_q = \lambda 0 - 0$, and it is clear that the subspace $\mathbb{C}^n$ is $(\lambda B_q - A_q)$-reducing. On the other hand, if $\nu(\lambda B_q - A_q) = n - 1$, then either $\operatorname{rank}(A_q) = 1$ or $\operatorname{rank}(B_q) = 1$. Therefore

$$1 \ge \operatorname{rank}(A_q I_n, B_q I_n) \ge \max\{\operatorname{rank}(A_q), \operatorname{rank}(B_q)\} = 1 \Rightarrow \operatorname{rank}(A_q I_n, B_q I_n) = 1,$$

hence $\mathbb{C}^n$ is $(\lambda B_q - A_q)-$reducing, and so $\mathbb{C}^n$ is $(\lambda B - A)$-stable.

7.2 At least a column minimal index is is different than zero.

To simplify the exposition, in this case, suppose that $\lambda B - A \in \mathcal{P}[\lambda]^{(m+1)\times n}$. Now, as this pencil only has one row minimal index which is equal to zero and at least a nonzero column minimal index, then this pencil is strictly equivalent to a pencil of the form

$$\lambda \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} E & F \\ 0 & 0 \end{pmatrix}, \tag{72}$$

with $(E, F) \in \mathbb{C}^{m\times m} \times \mathbb{C}^{m\times p}$ a controllable pair and $n = m + p$. Hence, by Remark 26, to study the stability of the subspace $\mathbb{C}^n$ there is no loss of generality if we suppose that $\lambda B - A$ is in the form (72).

Now let $\lambda B_q - A_q$ be a sequence that converges to $\lambda B - A$. Then, since $\operatorname{nrank}(\lambda B - A) = m$, we have $\operatorname{nrank}(\lambda B_q - A_q) = m+1$ or m. That is, we have two subcases: $\nu(\lambda B_q - A_q) = n - m - 1$ or $\nu(\lambda B_q - A_q) = n - m$. We analyze them separately.

Subcase $\nu(\lambda B_q - A_q) = n-m-1$. As in this case $\operatorname{nrank}(\lambda B_q - A_q) = m+1$ then this pencil is left regular; that is, it has no row minimal indices and, therefore, by Theorem 20, $\mathbb{C}^n = \mathcal{D}^n_{\lambda B_q - A_q}$ is a $(\lambda B_q - A_q)$-reducing subspace.

Subcase $\nu(\lambda B_q - A_q) = n - m \Leftrightarrow \operatorname{nrank}(\lambda B_q - A_q) = m$.

To analyze this other subcase we need the following result.

Lemma 37. *Consider the matrix pencil*

$$\lambda D - C = \lambda \begin{pmatrix} I_m & 0 \\ \alpha & 0 \end{pmatrix} - \begin{pmatrix} G & H \\ 0 & 0 \end{pmatrix} \in \mathcal{P}[\lambda]^{(m+1)\times(m+p)},$$

with $(G, H) \in \mathbb{C}^{m\times m} \times \mathbb{C}^{m\times p}$ *a controllable pair. Assume that* $\operatorname{nrank}(\lambda D - C) = m$. *Then* $\alpha = 0$.

Proof. Let $r_1, r_2, \ldots, r_h$ be the sequence of nonzero column minimal indices of $\lambda D - C$. Then ([5], Theorem 6.2.5, p. 196) there exist invertible matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{p\times p}$ and a matrix $R \in \mathbb{C}^{p\times m}$ such that

$$P(G, H) \begin{pmatrix} P^{-1} & 0 \\ R & Q \end{pmatrix} = (\overline{G}, \overline{H}),$$

with $\overline{G} = (\operatorname{diag}(G_1, \ldots, G_h)$ and $\overline{H} = (\operatorname{diag}(H_1 \ldots, H_s), 0)$, where

$$G_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{r_i \times r_i}, \; H_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^{r_i \times 1}.$$

Hence, as

$$\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} (\lambda D - C) \begin{pmatrix} P^{-1} & 0 \\ R & Q \end{pmatrix} = \begin{pmatrix} \lambda I_m - \overline{G} & -\overline{H} \\ \lambda \alpha P^{-1} & 0 \end{pmatrix},$$

to prove the lemma we can assume that $(G, H) = (\overline{G}, \overline{H})$. Denote $\alpha := (a_1, a_2, \cdots, a_h)$ with $a_i \in \mathbb{C}^{1\times r_i}$. Then, because $\mathrm{nrank}(\lambda D - C) = m$, we have

$$\mathrm{nrank}\begin{pmatrix} \lambda I_{r_1} - G_1 & -H_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{r_2} - G_2 & -H_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I_{r_h} - G_h & -H_h \\ a_1 & 0 & a_2 & 0 & \dots & a_h & 0 \end{pmatrix} = m. \tag{73}$$

Now, as

$$(\lambda I_{r_i} - G_i, -H_i) = \begin{pmatrix} \lambda & -1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -1 \end{pmatrix},$$

if we denote by $a_i := (b_{i1}, b_{i2}, \dots, b_{ir_i})$, making transformations by columns in the matrix of (73), we deduce that

$$\mathrm{nrank}\begin{pmatrix} 0 & -I_{r_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -I_{r_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -I_{r_h} \\ p_1(\lambda) & \star & p_2(\lambda) & \star & \dots & p_h(\lambda) & \star \end{pmatrix} = r_1 + r_2 + \cdots + r_h = m,$$

with $p_i(\lambda) = b_{i1} + b_{i2}\lambda + \cdots + b_{ir_i}\lambda^{r_i - 1}$. Therefore $p_i(\lambda) = 0$, or equivalently $\alpha = 0$. □

Consider now a sequence $\lambda B_q - A_q \to \lambda B - A$. From (72), we can assume that for every q,

$$\lambda B_q - A_q = \lambda \begin{pmatrix} I_m^q & \beta_q \\ \alpha_q & \delta_q \end{pmatrix} - \begin{pmatrix} E_q & F_q \\ \eta_q & \theta_q \end{pmatrix}.$$

Now, as $m = \mathrm{nrank}(\lambda B_q - A_q) \geq \mathrm{rank}(A_q) \geq \mathrm{rank}(E_q, F_q) \geq \mathrm{rank}(E, F) = m$, it is immediate to see that there exists a sequence of matrices

$$P_q = \begin{pmatrix} I_m & 0 \\ \xi_q & 1 \end{pmatrix} \to I_{m+1},$$

such that for every q

$$P_q A_q = \begin{pmatrix} E_q & F_q \\ 0 & 0 \end{pmatrix}.$$

Hence, by Proposition 27, it suffices to consider sequences of the form $P_q(\lambda B_q - A_q)$ to study the stability of the subspace $\mathbb{C}^n$, that is,

$$\lambda B_q - A_q = \lambda \begin{pmatrix} I_m^q & \beta_q \\ \alpha_q & \delta_q \end{pmatrix} - \begin{pmatrix} E_q & F_q \\ 0 & 0 \end{pmatrix} \to \lambda B - A. \tag{74}$$

On the other hand, as the sequence of matrices

$$Q_q = \begin{pmatrix} (I_m^q)^{-1} & -(I_m^q)^{-1}\beta_q \\ 0 & 1 \end{pmatrix} \to I_{m+1},$$

by Proposition 27 it is sufficient to consider sequences of the form $(\lambda B_q - A_q)Q_q$; that is, from (74),

$$\lambda B_q - A_q = \lambda \begin{pmatrix} I_m & 0 \\ \alpha_q & \delta_q \end{pmatrix} - \begin{pmatrix} E_q & F_q \\ 0 & 0 \end{pmatrix} \to \lambda B - A.$$

But, since $m = \operatorname{nrank}(\lambda B_q - A_q) \geq \operatorname{rank}(B_q) \geq m$, we have $\delta_q = 0$, for every q.
,

$$\lambda B_q - A_q = \lambda \begin{pmatrix} I_m & 0 \\ \alpha_q & 0 \end{pmatrix} - \begin{pmatrix} E_q & F_q \\ 0 & 0 \end{pmatrix}.$$

Now, since (E_q, F_q) is controllable and $\operatorname{nrank}(\lambda B - A) = m$, by Lemma 37 it follows that $\alpha_q = 0$ for every q. Hence, it suffices to consider sequences of the form

$$\lambda B_q - A_q = \lambda \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} E_q & F_q \\ 0 & 0 \end{pmatrix}.$$

Finally, as $\operatorname{rank}(A_q I_n, B_q I_n) = m = n - \nu(\lambda B_q - A_q)$, then the subspace $\mathbb{C}^n$ is $(\lambda B_q - A_q)$-reducing and, , $\mathbb{C}^n$ is $(\lambda B - A)$-stable.

8 Proof of Assertion (6) of Theorem 4

In this section we will prove that if the pencil $\lambda B - A \in \mathcal{P}[\lambda]^{m \times n}$ has only one row minimal index which is different than zero, at least two column minimal indices and no eigenvalues, then it has not any stable reducing subspace. First note that from Lemma 35, in an analogous way as in Remark 36, we can assume that $\lambda B - A$ only has two column minimal indices. Hence we consider three subcases: (a) two column minimal indices which are equal to zero; (b) one column minimal index which is equal to zero and another column minimal index which is different than zero; (c) both column minimal indices which are different than zero.

8.1 Two column minimal indices which are equal to zero

Denote $D := \left(\begin{smallmatrix} I_k \\ 0 \end{smallmatrix}\right), C := \left(\begin{smallmatrix} 0 \\ I_k \end{smallmatrix}\right) \in \mathbb{C}^{(k+1) \times k}$, in this case we can assume that the pencil $\lambda B - A$ and the unique reducing subspace $\mathcal{N}$ are of the form

$$\lambda B - A = \lambda(0, D) - (0, C) \in \mathcal{P}[\lambda]^{(k+1) \times (2+k)},\ \mathcal{N} = \left\langle \begin{pmatrix} I_2 \\ 0 \end{pmatrix} \right\rangle,$$

respectively.

Consider the sequences of matrices

$$a_q = \begin{pmatrix} 1/q & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, b_q = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1/q \end{pmatrix} \in \mathbb{C}^{(k+1) \times 2},$$

the sequence $\lambda B_q - A_q = \lambda(b_q, D) - (a_q, C)$ converges to $\lambda B - A$ and moreover $\nu(\lambda B_q - A_q) = 1$. Now, if $\mathcal{N}$ is $(\lambda B - A)$-stable, by Proposition 24 there exists a sequence of matrices $X_q \to 0 \in \mathbb{C}^{k \times 2}$ such that for every q the subspace

$\mathcal{N}_q = \left\langle \left(\begin{smallmatrix} I_2 \\ X_q \end{smallmatrix} \right) \right\rangle$ is $(\lambda B_q - A_q)$-reducing; that is, $\dim(A_q(\mathcal{N}_q) + B_q(\mathcal{N}_q)) = 1$. Therefore, if we define $X_q := (x_{ij}^q)_{1\le i\le k, j=1,2}$ so that $\mathcal{N}_q$ can be $(\lambda B_q - A_q)$-reducing, it must be satisfied

$$\operatorname{rank} \begin{pmatrix} 1/q & 0 & x_{11}^q & x_{12}^q \\ x_{11}^q & x_{12}^q & x_{21}^q & x_{22}^q \\ \vdots & \vdots & \vdots & \vdots \\ x_{k-1,1}^q & x_{k-1,2}^q & x_{k1}^q & x_{k2}^q \\ x_{k1}^q & x_{k2}^q & 0 & 1/q \end{pmatrix} = 1.$$

Hence $x_{ij}^q = 0$. Thus, we conclude that $\operatorname{rank} \left(\begin{smallmatrix} 1/q & 0 \\ 0 & 1/q \end{smallmatrix} \right) = 1$, which is a contradiction. In conclusion, $\mathcal{N}$ is not $(\lambda B - A)$-stable.

8.2 One column minimal index which is equal to zero and another column minimal index which is different than zero

Define $\lambda D - C := \lambda[I_n, 0] - [0, I_n]$ and $\lambda F - E := \lambda \left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix} \right) - \left(\begin{smallmatrix} 0 \\ I_m \end{smallmatrix} \right)$, in this case it follows that the pencil $\lambda B - A$ and its unique reducing subspace $\mathcal{N}$ have the form

$$\lambda B - A = \begin{pmatrix} 0 & \lambda D - C & 0 \\ 0 & 0 & \lambda F - E \end{pmatrix}, \mathcal{N} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & I_{n+1} \\ 0 & 0 \end{pmatrix} \right\rangle,$$

respectively.

Consider the sequences

$$a_q = \begin{pmatrix} 1/q \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{(m+1)\times 1},\ b_q = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1/q \end{pmatrix} \in \mathbb{C}^{(m+1)\times(n+1)}.$$

Then

$$\lambda B_q - A_q = \lambda \begin{pmatrix} 0 & D & 0 \\ 0 & b_q & F \end{pmatrix} - \begin{pmatrix} 0 & C & 0 \\ a_q & 0 & E \end{pmatrix} \to \lambda B - A,$$

and moreover $\nu(\lambda B_q - A_q) = 1$. Hence, if $\mathcal{N}$ is $(\lambda B - A)$-stable, there exist sequences of matrices $X_q \to 0 \in \mathbb{C}^{m\times 1}$ and $Y_q \to 0 \in \mathbb{C}^{m\times(n+1)}$ such that the subspace

$$\mathcal{N}_q := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & I_{n+1} \\ X_q & Y_q \end{pmatrix} \right\rangle,$$

is $(\lambda B_q - A_q)$-reducing; that is, $\dim(A_q(\mathcal{N}_q) + B_q(\mathcal{N}_q)) = n + 1$. Thus

$$\operatorname{rank} \begin{pmatrix} 0 & C & 0 & D \\ a_q + EX_q & EY_q & FX_q & b_q + FY_q \end{pmatrix} = n + 1. \qquad (75)$$

Define $X_q := (x_1^q, x_2^q, \dots, x_m^q)^T$. Then as $\operatorname{rank} C = n$, from (75) we have

$$1 \ge \operatorname{rank}(a_q + EX_q) = \operatorname{rank} \begin{pmatrix} 1/q & x_1^q & \cdots & x_{m-1}^q & x_m^q \\ x_1^q & x_2^q & \cdots & x_m^q & 0 \end{pmatrix}^T.$$

Therefore $x_i^q = 0$ and $X_q = 0$. Now, denote by $Y_{n+1}^q := (y_1^q, y_2^q, \ldots, y_m^q)^T$ the last column of Y_q. As $\operatorname{rank} D = n$, from (75) we see that

$$1 \geq \operatorname{rank}(a_q, b_q + FY_{n+1}^q) = \operatorname{rank} \begin{pmatrix} 1/q & 0 & \cdots & 0 & 0 \\ y_1^q & y_2^q & \cdots & y_m^q & 1/q \end{pmatrix}^T,$$

which is a contradiction. Thus, $\mathcal{N} = \mathbb{C}^{n+1}$ is not $(\lambda B - A)$-stable.

8.3 Two column minimal indices which are different than zero.

Define $\lambda D - C := \lambda[I_n, 0] - [0, I_n]$, $\lambda F - E := \lambda[I_p, 0] - [0, I_p]$ and $\lambda H - G := \lambda \left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right)$, in this case we infer that the pencil $\lambda B - A$ and its unique reducing subspace $\mathcal{N}$ have the form

$$\lambda B - A = \begin{pmatrix} \lambda D - C & 0 & 0 \\ 0 & \lambda F - E & 0 \\ 0 & 0 & \lambda H - G \end{pmatrix}, \mathcal{N} = \left\langle \begin{pmatrix} I_{n+1} & 0 \\ 0 & I_{p+1} \\ 0 & 0 \end{pmatrix} \right\rangle,$$

respectively.

Consider the sequences

$$a_q = \begin{pmatrix} 1/q & 0 \cdots & 0 & \\ 0 & 0 \cdots & 0 & \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 \cdots & 0 & \end{pmatrix} \in \mathbb{C}^{(m+1)\times(n+1)},\ b_q = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1/q \end{pmatrix} \in \mathbb{C}^{(m+1)\times(p+1)}.$$

Then

$$\lambda B_q - A_q = \lambda \begin{pmatrix} D & 0 & 0 \\ 0 & F & 0 \\ 0 & b_q & H \end{pmatrix} - \begin{pmatrix} C & 0 & 0 \\ 0 & E & 0 \\ a_q & 0 & G \end{pmatrix} \to \lambda B - A,$$

and moreover, $\nu(\lambda B_q - A_q) = 1$. Hence, if $\mathcal{N}$ is $(\lambda B - A)$-stable, there exist sequences of matrices $X_q \to 0 \in \mathbb{C}^{m\times(n+1)}$ and $Y_q \to 0 \in \mathbb{C}^{m\times(p+1)}$ such that the subspace

$$\mathcal{N}_q := \left\langle \begin{pmatrix} I_{n+1} & 0 \\ 0 & I_{p+1} \\ X_q & Y_q \end{pmatrix} \right\rangle$$

is $(\lambda B_q - A_q)$-reducing; that is, $\dim(A_q(\mathcal{N}_q) + B_q(\mathcal{N}_q)) = n + p + 1$. Therefore

$$\operatorname{rank} \begin{pmatrix} C & 0 & D & 0 \\ 0 & E & 0 & F \\ a_q + GX_q & GY_q & HX_q & b_q + HY_q \end{pmatrix} = n + p + 1.$$

Denote by $X_q = (x_{ij})$ and $Y_q = (y_{kl})$, from the previous equality it follows that

$$\operatorname{rank} \begin{pmatrix} 0 & I_n & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_p & 0 & 0 & I_p & 0 \\ 1/q & 0\cdots 0 & 0 & 0\cdots 0 & x_{11}\cdots x_{1n} & x_{1,n+1} & y_{11}\cdots y_{1p} & y_{1,p+1} \\ x_{11} & x_{12}\cdots x_{1,n+1} & y_{11} & y_{12}\cdots y_{1,p+1} & x_{21}\cdots x_{2n} & x_{2,n+1} & y_{21}\cdots y_{2p} & y_{2,p+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2}\cdots x_{m,n+1} & y_{m1} & y_{m2}\cdots y_{m,p+1} & 0\cdots 0 & 0 & 0\cdots 0 & 1/q \end{pmatrix} = n{+}p{+}1.$$

Observe now that, choosing the submatrix formed by the $n+p+2$ first columns, we deduce immediately that $y_{i1}=0$ for $i=1,2,\ldots,m$. In the same way, with the $n+p+2$ last columns we see that $x_{i,n+1}=0$ for $i=1,2,\ldots,m$. Hence with the entries 1 corresponding to the places $(n,n+1)$ and $(n+1,2n+p+4)$ we can reduce the previous matrix to one on the same form, but reducing the sizes from n to $n-1$ and from p to $p-1$, and whose rank is $n+p-1$. Following this process we reach the case where at least one column minimal index is equal to zero, which is already solved in Subsections 8.1 and 8.2.

9 Proof of Assertion (7) of Theorem 4

In this section we will analyze the case of a matrix pencil with only one row minimal index which is different than zero, and one column minimal index which is equal to zero. Previously we will introduce some auxiliary results. We begin by stating some bounds about the maximum modulus of a root of a polynomial, that can be seen in [14], Section 8, pp. 243–247.

Lemma 38. *Let $f(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}+z^n$ be a polynomial of degree n with coefficients in $\mathbb{C}$ distinct from the polynomial z^n. Denote*

$$\alpha := \max_{0\le k\le n-1}\left(\binom{n}{k}^{-1}|a_k|\right)^{1/(n-k)}.$$

Assume that z_n is a root of maximum modulus of $f(z)$. Then

$$(2^{1/n}-1)\alpha < |z_n| \le (2^{1/n}-1)^{-1}\alpha.$$

In order to prove Lemma 41, we need the following two lemmas. The first one, Lemma 39, is deduced immediately from (34).

Lemma 39. *Consider the matrix pencil $\lambda B-A\in\mathcal{P}[\lambda]^{m\times n}$ and the matrix $T^k_{\lambda B-A}$ defined in* (28)*. Then*

(i) *If $\nu(T^k_{\lambda B-A})=0$, then $\lambda B-A$ has not any column minimal indices $\le k-1$.*

(ii) *If $\lambda B-A=\lambda(I_k,0)-(0,I_k)\in\mathcal{P}[\lambda]^{k\times(k+1)}$, then*

$$\nu(T^p_{\lambda B-A})=\begin{cases}0 & \text{if } p\le k\\ 1 & \text{if } p\ge k+1.\end{cases}$$

The second one, Lemma 40, can be seen in [5], Theorem 13.5.1, p. 406.

Lemma 40. *Let $F\in\mathbb{C}^{p\times q}$ and let $X\in\mathbb{C}^{q\times r}$ be a basis matrix of* $\operatorname{Ker}F$*. Consider a sequence $F_q\to F$ such that, for every q, $\nu(F_q)=\nu(F)$. Then there exist a sequence $X_q\to X$ and a positive constant K_1 such that, for every q, X_q is a basis matrix of* $\operatorname{Ker}F_q$ *and*

$$\|X_q-X\|\le K_1\|F_q-F\|.$$

Based on these results we will prove the following lemma.

Lemma 41. *Consider the pencil* $\lambda B - A = \lambda(I_k, 0) - (0, I_k) \in \mathcal{P}[\lambda]^{k\times(k+1)}$. *Then for each sequence* $\lambda B_q - A_q \to \lambda B - A$ *there exist two sequences of matrices* $P_q \to I_k$ *and* $Q_k \to I_{k+1}$ *such that, for every* q, *we have* $P_q^{-1}(\lambda B_q - A_q)Q_q = \lambda B - A$; *moreover, there exists a constant* $K > 0$ *that satisfies*

$$\max\{\|P_q - I_k\|, \|Q_q - I_{k+1}\|\} \leq K(\|A_q - A\| + \|B_q - B\|).$$

Proof. Note first that by Lemma 39, $\nu(T^{k+1}_{\lambda B-A}) = 1$. Denote by $\{e_1, e_1, \ldots, e_{k+1}\}$ the vectors of the canonical basis of $\mathbb{C}^{k+1}$, it is clear that

$$\mathrm{Ker}(T^{k+1}_{\lambda B-A}) = \left\langle \begin{pmatrix} e_{k+1} \\ e_k \\ \vdots \\ e_1 \end{pmatrix} \right\rangle. \tag{76}$$

Now consider a sequence $\lambda B_q - A_q \to \lambda B - A$. Since $\mathrm{nrank}(\lambda B_q - A_q) = k$, it follows that the pencil $\lambda B_q - A_q$ has at least a column minimal index. Moreover, for every p we have $\nu(T^p_{\lambda B_q - A_q}) \leq \nu(T^p_{\lambda B-A})$, by Lemma 39, it follows that the pencil $\lambda B_q - A_q$ has not any column minimal indices $< k$. That is, it has one column minimal index which is equal to k; hence $\nu(T^{k+1}_{\lambda B_q-A_q}) = \nu(T^{k+1}_{\lambda B-A})$. Since $T^{k+1}_{\lambda B_q-A_q} \to T^{k+1}_{\lambda B-A}$ and for every q the matrices $T^{k+1}_{\lambda B_q-A_q}$ have the same nullity, it follows from (76) and Lemma 40 that there exists a basis matrix of the subspace $\mathrm{Ker}(T^{k+1}_{\lambda B_q-A_q})$

$$\begin{pmatrix} x^q_{k+1} \\ x^q_k \\ \vdots \\ x^q_1 \end{pmatrix} \quad \text{converging to} \quad \begin{pmatrix} e_{k+1} \\ e_k \\ \vdots \\ e_1 \end{pmatrix}$$

such that $\|x^q_i - e_i\| \leq K_1(\|A_q - A\| + \|B_q - B\|)$. Now let

$$P_q := (A_q x^q_2, \cdots, A_q x^q_{k+1}),\ Q_q := (x^q_1, \cdots, x^q_{k+1}).$$

It is obvious that $P_q^{-1}(\lambda B_q - A_q)Q_q = \lambda B - A$ and $\|Q_q - I_{k+1}\| \leq K_1(\|A_q - A\| + \|B_q - B\|)$. It suffices to demonstrate that $\|P_q - I_k\| \leq K_2(\|A_q - A\| + \|B_q - B\|)$ to conclude the proof of the lemma.

In fact, denoting by $(f_1, f_2, \ldots, f_k)$ the canonical basis of $\mathbb{C}^k$, it follows that

$$\|P_q - I_k\| \leq \sum_{i=2}^{k+1} \|A_q x^q_i - f_{i-1}\| = \sum_{i=2}^{k+1} \|A_q x^q_i - A e_i\|.$$

Now

$$\begin{aligned} \|A_q x^q_i - A e_i\| \leq\ & \|A_q x^q_i - A_q e_i\| + \|A_q e_i - A e_i\| \leq \|A_q\|\|x^q_i - e_i\| + \|A_q - A\|\|e_i\| \\ \leq\ & (\|A_q - A\| + \|A\|)\|x^q_i - e_i\| + \|A_q - A\| \leq K_2(\|A_q - A\| + \|B_q - B\|), \end{aligned}$$

□

With these previous results we are ready to prove Assertion (7) of Theorem 4.

Proof of Assertion (7) of Theorem 4

Define $D := \left(\begin{smallmatrix} I_n \\ 0 \end{smallmatrix}\right)$ and $C := \left(\begin{smallmatrix} 0 \\ I_n \end{smallmatrix}\right)$, both matrices of $\mathbb{C}^{(n+1)\times n}$. Hence,

$$\lambda B - A = \lambda(0, D) - (0, C) \in \mathcal{P}[\lambda]^{(n+1)\times(n+1)}.$$

The unique reducing subspace of $\lambda B - A$ is $\mathcal{N} = \langle e_1 \rangle$, with e_1 the first canonical vector of $\mathbb{C}^{n+1}$. Note that $\nu(\lambda B - A) = 1$. Now consider a sequence $(\lambda B_q - A_q) \to (\lambda B - A)$. Then, by Lemma 40 and by Proposition 27, when studying the Lipschitz stability of the subspace $\mathcal{N}$, no generality is lost if we only consider sequences of the form $\lambda B_q - A_q = \lambda(\varepsilon_q, D) - (\delta_q, C)$. Operating with the columns of D, by Proposition 27, we can assume that

$$\lambda B_q - A_q = \lambda \left(\begin{array}{c|c} 0 & I_n \\ \hline a_q & 0 \end{array}\right) - \left(\begin{array}{c|c} b_q & 0 \\ \hline H_q & I_n \end{array}\right), \tag{77}$$

with $H_q = (c_1^q, c_2^q, \dots, c_n^q)^T \in \mathbb{C}^n$.

Note that making row operations it is immediate to see that

$$\det(\lambda B_q - A_q) = a_q \lambda^{n+1} - \sum_{i=1}^{n} c_i^q \lambda^i - b_q.$$

Therefore, $\nu(\lambda B_q - A_q) = 1$ if and only if $a_q = b_q = c_i^q = 0$, which is equivalent to $\lambda B_q - A_q = \lambda B - A$. For this case, it is clear that $\mathcal{N}_q = \mathcal{N}$ is a reducing subspace for $\lambda B_q - A_q$. Thus, from here on, we will assume that $\nu(\lambda B_q - A_q) = 0$. In order to prove that $\mathcal{N}$ is Lipschitz stable, it suffices to find sequences of complex numbers x_i^q, $i = 1, 2, \dots, n$, such that for every q, the subspace

$$\mathcal{N}_q := \left\langle \begin{pmatrix} 1 \\ x_1^q \\ x_2^q \\ \vdots \\ x_n^q \end{pmatrix} \right\rangle$$

is $(\lambda B_q - A_q)$–reducing; that is, since $\nu(\lambda B_q - A_q) = 0$, it follows that $\dim A_q(\mathcal{N}_q) + B_q(\mathcal{N}_q) = 1$ holds. Or, which is the same, from (77)

$$\operatorname{rank} \begin{pmatrix} b_q & x_1^q \\ c_1^q + x_1^q & x_2^q \\ c_2^q + x_2^q & x_3^q \\ \vdots & \vdots \\ c_{n-1}^q + x_{n-1}^q & x_n^q \\ c_n^q + x_n^q & a_q \end{pmatrix} = 1, \tag{78}$$

and, moreover, that there exists a constant $K > 0$ such that,

$$|x_i^q| \le K(\|B_q - B\| + \|A_q - A\|),\ i = 1, 2, \dots, n. \tag{79}$$

Note first that if $a_q = 0$, it suffices to take $x_i^q = 0$ for each i. On the other hand, if $b_q = 0$, it is sufficient to choose $x_i^q = -c_i^q$ for each i. Hence, we will assume that $a_q b_q \neq 0$. In order for (78) to hold, since $a_q \neq 0$, we search for the x_i^q in such a way that the first column is proportional to the second one. Note that the proportionality factor is b_1^q/x_1^q. Now, doing operations in (78), by a induction process it is proved that

$$x_k^q = \frac{(x_1^q)^k + \displaystyle\sum_{i=1}^{k-1} c_i^q b_q^{i-1} (x_1^q)^{k-i}}{b_q^{k-1}}, \; k = 2, 3, \ldots, n, \tag{80}$$

and for x_1^q we have

$$(x_1^q)^{n+1} + \sum_{i=1}^{n} c_i^q b_q^{i-1} (x_1^q)^{n-i+1} - a_q b_q^n = 0. \tag{81}$$

Consider the polynomial

$$f_q(z) := z^{n+1} + \sum_{i=1}^{n} c_i^q b_q^{i-1} z^{n-i+1} - a_q b_q^n.$$

We find a bound for the maximum modulus of its roots. Define $B_{n+1}^k := \binom{n+1}{k}^{-1/(n-k+1)}$, by Lemma 38,

$$\begin{aligned} \alpha = \; & \max\{B_{n+1}^0 \mid a_q b_q^n \mid^{1/(n+1)}, B_{n+1}^1 \mid c_n^q b_q^{n-1} \mid^{1/n}, \ldots, \\ & \ldots, \mid B_{n+1}^k \mid c_{n-k+1}^q b_q^{n-k} \mid^{1/(n-k+1)}, \ldots, B_{n+1}^n \mid c_1^q \mid\}. \end{aligned} \tag{82}$$

After that, we choose x_1^q as one of the roots of $f(z)$ that have maximum modulus. By Lemma 38 and (82) it is clear that x_1^q satisfies (79).

Let $k \in \{2, 3, \ldots, n\}$. Then, combining (80) and (81) we infer that

$$x_k^q = -\frac{\displaystyle\sum_{i=k}^{n} c_i^q b_q^{i-1} (x_1^q)^{n-i+1} - a_q b_q^n}{b_q^{k-1} (x_1^q)^{n+1-k}} = -\sum_{i=k}^{n} \frac{c_i^q b_q^{i-k}}{(x_1^q)^{i-k}} + \frac{a_q b_q^{n-k+1}}{(x_1^q)^{n-k+1}}. \tag{83}$$

In order to conclude this case, it suffices to see that each summand of (83) is bounded by $K(\|B_q - B\| + \|A_q - A\|)$, for a positive constant K.

First, by Lemma 38 and (82) it follows that there exists a positive constant L such that $|x_1^q|^{-1} \leq L |a_q b_q^n|^{-1/(n+1)}$. Therefore,

$$\left| \frac{a_q b_q^{n-k+1}}{(x_1^q)^{n-k+1}} \right| \leq L \left| \frac{a_q b_q^{n-k+1}}{(a_q b_q^n)^{(n-k+1)/(n+1)}} \right| = L \left| a_q^{k/(n+1)} b_q^{(n-k+1)/(n+1)} \right| \leq K(\|B_q - B\| + \|A_q - A\|).$$

Second, following Lemma 38 and (82) again, we see that there exists a positive constant L_i such that $|x_1^q|^{-1} \leq L_i |c_i^q b_q^{i-1}|^{-1/i}$. Thus,

$$\left| \frac{c_i^q b_q^{i-k}}{(x_1^q)^{i-k}} \right| \leq L_i \left| \frac{c_i^q b_q^{i-k}}{(c_i^q b_q^{i-1})^{(i-k)/i}} \right| = L_i \left| (c_i^q)^{k/i} b_q^{(i-k)/i} \right| \leq K(\|B_q - B\| + \|A_q - A\|).$$

□

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