The reverse self-dual serial cost-sharing rule

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Abstract. In this study we define a cost sharing rule for cost sharing problems. This rule is related to the serial cost-sharing rule defined by Moulin and Shenker (1992). We give some formulas and axiomatic characterizations for the new rule. The axiomatic characterizations are related to some previous ones provided by Moulin and Shenker (1994) and Albizuri (2010).

JEL classification: C71

Key words: Cost sharing, serial cost-sharing rule, self-duality

1. Introduction

In this paper we deal with cost-sharing problems where there is a process to produce a private good which is shared by n agents. Each agent demands a quantity q_i of the good. The cost function is denoted C and a cost sharing rule allocates the total production cost, that is, $C\left(\sum_i q_i\right)$, among all the agents.

We define and study a new rule which is related to the serial cost-sharing rule defined by Moulin and Shenker (1992). First we present the serial cost-sharing rule and some others to better understand the rule we define in this paper.

The serial cost-sharing rule is as follows. Suppose there are only two agents i and j, and $q_i \leq q_j$. When the production starts, each unit of the good is equally divided among the two agents, who share equally the incurred cost. This continues until $2q_i$ is produced, that is, until agent i is given q_i. At this point agent i leaves the system and the process continues as before, that is, agent j receives the remaining quantity and pays the associated cost. Consequently, agent i pays $C(2q_i)/2$ and j pays the rest, that is, $C(q_j + q_i) - C(2q_i)/2$. In Fig. 1 we draw the associated production path. The serial cost-sharing rule is obtained when we generalize this process to n agents.

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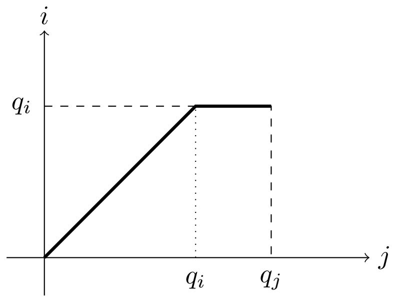

Figure 1

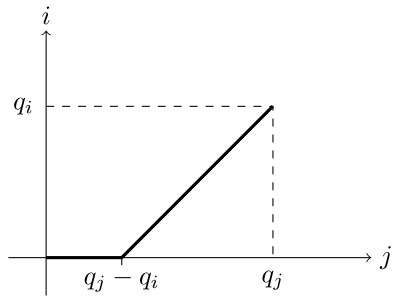

Figure 2

Albizuri and Zarzuelo (2007) define the dual serial cost-sharing which equalizes the quantities left to be allocated to agents. So when the good production starts each unit goes to agent j, that is, the agent with the highest demand, who pays the incurred cost. When agent j is served $q_j - q_i$ units, that is, when both i and j are short of the same quantity q_i, agent j pays $C(q_j - q_i)$ and agent i enters the picture. The production process continues and both agents are served simultaneously and pay equally the cost. Hence, each of the agents pays $\frac{C(q_j+q_i)-C(q_j-q_i)}{2}$. In Fig. 2 we see the associated path. Generalizing this procedure to n agents the dual serial cost-sharing rule is obtained.

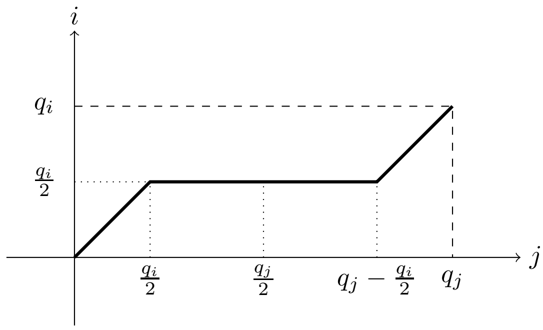

Figure 3

Notice that in the first case agent i pays only the cost of lowest increments and in the second case only the cost of highest increments. Albizuri (2010) defines a rule which allocates agent i both the cost of lowest increments and the cost of highest increments. This rule is obtained as follows. When the production starts each unit is equally divided until each agent gets half of his demand, who share equally the incurred cost. Thereafter units are given equalizing quantities left to be allocated and agents continue sharing equally the incurred cost (see Fig. 3). So when q_i is produced each agent receives $q_i/2$ and pays $C(q_i)/2$. Since agent i is given half of his demand agent j is the only one who pays the rest until $\frac{q_j+q_i}{2}$ is produced, that is, he pays $C\left(\frac{q_j+q_i}{2}\right) - C(q_i)$. From now on quantities left to be allocated are equalized. Therefore, agent j receives $\frac{q_j-q_i}{2}$ and pays the associated cost increment, that is, $C(q_j) - C\left(\frac{q_j+q_i}{2}\right)$. Thereafter agents i and j are served simultaneously and pay equally the incurred cost, so each of i and j receives $q_i/2$ and pays $\frac{C(q_j+q_i)-C(q_j)}{2}$. If we generalize this procedure to n agents we obtain the self-dual serial cost-sharing rule. This rule gives the same cost shares in a problem and in its dual one.

But there is also another way to allocate cost by not taking the cost of lowest increments or the cost of highest increments to determine the allocation for agent i. It is by taking cost increments which are in the middle, that is, instead of mixing high and low cost increments, we take middle cost increments. This new rule is obtained by equalizing first the demands left to be allocated until each agent receives half of his demand and thereafter units of good are given equally until all the demands are met. Therefore, first the agent with the highest demand, that is, agent j, receives $\frac{q_j-q_i}{2}$ units and pays $C\left(\frac{q_j-q_i}{2}\right)$. After that, agent i enters the system, each of the agents is given $q_i/2$ and pays $\frac{C\left(\frac{q_j+q_i}{2}\right)-C\left(\frac{q_j-q_i}{2}\right)}{2}$. Since they have met half of their demands, from now on they receive equally the units and pay the incurred cost until the demand is met. Hence, each agent is given $q_i/2$ and pays $\frac{C\left(\frac{q_j+q_i}{2}+q_i\right)-C\left(\frac{q_j+q_i}{2}\right)}{2}$. Since agent i has met his demand he leaves the picture and agent j is the only one who pays the rest, that is, $C(q_j+q_i) - C\left(\frac{q_j+q_i}{2}+q_i\right)$. We can see in Fig. 4 the associated path. Generalizing the procedure to n agents we obtain the new rule. As the self-dual serial cost-sharing rule does, this rule gives the same allocations in a problem and in its dual one.

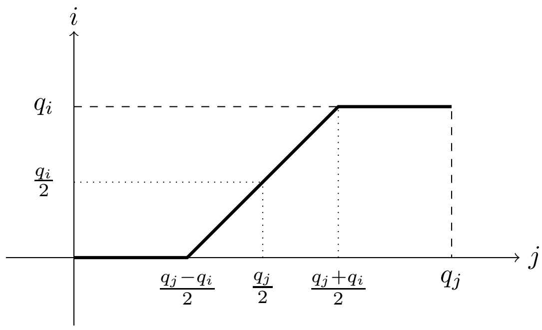

FIGURE 4

In this paper we define and give some formulas for this rule. And we give three axiomatic characterizations. Two of them are related to axiomatic characterizations given by Albizuri (2010) for the self-dual serial cost-sharing rule. The other is related to the characterization for the serial cost-sharing rule provided by Moulin and

Shenker (1994), where they characterize the serial cost-sharing rule by employing some standard axioms and free lunch, a kind of consistency axiom. Free lunch deals with cost functions that vanish identically at the beginning of production process of the good. We see in this paper that if we consider cost sharing functions which are flat in the middle (production has not cost in the middle), we obtain the rule defined in this paper.

Moulin (2002) shows that the set of monotonic rationing methods is linearly isomorphic to that of additive cost sharing rules. This author gives a list of rationing methods and cost sharing rules matched by this linear isomorphism. The Uniform Gains methods gives rise to the serial cost-sharing rule (Moulin and Shenker, 1992) and the Uniform Loses to the "dual" serial rule. He writes the resulting cost-sharing rule from the Talmudic rationing method when there are two or three agents. It turns out that the self-dual serial cost-sharing rule is the cost sharing rule that matches with the Talmudic rationing method. Recall that the Talmudic rationing method is a mixture of the Uniform Gains rule and the Uniform Loses rule. When mixture is done in the opposite way the reverse Talmud rationing method is obtained (see Thomson (2008)). And the cost sharing rule that matches with it is the rule defined in this paper. That is why we call it the reverse self-dual serial cost-sharing rule.

We mention that there are other variations of the serial cost-sharing rule which allocate cost in decreasing order of demands (see, for instance, Frutos (1998) and Leroux (2005)).

The paper is structured as follows. Section 2 is a preliminary one. In Section 3 we give the definition of the new rule and a formula for it. In Section 4 we provide other formulas for both the reverse self-dual serial cost-sharing rule and the self-dual serial cost-sharing rule. In Section 5 we give a characterization for the reverse self-dual serial cost-sharing rule which is related to the characterization for the serial cost-sharing rule provided by Moulin and Shenker (1994). In Section 6 we provide axiomatic characterizations for the new rule which are related to the ones given by Albizuri (2010) for the self-dual serial cost-sharing rule. Finally, in Section 7 some examples are presented.

2. Preliminaries

Let U denote a set of potential *agents*. Given a non-empty finite subset N of U, by $\mathbb{R}^N$ we write the $|N|$-dimensional euclidean space whose axes are labelled with the members of N, $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_i \geq 0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. We denote $\mathbf{1}_N$ the vector in $\mathbb{R}^N_+$ such that $(\mathbf{1}_N)_i = 1$ for all $i \in N$. If $x \in \mathbb{R}$ then $x_+ = \max\{x, 0\}$. Given $q \in \mathbb{R}^N_+$, we denote $Q = \sum_{i \in N} q_i$, and if $N = \{1, 2, \ldots, n\}$ and $q_1 \leq q_2 \leq \cdots \leq q_n$, we write $q^0 = 0$ and $q^j = (n - j + 1)\, q_j + q_{j-1} + ... + q_1$ for every $j \in N$. And if $S \subseteq N$ then $q_S \in \mathbb{R}^S_+$ satisfies $(q_S)_i = q_i$ for all $i \in S$.

A triple (N, q, C) is called a *cost sharing problem*, if N is a non-empty finite subset of U (the set of agents involved in the problem), $q \in \mathbb{R}_+^N$ (the demand profile of the cost sharing problem) and C is a nondecreasing function defined on $[0, Q]$ such that $C(0) = 0$ (the cost function of the cost sharing problem).

Let Γ_U denote the set of all cost sharing problems with the foregoing properties.

A *cost sharing rule* σ on a subset Γ of Γ_U associates with each $(N, q, C) \in \Gamma$ a vector $\sigma(N, q, C) \in \mathbb{R}_+^N$ satisfying

$$\sum_{i \in N} \sigma_i(N, q, C) = C(Q) \qquad \text{(efficiency)}.$$

Hence, a cost sharing rule allocates total cost among the n agents.

Along the paper, if not say otherwise, cost sharing rules are defined on Γ_U.

Moulin and Shenker (1992) define *the serial cost-sharing rule*. To present an explicit formula of this rule (and the following ones) assume that $N = \{1, 2, \ldots, n\}$ and $q_1 \leq q_2 \leq \cdots \leq q_n$. The serial cost-sharing rule of (N, q, C), denoted φ, is defined by

$$\varphi_i(N, q, C) = \sum_{j=1}^{i} \frac{C_j^q - C_{j-1}^q}{n - j + 1} \tag{1}$$

for all $i \in \{1, \ldots, n\}$, where

$$C_j^q = C\left(q^j\right) \tag{2}$$

for all $j \in \{0, ..., i\}$. Moulin and Shenker (1994) characterize the serial cost-sharing rule on Γ_U by means of five properties: additivity, ranking, separable costs, free lunch and continuity. The first axiom is a well known axiom. Ranking requires that the cost sharing rule should respect the ordering of the demands. Under separable costs, if costs are separable they should be so allocated. Free lunch is a kind of consistency axiom which we describe in more details in Section 5 and the fifth property requires the continuity for the topology of pointwise convergence.

Albizuri and Zarzuelo (2007) define *the dual serial cost-sharing rule*. The dual serial cost-sharing rule of (N, q, C), denoted φ^*, is defined by

$$\varphi_i^*(N, q, C) = \sum_{j=1}^{i} \frac{\widehat{C}_{j-1}^q - \widehat{C}_j^q}{n - j + 1} \tag{3}$$

for all $i \in \{1, \ldots, n\}$, where

$$\widehat{C}_j^q = C\left(Q - q^j\right) \tag{4}$$

for all $j \in \{0, ..., i\}$.

Given a problem (N, q, C), we define its dual cost function by $D^q C(t) = C(Q) - C(Q-t)$ [1] and its dual problem by $(N, q, D^q C)$. It follows that

$$\varphi^*(N, q, C) = \varphi(N, q, D^q C),$$

that is, the dual serial cost sharing-rule of a cost sharing problem is the serial cost-sharing rule of the dual problem.

Finally, Albizuri (2010) defines the *self-dual serial cost-sharing rule.* The self-dual serial cost-sharing rule of (N, q, C), denoted φ^S, is defined by

$$\varphi_i^S(N, q, C) = \sum_{j=1}^{i} \frac{C_j^{q/2} - C_{j-1}^{q/2}}{n - j + 1} + \sum_{j=1}^{i} \frac{\widehat{C}_{j-1}^{q/2} - \widehat{C}_j^{q/2}}{n - j + 1} \tag{5}$$

for all $i \in \{1, \ldots, n\}$, where

$$C_j^{q/2} = C\left(\frac{q^j}{2}\right) \tag{6}$$

and

$$\widehat{C}_j^{q/2} = C\left(Q - \frac{q^j}{2}\right) \tag{7}$$

for all $j \in \{0, ..., i\}$.

Albizuri (2010) gives two axiomatic characterizations of the self-dual serial cost-sharing rule. On the one hand, it is characterized by means of additivity, self duality and partially serial. The second axiom says that the allocations are the same in a problem and in its dual one. And partially serial requires that if the cost up to total demand Q is less than the cost of half of the demands then the cost shares to be equal to the cost shares determined by means of the serial cost-sharing rule when half of the demands are required.

On the other hand, the self-dual serial cost-sharing rule is characterized by means of anonymity, independence of null demands and translated equal changes in payoff. Anonymity requires the cost shares associated with two demands to be permuted if the two demands are permuted. Independence of null demands says that the payoffs of the agents are independent of the agents who demand nothing. The third axiom requires the change of the cost shares to be the same for all the agents if we consider some translated axiom instead of the original one.

[1] We have defined cost function C on $[0, Q]$ not to define arbitrarily this cost function beyond Q. We could have chosen to define C and $D^q C$ on $\mathbb{R}_+$ and all the results in this paper would be valid. The same happens for the definition of $\widetilde{C}^Q$ in Section 4.

3. The reverse self-dual serial cost-sharing rule

We describe first the serial cost-sharing rule of Moulin and Shenker (1992). Let (N, q, C) be a cost sharing problem with $q_1 \leq q_2 \leq \cdots \leq q_n$. When the production starts, each unit of the good is equally divided between the agents, who share equally the incurred cost. When quantity q^1 is produced, since agent 1 has met all his demand, he stops receiving the good and leaves the picture. And the process goes on in the same way. The production continues and each additional unit is divided equally among the remaining $n-1$ agents, who share equally the incurred cost. When agent 2 has met his demand, that is, when quantity q^2 is produced, agent 2 stops receiving the good, he leaves the picture and the production continues for the remaining agents. These agents pay equally until agent 3 has met his demand and so on.

The dual serial cost-sharing rule defined by Albizuri and Zarzuelo (2007) is as follows. This rule equalizes quantities left to be allocated to agents, who share equally the incurred cost. When the production starts, each unit is given to agent n, the agent with the highest demand. When agent n is given $q_n - q_{n-1}$ units, that is, $Q - q^{n-1}$ units, then both n and $n-1$ are short of the same quantity. Agent n pays the incurred cost, that is, $C\left(Q - q^{n-1}\right)$, and the production process continues by sharing the good equally among agents n and $n-1$, who pay equally the incurred cost. When each of them is given $q_{n-1} - q_{n-2}$ then agents n, $n-1$ and $n-2$ are left to be allocated the same quantity of good. So n and $n-1$ share equally the corresponding cost, that is, each one pays $\frac{C\left(Q-q^{n-2}\right)-C\left(Q-q^{n-1}\right)}{2}$ and the process continues in the same way. Units are given simultaneously to agents n, $n-1$ and $n-2$ until they are short of the same quantity of demand and so on.

As pointed in the Introduction, when Albizuri (2010) defines the self-dual cost-sharing rule agents with low demands not only pay cost increments associated with low demands as it happens with the serial cost-sharing rule but also to high demands. And hence, agents with high demands are not the only ones to pay cost increments associated with high demands. And by comparing with the dual serial cost-sharing rule, agents with low demands not only pay cost increments associated with high demands but also to low demands. So agents with high demands are not the only ones to pay cost increments associated with low demands.

Think for example of a convex cost function. The self-dual serial cost-sharing rule makes agents with low demands pay both low increments (as the serial cost-sharing rule does) and high increments (as the dual serial cost-sharing rule does). But notice that there is also another way to make agents with lower demands pay not so few (as it could happen with the serial cost-sharing rule) and hence agents with higher demands pay less. Or make agents with lower demand pay not so much (as it could happen with the dual serial cost-sharing rule) and therefore agents with higher demands pay more. It can be done by taking for agents with low demands middle increments instead of mixing low and high increments.

To define the new rule we equalize first the demands left to be allocated until each agent is given half of his demand and then units of good are given equally until all the demands are met. So, first the agent with the highest demand, agent n, receives $\frac{q_n - q_{n-1}}{2}$ units, that is, $\frac{Q}{2} - \frac{q^{n-1}}{2}$ units, and pays $C\left(\frac{Q}{2} - \frac{q^{n-1}}{2}\right)$. Since n and $n-1$ are short of the same quantity of demand with respect to their half demands ($q_{n-1}/2$ each of them) then agent $n-1$ enters the picture. Both n and $n-1$ are served simultaneously and pay equally until agents n, $n-1$ and $n-2$ are left to be allocated the same quantity of good with respect to their half demands. That happens when each of n and $n-1$ is given $\frac{q_{n-1} - q_{n-2}}{2}$ units of good. They share equally the incurred cost, that is, each of n and $n-1$ pays

$$\frac{C\left(\frac{Q}{2} - \frac{q^{n-2}}{2}\right) - C\left(\frac{Q}{2} - \frac{q^{n-1}}{2}\right)}{2}.$$

Afterwards, agent $n-2$ enters the system. Agents n, $n-1$ and $n-2$ are served simultaneously and share equally the incurred cost until they are short of the same demand with respect their half demands and so on. Agents incorporate one by one until $n, n-1, ..., 1$ are given half of their demands and the corresponding cost is paid. Now all the agents are served simultaneously and pay the incurred cost until the entire demands are met. Hence, first each agent is given $q_1/2$ and pays

$$\frac{C\left(\frac{Q}{2} + \frac{q^1}{2}\right) - C\left(\frac{Q}{2}\right)}{n}.$$

At this point agent 1 has met his demand and leaves the production process. And agents $2, ..., n$ are served simultaneously and share equally the incurred cost until agent 2 has met his demand. It happens when each of them is given $\frac{q_2 - q_1}{2}$. So each of $2, ..., n$ pays

$$\frac{C\left(\frac{Q}{2} + \frac{q^2}{2}\right) - C\left(\frac{Q}{2} + \frac{q^1}{2}\right)}{n-1}$$

and agent 2 leaves the system. The process continues in the same way until all the agents meet their demands.

We call the resulting rule the *reverse self-dual serial cost-sharing rule.* If we denote it by φ^R, then

$$\varphi_i^R (N, q, C) = \sum_{j=1}^{i} \frac{\widehat{RC}_{j-1}^{q/2} - \widehat{RC}_j^{q/2}}{n-j+1} + \sum_{j=1}^{i} \frac{RC_j^{q/2} - RC_{j-1}^{q/2}}{n-j+1} \tag{8}$$

for all $i \in \{1, ..., n\}$, where

$$RC_j^{q/2} = C\left(\frac{Q}{2} + \frac{q^j}{2}\right) \quad (9)$$

and

$$\widehat{RC}_j^{q/2} = C\left(\frac{Q}{2} - \frac{q^j}{2}\right) \quad (10)$$

for all $j \in \{0, ..., i\}$.

In the following proposition we give an equality for the allocations given by the reverse self-dual serial cost-sharing rule that involves the cost function of the cost sharing problem and its dual cost function. More precisely, it is shown that the cost share determined by the reverse self-dual serial cost-sharing rule for an agent in a cost sharing problem coincides with the sum of the cost share determined by the dual serial cost-sharing rule with the same cost function and half of the demands and the cost share determined by the dual serial cost-sharing rule with the dual cost function (associated with q) and half of the demands.

Proposition 1.

$$\varphi_i^R(N, q, C) = \varphi_i^*(N, q/2, C) + \varphi_i^*(N, q/2, D^qC),$$

where $D^qC(t) = C(Q) - C(Q - t)$.

Proof. By expressions (3), (4) and (10), it is clear that $\varphi_i^*(N, q/2, C)$ coincides with the first sum in expression (8). So we only have to prove that $\varphi_i^*(N, q/2, D^qC)$ coincides with the second sum in (8), that is,

$$\sum_{j=1}^{i} \frac{\left(\widehat{D^qC}\right)_{j-1}^{q/2} - \left(\widehat{D^qC}\right)_j^{q/2}}{n-j+1} = \sum_{j=1}^{i} \frac{RC_j^{q/2} - RC_{j-1}^{q/2}}{n-j+1},$$

where $\left(\widehat{D^qC}\right)_{j-1}^{q/2}$ and $\left(\widehat{D^qC}\right)_j^{q/2}$ are given by (4), and $RC_j^{q/2}$ and $RC_{j-1}^{q/2}$ by (9).

So it is sufficient to prove that

$$\left(\widehat{D^qC}\right)_{j-1}^{q/2} - \left(\widehat{D^qC}\right)_j^{q/2} = RC_j^{q/2} - RC_{j-1}^{q/2}$$

for each $j \in \{1, ..., i\}$.

We have

$$\left(\widehat{D^qC}\right)_{j-1}^{q/2} - \left(\widehat{D^qC}\right)_{j}^{q/2} = (D^qC)\left(\frac{Q}{2} - \frac{q^{j-1}}{2}\right) - (D^qC)\left(\frac{Q}{2} - \frac{q^j}{2}\right)$$
$$= C\left(\frac{Q}{2} + \frac{q^j}{2}\right) - C\left(\frac{Q}{2} + \frac{q^{j-1}}{2}\right) = RC_j^{q/2} - RC_{j-1}^{q/2},$$

where in the first equality we have taken into account (4), in the second equality the definition of D^qC and in the third one equality (9). And the proof is complete. ■

Observe that if we substitute in the equality of Proposition 1 the dual serial cost-sharing rule by the serial cost-sharing rule, then we obtain Proposition 1 by Albizuri (2010).

4. THE REVERSE SELF-DUAL SERIAL COST-SHARING RULE AS A SERIAL RULE

In this section we show that the cost shares determined by the reverse self-dual cost-sharing rule in a cost sharing problem coincide with the cost shares determined by the serial-cost sharing rule in an associated cost sharing problem. We also prove that the dual serial cost-sharing rule of that associated problem coincides with the self-dual cost-sharing rule of the original one.

Given a cost sharing problem (N, q, C), the cost sharing problem we associate with is $\left(N, q, \widetilde{C}^Q\right)$, where

$$\widetilde{C}^Q(t) = C\left(\frac{Q}{2} + \frac{t}{2}\right) - C\left(\frac{Q}{2} - \frac{t}{2}\right). \quad (11)$$

As we can see in Fig. 5, $\widetilde{C}^Q(t)$ measures the cost of t units which are just in the middle of production process, that is, when the quantity to be produced is the same as the quantity produced.

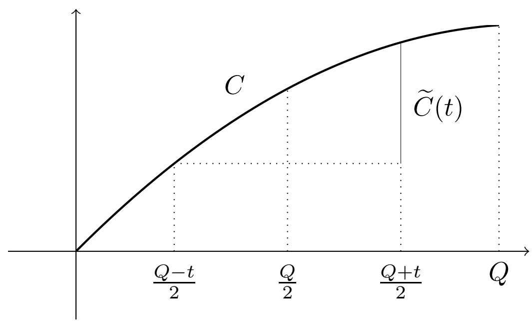

FIGURE 5

Proposition 2.

$$\varphi_i^R(N,q,C)=\varphi_i\left(N,q,\widetilde{C}^Q\right),$$

where $\widetilde{C}^Q$ *is defined in* (11).

Proof. By expressions (1) and (8), we have to prove that

$$\sum_{j=1}^{i}\frac{\widehat{RC}_{j-1}^{q/2}-\widehat{RC}_{j}^{q/2}+RC_{j}^{q/2}-RC_{j-1}^{q/2}}{n-j+1}=\sum_{j=1}^{i}\frac{\left(\widetilde{C}^Q\right)_j^q-\left(\widetilde{C}^Q\right)_{j-1}^q}{n-j+1},$$

where $\widehat{RC}_{j-1}^{q/2}$ and $\widehat{RC}_{j}^{q/2}$ are given by expression (10), $RC_{j}^{q/2}$ and $RC_{j-1}^{q/2}$ are given by (9), and $\left(\widetilde{C}^Q\right)_j^q$ and $\left(\widetilde{C}^Q\right)_{j-1}^q$ by (2).

So it suffices to prove that

$$\widehat{RC}_{j-1}^{q/2}-\widehat{RC}_{j}^{q/2}+RC_{j}^{q/2}-RC_{j-1}^{q/2}=\left(\widetilde{C}^Q\right)_j^q-\left(\widetilde{C}^Q\right)_{j-1}^q$$

for all $j\in\{1,...,i\}$.

We have

$$\begin{aligned}&\widehat{RC}_{j-1}^{q/2}-\widehat{RC}_{j}^{q/2}+RC_{j}^{q/2}-RC_{j-1}^{q/2}\\&=C\left(\frac{Q}{2}-\frac{q^{j-1}}{2}\right)-C\left(\frac{Q}{2}-\frac{q^{j}}{2}\right)+C\left(\frac{Q}{2}+\frac{q^{j}}{2}\right)-C\left(\frac{Q}{2}+\frac{q^{j-1}}{2}\right)\\&=\widetilde{C}^Q\left(q^j\right)-\widetilde{C}^Q\left(q^{j-1}\right)=\left(\widetilde{C}^Q\right)_j^q-\left(\widetilde{C}^Q\right)_{j-1}^q,\end{aligned}$$

as was to be proved. We have employed (9) and (10) in the first equality, the definition of $\widetilde{C}^Q$ in the second equality and (2) in the third one. ■

And as written above, if we take the dual serial cost-sharing rule instead of the serial cost sharing-rule, then the self-dual serial cost-sharing rule is obtained.

Proposition 3.

$$\varphi_i^S(N,q,C)=\varphi_i^*\left(N,q,\widetilde{C}^Q\right),$$

where $\widetilde{C}^Q$ *is defined in* (11).

Proof. Taking into account expressions (3) and (5), we have to show

$$\sum_{j=1}^{i} \frac{C_j^{q/2} - C_{j-1}^{q/2} + \widehat{C}_{j-1}^{q/2} - \widehat{C}_j^{q/2}}{n-j+1} = \sum_{j=1}^{i} \frac{\widehat{\left(\widetilde{C}^Q\right)}_{j-1}^{q} - \widehat{\left(\widetilde{C}^Q\right)}_{j}^{q}}{n-j+1},$$

where $C_j^{q/2}$ and $C_{j-1}^{q/2}$ are given by (6), $\widehat{C}_{j-1}^{q/2}$ and $\widehat{C}_j^{q/2}$ are given by (7), and $\widehat{\left(\widetilde{C}^Q\right)}_{j-1}^{q}$ and $\widehat{\left(\widetilde{C}^Q\right)}_{j}^{q}$ are given by (4).

Therefore, it is sufficient to prove that

$$C_j^{q/2} - C_{j-1}^{q/2} + \widehat{C}_{j-1}^{q/2} - \widehat{C}_j^{q/2} = \widehat{\left(\widetilde{C}^Q\right)}_{j-1}^{q} - \widehat{\left(\widetilde{C}^Q\right)}_{j}^{q}$$

for all $j \in \{1, ..., i\}$.

We have

$$C_j^{q/2} - C_{j-1}^{q/2} + \widehat{C}_{j-1}^{q/2} - \widehat{C}_j^{q/2} = C\left(\frac{q^j}{2}\right) - C\left(\frac{q^{j-1}}{2}\right) + C\left(Q - \frac{q^{j-1}}{2}\right) - C\left(Q - \frac{q^j}{2}\right)$$

$$= \widetilde{C}^Q\left(Q - q^{j-1}\right) - \widetilde{C}^Q\left(Q - q^j\right) = \widehat{\left(\widetilde{C}^Q\right)}_{j-1}^{q} - \widehat{\left(\widetilde{C}^Q\right)}_{j}^{q},$$

where in the first equality we have taken into account (6) and (7), in the second equality the definition of $\widetilde{C}^Q$ and in the third one equality (4). And the proof is complete. ■

5. A CHARACTERIZATION FOR THE REVERSE SELF-DUAL SERIAL COST-SHARING RULE

In this section we provide a characterization for the new rule which is related to the one given by Moulin and Shenker (1994). As written in the Introduction, they characterize the serial cost-sharing rule by means of five properties: continuity, additivity, ranking, separable costs and free lunch. The first axiom requires continuity for the topology of pointwise convergence and the other ones are formalized as follows.

Let σ be a cost sharing rule.

Additivity:

$$\sigma_i\left(N, q, C_1 + C_2\right) = \sigma_i\left(N, q, C_1\right) + \sigma_i\left(N, q, C_2\right) \text{ for all } (N, q, C_1), (N, q, C_2) \in \Gamma_U.$$

Ranking requires the order of cost shares to coincide with the order of demands.

Ranking: If $q_i \leq q_j$, then

$$\sigma_i (N, q, C) \leq \sigma_j (N, q, C)$$

for all q and all $i, j \in N$.

According to the following axiom, if costs are separable then they are so allocated.

Separable costs: If there exists $\lambda \geq 0$ such that $C(t) = \lambda t$ for all $t \geq 0$ then

$$\sigma_i (N, q, C) = \lambda q_i$$

for all $i \in N$.

And the last axiom is a kind of consistency axiom. To write it we give the following notation. Given a cost function C and $a, \delta \in \mathbb{R}_+$, we define the cost function $C^{a,\delta}$ by

$$C^{a,\delta}(t) = \begin{cases} C(t) & \text{if } t \leq a, \\ C(t+\delta) - C(a+\delta) + C(a) & \text{otherwise.} \end{cases}$$

As we can see in Fig. 6, $C^{a,\delta}$ is the cost function which results from C when δ units from a to $a+\delta$ have been removed together with their associated cost, that is, $C(a+\delta) - C(a)$.

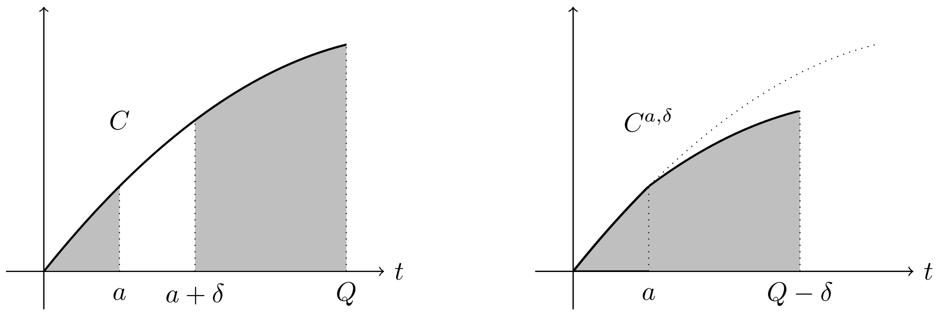

FIGURE 6

Free lunch: If $C(nq_i) = 0$ for some $i \in N$, then $\sigma_i (N, q, C) = 0$ and

$$\sigma_j (N, q, C) = \sigma_j \left(N \backslash \{i\}, q_{N\backslash\{i\}}, C^{0,q_i}\right)$$

for all $j \in N \backslash \{i\}$.

According to free lunch, if nq_i units of the good have no cost, then any agent whose demand is q_i pays nothing and if that agent leaves the system the cost shares for the remaining agents do not change. Notice that when q_i goes out then the new cost function is C^{0,q_i} since agent i is given q_i and pays $C(q_i)$, that is, zero.

To characterize the reverse self-dual serial cost-sharing rule we also consider continuity, additivity, ranking and separable costs. But we do not require free lunch. First notice that in free lunch the authors require the cost of the first nq_i units to be zero and agent i to be given the first q_i units. But this might not be the case. Agent i might be given q_i units which are not at the beginning. We propose an axiom in which we also suppose that the cost of nq_i units is zero, but not the cost of the first nq_i units, but the one of those in the middle of production. Graphically see Fig. 7.

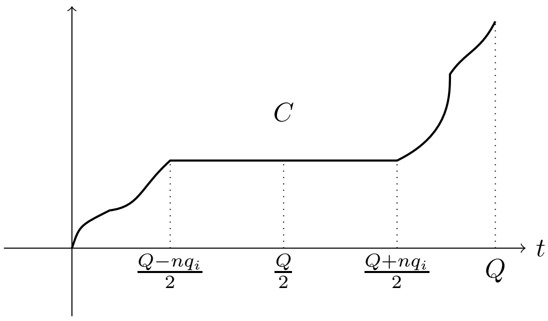

FIGURE 7

When agent i leaves the system he is given q_i units which are in the middle of production and does not pay anything for them. And the allocations for the remaining agents do not change.

We formalize the property as follows.

Free middle: If
$$C\left(\frac{Q}{2}+\frac{nq_i}{2}\right)-C\left(\frac{Q}{2}-\frac{nq_i}{2}\right)=0$$
for some $i \in N$, then $\sigma_i(N,q,C)=0$ and
$$\sigma_j(N,q,C)=\sigma_j\left(N\setminus\{i\},q_{N\setminus\{i\}},C^{\frac{Q-q_i}{2},q_i}\right)$$
for all $j \in N\setminus\{i\}$.

Notice that in this case $C^{\frac{Q-q_i}{2},q_i}(t)=\begin{cases} C(t) & \text{if } t \le Q/2-q_i/2, \\ C(t+q_i) & \text{otherwise.}\end{cases}$

Observe that in free middle agent i has to satisfy $Q-nq_i \geq 0$, so in general it cannot be applied to all agents. Obviously it can be applied to the agent with the lowest demand. We could have supposed $C\left(\frac{Q}{2}+\frac{q^i}{2}\right)-C\left(\frac{Q}{2}-\frac{q^i}{2}\right)=0$ in free middle, that is, taken q^i instead of nq_i, and all the results in this paper would be valid. But we have preferred to be closer to the writing of free lunch by Moulin and Shenker (1994). We point out that it can be easily checked that in their characterization it is sufficient to satisfy free lunch for the agent with the lowest demand. We employ that fact to prove the following characterization for the reverse self-dual serial cost-sharing rule.

We show below that the reverse self-dual serial cost-sharing rule satisfies free middle. In fact this rule is the unique one which satisfies continuity, additivity, ranking, separable costs and free middle. Notice that we do not require self duality, but it is implied by the other axioms.

First we show free middle is satisfied and a previous lemma, and then the characterization theorem.

Lemma 4. *The reverse self-dual serial cost-sharing rule satisfies free middle.*

Proof. If

$$C\left(\frac{Q}{2}+\frac{nq_i}{2}\right)-C\left(\frac{Q}{2}-\frac{nq_i}{2}\right)=0$$

for some $i \in N$, then $\widetilde{C}^Q(nq_i)=0$. Since φ satisfies free lunch we get $\varphi_i\left(N,q,\widetilde{C}^Q\right)=0$ and

$$\varphi_j\left(N,q,\widetilde{C}^Q\right)=\varphi_j\left(N\backslash\{i\},q_{N\backslash\{i\}},\left(\widetilde{C}^Q\right)^{0,q_i}\right)$$

for all $j \in N\backslash\{i\}$. Applying Proposition 2 we have $\varphi_i^R(N,q,C)=0$ and

$$\varphi_j^R(N,q,C)=\varphi_j\left(N\backslash\{i\},q_{N\backslash\{i\}},\left(\widetilde{C}^Q\right)^{0,q_i}\right)$$

for all $j \in N\backslash\{i\}$. And taking into account

$$\left(\widetilde{C}^Q\right)^{0,q_i}=\left(\widetilde{C^{\frac{Q-q_i}{2},q_i}}\right)^{Q-q_i},$$

we write the equality above as follows

$$\varphi_j^R(N,q,C)=\varphi_j\left(N\backslash\{i\},q_{N\backslash\{i\}},\left(\widetilde{C^{\frac{Q-q_i}{2},q_i}}\right)^{Q-q_i}\right).$$

And Proposition 2 implies

$$\varphi_j^R(N,q,C)=\varphi_j^R\left(N\backslash\{i\},q_{N\backslash\{i\}},C^{\frac{Q-q_i}{2},q_i}\right),$$

as was to be proved. ■

In the following lemma we employ cost-sharing problems $(N, q, \lambda\Lambda_a), (N, q, \lambda\Lambda'_a)$, where $\lambda \in \mathbb{R}_+$, $a \in [0, Q]$, $\Lambda_a(t) = (t - a)_+$ and $\Lambda'_a(t) = \min\{t, a\}$.

Lemma 5. *If a cost-sharing rule σ satisfies additivity, ranking, separable costs and free middle, then*

$$\sigma\left(N, q, \lambda\Lambda_{Q/2}\right) = \frac{\lambda q}{2} = \sigma\left(N, q, \lambda\Lambda'_{Q/2}\right).$$

Proof. Let $q \in \mathbb{R}^N_+$ and suppose without loss of generality that $N = \{1, ..., n\}$ and $q_1 \leq q_2 \leq \cdots \leq q_n$. We prove this lemma by induction on $i = 1, ..., n$. Define

$$\Delta_{\frac{nq_1}{2}} = \Lambda_{\frac{Q}{2}} - \Lambda_{\frac{Q+nq_1}{2}}.$$

By efficiency we have

$$\sum_{i=1}^{n} \sigma_i\left(N, q, \lambda\Delta_{\frac{nq_1}{2}}\right) = \frac{n\lambda q_1}{2},$$

and ranking implies $\sigma_1\left(N, q, \lambda\Delta_{\frac{nq_1}{2}}\right) \leq \frac{\lambda q_1}{2}$. Taking into account free middle we get

$$\sigma_1\left(N, q, \lambda\Delta_{\frac{nq_1}{2}}\right) = \sigma_1\left(N, q, \lambda\Lambda_{\frac{Q}{2}}\right),$$

and therefore,

$$\sigma_1\left(N, q, \lambda\Lambda_{\frac{Q}{2}}\right) \leq \frac{\lambda q_1}{2}. \tag{12}$$

If we define

$$\Delta'_{\frac{nq_1}{2}} = \Lambda'_{\frac{Q}{2}} - \Lambda'_{\frac{Q-nq_1}{2}},$$

reasoning as above we get

$$\sigma_1\left(N, q, \lambda\Lambda'_{\frac{Q}{2}}\right) \leq \frac{\lambda q_1}{2}. \tag{13}$$

Since σ satisfies additivity and separable costs it follows that

$$\sigma_1\left(N, q, \lambda\Lambda_{\frac{Q}{2}}\right) + \sigma_1\left(N, q, \lambda\Lambda'_{\frac{Q}{2}}\right) = \sigma_1\left(N, q, \lambda\Lambda_0\right) = \lambda q_1.$$

And taking into account expressions (12) and (13) we can deduce

$$\sigma_1\left(N, q, \lambda\Lambda_{\frac{Q}{2}}\right) = \frac{\lambda q_1}{2} = \sigma_1\left(N, q, \lambda\Lambda'_{\frac{Q}{2}}\right).$$

Now suppose that if $i < j$ then

$$\sigma_i \left(N, q, \lambda \Lambda_{\frac{Q}{2}} \right) = \frac{\lambda q_i}{2} = \sigma_i \left(N, q, \lambda \Lambda'_{\frac{Q}{2}} \right),$$

and let us prove the two equalities for agent j.

Define

$$\Delta_{\frac{q^j}{2}} = \Lambda_{\frac{Q}{2}} - \Lambda_{\frac{Q+q^j}{2}}.$$

Since σ satisfies efficiency, then

$$\sum_{i=1}^{n} \sigma_i \left(N, q, \lambda \Delta_{\frac{q^j}{2}} \right) = \frac{\lambda q^j}{2}. \tag{14}$$

Let us prove that

$$\sigma_i \left(N, q, \lambda \Lambda_{\frac{Q+q^j}{2}} \right) = 0 \tag{15}$$

if $i \in \{1, ..., j\}$. Free middle implies $\sigma_1 \left(N, q, \lambda \Lambda_{\frac{Q+q^j}{2}} \right) = 0$ and

$$\sigma_i \left(N, q, \lambda \Lambda_{\frac{Q+q^j}{2}} \right) = \sigma_i \left(N \backslash \{1\}, q_{N \backslash \{1\}}, \lambda \Lambda_{\frac{Q-q_1}{2} + \frac{q^j - q_1}{2}} \right)$$

if $i \in \{2, ..., j\}$. Applying again free middle, on the one hand we obtain

$$\sigma_2 \left(N \backslash \{1\}, q_{N \backslash \{1\}}, \lambda \Lambda_{\frac{Q-q_1}{2} + \frac{q^j - q_1}{2}} \right) = 0,$$

and thus $\sigma_2 \left(N, q, \lambda \Lambda_{\frac{Q+q^j}{2}} \right) = 0$, and on the other hand

$$\sigma_i \left(N \backslash \{1\}, q_{N \backslash \{1\}}, \lambda \Lambda_{\frac{Q-q_1}{2} + \frac{q^j - q_1}{2}} \right) = \sigma_i \left(N \backslash \{1, 2\}, q_{N \backslash \{1,2\}}, \lambda \Lambda_{\frac{Q-q_1-q_2}{2} + \frac{q^j - q_1 - q_2}{2}} \right)$$

if $i \in \{3, ..., j\}$. Applying free middle repeatedly, if $i \leq j$ it follows that

$$\sigma_i \left(N, q, \lambda \Lambda_{\frac{Q+q^j}{2}} \right) =$$

$$= \cdots = \sigma_i \left(N \backslash \{1, 2, ..., i-1\}, q_{N \backslash \{1,2,...,i-1\}}, \lambda \Lambda_{\frac{Q-q_1-\cdots-q_{i-1}}{2} + \frac{q^j - q_1 - \cdots - q_{i-1}}{2}} \right) = 0,$$

and equality (15) is proved.

From additivity, equality (14) and equality (15) for $i \in \{1, ..., j-1\}$ we have

$$\sum_{i=1}^{j-1} \sigma_i \left(N, q, \lambda \Lambda_{\frac{Q}{2}}\right) + \sum_{i=j}^{n} \sigma_i \left(N, q, \lambda \Delta_{\frac{q^j}{2}}\right) = \frac{\lambda q^j}{2}.$$

And by induction hypothesis

$$\sum_{i=j}^{n} \sigma_i \left(N, q, \lambda \Delta_{\frac{q^j}{2}}\right) = \frac{(n-(j-1))\, \lambda q_j}{2}.$$

Taking into account ranking we can deduce

$$\sigma_j \left(N, q, \lambda \Delta_{\frac{q^j}{2}}\right) \leq \frac{\lambda q_j}{2}.$$

And applying additivity and equality (15) for $i = j$ it follows that

$$\sigma_j \left(N, q, \lambda \Lambda_{\frac{Q}{2}}\right) \leq \frac{\lambda q_j}{2}. \tag{16}$$

If we take $\Delta'_{\frac{q^j}{2}} = \Lambda'_{\frac{Q}{2}} - \Lambda'_{\frac{Q-q^j}{2}}$, reasoning in a similar way we get

$$\sigma_j \left(N, q, \lambda \Lambda'_{\frac{Q}{2}}\right) \leq \frac{\lambda q_j}{2}.$$

And additivity, separable costs and (16) imply

$$\sigma_j \left(N, q, \lambda \Lambda_{\frac{Q}{2}}\right) = \frac{\lambda q_j}{2} = \sigma_j \left(N, q, \lambda \Lambda'_{\frac{Q}{2}}\right),$$

as was to be proved. ■

Theorem 6. *The reverse self-dual serial cost-sharing rule is the unique cost-sharing rule that satisfies continuity, additivity, ranking, separable costs and free middle.*

Proof. By Lemma 4 the reverse self-dual serial cost-sharing rule satisfies free middle. And it is straightforward to prove the other axioms.

For uniqueness we consider an allocation rule σ satisfying the five properties above. Given (N, q, C), we define

$$C_1(t) = \begin{cases} C(t) & \text{if } t \leq Q/2, \\ C(Q/2) & \text{otherwise} \end{cases} \tag{17}$$

and

$$C_2(t) = \begin{cases} 0 & \text{if } t \leq Q/2, \\ C(t) - C(Q/2) & \text{otherwise.} \end{cases} \tag{18}$$

It is clear that (N, q, C_1) and (N, q, C_2) are cost sharing problems, and moreover $C = C_1 + C_2$. By additivity it suffices to show uniqueness for (N, q, C_1) and (N, q, C_2). Notice that C_2 vanishes identically from 0 to $Q/2$ and that C_1 is constant from $Q/2$ to Q. Therefore, we have to prove uniqueness on that kind of cost functions.

To prove uniqueness on cost functions which vanish identically from 0 to $Q/2$ let us define cost sharing rule σ' on Γ_U by

$$\sigma'(N, q, C) = \sigma\left(N, q, C^Q\right),$$

where

$$C^Q(t) = \begin{cases} 0 & \text{if } t \leq Q/2, \\ C\left(2\left(t - \frac{Q}{2}\right)\right) & \text{otherwise.} \end{cases}$$

Since σ satisfies continuity, additivity, and ranking, then σ' also satisfies these axioms. Let us show σ' satisfies separable costs and free lunch.

To show σ' satisfies separable costs we have to prove $\sigma'(N, q, C) = \lambda q$ when $C(t) = \lambda t$ for some $\lambda \geq 0$. Notice that

$$C^Q(t) = 2\lambda\left(t - \frac{Q}{2}\right)_+.$$

Applying Lemma 5 we get $\sigma\left(N, q, C^Q\right) = \lambda q$, and hence $\sigma'(N, q, C) = \lambda q$.

Finally, let us prove that σ' satisfies satisfies free lunch for agent i such that $Q - nq_i \geq 0$. Let (N, q, C) be a cost sharing problem which satisfies $C(nq_i) = 0$ for some $i \in N$ such that $Q - nq_i \geq 0$. Then by definition of C^Q it follows that

$$C^Q\left(\frac{Q}{2} + \frac{nq_i}{2}\right) - C^Q\left(\frac{Q}{2} - \frac{nq_i}{2}\right) = 0.$$

And $\sigma_i\left(N, q, C^Q\right) = 0$ for σ satisfies free middle. That is, $\sigma'_i(N, q, C) = 0$. Moreover,

$$\sigma_j\left(N, q, C^Q\right) = \sigma_j\left(N \backslash \{i\}, q_{N\backslash\{i\}}, \left(C^Q\right)^{\frac{Q-q_i}{2}, q_i}\right)$$

for all $j \in N \backslash \{i\}$. If we take into account that

$$\left(C^Q\right)^{\frac{Q-q_i}{2}, q_i} = \left(C^{0,q_i}\right)^Q,$$

the equality above turns into

$$\sigma_j\left(N, q, C^Q\right) = \sigma_j\left(N \backslash \{i\}, q_{N\backslash\{i\}}, \left(C^{0,q_i}\right)^Q\right).$$

Therefore, $\sigma'_j(N,q,C) = \sigma'_j\left(N\backslash\{i\}, q_{N\backslash\{i\}}, C^{0,q_i}\right)$ and free lunch is satisfied restricted to agents i such that $Q - nq_i \geq 0$.

Since σ' satisfies continuity, additivity, ranking, separable costs and the restricted free lunch, then it is determined for all cost sharing problems (it coincides with the serial cost-sharing rule). And the definition of σ' implies σ is determined for all cost sharing problems in which cost function vanishes from 0 to $Q/2$, for all of them coincides with some C^Q.

To prove uniqueness on cost functions which are constant from $Q/2$ to Q we define cost-sharing rule σ^* on Γ_U by

$$\sigma^*(N,q,C) = \sigma\left(N,q,D^qC^Q\right).$$

Observe that if $C(t) = \lambda t$ for some $\lambda \geq 0$ then $D^qC^Q(t) = 2\lambda\min\{t,Q/2\}$. It can be proved in a similar way as above that σ^* satisfies continuity, additivity, ranking, separable costs and the restricted free lunch. Hence, σ is determined for all cost sharing problems in which cost function is constant from $Q/2$ to Q, and the proof is complete. ■

Remark 1. *Moulin and Shenker (1994) also give a characterization for the serial cost sharing-rule on a smaller domain than Γ_U: the domain of differences of convex cost functions. For that characterization continuity is not needed. Continuity allows to characterize the serial cost-sharing rule for all cost sharing-rules. In our case we cannot restrict the domain to differences of convex cost functions (or to concave cost functions) since in the proof not all cost functions are convex (or concave).*

6. OTHER CHARACTERIZATIONS FOR THE REVERSE SELF-DUAL SERIAL COST-SHARING RULE

We propose two characterizations for the reverse self-dual serial cost-sharing rule which are related to the ones provided by Albizuri (2010) for the self-dual serial cost-sharing rule.

First we characterize the reverse self-dual cost-sharing rule by means of additivity and the following two axioms. Let σ be a cost sharing rule and (N,q,C) be a cost sharing problem.

Self duality:

$$\sigma_i(N,q,C) = \sigma_i(N,q,D^qC),$$

where $D^qC(t) = C(Q) - C(Q-t)$.

Partially dual serial: If $C(t) \leq C(Q/2)$ when $0 \leq t \leq Q$ then

$$\sigma_i(N, q, C) = \varphi_i^*(N, q/2, C).$$

Self duality requires the cost shares in a problem to be the same in the problem and in its dual one. And the third axiom requires that if the cost is always less than the cost of half of the demands then the cost shares to be equal to the cost shares determined by the dual serial cost-sharing rule when half of the demands are required.

Theorem 7. *The reverse self-dual serial cost-sharing rule is the unique cost-sharing rule that satisfies additivity, self duality and partially dual serial.*

Proof. Let us prove φ^R satisfies the foregoing axioms.

Expression (8) implies φ^R satisfies additivity.

By Proposition 1, we get

$$\varphi_i^R(N, q, D^qC) = \varphi_i^*(N, q/2, D^qC) + \varphi_i^*(N, q/2, D^q(D^qC)).$$

And since $D^q(D^qC) = C$, employing again Proposition 1 self duality is obtained.

Finally, if $C(t) \leq C(Q/2)$ when $0 \leq t \leq Q$, then $D^qC(q) = 0$ when $0 \leq t \leq Q/2$. Hence, $\varphi_i^*(N, q/2, D^qC) = 0$ and Proposition 1 implies $\varphi_i^R(N, q, C) = \varphi_i^*(N, q/2, C)$.

Now we prove uniqueness. Let σ be an allocation rule satisfying the three properties. Given (N, q, C) let C_1 and C_2 be defined respectively by expressions (17) and (18). We get $C = C_1 + C_2$. Hence, by additivity it only suffices to prove uniqueness for (N, q, C_1) and (N, q, C_2).

On the one hand, partially dual serial implies

$$\sigma(N, q, C_1) = \varphi^*(N, q/2, C_1).$$

On the other hand, from self duality it follows that

$$\sigma(N, q, C_2) = \sigma(N, q, D^qC_2).$$

Since $D^qC_2(t) = C_2(Q)$ when $\frac{Q}{2} \leq t \leq Q$, partially dual serial implies

$$\sigma(N, q, C_2) = \varphi^*(N, q/2, D^qC_2)$$

and the proof is complete. ■

As written in Preliminaries, Albizuri (2010) characterizes the self-dual cost-sharing rule by means of additivity, self duality and partially serial. So if we compare our axiomatic characterization with that one, partially serial is considered instead of partially dual serial.

In the second characterization we use the following three axioms. Let σ be a cost sharing rule and (N, q, C) be a cost sharing problem.

Anonymity: Let $\pi : N \to N$ be a one-to-one mapping and $\pi q \in \mathbb{R}_+^N$ such that $(\pi q)_i = q_{\pi^{-1}(i)}$ for all $i \in N$. Then

$$\sigma_i (N, \pi q, C) = \sigma_{\pi^{-1}(i)} (N, q, C)$$

for all $i \in N$.

Independence of null demands: If $q_i = 0$ for some $i \in N$, then

$$\sigma_j (N, q, C) = \sigma_j \left(N \backslash \{i\}, q_{N \backslash \{i\}}, C\right)$$

for all $j \in N \backslash \{i\}$.

Both axioms are well known. By anonymity the identity of agents is irrelevant, the individuals are not distinguished by anything other than their demands. Under independence of null demands the payoffs of the agents do not depend on the demands of the agents who demand nothing. These two axioms have been employed in the characterizations of the self-dual serial cost-sharing rule by Albizuri (2010).

And the last axiom is the following one.

Middle equal changes in payoff: Let $T \in \mathbb{R}_+$ be such that $T \leq \min \{q_i\}_{i \in N}$. Then

$$\sigma_i (N, q, C) - \sigma_i \left(N, q - T \cdot \mathbf{1}_N, C^{\frac{Q}{2} - \frac{nT}{2}, nT}\right)$$

$$= \sigma_j (N, q, C) - \sigma_j \left(N, q - T \cdot \mathbf{1}_N, C^{\frac{Q}{2} - \frac{nT}{2}, nT}\right)$$

for all $i, j \in N$.

According to this axiom, if every agent is given T units and they pay

$$C \left(\frac{Q}{2} + \frac{nT}{2}\right) - C \left(\frac{Q}{2} - \frac{nT}{2}\right),$$

that is, they pay the cost increment from $\frac{Q}{2} - \frac{nT}{2}$ to $\frac{Q}{2} + \frac{nT}{2}$ (the cost of nT units in the middle of production process), and we calculate what they still have to pay, then the allocations change in the same quantity for every agent.

Albizuri (2010) characterizes the serial cost-sharing rule by means of anonymity, independence of higher demands and translated equal changes in payoff. We obtain translated equal changes in payoff if we substitute in middle equal changes in payoff

$C^{\frac{Q}{2}-\frac{nT}{2},nT}$ by $C^{0,nT}$. That is, if we remove the amount nT from the beginning of production and not from the middle. To prove Proposition 8, we employ the fact that the serial cost-sharing rule satisfies translated equal changes in payoff . On the other hand, Albizuri (2010) characterizes the self-dual serial cost-sharing rule by a translation axiom in which the new cost function is obtained when half of the amount nT has been removed at the beginning of production process and the other half of nT at the end.

We provide the axiomatic characterization in two steps.

Proposition 8. *The reverse self-dual serial cost-sharing rule satisfies anonymity, independence of null demands and middle equal changes in payoff.*

Proof. It is clear that the reverse self-dual serial cost-sharing rule satisfies anonymity and independence of null demands. To prove that it also satisfies middle equal changes in payoff let $T \in \mathbb{R}_+$ be such that $T \leq \min\{q_i\}_{i \in N}$. If we denote $C_0 = C^{\frac{Q}{2}-\frac{nT}{2},nT}$ and $i, j \in N$ we have

$$\begin{aligned}
\varphi_i^R(N,q,C) - \varphi_i^R(N,q-T\cdot\mathbf{1}_N,C_0) &= \varphi_i\left(N,q,\widetilde{C}^Q\right) - \varphi_i\left(N,q-T\cdot\mathbf{1}_N,\widetilde{C_0}^{Q-nT}\right) \\
&= \varphi_i\left(N,q,\widetilde{C}^Q\right) - \varphi_i\left(N,q-T\cdot\mathbf{1}_N,\left(\widetilde{C}^Q\right)^{0,nT}\right) \\
&= \varphi_j\left(N,q,\widetilde{C}^Q\right) - \varphi_j\left(N,q-T\cdot\mathbf{1}_N,\left(\widetilde{C}^Q\right)^{0,nT}\right) \\
&= \varphi_j^R(N,q,C) - \varphi_j^R(N,q-T\cdot\mathbf{1}_N,C_0),
\end{aligned}$$

where the first equality holds by Proposition 2, the second since $\widetilde{C_0}^{Q-nT} = \left(\widetilde{C}^Q\right)^{0,nT}$ and the third because φ satisfies translated equal changes in payoff. ■

Theorem 9. *The reverse self-dual serial cost-sharing rule is the unique cost-sharing rule that satisfies anonymity, independence of null demands and middle equal changes in payoff.*

Proof. By Proposition 8 the reverse self-dual serial cost-sharing rule satisfies the foregoing axioms. We show uniqueness by induction on $|\{i \in N : q_i \neq 0\}|$.

Let σ be a cost sharing rule that satisfies the three axioms.

If $|\{i \in N : q_i \neq 0\}| = 0$, by anonymity and efficiency $\sigma_i(N,q,C) = 0$ for all $i \in N$.

Suppose that $\sigma(N,q,C)$ is determined when $|\{i \in N : q_i \neq 0\}| < m$ and suppose $|\{i \in N : q_i \neq 0\}| = m$.

Let $j = \min \{i \in N : q_i \neq 0\}$ and $S = \{1, ..., j-1\}$. Independence of null demands implies

$$\sigma_i (N, q, C) = \sigma_i \left(N \backslash S, q_{N \backslash S}, C\right)$$

for all $i \in N \backslash S$.

Thus, by efficiency and anonymity it only suffices to show $\sigma_i \left(N \backslash S, q_{N \backslash S}, C\right)$ is determined for all $i \in N \backslash S$.

Let $i \in N \backslash S$. By middle equal changes in payoff we have

$$\sigma_i \left(N \backslash S, q_{N \backslash S}, C\right) - \sigma_i \left(N \backslash S, q_{N \backslash S} - q_j \cdot \mathbf{1}_{N \backslash S}, C^{\frac{Q}{2} - \frac{(n-|S|)q_j}{2}, (n-|S|)q_j}\right)$$
$$= \sigma_j \left(N \backslash S, q_{N \backslash S}, C\right) - \sigma_j \left(N \backslash S, q_{N \backslash S} - q_j \cdot \mathbf{1}_{N \backslash S}, C^{\frac{Q}{2} - \frac{(n-|S|)q_j}{2}, (n-|S|)q_j}\right). \tag{19}$$

If we write $\alpha = \sigma_j \left(N \backslash S, q_{N \backslash S}, C\right) - \sigma_j \left(N \backslash S, q_{N \backslash S} - q_j \cdot \mathbf{1}_{N \backslash S}, C^{\frac{Q}{2} - \frac{(n-|S|)q_j}{2}, (n-|S|)q_j}\right)$, equality (19) implies

$$\sigma_i \left(N \backslash S, q_{N \backslash S}, C\right) - \sigma_i \left(N \backslash S, q_{N \backslash S} - q_j \cdot \mathbf{1}_{N \backslash S}, C^{\frac{Q}{2} - \frac{(n-|S|)q_j}{2}, (n-|S|)q_j}\right) = \alpha \tag{20}$$

for all $i \in N \backslash S$.

By adding all the equalities in (19) we get

$$\sum_{i \in N \backslash S} \sigma_i \left(N \backslash S, q_{N \backslash S}, C\right) - \sum_{i \in N \backslash S} \sigma_i \left(N \backslash S, q_{N \backslash S} - q_j \cdot \mathbf{1}_{N \backslash S}, C^{\frac{Q}{2} - \frac{(n-|S|)q_j}{2}, (n-|S|)q_j}\right)$$
$$= (n - |S|) \cdot \alpha.$$

By efficiency this equality turns into

$$\alpha = \frac{C \left(\sum_{i \in N \backslash S} q_i\right) - C^{\frac{Q}{2} - \frac{(n-|S|)q_j}{2}, (n-|S|)q_j} \left(\sum_{i \in N \backslash S} (q_i - q_j)\right)}{n - |S|}.$$

And given that $\sum_{i \in N \backslash S} (q_i - q_j) = Q - (n - |S|) q_j \geq Q/2 - (n - |S|) q_j/2$, then

$$\alpha = \frac{C \left(\sum_{i \in N \backslash S} q_i\right) - C \left(\sum_{i \in N \backslash S} q_i\right) + \widetilde{C}^Q \left((n - |S|) q_j\right)}{n - |S|} = \frac{\widetilde{C}^Q \left((n - |S|) q_j\right)}{n - |S|}.$$

Thus, in equality (20) parameter α is determined and by induction

$$\sigma_i\left(N\backslash S, q_{N\backslash S} - q_j \cdot \mathbf{1}_{N\backslash S}, C^{\frac{Q}{2}-\frac{(n-|S|)q_j}{2},(n-|S|)q_j}\right)$$

is determined. Therefore, so $\sigma_i\left(N\backslash S, q_{N\backslash S}, C\right)$ is and the proof is complete. ■

7. Examples

Under convex cost functions agents with lower demands are better with the serial cost-sharing rule than with the dual serial cost-sharing rule. And on the contrary, with concave cost functions the dual serial cost-sharing rule is better for agents with higher demands. If we employ both the reverse self-dual serial cost-sharing rule and the self-dual serial cost-sharing rule, we see that under convex cost functions agents with low demands are not as well as with the serial cost-sharing rule and that with concave cost functions agents with higher demands are not as well as with the dual serial cost-sharing rule. As an illustration we give some examples which involve all the rules. In these examples we can see that in some cases agents with lower (resp. higher) demands are better (resp. worse) with the reverse self-dual serial cost-sharing rule than with the self-dual rule and that in other cases the opposite holds. The first happens if the middle cost increments are smaller than the average of the cost increments at the beginning and at the end of the production, and the second does in the opposite case. Observe also that not always the reverse self-dual serial cost-sharing rule and the self-dual serial cost-sharing rule give cost shares in between the cost shares given by the serial and the dual serial cost-sharing rule.

Consider $N = \{1, ..., 5\}$, $q = (5, 15, 20, 50, 70)$ and cost functions $C_1(t) = \sqrt{t}$ and $C_2(t) = t^3/1000$. The following table summarizes the allocations prescribed by the serial cost-sharing rule, the dual serial cost-sharing rule, the self-dual serial cost-sharing rule and the reverse self-dual serial cost-sharing rule. Note that the first cost function is concave and the other convex.

	C_1	C_2
φ	(1, 1.77, 2.06, 3.50, 4.32)	(3.13, 67.88, 147, 1263, 2615)
φ^*	(0.21, 0.67, 0.94, 3.18, 7.65)	(327.13, 727. 88, 843, 1095, 1103)
φ^S	(0.81, 1.56, 1.88, 3.64, 4.76)	(177.78, 469.97, 594.75, 1233.75, 1619.75)
φ^R	(0.28, 0.86, 1.16, 3.39, 6.96)	(96.78, 304.97, 420.75, 1275.75, 1997.75)

And for cost functions $C_3(t) = \min\{t, 100\}$ and $C_4(t) = t^{1.5}$, respectively concave and convex,

	C_3	C_4
φ	(5, 15, 20, 30, 30)	(25, 124.76, 188.59, 659.07, 1026.43)
φ^*	(0, 1.25, 6.25, 36.25, 56.25)	(91.06, 251.71, 321.85, 634.90, 724.34)
φ^S	(2.5, 7.5, 10, 30, 50)	(55.33, 178.53, 242.81, 639.51, 907.67)
φ^R	(5, 11.875, 14.375, 29.375, 39.375)	(67.01, 199.71, 264.94, 636.91, 855.28)

Notice that for C_3 and C_4, rules φ^S and φ^R order the allocations in the opposite way as they do for C_1 and C_2.

8. References

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