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*Selectivity, pulse fishing and endogenous
lifespan in Beverton-Holt models*

Selectivity, pulse fishing and endogenous lifespan in Beverton-Holt models

ABSTRACT

Optimal management in a multi-cohort Beverton-Holt model with any number of age classes and imperfect selectivity is equivalent to finding the optimal fish lifespan by chosen fallow cycles. Optimal policy differs in two main ways from the optimal lifespan rule with perfect selectivity. First, weight gain is valued in terms of the whole population structure. Second, the cost of waiting is the interest rate adjusted for the increase in the pulse length. This point is especially relevant for assessing the role of selectivity. Imperfect selectivity reduces the optimal lifespan and the optimal pulse length. We illustrate our theoretical findings with a numerical example. Results obtained using global numerical methods select the optimal pulse length predicted by the optimal lifespan rule.

JEL classification: O1, AMS 91B76, 92D25.

Keywords: optimisation in age-structured models, pulse fishing.

1 Introduction

Clark *et al.* (1973) and Hannesson’s pioneering article (1975) show that optimal fishing mortality trajectories that maximises net present profits in Beverton-Holt multi-cohort models may lead to pulse fishing. That is, the optimal solution of the management problem is a periodic cycle of fishing followed by fallow periods to enable stocks to recover.

Although after Hannesson and Clark *et al.* many other papers have found, by using numerical methods, that pulse fishing is the optimal policy in some fisheries modelled with Beverton-Holt models (Horwood, 1987; Bjørndal and Brasão, 2006; Bjørndal *et al.* 2004a, 2004b; Stage, 2006; Da Rocha *et al.*, 2012), the first analytical results were only recently derived. Tahvonen (2009), proves that in a 2-age structured fisheries model with endogenous recruitment and harvesting costs, optimal harvesting consists of pulse fishing under specific conditions such as nonselective gear. Steinshamn (2011) extends Beverton-Holt models by incorporating density dependence growth. Skonhoft *et al.* (2012), derive the optimal harvesting when the fleet can choose different fishing gear with different fishing selectivity by paying different costs. Moreover, Tahvonen (2008) shows how fishery management based on age structured models diverges from surplus production models.¹

In this article we apply Tahvonen’s (2009) analysis to the Hannesson’s multi-cohort Beverton-Holt model with any number of age classes. We show that in this context managers faces a fishing delay problem whose solution is a policy that endogenously characterises the optimal resource lifespan as a function of the growth rate, discount factor and fishing technology. As a result optimal management in age structured models with nonselective gear is found to be closely related to the pioneering Clark and Hannesson of the optimal harvesting of a single year class fishery.

Suppose, as in Hannesson (1975), that one fish, with a lifespan of A , is “impounded in a bay in order to be raised to an optimal size and age at which it will be fished. No more fish enter the ‘pound’ until those now present have been fished.” This assumption is equivalent to considering a perfect selectivity gear that allows users to discriminate perfectly which fish are harvested and which ones are not. With this technology in each period, there are A cohorts of different ages and it is possible to harvest only those individuals of the maximum age.

However this reasoning is not valid when there is nonselective gear. In this case, harvesting the maximum number of fish of age A implies also harvesting fish younger

¹For a very useful survey of bioeconomic age-structured optimisation models, see Tahvonen (2010).

than A . So, in this context of imperfect selectivity it is not clear that harvesting only fish of maximum age is optimal.

In particular, we show that under imperfect selectivity optimal management is equivalent to selecting the optimal pulse length. We also show that to calculate the optimal fish lifespan it suffices to compare the value of harvesting under different pulses. Our findings differ in two main ways from the optimal age rule under perfect selectivity. First, with imperfect selectivity, weight gain is valued in terms of the whole population structure while with perfect selectivity it is valued in terms of the population of the previous age. Second, under imperfect selectivity the cost of waiting is given by the interest rate adjusted for the increase in the pulse length. This point is specially relevant for assessing the role of selectivity. We prove that if the value of weight gain is decreasing with age, imperfect selectivity reduces the optimal lifespan and the optimal pulse length.

We also show that there is a relationship between the non-concavity of the manager's objective function and the non-continuity of the optimal solution. In particular, by focusing on the second order conditions we find sufficient conditions that guarantee that pulse fishing is the optimal solution to the problem.

Finally, we illustrate our theoretical findings with a numerical example. We apply our theoretical algorithm to the Northern Stock of Hake (NSH) (*Merluccius merluccius*) and compare the result with that obtained using global numerical methods. The same optimal pulse length is selected in both cases.

The rest of the paper is organised as follows. We start out by solving a very simple 2-age class model in Section 2. In Section 3 we extend the results to any number of age classes to show that lifespan is endogenously determined. Section 4 presents a numerical application of our findings to the European Northern Hake Stock. We end with some concluding remarks.

2 The simplest fishery with two age classes

Consider the Hannesson's 1975 fishery based in the Beverton-Holt model with two age classes, juveniles and adults. Let N_t^1 , and N_t^2 be the populations of juveniles and adults in period t , respectively. The population dynamic is very simple. Each year, t , an exogenous number of juvenile fish are born; without loss of generality we consider that $N_t^1 = 1$. Only some of these juveniles become adults in the next period; formally

$$N_t^2 = e^{-p^1 F_{t-1} - m},$$

Table 1: Population dynamic in a 2 age-class model

	Period t-1	Period t	Period t+1
age 1	$N_{t-1}^1 = 1$	$N_t^1 = 1$	$N_{t+1}^1 = 1$
age 2		$N_t^2 = e^{-(p^1 F_{t-1} + m)}$	$N_{t+1}^2 = e^{-(p^1 F_t + m)}$

where m is the natural mortality and p^1 is the selectivity parameter that indicates how the fishing mortality rate F affects juveniles. Figure 1 illustrates the dynamic evolution of this fishery population. The yield in value for year t , is given by Baranov's equation (1918)

$$Y_t^a = y_t^a(F_t)N_t^a,$$

where,

$$y_t^a(F_t) = pr^a \omega^a \frac{p^a F_t}{m + p^a F_t} [1 - e^{-(p^a F_t + m)}],$$

is the yield in value per unit of fish and pr^a and ω^a are the price and weight of the a -age class, respectively. Therefore, the net present value of the fishery's yield is

$$\sum_{t=0}^{\infty} \beta^t [y_t^1(F_t) + y_t^2(F_t)N_t^2(F_{t-1})],$$

where $0 < \beta < 1$ is the discount factor.²

First, consider that there is perfect selectivity and p^1 , p^2 and F_t can be selected by the regulator. If the value of adults, $pr^2\omega^2$, is greater than the value of juveniles $pr^1\omega^1$, the optimal policy consists of letting all juveniles become adults and catching only adults³. Formally we set $p^2 = 1$, $p^1 = 0$ and $F_t = F_{t+1} = \infty$. Therefore, each year we harvest e^{-m} adults, and the net present value is

$$V^{PS} = \frac{1}{1 - \beta} pr^2 \omega^2 e^{-m},$$

²This is the discrete version of a model where fishing mortality is continuous throughout the season and weights are non density dependent. In such cases, the dynamics follow the McKendrick-von Foerster partial differential equation (Von Foerster, 1959; McKendrick, 1926). In general terms $\frac{\partial n(a,t)}{\partial t} = -\frac{\partial n(a,t)}{\partial a} - [m(a) + p(a)F(t)]n(a,t)$, where $n(a,t)$ is the number of fish of age a at time t . This equation shows that the rate of change of the number of fish in a given age interval, $\partial n(a,t)/\partial t$, is equal to the net rate of departure, $\partial n(a,t)/\partial a$, less the rate of deaths. $[m(a) + p(a)F(t)]n(a,t)$. For more details about the continuous version of the model see Da Rocha and Gutiérrez (2012). For the density dependence model, see Steinshamn (2011).

³Reed (1980) and Skonhøft *et al.* (2012) show that with endogenous recruitment and harvesting costs it may not be optimal to harvest only one age group.

where superscript PS stands for perfect selectivity.

Now consider that there is imperfect selectivity. That is p^1, p^2 are given by the fishing technology. This means that is not technically possible to catch adults without catching juveniles. Moreover, like Tahvonen (2009), assume that there is “growth overfishing”, that is $pr^1 = 0$. The optimal policy is the solution to the problem

$$\begin{aligned} \max_{\{F_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t y_t^2(F_t) N_t^2(F_{t-1}), \\ \text{s.t.} \quad 0 \leq F_t. \end{aligned} \quad (1)$$

The first order conditions of this maximization problem are given by

$$\frac{\partial y_t^2(F_t)}{\partial F_t} N_t^2(F_{t-1}) + \beta y_{t+1}^2(F_{t+1}) \frac{\partial N_{t+1}^2(F_t)}{\partial F_t} + \mu_t = 0, \quad (2)$$

$$\mu_t F_t = 0 \quad (3)$$

where $\mu_t \geq 0$ is the Lagrange multiplier associated with the inequality restriction $F_t \geq 0$.

As in the perfect selectivity context, the optimal solution to (1) may be expected to consist of harvesting the maximum number of fish of the highest value. However, the imperfectness of the selectivity does not allow adults to be caught continuously without catching juveniles. Nevertheless this aim can be achieved indirectly if fishing is carried out in a cyclical manner. In this 2 age-class model, the natural periodic solution consists of one fallow cycle. First, the fishery is allowed to lie fallow for one year -by closing it- and then in the next year the whole population is harvested. Formally we are describing pulse fishing characterised by $F_t = F_{t+2} = \dots = 0$ and $F_{t+1} = F_{t+3} = \dots = \infty$. The following proposition proves that this periodic pulse is a solution that satisfies the first order conditions (2)-(3). The superscript IS stands for imperfect selectivity.

Proposition 1. *In a 2 age-class model, a periodic pulse two periods long such that $F_t = F_{t+2} = \dots = 0$ and $F_{t+1} = F_{t+3} = \dots = \infty$, is a local optimum for problem (1). Moreover, its net present value is*

$$V_{pulse}^{IS} = \frac{\beta}{1 - \beta^2} pr^2 \omega^2 e^{-m}. \quad (4)$$

Proof: See Appendix.

Comparing V^{PS} and V_{pulse}^{IS} , it is clear that the net present value of the *IS* solution is lower than that of the *PS* solution. The intuition is straightforward: with perfect selectivity it is possible to harvest adults continuously without catching juveniles. However with imperfect selectivity, the impossibility of keeping the population structure that exists under the perfect selectivity makes it necessary to close of the fishery for one year so as to maximise the stock in the next period. This closure implies a lower value for the fishery. This result is in line with Skonhoft *et al.* (2012, result 4)

Is periodic fishing (one fallow cycle) the optimal harvesting rule? Tahvonen (2009) shows in a similar context that there is also a stationary solution to maximise problem (1). Let F_{ss} be the stationary solution such that $F_t = F_{t+1} = F_{ss}$ associated with $\mu_t = 0$. For this case, the optimal condition (2) can be expressed as the following Lerner rule

$$\varepsilon_{ss} = \beta p^1 F_{ss}, \quad (5)$$

where $\varepsilon_{ss}^a = \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{F_{ss}}{y_{ss}^a}$ is the fishing effort elasticity of the yield in the stationary solution and βp^1 is the future marginal cost per unit of adult fish

$$-p^1 = \frac{\partial N_{ss}^2}{\partial F_{ss}} \frac{1}{N^2}.$$

The economic interpretation of the Lerner rule (5) is high intuitive. In the optimal stationary solution, an increase in the mortality rate leads to an increase in the current yield per unit of fish (left hand side, in percentage terms) that is offset by the decrease in adult population per unit of fish in the next period (right hand side). The net present value associated with F_{ss} is

$$V_{ss}^{IS} = \frac{1}{1 - \beta} y_{ss}^2 e^{-(p^1 F_{ss} + m)}, \quad (6)$$

where $y_{ss}^2 = pr^2 \omega^2 \frac{p^2 F_{ss}}{p^2 F_{ss} + m} \left[1 - e^{-(p^2 F_{ss} + m)} \right]$.

Notice that the stationary solution implies a distortion of the population structure generated with perfect selectivity. As a result the value of the fishery with the stationary solution is lower than with perfect selectivity, $V_{ss}^{IS} < V^{PS}$.⁴

However, comparing (6) and (4) it is not clear whether or not the pulse solution is better than the stationary solution, $V_{ss}^{IS} \leq V_{pulse}^{IS}$. Nevertheless, it can be proved that the more imperfect the selectivity technology is, the lower the fishing effort applied in the stationary solution F_{ss} is. This means that, given the adult selectivity, if the juvenile selectivity parameter is high enough, pulse fishing dominates the smooth

⁴Notice that $\frac{p^2 F_{ss}}{m + p^2 F_{ss}} \left[1 - e^{-(p^2 F_{ss} + m)} \right] e^{-p^1 F_{ss}} < 1$.

stationary solution. Formally, a lower threshold is found for the juvenile selectivity parameter. When it is exceeded the periodic pulse provides a higher net present value than the interior stationary solution. The following proposition states this result.

Proposition 2. *Normalise p^2 to one. Then the pulse solution (stationary solution) is the optimal solution to the maximization problem (1) if p^1 is higher (lower) than \underline{p}^1 , with \underline{p}^1 being the value of the juvenile selectivity parameter that satisfies the following equality*

$$y_{ss}^2 e^{-\underline{p}^1 F_{ss}(\underline{p}^1)} = \frac{\beta}{1 + \beta} p r^2 \omega^2.$$

Proof: See Appendix.

Figure 1 shows the optimal solutions for a benchmark 2-age fishery. For each juvenile selectivity parameter, p^1 , and stationary fishing mortality, the net present value of the fishery, V_{ss}^{IS} , is calculated. The blue-red bell surface represents this function. The figure also represents the value of the fishery for the pulse solution, V_{pulse}^{IS} , through the yellow horizontal hyperplane. Comparing the bell function with the horizontal hyperplane it can be seen that for values of $p^1 \lesssim 0.26$ the stationary solution dominates the pulse solution. However for values of $p^1 \gtrsim 0.26$ the hyperplane is higher than the bell function, so the pulse solution is better than the stationary solution.

Figure 2 shows the net present value of the fishery for any combination of (F_t, F_{t+1}) for two values of the juvenile selectivity parameter, low p^1 (left plot) and high p^1 (right plot). A point on the horizontal diagonal represents the value of the fishery for a stationary path $\{F_{ss}\}$. It is clear that the stationary solution and the pulse solution are always local optimums. However while the stationary solution is the global maximum when p^1 is low, the pulse fishing is the global maximum when p^1 is high. Moreover when the pulse fishing is the global maximum, the function is not concave and the stationary solution is a saddle path rather than a local maximum.

The role of the discount factor

Pulse fishing literature claims that higher discounting rates (factors) decrease (increase) the advantage of pulse fishing (Hannesson, 1975). This claim is based on the impact of the discount factor on the pulse solution. Because the value of the fishery is the present value of a constant yield, $pr^2\omega^2e^{-m}$, increasing the discount factor raises the net present value of pulse fishing.

However, increasing the discount factor also increases the net present value of the stationary solution. First, it is well known that as $\beta \rightarrow 1$, $F_{ss} \rightarrow F_{max}$ (Da Rocha and Gutiérrez, 2011). That is, the stationary solution without discounting is equal to

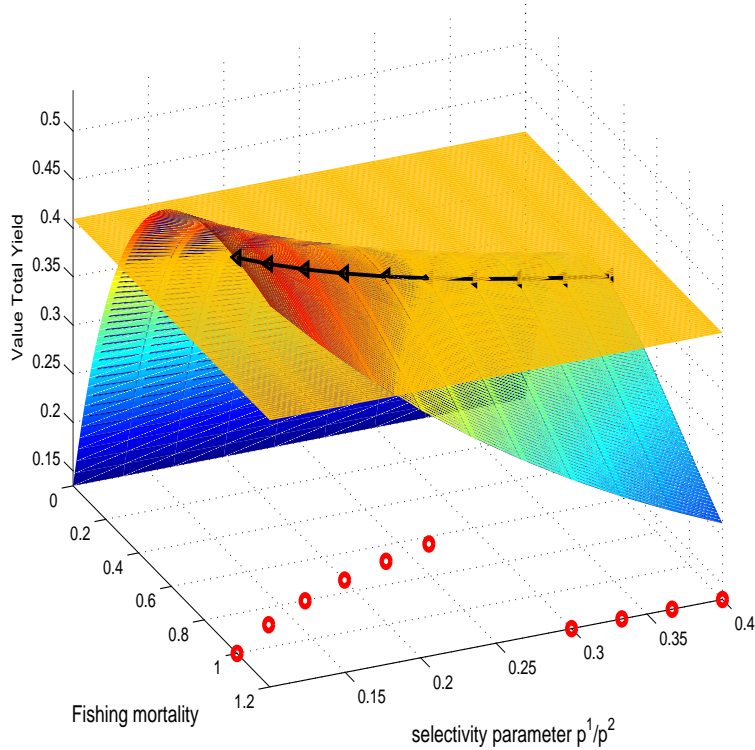


Figure 1: Simulation of the net present value of the fishery under the stationary and pulse solutions. Benchmark 2-ages fishery ($p^2 = 1, m = 0.2, pr^2 = 0$, and $pr^2\omega^2 = 1$). The blue-red shape bell represents the net present value of the fishery for each pair of juvenile selectivity parameters and stationary fishing rate. The yellow horizontal hyperplane illustrates the value of the fishery for the pulse solution.

the F_{max} . Second, it is easy to prove that $\partial F_{ss}/\partial\beta < 0$. Therefore, as β increases, F_{ss} decreases and the stationary yield rises. At first glance it is not clear whether or not this increase in the value of the stationary fishery is lower than the increase in value of the pulse fishery. However the following proposition shows that if the discount factor is high enough then any increase in it raises the advantages of the periodic solution.

Proposition 3. *In a 2-age class model, if for a discount factor β such that $1 < \beta(2 - \beta^2)$ periodic fishing and stationary fishing yield the same present value of the fishery, a slight increase (decrease) in the discount factor leads the present value of periodic fishing to higher (lower) values than the present value of stationary fishing.*

Proof: See Appendix.

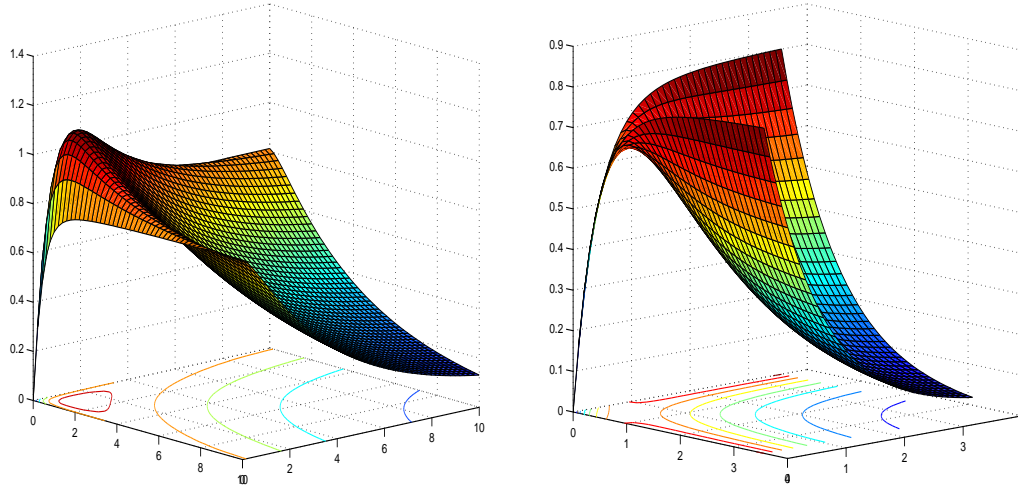


Figure 2: Net present value of the fishery for any combination of (F_t, F_{t+1}) for two values of the juvenile selectivity parameter, p^1 : left hand side, low p^1 ; right hand side, high p^1 . Benchmark 2-age fishery ($p^2 = 1, m = 0.2, pr^2 = 0$, and $pr^2\omega^2 = 1$)

3 Endogenous Lifespan

In the previous section it was not possible to choose the fish lifespan. In both the pulse and stationary solutions the fish lifespan is two years. We now extend the 2-ages model of the previous section to a more realistic framework with n -ages, to show that the fish lifespan is an endogenous variable.

Note, first, that if there were perfect selectivity optimal management would consist of finding the most profitable age at which to catch fish, let them grow until they reach taht optimal age and harvest them all at that age. Formally, assume that fish are allowed to grow to age L . Then the value of the biomass if it is harvested is $pr^L w^L e^{-(L-1)m}$. It is worth waiting one period before harvesting if $\beta pr^{L+1} w^{L+1} e^{-Lm} > pr^L w^L e^{-(L-1)m}$.

Define the value of weight gain for waiting from period L to period $L + 1$ as $\Delta_L^{L+1} = pr^{L+1} w^{L+1} e^{-Lm} - pr^L w^L e^{-(L-1)m}$. The condition under which it is worth waiting one more period in terms of returns can then be describe as follows:

$$\text{Perfect selectivity: } \frac{\Delta_L^{L+1}}{pr^L w^L e^{-(L-1)m}} \geq \frac{1 - \beta}{\beta}, \quad (7)$$

where $(1 - \beta)/\beta$ represents the return. In this perfect selectivity context, the fish are

harvested in a period L^{PS} such that

$$\frac{\Delta_{L^{PS}-1}^{L^{PS}}}{pr^{L^{PS}-1}w^{L^{PS}-1}e^{-(L^{PS}-2)m}} \geq \frac{1-\beta}{\beta} \geq \frac{\Delta_{L^{PS}}^{L^{PS}+1}}{pr^{L^{PS}}w^{L^{PS}}e^{-(L^{PS}-1)m}}.$$

After some manipulation this condition can be rewritten as

$$\frac{pr^{L^{PS}}w^{L^{PS}}}{pr^{L^{PS}-1}w^{L^{PS}-1}} \geq \frac{e^m}{\beta} \geq \frac{pr^{L^{PS}+1}w^{L^{PS}+1}}{pr^{L^{PS}}w^{L^{PS}}}.$$

Notice that selecting L^{PS} is equivalent to choosing the lifespan of fish endogenously. In fact to guarantee that L^{PS} exists some conditions have to be imposed. The left panel in Figure 3 shows a case for $A = 6$ in which L^{PS} exists and is unique. The step function represents the increase percentage of the value of weight from waiting one period before harvesting. If this function is decreasing in L , the intersection with e^m/β represents the optimal size of the pulse ($L^{PS} = 5$ in the case of Figure 3). It seems clear that the following are sufficient conditions to guarantee the existence of L^{PS} : *i*) the function $pr^{L+1}w^{L+1}/pr^Lw^L$ is a decreasing function on L ; *ii*) $pr^2w^2/pr^1w^1 \geq e^m/\beta$; and *iii*) $e^m/\beta \geq pr^nw^n/pr^{n-1}w^{n-1}$. Conditions *ii*) and *iii*) are satisfied in most cases since pr^1w^1 represents the weight value of the eggs and $pr^nw^n/pr^{n-1}w^{n-1} = 1 < e^m/\beta$ because for most species the weight value remains constant in old age.

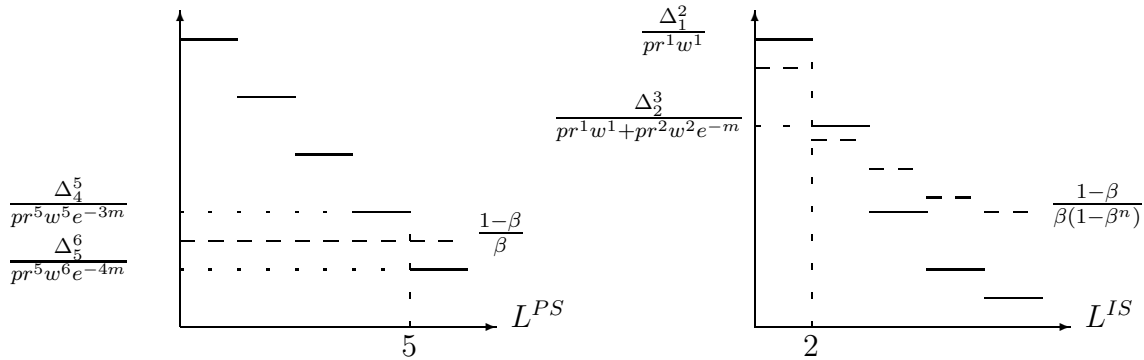


Figure 3: An example with $A = 6$ for determining pulses under perfect selectivity (left panel) and under imperfect selectivity (right panel). In this case there is a unique pulse in both scenarios. $L^{PS} = 5$ because $\frac{\Delta_4^5}{pr^4w^4e^{-3m}} \geq \frac{1-\beta}{\beta} \geq \frac{\Delta_5^6}{pr^5w^5e^{-4m}}$ and $L^{IS} = 2$ because $\frac{\Delta_1^2}{pr^1w^1} \geq \frac{1-\beta}{\beta} \frac{1}{1-\beta^2} \geq \frac{\Delta_2^3}{pr^1w^1+pr^2w^2e^{-m}}$

However, in a context of imperfect selectivity, harvesting fish of age L implies waiting $L - 1$ periods and as a collateral result fish of age $L - 1, L - 2, \dots$ are also harvested. Therefore to calculate the optimal fish lifespan the value of harvesting under different pulses must be compared.

Assume that fish are harvested with a pulse of length L . This means that $F_t = F_{t+1} = \dots = F_{t+L-1} = 0$ and $F_{t+L} = \infty$. So the value of the harvest is given by

$$V_L^{IS} = \frac{\beta^{L-1}}{1 - \beta^L} \sum_{a=1}^L pr^a w^a e^{-(a-1)m}. \quad (8)$$

It will be worth harvesting every $L + 1$ periods if $V_{L+1}^{IS} > V_L^{IS}$. In this imperfect selectivity context the condition under which it is worth waiting from period L to period $L + 1$ can be expressed in terms of the returns as

$$\text{Imperfect selectivity:} \quad \frac{\Delta_L^{L+1}}{\sum_{a=1}^L pr^a w^a e^{-(a-1)m}} \geq \frac{1 - \beta}{\beta} \frac{1}{1 - \beta^L}. \quad (9)$$

Therefore, fish are harvested in a period L^{IS} such that

$$\frac{\Delta_{L^{IS}-1}^{L^{IS}}}{\sum_{a=1}^{L^{IS}-1} pr^a w^a e^{-(a-1)m}} \geq \frac{1 - \beta}{\beta} \frac{1}{1 - \beta^{L^{IS}}} \geq \frac{\Delta_{L^{IS}+1}^{L^{IS}+1}}{\sum_{a=1}^{L^{IS}} pr^a w^a e^{-(a-1)m}}.$$

The right panel in Figure 3 shows a case for $N = 6$ in which L^{IS} exists and is unique. The step continuous line represents the function $\Delta_{L-1}^L / \sum_{a=1}^{L-1} pr^a w^a e^{-(a-1)m}$. The step discontinuous line represents the function $\frac{1-\beta}{\beta} \frac{1}{1-\beta^L}$ which is decreasing on L and is bounded, $\frac{1-\beta}{\beta} \frac{1}{1-\beta^L} \in [1/\beta, (1-\beta)/\beta]$. The intersection between the two functions represents the optimal size of the pulse ($L^{PS} = 2$ in the case of Figure 3). Given the properties of the function, it seems clear that the following are sufficient conditions to guarantee the existence of L^{IS} : *i*) the function $\Delta_{L-1}^L / \sum_{a=1}^{L-1} pr^a w^a e^{-(a-1)m}$ is an decreasing function on L ; *ii*) $\Delta_1^2 / pr^1 w^1 \geq \beta^{-1}$; and *iii*) $\frac{1-\beta}{\beta} \leq \Delta_{n-1}^n / \sum_{a=1}^{n-1} pr^a w^a e^{-(a-1)m}$.

Formally L^{IS} can be selected as the result of

$$V_L^{IS} = \max_L \frac{\beta^{L-1}}{1 - \beta^L} \left[\sum_{a=1}^L pr^a w^a e^{-(a-1)m} \right].$$

A comparison of condition (9) with condition (7) obtained under perfect selectivity shows two main differences. First, with imperfect selectivity, the value of weight gain, Δ_L^{L+1} , is valued in terms of the whole population structure while with perfect selectivity it is valued in terms of the population of the previous age. In terms of Figure 3, this means that function $\Delta_{L-1}^L / \sum_{a=1}^{L-1} pr^a w^a e^{-(a-1)m}$ is always lower than

function $pr^{L+1}w^{L+1}/pr^Lw^L$ and coincide for $L = 1$. Second, the cost of waiting is the interest rate adjusted for the increase in the pulse length. This point is especially relevant for assessing the role of selectivity. In terms of Figure 3, this means that function $\frac{1-\beta}{\beta} \frac{1}{1-\beta^L}$ is always greater than $1 - \beta/\beta$ and tends to be equal for $L \rightarrow \infty$. Therefore, imperfect selectivity gives a shorter optimal lifespan than perfect selectivity.⁵ The following proposition formalizes this finding.

Proposition 4. *If i) Δ_L^{L+1}/pr^Lw^L and $\Delta_L^{L+1}/\sum_{a=1}^L pr^a w^a e^{-(a-1)m}$ are decreasing functions on L ; ii) $\Delta_1^2/pr^1w^1 \geq \beta^{-1}$; and iii) $pr^n w^n < (1 - \beta)/\beta$ then the optimal fish lifespan and the pulse length under imperfect selectivity are lower than under perfect selectivity.*

Proof: See Appendix.

Finally, notice that with both perfect and imperfect selectivity, an increase in the discount factor results in an increase in the optimal lifespan of the resource because is more profitable to wait longer before harvesting.

Proposition 5. *If $pr^{L+1}w^{L+1}/pr^Lw^L$ and $\Delta_L^{L+1}/\sum_{a=1}^L pr^a w^a e^{-(a-1)m}$ are decreasing functions on L then any increase in the discount factor β either increases or leaves unchanged the optimal fish lifespan and the pulse length under perfect and imperfect selectivity, respectively.*

Proof: See Appendix.

It is worth mentioning that harvesting under a pulse of length L is a local optimum of the fishery management problem that maximizes the net present value of the fishery. In a framework with n -ages the optimal management problem can be expressed as

$$\begin{aligned} \max_{\{F_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{a=1}^n y_t^a(F_t) \phi_t^a, \\ \text{s.t.} \quad 0 \leq F_t, \end{aligned} \tag{10}$$

where

$$\phi_t^a = \phi(F_{t-1}, F_{t-2}, \dots, F_{t-(a-1)}) = \begin{cases} 1 & \text{for } a = 1, \\ \prod_{i=1}^{a-1} e^{-p^{a-i} F_{t-i-m}} & \text{for } a = 2, \dots, n, \end{cases}$$

⁵We thank the anonymous referee for opening our eyes to this relevant point, which has helped us to realize the role played by the monotonicity of $\Delta_L^{L+1}/\sum_{a=1}^L pr^a w^a e^{-(a-1)m}$ in the analysis.

is obtained by backward substitution of the population, $N_t^a = \phi_t^a N_{t-(a-1)}^1 = \phi_t^a$, and can be understood as the survival function that shows the probability of a recruit born in period $t - (a - 1)$ reaching age $a > 1$ for a given fishing mortality path. The following proposition proves that under sufficient conditions any periodic pulse of length L is a solution that satisfies the first order condition of the maximisation problem (10).

Proposition 6. *In an n age-class model, a periodic pulse with a length of $L \leq n$ such that $F_{t+kL} = \infty \forall k \in \mathbb{Z}$ and $F_j = 0$ for any other period is a local optimum for problem (10) if the following qualifying condition holds*

$$\frac{pr^j \omega^j}{pr^i \omega^i} > \frac{(1 - e^{-m})}{m} \left(\frac{e^m}{\beta} \right)^{j-i}, \quad (11)$$

for all $j > i \in \{1, 2, \dots, n\}$.

Proof: See Appendix.

It is worth mentioning that the qualifying conditions (11) are sufficient conditions to guarantee that the pulse satisfies the first order condition of the maximisation problem (10). Nevertheless, there may be cases in which the pulse does not satisfy the qualifying conditions but does satisfy the optimal conditions.

There is also a stationary solution, $F_t = F_{t+1} = F_{ss}$, that satisfies the first order condition of maximisation problem (10). As in the case of the 2-age model, the stationary solution can be expressed in terms of a Lerner rule. Specifically,

$$\sum_{a=1}^n \varepsilon_{ss}^a s^a = F_{ss} \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left(\sum_{j=1}^{a-1} \beta^{a-j} p^j \right)}{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a}, \quad (12)$$

where $s^a = y_{ss}^a \phi_{ss}^a / \sum_{a=1}^n y_{ss}^a \phi_{ss}^a$ is the share of yield in value of age a in the total value for all age classes and $\varepsilon_{ss}^a = \frac{\partial y_{ss}^a F_{ss}}{\partial F_{ss} y_{ss}^a}$ is the fishing effort elasticity of the yield of age a . The following proposition characterizes an interior stationary candidate for global optimum, F_{ss} .

Proposition 7. *In an n age-class model, the stationary solution F_{ss} satisfies the Lerner rule (12). Moreover, F_{ss} decreases as the selectivity parameter p^a and/or the discount factor β increases whenever $y_{ss}^a \phi_{ss}^a$ is an increasing function on $F_{ss} \forall a = 1, \dots, n$. Furthermore, the net present value associated with F_{ss} is*

$$V_{ss}^{IS} = \frac{1}{1 - \beta} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a,$$

where $\sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \sum_{a=1}^n pr^a \omega^a \frac{p^a F_{ss}}{p^a F_{ss} + m} (1 - e^{-p^a F_{ss} - m}) \prod_{i=1}^{a-1} e^{-(p^i F_{ss} + m)}$.

Proof: See Appendix.

It is well known that there is a relationship between the non-concavity of the objective function and the non-continuity of the optimal solution (Scarf, 1959, Stokey, Lucas and Prescott, 1989). For biomass fishery models this link was established by Dawid and Kopel (1997, 1999), who show that the optimal pulse length is related to the non-concavity of the return function.⁶

We prove that this link also exists for age-structured fishery models. In order to find this link, we analyse the (non-)concavity properties of the objective function of problem (10). In particular, by focusing on the second order conditions, we are able to find sufficient conditions that guarantee that the stationary solution is not the optimal solution to problem (10). The following proposition summarises this result.

Proposition 8. *A stationary solution is not the optimal solution to maximisation problem (10) whenever the following inequality holds*

$$H = \sum_{a=1}^n \left[\frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} + y_{ss}^a \left(\sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right) \right] \phi_{ss}^a \geq 0. \quad (13)$$

Proof: See Appendix.

H corresponds to the order one minor in the Hessian matrix associated with the second order conditions of the maximisation problem (10). Notice that $\partial^2 y_{ss}^a / \partial F_{ss}^2 < 0$ because the yield per unit of fish is a concave function. Moreover as $\beta \rightarrow 1$, inequality (13) is more likely that be positive. Notice that when $\beta \rightarrow 1$, $F_{ss} = F_{\max}$ (Da Rocha and Gutiérrez, 2011).

As in the 2-age model, we seek to determine the conditions under which an increase in the discount factor can promote pulse fishing compared to the stationary smooth solution. In particular, we find that a sufficient condition to guarantee that an increase in the discount factor promotes pulse fishing is for the stationary yield, $y_{ss}^a \phi_{ss}^a$, to be an increasing function with respect to F_{ss} . We state this result in the following proposition.

Proposition 9. *In an n age-class model, if for a discount factor β periodic fishing and stationary fishing yield the same present value of the fishery, a slight increase*

⁶Maroto and Moran (2008) also show that concavity plays a relevant role in the appearance of resource extinction which can be considered a non continuous solution.

(decrease) in the discount factor leads the present value of periodic fishing to higher (lower) values than the present value of stationary fishing whenever $y_{ss}^a \phi_{ss}^a$ is an increasing function on $F_{ss} \forall a = 1, \dots, n$.

Proof: See Appendix.

Finally, it can be concluded that to find the optimal fishing path with imperfect selectivity it suffices to compare the value of a fishery for a small number of possibilities. In particular, the value of the fishery for the pulses V_L^{IS} for $L = 2, 3, \dots, n$ and for the stationary solution V_{ss}^{IS} need to be calculated. The optimal solution will be the one that results in the maximum value of the fishery. In the next section we illustrate this numerically.

4 A numerical illustration

In this section we apply the results obtained in the previous section to the Northern Stock of Hake (NSH) (*Merluccius merluccius*) in order to select the optimal fishing path, i.e. we calculate the value of the fishery for the pulses V_L^{IS} for $L = 2, 3, \dots, n$ and the stationary solution V_{ss}^{IS} . We also compare this result with the result obtained using global numerical methods to solve the management problem (10). Similar results are obtained.

The NSH includes all fisheries in International Council for Exploitation for of the Sea (ICES) subareas VII and VIII and also some fisheries in Subareas IV and VI. Landings in 2008 were 47,800 tones, below the regulated TAC of 54,000 tones. Spain accounts for 53% of the total captures. France for 30%, the UK for 7%, Denmark for 3%, Ireland for 3% and other countries (Norway, Belgium, Netherlands, Germany, and Sweden) for smaller amounts (ICES 2009).

A recovery plan was drawn up in 2004 (EC 811/2004) for this stock. Its aim was to achieve a spawning stock biomass (SSB) of 140,000 tones by limiting fishing mortality to 0.25 and by allowing a maximum change in harvest between consecutive years of 15%. According to the ICES, the northern hake SSB for 2009 is estimated to be above the recovery plan target. Article 3 of the recovery plan prescribes that a management plan should be implemented when the target is reached in two consecutive years and the ICES considers SSB to have been approximately 140,000 tones in the last two years. Such a plan is now under development by the European Commission (ICES 2010).

Table 2: Biological Parameters for the Northern Stock of Hake.

Age	Abundance (N) ⁽¹⁾	Mortality (m)	Maturity μ	Weight (ω) ⁽²⁾	Selectivity (p)	Price (pr) ⁽³⁾
Northern Hake						
1	186,213	0.20	0.00	0.06	0.00	2.34
2	152,458	0.20	0.00	0.13	0.01	2.90
3	123,457	0.20	0.00	0.22	0.10	3.39
4	100,213	0.20	0.23	0.34	0.22	3.82
5	67,409	0.20	0.60	0.66	0.20	4.51
6	35,551	0.20	0.90	0.99	0.30	5.18
7	19,674	0.20	1.00	1.44	0.40	5.76
8	10,206	0.20	1.00	1.83	0.47	6.17
9	9,1503	0.20	1.00	2.68	0.47	6.86
10	4,080	0.20	1.00	2.68	0.47	6.86
11	1,821	0.20	1.00	2.68	0.47	6.86

Source: Meeting on Northern Hake Long-Term Management Plans (STECF/SGBRE-07-03). ⁽¹⁾ Thousand; ⁽²⁾ kg; ⁽³⁾ euro per kg

To calibrate the age structured model for this fishery two data sources have been used. Information regarding the biological parameters of the fishery comes from the Expert Working Group (STECF, 2008a). Most of the parameters emanate from the summary of XSA results from the 2006 update (ICES, 2007). Secondly, as the Spanish fleet accounts for most northern hake landings (59% of the total in 2006; ICES, 2007), we use 2007 daily sale prices for the Spanish fleet.

Table 2 shows, for each age, the number of fish at the initial conditions, the parameters of the population dynamics (selection pattern, weight and maturity) and the prices.

Using this data calibration and assuming $\beta = 0.95$ (equivalent to a 5% interest rate), we calculate the value of the fishery, V_L^{IS} , for $L = 2, 3, \dots, 11$ defined in (8) for the NHS. Table 3 shows the results. Observe that the fishery reaches the maximum value when a pulse of 9 periods is applied, that is the fishery is allowed to lie fallow for eight years -by closing it- and the whole population is harvested in the ninth year.

This numerical solution has been validated by obtaining the optimal harvest-

Table 3: Fishery value under pulse fishing with imperfect selectivity for the Northern Stock of Hake.

L (periods)	2	3	4	5	6	7	8	9	10	11
V_L^{IS} (€ million)	2.39	2.57	2.65	3.61	4.48	4.98	4.72	5.45	3.90	2.82

Source: Own calculations

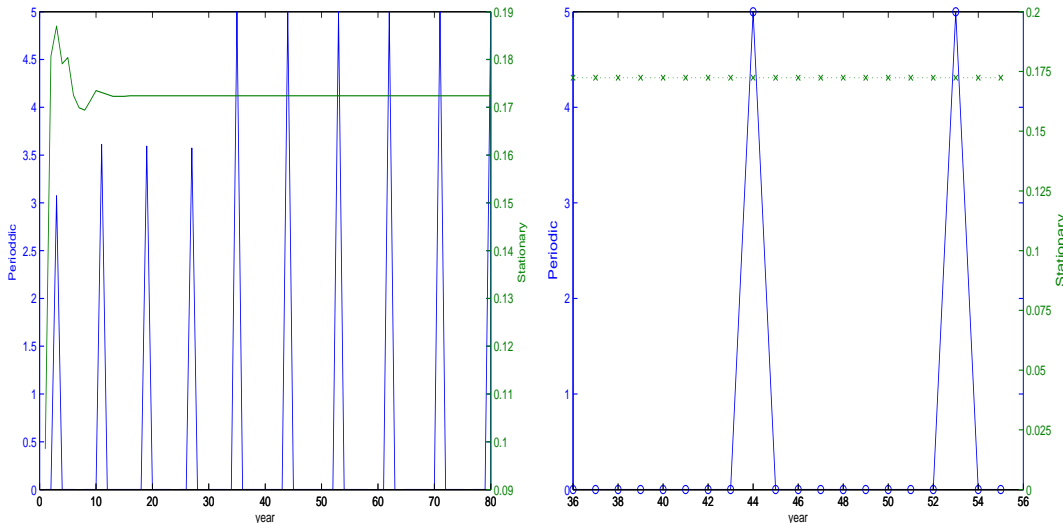


Figure 4: Optimal pulse fishing policy for the NSH obtained by solving (10) with global numerical methods

ing policy for this NSH stock that solves the management problem (10). Since the Beverton-Holt multi-cohort models used to assess the stock are not globally concave, the stationary solution described may be a local rather than a global optimum (Tahvonen, 2009). Because of this, standard global maximisation algorithms are used to determine the trajectory that drives the NSH fishery for the initial conditions to the global optimal solution of (10). Our main finding is that the global optimal solution consists in a pulse fishing every 9 years by applying a fishing rate $F = 5$ in the harvesting years and 8 consecutive fallow years.⁷ Figure 4 shows the evolution and magnitude of cited pulses. Moreover we have checked also that the stationary solution is not a global maximum by checking that condition (13) from Proposition 8 is satisfied; we find that $H = 16.7344 > 0$.

Finally, Table 4 shows the quantitative results associated with the stationary and pulse solutions using the global methods. Note that average price per kilo is higher

⁷We have checked that increasing this value does not change the results.

Table 4: Pulse fishing and stationary solution for the NSH using global numerical methods

	Stationary	Pulse
Yield		
weight ('000 t.)	53.6	567.2 (every 9 years)
value ('000 €)	299.1	3,325.1 (every 9 years)
Average price per kilo (€)	5.5802	5.8623
Profits		
net present value (€ millions)	5,799.1	6,734.2

in the pulse fishing solution than in the stationary solution. That is due to the fact that age distribution is biased towards higher ages in the pulse solution. It can be concluded that numerical global methods support the optimal pulse selected by comparing the valuation of the different pulses V_L^{IS} .

5 Conclusions

We apply the analysis of Tahvonen (2009) to Hanneson’s multi-cohort Beverton-Holt model with any number of age classes. We show that under imperfect selectivity, when optimal age cannot be chosen, optimal management is equivalent to choosing the pulse length that induces the optimal average fish lifespan. Therefore, with imperfect selectivity optimal lifespan is valued in terms of the whole population structure. Moreover, imperfect selectivity, reduces the optimal lifespan and the optimal pulse length. Finally, as in biomass models we show that pulse fishing is related to the non concavity of the return function.

In seeking simplification, unlike Tahvonen (2009) and Skonhøft *et al.* (2012), our analysis considers neither endogenous recruitment nor harvesting costs. Likewise our analysis avoids the presence of density dependence growth. Steinshamn (2011) shows that pulse fishing seems to become less and less economically profitable as one moves from uniformly distributed fish to schooling species. Moreover, the idea of considering the size of the fish instead of the age as the criterion for harvesting has not been developed in the relevant literature. All these considerations can be analyzed in future research.

On the other hand, most of the literature on fisheries assessment is focused on stationary policies. See Gröger *et al.* (2007), Grafton *et al.* (2007, 2010), Dichmont *et al.* (2010), Kompas *et al.* (2010), Da Rocha *et al.* (2010), Diekert *et al.* (2010) for

recent examples. This prevalence of stationary policies is due to practical reasons. As Hannesson (2011) points out “*The pulse fishing approach is not very practical. What it means is that a stock of fish is fished down heavily for a short period of time and then left to replenish itself for a longer period. But what does the industry do in the meantime?...The way I envisaged for making pulse fishing practical was that the fishing fleets could rotate between stocks of the same or similar species, a little bit like when timber is harvested from different lots at different times*”. Including this possibility of rotation in the analysis is one of the most important challenge for the researchers.

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A Appendix

A.1 Proof of Proposition 1

Valuing the optimal condition (2) in t and $t + 1$ and taking into account that $F_t = F_{t+2} = \dots = 0$ and $F_{t+1} = F_{t+3} = \dots = \infty$, the following is obtained

$$\begin{aligned}
\mu_t &= - \lim_{\substack{F_{t-1}=F_{t+1} \rightarrow \infty \\ F_t=0}} \left[\frac{\partial y_t^2(F_t)}{\partial F_t} N_t^2(F_{t-1}) + \beta y_{t+1}^2(F_{t+1}) \frac{\partial N_{t+1}^2(F_t)}{\partial F_t} \right] \\
&= - \lim_{\substack{F_{t-1}=F_{t+1} \rightarrow \infty \\ F_t=0}} \left\{ pr^2 \omega^2 e^{-p^1 F_{t-1}-m} \left[\frac{p^2 m}{(p^2 F_t + m)^2} (1 - e^{-p^2 F_t - m}) \right. \right. \\
&\quad \left. \left. + \frac{p^2 F_t}{p^2 F_t + m} e^{-p^2 F_t - m} \right] + \beta pr^2 \omega^2 \frac{p^2 F_{t+1}}{p^2 F_{t+1} + m} (1 - e^{-p^2 F_{t+1} - m}) (-p^1) e^{-p^1 F_t - m} \right\} \\
&= p^1 \beta pr^2 \omega^2 e^{-m} > 0, \\
\mu_{t+1} &= - \lim_{\substack{F_{t+1} \rightarrow \infty \\ F_t=F_{t+2}=0}} \left[\frac{\partial y_{t+1}^2(F_{t+1})}{\partial F_{t+1}} N_{t+1}^2(F_t) + \beta y_{t+2}^2(F_{t+2}) \frac{\partial N_{t+2}^2(F_{t+1})}{\partial F_{t+1}} \right] = \\
&\quad - \lim_{\substack{F_{t+1} \rightarrow \infty \\ F_t=F_{t+2}=0}} \left\{ pr^2 \omega^2 e^{-p^1 F_t - m} \left[\frac{p^2 m}{(p^2 F_{t+1} + m)^2} (1 - e^{-p^2 F_{t+1} - m}) \right. \right. \\
&\quad \left. \left. + \frac{p^2 F_{t+1}}{p^2 F_{t+1} + m} e^{-p^2 F_{t+1} - m} \right] + \beta pr^2 \omega^2 \frac{p^2 F_{t+2}}{p^2 F_{t+2} + m} (1 - e^{-p^2 F_{t+2} - m}) (-p^1) e^{-p^1 F_{t+1} - m} \right\} \\
&= 0.
\end{aligned}$$

Therefore a stationary pulse with $F_t = F_{t+2} = \dots = 0$ and $F_{t+1} = F_{t+3} = \dots = \infty$ satisfies the optimal conditions (2)-(3). Taking into account that in periods 0, 2, 4, ... the yield is zero and in periods 1, 3, 5, ... the fishing mortality tends to infinity, then the net present value of yield in value for the cycle is given by

$$\begin{aligned}
V_{pulse}^{IS} &= \sum_{t=0}^{\infty} \beta^t y_t^2(F_t) N^2(F_t) = \\
&= \sum_{t=0}^{\infty} \beta^{2t+1} pr^2 \omega^2 \lim_{F_{2t+1} \rightarrow \infty} \frac{p^2 F_{2t+1}}{p^2 F_{2t+1} + m} (1 - e^{-p^2 F_{2t+1} - m}) e^{-m^1} \\
&= pr^2 \omega^2 e^{-m^1} [\beta + \beta^3 + \beta^5 + \dots] \\
&= pr^2 \omega^2 e^{-m^1} \frac{\beta}{1 - \beta^2}. \quad \blacksquare
\end{aligned}$$

A.2 Proof of Proposition 2

Let \underline{p}^1 be the value that satisfies the following equality

$$y_{ss}^2 e^{-\underline{p}^1 F_{ss}(\underline{p}^1)} = \frac{\beta}{1 + \beta} pr^2 \omega^2.$$

First we prove that F_{ss} decreases as the selectivity parameter p^1 increases. Notice that if F_{ss} is an interior stationary solution then $\mu_t = 0$ and the optimal condition (2) valued in F_{ss} can be expressed as

$$\frac{\partial y_{ss}^2}{\partial F_{ss}} - \beta y_{ss}^2 p^1 = 0.$$

Total differentiation of this optimal condition implies

$$\frac{\partial F_{ss}}{\partial p^1} = \frac{\beta y_{ss}^2}{\frac{\partial^2 y_{ss}^2}{\partial F_{ss}^2} - \beta p^1 \frac{\partial y_{ss}^2}{\partial F_{ss}}} < 0.$$

So if $p^1 > \underline{p}^1$, then $y_{ss}^2 e^{-p^1 F_{ss}(p^1)} < y_{ss}^2 e^{-\underline{p}^1 F_{ss}(\underline{p}^1)}$. This implies that $y_{ss}^2 e^{-p^1 F_{ss}(p^1)} < \frac{\beta}{1+\beta} p r^2 \omega^2$. Therefore $V_{ss}^{IS} = \frac{1}{1-\beta} y_{ss}^2 (p^1) e^{-p^1 F_{ss}(p^1)-m} < \frac{\beta}{1-\beta^2} p r^2 \omega^2 e^{-m} = V_{pulse}^{IS}$. ■

A.3 Proof of Proposition 3

Taking the partial derivative in V_{pulse}^{IS} defined in Proposition 1

$$\frac{\partial V_{pulse}^{IS}}{\partial \beta} = \frac{1 + \beta^2}{(1 - \beta^2)^2} p r^2 \omega^2 e^{-m} > 0. \quad (14)$$

Doing the same over V_{ss}^{IS} defined in (6)

$$\begin{aligned} \frac{\partial V_{ss}^{IS}}{\partial \beta} &= \frac{1}{(1 - \beta)^2} y_{ss}^2 e^{-(p^1 F_{ss} + m)} \\ &+ \frac{1}{1 - \beta} \frac{\partial y_{ss}^2}{\partial F_{ss}} \frac{\partial F_{ss}}{\partial \beta} e^{-(p^1 F_{ss} + m)} - p^1 \frac{\partial F_{ss}}{\partial \beta} \frac{1}{1 - \beta} y_{ss}^2 e^{-(p^1 F_{ss} + m)} \\ &= \frac{1}{(1 - \beta)^2} y_{ss}^2 e^{-(p^1 F_{ss} + m)} + \frac{1}{1 - \beta} \frac{\partial F_{ss}}{\partial \beta} e^{-(p^1 F_{ss} + m)} \left[\frac{\partial y_{ss}^2}{\partial F} - p^1 y_{ss}^2 \right]. \end{aligned}$$

Given that F_{ss} satisfies the Lerner rule (5), this expression can be written as

$$\begin{aligned} \frac{\partial V_{ss}^{IS}}{\partial \beta} &= \frac{1}{(1 - \beta)^2} y_{ss}^2 e^{-(p^1 F_{ss} + m)} - \frac{\partial F}{\partial \beta} e^{-(p^1 F_{ss} + m)} p^1 y_{ss}^2 = \\ &= y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[\frac{1}{(1 - \beta)^2} - \frac{\partial F}{\partial \beta} p^1 \right] \\ &= y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[\frac{1}{(1 - \beta)^2} - \frac{(p^1)^2 y_{ss}^2}{\frac{\partial^2 y_{ss}^2}{\partial F_{ss}^2} - \beta p^1 \frac{y_{ss}^2}{\partial F_{ss}}} \right] \\ &= y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[\frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2 - \frac{\partial^2 y_{ss}^2}{\partial F_{ss}^2} \frac{1}{(p^1)^2 y_{ss}^2}} \right]. \end{aligned}$$

Given that y^2 is convex,

$$\frac{\partial V_{ss}^{IS}}{\partial \beta} < y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[\frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right]. \quad (15)$$

Now assume that for a given β , the two solutions are equivalent, $V_{pulse}^{IS} = V_{ss}^{IS}$. This implies that

$$y_{ss}^2 e^{-(p^1 F_{ss} + m)} = \frac{\beta}{1 + \beta} pr^2 \omega^2 e^{-m}.$$

Taking this into account this and (14), (15) can be expressed as

$$\begin{aligned} \frac{\partial V_{ss}^{IS}}{\partial \beta} &< y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[\frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right] = \left[\frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right] \frac{\beta}{1 + \beta} pr^2 \omega^2 e^{-m} \\ &= \left[\frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right] \frac{\beta}{1 + \beta} \frac{(1 - \beta^2)^2}{1 + \beta^2} \frac{\partial V_{pulse}^{IS}}{\partial \beta} \\ &= \frac{[1 - 2\beta(1 - \beta)](1 + \beta)}{\beta(1 + \beta^2)} \frac{\partial V_{pulse}^{IS}}{\partial \beta}. \end{aligned}$$

Note that $[1 - 2\beta(1 - \beta)](1 + \beta) < \beta(1 + \beta^2)$ implies $1 - 2\beta + \beta^3 < 0$. Therefore, whenever $1 < \beta(2 - \beta^2)$ is satisfied $\frac{\partial V_{ss}^{IS}}{\partial \beta} < \frac{\partial V_{pulse}^{IS}}{\partial \beta}$. ■

A.4 Proof of Proposition 4

Under *i*), *ii*) and *iii*) L^{PS} exists and is unique. Since $\Delta_L^{L+1} / \sum_{a=1}^L pr^a w^a e^{-(a-1)m} < \Delta_L^{L+1} / pr^L w^L$ for all L and $\frac{1-\beta}{\beta} \frac{1}{1-\beta^L} > \frac{1-\beta}{\beta}$ then $\Delta_L^{L+1} / \sum_{a=1}^L pr^a w^a e^{-(a-1)m} = \frac{1-\beta}{\beta} \frac{1}{1-\beta^L}$ always happens for a lower L^{IS} than L^{PS} .

A.5 Proof of Proposition 5

The optimal lifespan of the resource is given by an age L^{PS} and L^{IS} such that (7) and (9) hold, under perfect and imperfect selectivity, respectively. Since $pr^{L+1} w^{L+1} / pr^L w^L$ and $\Delta_L^{L+1} / \sum_{a=1}^L pr^a w^a e^{-(a-1)m}$ are increasing functions on L , to prove how the pulse length varies when the discount factor changes it suffices to analyse what happens to the right hand side of (7) and (9) (see right panel of Figure 3).

Since

$$\frac{\partial}{\partial \beta} \left[\frac{e^m}{\beta} \right] = -\frac{e^m}{\beta^2} < 0,$$

and

$$\frac{\partial}{\partial \beta} \left[\left(\frac{1}{\beta} - 1 \right) \frac{1}{1 - \beta^{L^{IS}}} \right] = \frac{1}{\beta} \left(\frac{1}{1 - \beta^{L^{IS}}} \right) \left[\frac{L^{IS} \beta^{L^{IS}-1} (1 - \beta)}{1 - \beta^{L^{IS}}} - \frac{1}{\beta} \right] < 0$$

for $L^{IS} > 1$ and $0 < \beta < 1$ then any increase in the discount factor β either reduces or leaves unchanged the optimal fish lifespan and the pulse length under perfect and imperfect selectivity. ■

A.6 Proof of Proposition 6

The Lagrangian associated with the maximisation problem (10) is given by

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ \sum_{a=1}^n y_t^a(F_t) N_t^a + \mu_t F_t \right\}.$$

The first order conditions of this maximisation problem can be written as

$$\frac{\partial L}{\partial F_t} = 0, \implies \sum_{a=1}^n \frac{\partial y_t^a(F_t)}{\partial F_t} \phi_t^a - \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^j y_{t+j}^{a+j}(F_{t+j}) \phi_{t+j}^{a+j} \right\} + \mu_t = 0, \quad (16)$$

$$\mu_t F_t = 0, \quad (17)$$

where we have taken into account that

$$\frac{\partial \phi_{t+j}^a}{\partial F_t} = \begin{cases} 0 & \text{for } j = 0, \\ -p^a \phi_{t+j}^a(F_{t+j-1}, F_{t+j-2}, \dots, F_{t+j-(a-1)}) & \text{for } j = 1, \dots, n-1. \end{cases}$$

In order to prove that a pulse of length $L \leq n$ satisfies the optimal conditions (16)-(17), we build a constructive proof. First, we prove that a pulse of length $L = n$ is optimal under the qualifying conditions (11). Then we prove that a pulse of length $L = n-1$, $L = n-2$ and so on are also optimal. Without loss of generality, consider the case of $n = 4$ and $L = n$. Condition (16) can be expressed as

$$\begin{aligned} \mu_t = & -\frac{\partial y_t^1(F_t)}{\partial F_t} - \frac{\partial y_t^2(F_t)}{\partial F_t} \phi_t^2(F_{t-1}) - \frac{\partial y_t^3(F_t)}{\partial F_t} \phi_t^3(F_{t-1}, F_{t-2}) - \frac{\partial y_t^4(F_t)}{\partial F_t} \phi_t^4(F_{t-1}, F_{t-2}, F_{t-3}) \\ & + p^1 \left[\beta y_{t+1}^2(F_{t+1}) \phi_{t+1}^2(F_t) + \beta^2 y_{t+2}^3(F_{t+2}) \phi_{t+2}^3(F_{t+1}, F_t) \right] \\ & + \beta^3 y_{t+3}^4(F_{t+3}) \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t) \\ & + p^2 \left[\beta y_{t+1}^3(F_{t+1}) \phi_{t+1}^3(F_t, F_{t-1}) + \beta^2 y_{t+2}^4(F_{t+2}) \phi_{t+2}^4(F_{t+1}, F_t, F_{t-1}) \right] \\ & + p^3 \left[\beta y_{t+1}^4(F_{t+1}) \phi_{t+1}^4(F_t, F_{t-1}, F_{t-2}) \right]. \end{aligned}$$

Assume a pulse length $L = n = 4$ such that $\vec{F}(4) = (F_{t+kL}, F_{t+kL+1}, F_{t+kL+2}, F_{t+kL+3}) = (0, 0, 0, \infty) \forall k \in \mathbb{Z}$. Notice that for this case $y_{t+j}^a(F_{t+j}) = 0$ whenever $j = 0, 1, 2$ and $\phi_t^a = 0 \forall a > 1$. Therefore valuing the optimal qualifying (16) in $\vec{F}(4)$ the following is obtained

$$\begin{aligned} \mu_t|_{L=n} & = \lim_{F \rightarrow \vec{F}(4)} \left\{ -\frac{\partial y_t^1(F_t)}{\partial F_t} + \beta^3 \left[p^1 y_{t+3}^4(F_{t+3}) \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t) \right] \right\} = \\ & = -\frac{\partial y_t^1(0)}{\partial F_t} + \beta^3 p^1 y_{t+3}^4(\infty) \phi_{t+3}^4(0, 0, 0) = \\ & = -p^1 p r^1 \omega^1 \frac{(1 - e^{-m})}{m} + p^1 p r^4 \omega^4 \beta^3 e^{-3m} = \\ & = p^1 \left[p r^4 \omega^4 \beta^3 e^{-3m} - p r^1 \omega^1 \frac{(1 - e^{-m})}{m} \right]. \end{aligned} \quad (18)$$

Notice that (18) is positive under the qualifying condition (11) for the case of $j - i = 3$.
Valuing the optimal condition (16) in $t + 1$

$$\begin{aligned}
\mu_{t+1} &= -\frac{\partial y_{t+1}^1(F_{t+1})}{\partial F_{t+1}} - \frac{\partial y_{t+1}^2(F_{t+1})}{\partial F_{t+1}} \phi_{t+1}^2(F_t) \\
&\quad - \frac{\partial y_{t+1}^3(F_{t+1})}{\partial F_{t+1}} \phi_{t+1}^3(F_t, F_{t-1}) - \frac{\partial y_{t+1}^4(F_{t+1})}{\partial F_{t+1}} \phi_{t+1}^4(F_t, F_{t-1}, F_{t-2}) \\
&\quad + p^1 \left[\beta y_{t+2}^2(F_{t+2}) \phi_{t+2}^2(F_{t+1}) + \beta^2 y_{t+3}^3(F_{t+3}) \phi_{t+3}^3(F_{t+2}, F_{t+1}) \right. \\
&\quad \quad \left. + \beta^3 y_{t+4}^4(F_{t+4}) \phi_{t+4}^4(F_{t+3}, F_{t+2}, F_{t+1}) \right] \\
&\quad + p^2 \left[\beta y_{t+2}^3(F_{t+2}) \phi_{t+2}^3(F_{t+1}, F_t) + \beta^2 y_{t+3}^4(F_{t+3}) \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t) \right] \\
&\quad + p^3 \left[\beta y_{t+2}^4(F_{t+2}) \phi_{t+2}^4(F_{t+1}, F_t, F_{t-1}) \right]
\end{aligned}$$

And for the case of $\vec{F}(4) = (F_{t+kL}, F_{t+kL+1}, F_{t+kL+2}, F_{t+kL+3}) = (0, 0, 0, \infty) \forall k \in \mathbb{Z}$

$$\begin{aligned}
\mu_{t+1}|_{L=n} &= \lim_{F \rightarrow \vec{F}(4)} \left\{ -\frac{\partial y_{t+1}^1(F_{t+1})}{\partial F_{t+1}} - \frac{\partial y_{t+1}^2(F_{t+1})}{\partial F_{t+1}} \phi_{t+1}^2(F_t) + \right. \\
&\quad \left. \beta^2 \left[p^1 y_{t+3}^3(F_{t+3}) \phi_{t+3}^3(F_{t+2}, F_{t+1}) + p^2 y_{t+3}^4(F_{t+3}) \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t) \right] \right\} \\
&= -\frac{\partial y_{t+1}^1(0)}{\partial F_{t+1}} - \frac{\partial y_{t+1}^2(0)}{\partial F_{t+1}} \phi_{t+1}^2(0) + \beta^2 \left[p^1 y_{t+3}^3(\infty) \phi_{t+3}^3(0, 0) + p^2 y_{t+3}^4(\infty) \phi_{t+3}^4(0, 0, 0) \right] \\
&= -p^1 pr^1 \omega^1 \frac{(1 - e^{-m})}{m} - p^2 pr^2 \omega^2 \frac{(1 - e^{-m})}{m} e^{-m} + \beta^2 \left[p^1 pr^3 \omega^3 e^{-2m} + p^2 pr^4 \omega^4 e^{-3m} \right] \\
&= p^1 \left[pr^3 \omega^3 \beta^2 e^{-2m} - pr^1 \omega^1 \frac{(1 - e^{-m})}{m} \right] + p^2 e^{-m} \left[pr^4 \omega^4 \beta^2 e^{-2m} - pr^2 \omega^2 \frac{(1 - e^{-m})}{m} \right] \quad (19)
\end{aligned}$$

Notice that (19) is positive under the qualifying condition (11) for the cases of $j - i = 2$.
Valuing the optimal condition (16) in $t + 2$

$$\begin{aligned}
\mu_{t+2} &= -\frac{\partial y_{t+2}^1(F_{t+2})}{\partial F_{t+2}} - \frac{\partial y_{t+2}^2(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^2(F_{t+1}) \\
&\quad - \frac{\partial y_{t+2}^3(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^3(F_{t+1}, F_t) - \frac{\partial y_{t+2}^4(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^4(F_{t+1}, F_t, F_{t-1}) \\
&\quad + p^1 \left[\beta y_{t+3}^2(F_{t+3}) \phi_{t+3}^2(F_{t+2}) + \beta^2 y_{t+4}^3(F_{t+4}) \phi_{t+4}^3(F_{t+3}, F_{t+2}) \right. \\
&\quad \quad \left. + \beta^3 y_{t+5}^4(F_{t+5}) \phi_{t+5}^4(F_{t+4}, F_{t+3}, F_{t+2}) \right] \\
&\quad + p^2 \left[\beta y_{t+3}^3(F_{t+3}) \phi_{t+3}^3(F_{t+2}, F_{t+1}) + \beta^2 y_{t+4}^4(F_{t+4}) \phi_{t+4}^4(F_{t+3}, F_{t+2}, F_{t+1}) \right] \\
&\quad + p^3 \left[\beta y_{t+3}^4(F_{t+3}) \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t) \right].
\end{aligned}$$

And for the case of $\vec{F}(4) = (F_{t+kL}, F_{t+kL+1}, F_{t+kL+2}, F_{t+kL+3}) = (0, 0, 0, \infty) \forall k \in \mathbb{Z}$

$$\begin{aligned}
\mu_{t+2}|_{L=n} &= \lim_{F \rightarrow \vec{F}(4)} \left\{ \begin{aligned} &-\frac{\partial y_{t+2}^1(F_{t+2})}{\partial F_{t+2}} - \frac{\partial y_{t+2}^2(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^2(F_{t+1}) - \frac{\partial y_{t+2}^3(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^3(F_{t+1}, F_t) \\ &+ \beta [p^1 y_{t+3}^3(F_{t+3}) \phi_{t+3}^3(F_{t+2}, F_{t+1}) + p^2 y_{t+3}^4(F_{t+3}) \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t)] \end{aligned} \right\} = \\
&= -\frac{\partial y_{t+2}^1(0)}{\partial F_{t+2}} - \frac{\partial y_{t+2}^2(0)}{\partial F_{t+2}} \phi_{t+2}^2(0) - \frac{\partial y_{t+2}^3(0)}{\partial F_{t+2}} \phi_{t+2}^3(0, 0) \\
&\quad + \beta^2 [p^1 y_{t+3}^2(\infty) \phi_{t+3}^2(0) + p^2 y_{t+3}^3(\infty) \phi_{t+3}^3(0, 0) + y_{t+3}^4(\infty) \phi_{t+3}^4(0, 0, 0)] \\
&= -p^1 pr^1 \omega^1 \frac{(1 - e^{-m})}{m} - p^2 pr^2 \omega^2 \frac{(1 - e^{-m})}{m} e^{-m} - p^3 pr^3 \omega^3 \frac{(1 - e^{-m})}{m} e^{-2m} \\
&\quad + \beta [p^1 pr^2 \omega^2 e^{-m} + p^2 pr^3 \omega^3 e^{-2m} + p^3 pr^4 \omega^4 e^{-3m}] \\
&= p^1 \left[pr^2 \omega^2 \beta e^{-m} - pr^1 \omega^1 \frac{(1 - e^{-m})}{m} \right] + p^2 e^{-m} \left[pr^3 \omega^3 \beta e^{-m} - pr^2 \omega^2 \frac{(1 - e^{-m})}{m} \right] \\
&\quad + p^3 e^{-2m} \left[pr^4 \omega^4 \beta e^{-m} - pr^3 \omega^3 \frac{(1 - e^{-m})}{m} \right]. \tag{20}
\end{aligned}$$

Notice that (20) is positive under the qualifying condition (11) for the case of $j - i = 1$.

Valuing the optimal condition (16) in $t + 3$

$$\begin{aligned}
\mu_{t+3} &= -\frac{\partial y_{t+3}^1(F_{t+3})}{\partial F_{t+3}} - \frac{\partial y_{t+3}^2(F_{t+3})}{\partial F_{t+3}} \phi_{t+3}^2(F_{t+2}) \\
&\quad - \frac{\partial y_{t+3}^3(F_{t+3})}{\partial F_{t+3}} \phi_{t+3}^3(F_{t+2}, F_{t+1}) - \frac{\partial y_{t+3}^4(F_{t+3})}{\partial F_{t+3}} \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t) \\
&\quad + p^1 \left[\begin{aligned} &\beta y_{t+4}^2(F_{t+4}) \phi_{t+4}^2(F_{t+3}) + \beta^2 y_{t+5}^3(F_{t+5}) \phi_{t+5}^3(F_{t+4}, F_{t+3}) \\ &+ \beta^3 y_{t+6}^4(F_{t+6}) \phi_{t+6}^4(F_{t+5}, F_{t+4}, F_{t+3}) \end{aligned} \right] \\
&\quad + p^2 [\beta y_{t+4}^3(F_{t+4}) \phi_{t+4}^3(F_{t+3}, F_{t+2}) + \beta^2 y_{t+5}^4(F_{t+5}) \phi_{t+5}^4(F_{t+4}, F_{t+3}, F_{t+2})] \\
&\quad + p^3 [\beta y_{t+4}^4(F_{t+4}) \phi_{t+4}^4(F_{t+3}, F_{t+2}, F_{t+1})]
\end{aligned}$$

And for the case of $\vec{F}(4) = (F_{t+kL}, F_{t+kL+1}, F_{t+kL+2}, F_{t+kL+3}) = (0, 0, 0, \infty) \forall k \in \mathbb{Z}$

$$\begin{aligned}
\mu_{t+3}|_{L=n} &= \lim_{F \rightarrow \vec{F}(4)} \left\{ \begin{aligned} &-\frac{\partial y_{t+3}^1(F_{t+3})}{\partial F_{t+3}} - \frac{\partial y_{t+3}^2(F_{t+3})}{\partial F_{t+3}} \phi_{t+3}^2(F_{t+2}) \\ &-\frac{\partial y_{t+3}^3(F_{t+3})}{\partial F_{t+3}} \phi_{t+3}^3(F_{t+2}, F_{t+1}) - \frac{\partial y_{t+3}^4(F_{t+3})}{\partial F_{t+3}} \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t) \end{aligned} \right\} \\
&= -\frac{\partial y_{t+3}^1(\infty)}{\partial F_{t+3}} - \frac{\partial y_{t+3}^2(\infty)}{\partial F_{t+3}} \phi_{t+3}^2(0) - \frac{\partial y_{t+3}^3(\infty)}{\partial F_{t+3}} \phi_{t+3}^3(0, 0) - \frac{\partial y_{t+3}^4(\infty)}{\partial F_{t+3}} \phi_{t+3}^4(0, 0, 0) = 0.
\end{aligned}$$

Without loss of generality it can be said that for any age n the Lagrange multipliers $\mu_t, \mu_{t+1}, \dots, \mu_{t+(n-2)}$ are positive whenever the qualifying conditions (11) hold. Furthermore, $\mu_{t+(n-1)} = 0$ in any case. So any pulse of length $L = n$ satisfies the optimal conditions (16)-(17). Now consider the case of a pulse of length $L = n - 1$. Continuing with the case of age $n = 4$, this pulse is such that $\vec{F}(3) = (F_{t+kL}, F_{t+kL+1}, F_{t+kL+2}) = (0, 0, \infty) \forall k \in \mathbb{Z}$.

For this case μ_t in optimal condition (18) can be expressed as

$$\begin{aligned}
\mu_t|_{L=n-1} &= \lim_{F \rightarrow \vec{F}(3)} \left\{ -\frac{\partial y_t^1(F_t)}{\partial F_t} + \beta^2 [p^1 y_{t+2}^3(F_{t+2}) \phi_{t+2}^3(F_{t+1}, F_t)] \right\} \\
&= -\frac{\partial y_t^1(0)}{\partial F_t} + \beta^2 p^1 y_{t+2}^3(\infty) \phi_{t+3}^3(0, 0) \\
&= -p^1 p r^1 \omega^1 \frac{(1 - e^{-m})}{m} + p^1 p r^3 \omega^3 \beta^2 e^{-2m} \\
&= p^1 \left[p r^3 \omega^3 \beta^2 e^{-2m} - p r^1 \omega^1 \frac{(1 - e^{-m})}{m} \right].
\end{aligned}$$

And μ_{t+1} in the optimal condition (19) can be expressed as

$$\begin{aligned}
\mu_{t+1}|_{L=n-1} &= \lim_{F \rightarrow \vec{F}(3)} \left[-\frac{\partial y_{t+1}^1(F_{t+1})}{\partial F_{t+1}} - \frac{\partial y_{t+1}^2(F_{t+1})}{\partial F_{t+1}} \phi_{t+1}^2(F_t) \right. \\
&\quad \left. + \beta^2 [p^1 y_{t+3}^3(F_{t+3}) \phi_{t+3}^3(F_{t+2}, F_{t+1}) + p^2 y_{t+3}^4(F_{t+3}) \phi_{t+3}^4(F_{t+2}, F_{t+1}, F_t)] \right] \\
&= -\frac{\partial y_{t+1}^1(0)}{\partial F_{t+1}} - \frac{\partial y_{t+1}^2(0)}{\partial F_{t+1}} \phi_{t+1}^2(0) + \beta [p^1 y_{t+2}^2(\infty) \phi_{t+2}^2(0) + p^2 y_{t+3}^3(\infty) \phi_{t+2}^3(0, 0)] \\
&= -p^1 p r^1 \omega^1 \frac{(1 - e^{-m})}{m} - p^2 p r^2 \omega^2 \frac{(1 - e^{-m})}{m} e^{-m} + \beta [p^1 p r^2 \omega^2 e^{-m} + p^2 p r^3 \omega^3 e^{-2m}] \\
&= p^1 \left[p r^2 \omega^2 \beta e^{-m} - p r^1 \omega^1 \frac{(1 - e^{-m})}{m} \right] + p^2 e^{-m} \left[p r^3 \omega^3 \beta e^{-m} - p r^2 \omega^2 \frac{(1 - e^{-m})}{m} \right].
\end{aligned}$$

Finally, μ_{t+2} in the optimal condition (20) can be expressed as

$$\begin{aligned}
\mu_{t+2}|_{L=n-1} &= \lim_{F \rightarrow \vec{F}(3)} \left\{ -\frac{\partial y_{t+2}^1(F_{t+2})}{\partial F_{t+2}} - \frac{\partial y_{t+2}^2(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^2(F_{t+1}) \right. \\
&\quad \left. - \frac{\partial y_{t+2}^3(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^3(F_{t+1}, F_t) - \frac{\partial y_{t+2}^4(F_{t+2})}{\partial F_{t+2}} \phi_{t+2}^4(F_{t+1}, F_t, F_{t-1}) \right\} \\
&= -\frac{\partial y_{t+3}^1(\infty)}{\partial F_{t+3}} - \frac{\partial y_{t+3}^2(\infty)}{\partial F_{t+3}} \phi_{t+3}^2(0) - \frac{\partial y_{t+3}^3(\infty)}{\partial F_{t+3}} \phi_{t+3}^3(0, 0) - \frac{\partial y_{t+3}^4(\infty)}{\partial F_{t+3}} \phi_{t+3}^4(0, 0, \infty) = 0.
\end{aligned}$$

So under the qualifying conditions (11), $\mu_t|_{L=n-1}$ and $\mu_{t+1}|_{L=n-1}$ are positive. So without loss of generality it can be said that for any age n any pulse of length $L = n - 1$ satisfies the optimal conditions (16)-(17).

In the same way it can be proved that any pulse of length $L \leq 4$ is optimal in the case of age $n = 4$. So without loss of generality it can be said that for any age n any pulse of length $L \leq n$ satisfies the optimal conditions (16)-(17).

A.7 Proof of Proposition 7

If F_{ss} is an interior stationary smooth solution then $\mu_t = 0$ and the optimal condition (16) valued in F_{ss} can be expressed as

$$\sum_{a=1}^n \frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a = \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^j y_{ss}^{a+j} \phi_{ss}^{a+j} \right\}. \quad (21)$$

Notice that

$$\begin{aligned}
\sum_{a=1}^{n-1} p^a \left(\sum_{j=1}^{n-a} \beta^j y_{ss}^{a+j} \phi_{ss}^{a,j} \right) &= p^1 [\beta y_{ss}^{2,j} \phi_{ss}^{2,j} + \beta^2 y_{ss}^{3,j} \phi_{ss}^{3,j} + \dots + \beta^{n-1} y_{ss}^n \phi_{ss}^n] \\
&\quad + p^2 [\beta y_{ss}^3 \phi_{ss}^3 + \beta^2 y_{ss}^4 \phi_{ss}^4 + \dots + \beta^{n-2} y_{ss}^n \phi_{ss}^n] + \dots \\
&\quad + p^{n-2} [\beta y_{ss}^{n-1} \phi_{ss}^{n-1} + \beta^2 y_{ss}^n \phi_{ss}^n] + p^{n-1} \beta y_{ss}^n \phi_{ss}^n \\
&= y_{ss}^n \phi_{ss}^n [p^1 \beta^{n-1} + p^2 \beta^{n-2} + \dots + p^{n-1} \beta] \\
&\quad + y_{ss}^{n-1} \phi_{ss}^{n-1} [p^1 \beta^{n-2} + p^2 \beta^{n-3} + \dots + p^{n-2} \beta] + \dots \\
&\quad + y_{ss}^3 \phi_{ss}^3 [p^1 \beta^2 + p^2 \beta] + y_{ss}^2 \phi_{ss}^2 p^1 \beta \\
&= \sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left(\sum_{j=1}^{a-1} \beta^{a-j} p^j \right).
\end{aligned}$$

Therefore, the first order condition valued in the stationary solution, (21), can be expressed as

$$\sum_{a=1}^n \frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a = \sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left(\sum_{j=1}^{a-1} \beta^{a-j} p^j \right).$$

Defining $\varepsilon_{ss}^a = \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{F_{ss}}{y_{ss}^a}$ as the fishing effort elasticity of the yield of age a in the stationary solution, the above expression can be written as

$$\sum_{a=1}^n \varepsilon_{ss}^a s^a = F_{ss} \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left(\sum_{j=1}^{a-1} \beta^{a-j} p^j \right)}{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a},$$

where

$$s^a = \frac{y_{ss}^a \phi_{ss}^a}{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a}.$$

Moreover, total differentiation of (21) implies

$$\begin{aligned}
\frac{\partial F_{ss}}{\partial \beta} &= \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left(\sum_{j=1}^{a-1} (a-j) \beta^{a-j-1} p^j \right)}{\sum_{a=1}^n \left[\frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{\partial \phi_{ss}^a}{\partial F_{ss}} - \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}} \left(\sum_{j=1}^{a-1} \beta^{a-j} p^j \right) \right]}, \\
\frac{\partial F_{ss}}{\partial p^a} &= \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left(\sum_{j=1}^{a-1} \beta^{a-j} \right)}{\sum_{a=1}^n \left[\frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{\partial \phi_{ss}^a}{\partial F_{ss}} - \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}} \left(\sum_{j=1}^{a-1} \beta^{a-j} p^j \right) \right]}.
\end{aligned}$$

Since y_{ss}^a is a concave function and the survival function ϕ_{ss}^a is a decreasing function, both expressions are negative if $y_{ss}^a \phi_{ss}^a$ is an increasing function.

Therefore, the net present value of yield for F_{ss} in the imperfect selectivity n-age model

is given by

$$\begin{aligned}
V_{ss}^{IS} &= \sum_{t=0}^{\infty} \beta^t \sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \\
&= \left[\sum_{a=1}^n pr^a \omega^a \frac{p^a F_{ss}}{p^a F_{ss} + m} (1 - e^{-p^a F_{ss} - m}) \prod_{i=1}^{a-1} e^{-p^{a-i} F_{ss} - m} \right] [1 + \beta + \beta^2 + \beta^3 + \dots] \\
&= \frac{1}{1 - \beta} \sum_{a=1}^n y_{ss}^a \prod_{i=1}^{a-1} e^{-p^{a-i} F_{ss} - m}
\end{aligned}$$

A.8 Proof of Proposition 8

The first order conditions of maximisation problem (10) are given by equations (16)-(17). Notice that the first condition can be understood as a difference equation on F_t of order $2(A - 1) + 1$ because $\{F_{t-(n-1)}, \dots, F_{t-1}, F_t, F_{t+1}, \dots, F_{t+(n-1)}\}$ appear in it. For interior solution this condition can be written as the following difference equation system, $\forall t$

$$\begin{aligned}
\Psi_t &= \sum_{a=1}^n pr^a \frac{\partial y_t^a(F_t)}{\partial F_t} \phi_t^a(F_{t-1}, F_{t-2}, \dots, F_{t-(a-1)}) \\
&\quad - \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^j y_{t+j}^{a+j}(F_{t+j}) \phi_{t+j}^{a+j}(F_{t+j-1}, F_{t+j-2}, \dots, F_{t+j-(a-1)}) \right\} = 0.
\end{aligned}$$

So second order conditions associated with maximisation problem (10) can be analysed using the Hessian matrix given by

$$\begin{bmatrix}
\frac{\partial \Psi_t}{\partial F_t} & \frac{\partial \Psi_t}{\partial F_{t+1}} & \dots & \frac{\partial \Psi_t}{\partial F_{t+(n-1)}} \\
\frac{\partial \Psi_{t+1}}{\partial F_t} & \frac{\partial \Psi_{t+1}}{\partial F_{t+1}} & \dots & \frac{\partial \Psi_{t+1}}{\partial F_{t+(n-1)}} \\
\dots & \dots & \dots & \dots \\
\frac{\partial \Psi_{t+(n-1)}}{\partial F_t} & \frac{\partial \Psi_{t+(n-1)}}{\partial F_{t+1}} & \dots & \frac{\partial \Psi_{t+(n-1)}}{\partial F_{t+(n-1)}}
\end{bmatrix}.$$

It is known that any solution satisfying first order condition (16) and for which the Hessian is not a negative definite cannot be a maximum.

In the stationary solution all the elements in the diagonal are given by

$$\left. \frac{\partial \Psi_{t+k}}{\partial F_{t+k}} \right|_{F_{ss}} = \beta^k \left. \frac{\partial \Psi_t}{\partial F_t} \right|_{F_{ss}} \quad \forall k = -(A - 1), \dots, 0, \dots, (A - 1)$$

where

$$\begin{aligned}
\left. \frac{\partial \varphi_t}{\partial F_t} \right|_{F_{ss}} &= \beta^t \left[\sum_{a=1}^n \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \sum_{j=1}^{n-1} \beta^j \sum_{a=j+1}^n y_{ss}^a (-p^{a-j})^2 \phi_{ss}^a \right] \\
&= \beta^t \left[\sum_{a=1}^n \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \sum_{a=1}^n y_{ss}^a \phi_{ss}^a \sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right] \\
&= \beta^t \sum_{a=1}^n \left[\frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} + y_{ss}^a \sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right] \phi_{ss}^a
\end{aligned}$$

Notice that if $\sum_{a=1}^n \left[\frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} + y_{ss}^a \sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right] \phi_{ss}^a \geq 0$, then the principal minor of order one in the Hessian is not negative and it can be stated that the stationary solution is not optimal. ■

A.9 Proof of Proposition 9

The value of the fishery harvested with a pulse of length L under imperfect selectivity is given by (8). Taking the partial derivative the following is obtained:

$$\frac{\partial V_L^{IS}}{\partial \beta} = \frac{(L-1)\beta^{L-2} + \beta^{2(L-1)}}{(1-\beta^L)^2} \sum_{a=1}^L pr^a w^a e^{-(a-1)m} > 0.$$

Doing the same for V_{ss}^{IS} defined in Proposition 7 results in

$$\begin{aligned}
\frac{\partial V_{ss}^{IS}}{\partial \beta} &= \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \left[\frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a + y_{ss}^a \frac{\partial \phi_{ss}^a}{\partial F_{ss}} \right] \\
&= \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \left[\frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a - p^a y_{ss}^a \phi_{ss}^a \right] \\
&= \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}}.
\end{aligned}$$

Assuming that $y_{ss}^a \phi_{ss}^a$ is an increasing function, proposition (5) shows that $\partial F_{ss} / \partial \beta < 0$. This implies

$$\frac{\partial V_{ss}^{IS}}{\partial \beta} = \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}} < \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a. \quad (22)$$

Now, let us assume that for a given β , the two solutions are equivalent, $V_{pulse}^{IS} = V_{ss}^{IS}$. This

implies that

$$\frac{1}{1-\beta} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \frac{\beta^{L-1}}{1-\beta^L} \sum_{a=1}^L pr^a w^a e^{-(a-1)m}.$$

Taking this into account (22) can be expressed as

$$\begin{aligned}
\frac{\partial V_{ss}^{IS}}{\partial \beta} &< \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \frac{1}{(1-\beta)^2} \frac{(1-\beta) \beta^{L-1}}{1-\beta^L} \sum_{a=1}^L pr^a w^a e^{-(a-1)m} \\
&= \frac{1}{(1+\beta)} \frac{\beta^{L-1}}{1-\beta^L} \frac{(1-\beta^L)^2}{(L-1)\beta^{L-2} + \beta^{2(L-1)}} \frac{\partial V_L^{IS}}{\partial \beta} \\
&= \frac{\beta^{L-1} (1-\beta^L)}{(1+\beta) [(L-1)\beta^{L-2} + \beta^{2(L-1)}]} \frac{\partial V_L^{IS}}{\partial \beta}.
\end{aligned}$$

It can be proved that the factor multiplying $\partial V_L^{IS}/\partial \beta$ is lower than 1 by contradiction. Assume that it is greater than 1; in that case it should hold that

$$\begin{aligned}
\beta^{L-1} (1-\beta^L) &> (1+\beta) [(L-1)\beta^{L-2} + \beta^{2(L-1)}], \\
\beta^{L-1} - \beta^{2L-1} &> (L-1)\beta^{L-2} + \beta^{2(L-1)} + (L-1)\beta^{L-1} + \beta^{2L-1}, \\
(2-L)\beta^{L-1} - 2\beta^{2L-1} - \beta^{2(L-1)} - (L-1)\beta^{L-2} &> 0.
\end{aligned}$$

But this is not true because $1 < L < n$.

Therefore, whenever $y_{ss}^a \phi_{ss}^a$ is an increasing function $\frac{\partial V_{ss}^{IS}}{\partial \beta} < \frac{\partial V_{pulse}^{IS}}{\partial \beta}$. ■