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**A MONOTONIC CORE CONCEPT
FOR CONVEX GAMES:
THE SD-PRENUCLEOLUS**

by

J. Arin and I. Katsev

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University of the Basque Country

A monotonic core concept for convex games: The SD-prenucleolus

J. Arin* and I. Katsev†

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Abstract

We prove that the SD-prenucleolus satisfies monotonicity in the class of convex games. The SD-prenucleolus is thus the only known continuous core concept that satisfies monotonicity for convex games. We also prove that for convex games the SD-prenucleolus and the SD-prekernel coincide.

Keywords: TU games, prenucleolus, monotonicity.

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*Dpto. Ftos. A. Económico I, University of the Basque Country, L. Agirre 83, 48015 Bilbao, Spain. Email: jeparagj@bs.ehu.es. J. Arin acknowledges the support of the Spanish Ministerio de Ciencia e Innovación under projects ECO2012-31346 and by Basque Government funding for Grupo Consolidado GIC07/146-IT-377-07.

†St Petersburg Institute for Economics and Mathematics. Email: katsev@yandex.ru

1 Introduction

How to divide the outcome obtained by agents that cooperate is one of the main issues analyzed in the literature of coalitional game theory. One approach to dealing with the problem consists of proposing rules or solutions that are used to solve the game. In this approach, the Shapley value (Shapley, 1953) and the prenucleolus (Schmeidler, 1969) stand out as the most well-known, widely analyzed single-valued solutions for coalitional games with transferable utility (TU games). One of the main reasons for the attractiveness of the Shapley value lies in the fact that it respects the principle of monotonicity, i.e. if a new TU game w is obtained from a given TU game v by increasing the worth of a coalition S then the members of S receive a payoff in game w that is no lower than in game v . On the other hand, the prenucleolus respects the core stability principle, i.e. the prenucleolus selects a core allocation whenever the game is balanced. A core allocation provides each coalition with at least the worth of the coalition, the amount that the members of the coalition can obtain by themselves. It seems very attractive to ask for a solution that fulfils both principles, since they share a kind of incentive compatibility principle that can be summarized in the following idea: the higher the worth of a coalition the better for its members. However, in the class of balanced games they are not compatible (Young, 1985) and therefore the Shapley value does not respect core stability and the prenucleolus fails to satisfy monotonicity. The two principles are compatible in other domains or subclasses of TU games. For example, the Shapley value satisfies core stability in the class of convex games and the SD-prenucleolus, a lexicographic value introduced by Arin and Katsev in 2011, satisfies core stability and monotonicity in the class of veto balanced games (in this class, the Shapley value does not satisfy core stability). These results motivate the question that this paper seeks to solve: Is the SD-prenucleolus monotonic in the class of convex games? The answer is yes, so the SD-prenucleolus arises as the only known solution that satisfies monotonicity in the class of convex games and in the class of veto balanced games while respecting the principle of core stability.

Another major contribution of this paper is to prove that in the class

of convex games the SD-prekernel (Arin and Katsev, 2011) and the SD-prenucleolus coincide. The coincidence of the prekernel and the prenucleolus in the class of convex games was established by Maschler, Peleg and Shapley (1971).

The rest of the paper is organized as follows:

Section 2 introduces TU games, solutions and properties. Section 3 provides a detailed introduction to the definition of SD-prenucleolus of a game, the notion of “relevant coalition” and the concept of “SD-reduced game property”. This section is based on Arin and Katsev (2011). In Section 4 we analyze the monotonicity of the SD-prenucleolus when considering SD-relevant games. Section 5 deals with the class of convex games: we show that they are SD-relevant games. We also prove that the SD-prekernel and the SD-prenucleolus coincide for convex games.

2 Preliminaries: TU games

A *cooperative n -person game in characteristic function form* is a pair (N, v) , where N is a finite set of n elements and $v : 2^N \rightarrow \mathbb{R}$ is a¹ real-valued function in the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of N is called a *coalition*. Singletons are coalitions that contain only one player. A game is *monotonic* if whenever $T \subset S$ then $v(T) \leq v(S)$. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \setminus S$ the set of players of N that are not in S . A distribution of $v(N)$ among the players, an allocation, is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i . A distribution satisfying $\sum_{i \in N} x_i = v(N)$ is called an *efficient allocation* and the set of efficient allocations is denoted by $X(v)$. We denote $\sum_{i \in S} x_i$ by $x(S)$. The core of a game is the set of allocations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

¹See next section for a formal definition of these concepts.

It has been shown that a game with a non-empty core is balanced² and therefore games with non-empty core are called balanced games. *Player i is a veto player if $v(S) = 0$ for all S where player i is not present.* A balanced game with at least one veto player is called a veto balanced game. We denote by Γ_{VB} the class of balanced games and by Γ_{VB} the class of veto balanced games.

We say that a game (N, v) is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subset N$. We denote by Γ_C the class of convex games.

A solution φ in a class of games Γ_0 is a correspondence that associates a set $\varphi(N, v)$ in \mathbb{R}^N with each game (N, v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N, v)$. This solution is *efficient* if this inequality holds with equality. The solution is *single-valued* if the set contains a *single* element for each game in the class.

We say that the vector x *weakly lexicographically dominates* the vector y (denoted by $x \preceq_L y$) if either $\tilde{x} = \tilde{y}$ or there exists k such that $\tilde{x}_i = \tilde{y}_i$ for all $i \in \{1, 2, \dots, k-1\}$ and $\tilde{x}_k > \tilde{y}_k$ where \tilde{x} and \tilde{y} are the vectors with the same components as the vectors \tilde{x}, \tilde{y} , but rearranged in a non decreasing order ($i > j \Rightarrow \tilde{x}_i \leq \tilde{x}_j$).

Given $x \in \mathbb{R}^N$ the *satisfaction of a coalition S with respect to x* in a game v is defined as $e(S, x) := x(S) - v(S)$. Let $\theta(x)$ be the vector of all satisfactions at x arranged in non decreasing order. Schmeidler (1969) introduced the *prenucleolus* of a game v , denoted by $PN(v)$, as the unique allocation that lexicographically maximizes the vector of non decreasingly ordered satisfactions over the set of allocations. In formula:

$$PN(N, v) = \{x \in X(N, v) \mid \theta(x) \preceq_L \theta(y) \text{ for all } y \in X(N, v)\}.$$

For any game v the prenucleolus is a single-valued solution, is contained in the prekernel and lies in the core provided that the core is non-empty.

The per capita prenucleolus (Groote, 1970) is defined analogously by using the concept of per capita satisfaction instead of excess. Given S and x the per capita satisfaction of S at x is

$$e^{pc}(S, x) := \frac{x(S) - v(S)}{|S|}$$

²See Peleg and Sudholter (2007).

Other weighted prenucleoli can be defined in a similar way whenever a weighted excess function is defined. The same solution concepts can be analogously defined using the notion of satisfaction instead of excess. Given $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game (N, v) is defined as $f(S, x) := x(S) - v(S)$. In this paper we use the notion of satisfaction in defining the new solution.

Some convenient and well-known properties of a solution concept φ on Γ_0 are the following.

- φ satisfies **core stability** if it selects core allocations whenever the game is balanced.

The following properties are defined for single-valued solutions.

- φ satisfies **coalitional monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq T$, $v(S) = w(S)$ and $v(T) < w(T)$, then for all $i \in T$, $\varphi_i(v) \leq \varphi_i(w)$.
- φ satisfies **aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) \geq 0$.
- φ satisfies **strong aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) = \varphi_j(w) - \varphi_j(v) \geq 0$.

Young (1985) proves that no solution satisfies coalitional monotonicity and core stability. However there are solutions, including the per capita prenucleolus and the SD-prenucleolus, that satisfy core stability and strong aggregate monotonicity. Meggido (1974) proves that the nucleolus does not satisfy aggregate monotonicity. Clearly, strong aggregate monotonicity implies aggregate monotonicity. The prenucleolus does not satisfy aggregate-monotonicity in the class of convex games (Hokari, 2000). The per capita prenucleolus does not satisfy monotonicity in the class of convex games (see Arin, 2013).

The following notation is widely used in this work. We denote by (N, u_S) the game:

$$u_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

3 The SD-Prenucleolus

3.1 Definition and some properties

In 2011, Arin and Katsev introduced and characterized the SD-prenucleolus, a solution concept for TU games. In this section we recall some definitions and results that are needed in the present paper.

The definition of the SD-prenucleolus is based in the concept of satisfaction of a coalition given an allocation. Given a game (N, v) and an allocation x we calculate a satisfaction vector $\{F(S, x)\}_{S \subseteq N}$. The components of this vector are obtained recursively by defining an algorithm.

The algorithm has several steps (at most $2^n - 2$) and at each step we identify the collection of coalitions that has obtained the satisfaction. This collection of coalitions is denoted by \mathcal{H} . In the first step this collection \mathcal{H} is empty. The algorithm ends when $\mathcal{H} = 2^N$.

For a collection \mathcal{H} and a function $F : \mathcal{H} \rightarrow \mathbb{R}$ the function $F_{\mathcal{H}} : 2^N \rightarrow \mathbb{R}$ is defined. To that end, we introduce some notation. Denote by $\mathcal{H} \subset 2^N$

$$\sigma_{\mathcal{H}}(S) = \bigcup_{T \in \mathcal{H}, T \subset S} T$$

and also for a collection $\mathcal{H} \subset 2^N$ and a function $F : \mathcal{H} \rightarrow \mathbb{R}$ we denote by $f_{\mathcal{H}, F}(i, S)$ the satisfaction of player i with respect to a coalition S and a collection \mathcal{H} ($i \in \sigma_{\mathcal{H}}(S)$):

$$f_{\mathcal{H}, F}(i, S) = \min_{T: T \in \mathcal{H}, i \in T \subset S} F(T)$$

Note that this definition can only be used in a situation when the function $F(S)$ is defined for all $S \in \mathcal{H}$.

Now we define a function $F_{\mathcal{H}} : 2^N \rightarrow \mathbb{R}$. We consider two cases (since it is evident that $\sigma_{\mathcal{H}}(S) \subset S$):

1. *Relevant coalitions.* $\sigma_{\mathcal{H}}(S) \neq S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i, S)}{|S| - |\sigma_{\mathcal{H}}(S)|}$$

Note that if the collection \mathcal{H} is empty then the current satisfaction of the coalition S coincides with its *per capita satisfaction*:

$$F_{\emptyset}(S) = \frac{x(S) - v(S)}{|S|}$$

2. *Non relevant coalitions.* $\sigma_{\mathcal{H}}(S) = S$. In this case the current satisfaction of S is

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i \in S} f_{\mathcal{H},F}(i, S) + \max_{i \in S} f_{\mathcal{H},F}(i, S)$$

Therefore for any function $F : \mathcal{H} \rightarrow \mathbb{R}$ the value $f_{\mathcal{H},F}(i, S)$ can be calculated for every coalition S and player $i \in \sigma_{\mathcal{H}}(S)$. Also if a function $f_{\mathcal{H},F}(\cdot, \cdot)$ is defined for each $S \subsetneq N$ and $i \in \sigma_{\mathcal{H}}(S)$ then the function $F_{\mathcal{H}}$ can be defined.

The algorithm for the satisfaction vector is defined as follows:.

Algorithm 1 Consider a game (N, v) and an allocation $x \in X(N, v)$.

Step 1: Set $k = 0$, $\mathcal{H}_0 = \emptyset$. Go to Step 2.

Step 2: Set

$$\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{S \notin \mathcal{H}_k : F_{\mathcal{H}_k}(S) = \min_{T \notin \mathcal{H}_k} F_{\mathcal{H}_k}(T)\}$$

Step 3: Define for each $S \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$:

$$F(S) = F_{\mathcal{H}_k}(S)$$

Step 4: If $\mathcal{H}_{k+1} \neq 2^N \setminus \{N\}$ then let $k = k + 1$ and go to Step 2, else go to Step 5.

Step 5: Stop. Return the vector

$$\{F(S), S \subsetneq N\}$$

For the sake of simplicity we use the notation $F(S)$ instead of $F(S, x)$.

Note that for a relevant coalition ($\sigma_{\mathcal{H}}(S) \neq S$) it holds that

$$x(S) - v(S) = (|S| - |\sigma_{\mathcal{H}}(S)|)F_{\mathcal{H}}(S) + \sum_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i, S) = \sum_{i \in S} f_{\mathcal{H},F}(i, S)$$

which can be interpreted as a distribution of the total surplus of coalition S among its members.

We define the SD-prenucleolus as a lexicographic value in the set of vectors of satisfactions. We denote the SD-prenucleolus of game (N, v) by $SD(N, v)$.

We say that the vector x belongs to the SD-prenucleolus if its satisfaction vector dominates (or weakly dominates) every other satisfaction vector.

Definition 2 (*Arin and Katsev, 2011*) *Let (N, v) be a TU game. Then $x \in SD(N, v)$ if and only if for any $y \in X(N, v)$ it holds that $F^x \succeq_L F^y$.*

The SD-prenucleolus satisfies nonemptiness and single-valuedness in the class of all TU games.

Let (N, v) be a TU game and x be an allocation. Denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction at x . Given an allocation x and a real number α we define the following set of coalitions

$$\mathcal{B}_{\alpha} = \{S \subsetneq N : F(S, x) \leq \alpha\}.$$

Next theorem may be used to check whether an allocation is the SD-prenucleolus of a game or not.

Theorem 3 (*Arin and Katsev, 2011*) *Let (N, v) be a TU game and x be an allocation. Then $x = SD(N, v)$ if and only if the collection of sets \mathcal{B}_{α} is empty or balanced³ for every α .*

The notion of “relevant coalition” plays a central role in this paper. The following 3-person game is used to illustrate this concept. Let (N, v) be a

³See Peleg and Sudholter (2007) for the definition of a balanced collection of sets.

game where $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 4 & \text{if } S \in \{\{1, 3\}, \{1, 2\}\} \\ -10 & \text{if } S = \{2, 3\} \\ 6 & \text{if } S = N. \end{cases}$$

Consider the allocation $x = (5, 1, 0)$. Applying the algorithm the following is obtained:

Coalition	Satisfaction
$\{3\}$	0
$\{2\} \{1, 2\} \{1, 3\}$	1
$\{1\}$	5
$\{2, 3\}$	11.

Coalition $\{2, 3\}$ is a non relevant coalition. The rest of the coalitions are relevant coalitions. Consider the satisfaction of coalition $\{1, 3\}$. This coalition has a subset (coalition $\{3\}$) that has already obtained its satisfaction. This fact is incorporated into the computation of the satisfaction of coalition $\{1, 3\}$ since $\sigma_{\mathcal{H}}(\{1, 3\}) = \{3\}$. Therefore

$$F_{\mathcal{H}}(\{1, 3\}, x) = \frac{x(\{1, 3\}) - v(\{1, 3\}) - \sum_{i \in \sigma_{\mathcal{H}}(\{1, 3\})} f_{\mathcal{H}, F}(i, \{1, 3\})}{|\{1, 3\}| - |\sigma_{\mathcal{H}}(\{1, 3\})|} = \frac{5 - 4 - 0}{2 - 1}.$$

The total surplus of the coalition is divided as follows: player 1 gets 1 and player 3 gets 0.

The case of non relevant coalitions is different. If a coalition is non relevant for any player in the coalition there exists a subset of the coalition with a lower satisfaction and that subset determines the individual satisfaction of the player in the non relevant coalition. Note that

$$x(\{2, 3\}) - v(\{2, 3\}) = 11 > \sum_{i \in \sigma_{\mathcal{H}}(\{2, 3\})} f_{\mathcal{H}, F}(i, \{2, 3\}) = 1 + 0.$$

Theorem 3 may be used to check that $SD(N, v) \neq x$ since the collection of sets with minimal satisfaction with respect to x is not balanced. Theorem 3 may be used to check that $SD(N, v) = (\frac{14}{3}, \frac{2}{3}, \frac{2}{3})$.

In the proof of the main theorem we need to use the fact that in the class of all TU games the SD-prenucleolus satisfies the SD-reduced game property⁴. Arin and Katsev (2011) introduce the SD-reduced game.

Definition 4 Let (N, v) be a TU game, $S \subset N$ and $x \in X(N)$. A game (S, v_S^x) is the SD-reduced game with respect to S and x if

1. $v_S^x(S) = v(N) - x(N \setminus S)$
2. for every $T \subsetneq S$

$$F^{(S, v_S^x)}(T, x_S) = \min_{U \in N \setminus S} F^{(N, v)}(U \cup T, x).$$

For any game (N, v) and any allocation x the SD-reduced game exists and is unique.

We say that a solution ϕ satisfies the *SD-reduced game property*⁵ on Γ , *SD-RGP*, if for every game $v \in \Gamma$, for all nonempty coalitions S and for all $x \in \phi(v)$, $x^S \in \phi((S, v_S^x))$.

This property plays a determinant role in the proof of the main theorem (the monotonicity of the SD-prenucleolus in the class of convex games).

The SD-prenucleolus satisfies the SD-reduced game property.

3.2 Antipartitions

The notion of antipartition (Arin and Inarra, 1998) also plays a central role in the main results of this paper.

A collection of sets $\mathcal{C} = \{S : S \subset N\}$ is called *antipartition* if the collection of sets $\{N \setminus S : S \in \mathcal{C}\}$ is a partition of N . Note that in order to balance an antipartition \mathcal{Q} each coalition receives the same weight, i.e. $\frac{1}{|\mathcal{C}|-1}$.

For any game (N, v) the *satisfaction of an antipartition* \mathcal{C} with weights $(\frac{1}{|\mathcal{C}|-1})_{S \in \mathcal{C}}$ is defined by

$$F(\mathcal{C}) := \frac{v(N) - \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S)}{|N|}.$$

⁴The SD-prekernel also satisfies the SD-reduced game property.

⁵This property states that if x is an element in the multi-valued function ϕ of a game $v \in \Gamma$, then for any non-trivial coalition T the projection of x into $N \setminus T$ belongs to the multi-valued function ϕ of the reduced game v for coalition $N \setminus T$ with respect to x .

Theorem 5 *Given a game (N, v) and an allocation x , if the collection of sets with minimal satisfaction $\mathcal{B}(x)$ contains an antipartition \mathcal{C} then $F(S, x) = F(\mathcal{C})$ for all S belonging to $\mathcal{B}(x)$.*

Proof. Let x be an allocation and let \mathcal{C} be an antipartition in $\mathcal{B}(x)$. Note that for $S \in \mathcal{C}$ it results that $F(S, x) = \frac{x(S) - v(S)}{|S|} = \alpha$.

Since \mathcal{C} is balanced

$$\sum_{S \in \mathcal{C}} \lambda_S x(S) = \sum_{i \in N} x_i = v(N).$$

From the definition above the following emerges:

$$|N| F(\mathcal{C}) = v(N) - \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}| - 1} v(S).$$

From the balancedness of \mathcal{C} it holds that

$$\begin{aligned} v(N) - \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}| - 1} v(S) &= \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}| - 1} (x(S) - v(S)) = \\ \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}| - 1} |S| F(S, x) &= \alpha \sum_{S \in \mathcal{C}} \frac{1}{|\mathcal{C}| - 1} |S| = \alpha |N|. \end{aligned}$$

Last equality is a direct consequence of the fact that each player is present in all coalitions of the antipartition but one. ■

Note that if the set of coalitions with minimal satisfaction with respect to the SD-pre-nucleolus of the game contains an antipartition then the satisfaction of these coalitions only depends on the characteristic function of the game.

4 SD-relevant games: Monotonicity of the SD-pre-nucleolus

In this section we introduce and study the class of SD-relevant TU games. We show that convex games are SD-relevant games and by following Arin and Katsev (2011) it is not difficult to prove that monotonic games with

veto players are also SD-relevant games. These two facts make this new class⁶ of games interesting.

A TU game is SD-relevant if given the SD-prenucleolus of the game all its coalitions are relevant.

Definition 6 *We say that a TU game (N, v) is SD-relevant if any $S, S \subset N$, is relevant with respect to $SD(N, v)$.*

The game (N, v) where $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 4 & \text{if } S \in \{\{1, 3\}, \{1, 2\}\} \\ -10 & \text{if } S = \{2, 3\} \\ 6 & \text{if } S = N \end{cases}$$

is not SD-relevant. Recall that $SD(N, v) = (\frac{14}{3}, \frac{2}{3}, \frac{2}{3})$ and clearly, coalition $\{2, 3\}$ is non relevant.

In what follows we provide an alternative way of computing the SD-reduced games of an SD-relevant game which bears strong similarities to the well known Davis Maschler reduced game.

The lemma below proves that the surplus of relevant coalitions is fully divided among the members of the coalitions. This is not the case with non relevant coalitions, where the surplus of the coalition is higher than the sum of the surpluses of the members of the coalition.

Lemma 7 *For every game (N, v) and allocation $x \in X(v)$ it holds that*

1. *For every relevant coalition $S \subset N$*

$$\sum_{i \in S} f(i, S) = x(S) - v(S).$$

2. *For every non relevant coalition $S \subset N$*

$$\sum_{i \in S} f(i, S) < x(S) - v(S).$$

⁶The linear convex combination of two SD-relevant games is not necessarily a SD-relevant game.

3. For every coalition $S \subset T \subset N$

$$f(i, S) \geq f(i, T) \quad \text{for any } i \in S.$$

Proof. 1. By the definition of satisfaction of a relevant coalition it holds that

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i, S)}{|S| - |\sigma_{\mathcal{H}}(S)|}.$$

Therefore

$$\begin{aligned} x(S) - v(S) &= F_{\mathcal{H}}(S)(|S| - |\sigma_{\mathcal{H}}(S)|) + \sum_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i, S) = \\ &= \sum_{i \in S \setminus \sigma_{\mathcal{H}}(S)} f(i, S) + \sum_{i \in \sigma_{\mathcal{H}}(S)} f(i, S) = \sum_{i \in S} f(i, S) \end{aligned}$$

2. By the definition of satisfaction of a non relevant coalition it holds that

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i \in S} f_{\mathcal{H},F}(i, S) + \max_{i \in S} f_{\mathcal{H},F}(i, S).$$

By Lemma 3 from Arin and Katsev (2011) $F_{\mathcal{H}}(S) > \max_{i \in S} f_{\mathcal{H},F}(i, S)$ and therefore

$$\sum_{i \in S} f(i, S) < x(S) - v(S).$$

3. It is immediately apparent. ■

If a game is SD-relevant then any SD-reduced game with respect to the SD-prenucleolus of the game is also SD-relevant. Therefore in the class of SD-relevant games all the SD-reduced games with respect to the SD-prenucleolus belong to this class.

Lemma 8 *Let (N, v) be an SD-relevant TU game. Let (S, v^{SD}) be an SD-reduced game with respect to the SD-prenucleolus of (N, v) . Then (S, v^{SD}) is an SD-relevant TU game.*

Proof. Let P and M two subsets of S such that $F(M, SD(S, v^{SD})) \geq F(P, SD(S, v^{SD}))$ and $M \cup P \neq N$. We seek to prove that

$$F(M \cup P, SD(S, v^{SD})) \leq \max \{F(M, SD(S, v^{SD})), F(P, SD(S, v^{SD}))\}.$$

Let $F(M, SD(S, v^{SD})) = F(M \cup Q, SD(N, v^{SD}))$ for some $Q \subseteq N \setminus S$ and let $F(P, SD(S, v^{SD})) = F(P \cup T, SD(N, v^{SD}))$ for some $T \subseteq N \setminus S$. Clearly, $(M \cup Q) \cup (P \cup T) \neq N$.

Since all coalitions in the game (N, v) are relevant It holds that

$$\begin{aligned} F((M \cup Q) \cup (P \cup T), SD(N, v^{SD})) &\leq \\ \max \{F(M \cup Q, SD(N, v^{SD})), F(P \cup T, SD(N, v^{SD}))\} &= \\ F(M \cup Q, SD(N, v^{SD})). & \end{aligned}$$

Note that $(M \cup Q) \cup (P \cup T) = (M \cup P) \cup (Q \cup T)$ and therefore

$$\begin{aligned} F(M \cup P, SD(S, v^{SD})) &\leq F((M \cup Q) \cup (P \cup T), SD(N, v^{SD})) \leq \\ &\leq F(M \cup Q, SD(N, v^{SD})). \end{aligned}$$

Therefore the SD-reduced game of an SD-relevant game is SD-relevant.

■

Arin and Inarra (1998) prove that, given a convex game, the collection of coalitions with minimal satisfaction with respect to the prenucleolus of the game contains either a partition or an antipartition. In the case of the SD-prenucleolus of an SD-relevant game only antipartitions should be considered, as the following theorem shows.

Lemma 9 *Let (N, v) be an SD-relevant TU game. Then the collection of sets with minimal satisfactions with respect to $SD(N, v)$ contains an antipartition.*

Proof. Let $x = SD(N, v)$ and let $\mathcal{B}(x)$ be the set of coalitions with minimal satisfaction with respect to x . Let S be a maximal coalition in $\mathcal{B}(x)$, that is, there is no coalition T in $\mathcal{B}(x)$ such that $S \subset T$. Since $\mathcal{B}(x)$ is balanced for each $i \in S$ there exists a coalition, T^i , such that $i \notin T^i$ and $T^i \in \mathcal{B}(x)$. Since (N, v) is SD-relevant the maximality of S implies that

$N \setminus S \subset T^i$. Let $\{T^i : i \in S\}$ be the set of maximal coalitions for each i in S ((perhaps the case in which for two players i, j it holds that $T^i = T^j$). Then $\{T^i : i \in S\} \cup \{S\}$ is an antipartition. It is immediately apparent that $(N \setminus T^i) \cap (N \setminus S)$ is empty. If for any $i, j \in S$ it holds that $(N \setminus T^i) \cap (N \setminus T^j)$ is nonempty it is clear that $T^i \cup T^j \neq N$ which contradicts the maximality of T^i and T^j since the fact that (N, v) is SD-relevant implies that $T^i \cup T^j$ is an element of the set $\mathcal{B}(x)$. ■

The above results allow for a different interpretation of the SD-reduced game of an SD-relevant game. The SD-reduced games with respect to the SD-prenucleolus can be easily computed according to the result established by the following lemma.

Lemma 10 *Let (N, v) be an SD-relevant TU game, $S \subset N$ and $x = SD(N, v)$. Consider the SD-reduced game (S, v_S^x) . Then*

$$v_S^x(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} z_i(T \cup (N \setminus S)) = \sum_{i \in T} z_i(T \cup (N \setminus S)).$$

$$\text{where } z_i(T \cup (N \setminus S)) = x_i - f_i(T \cup (N \setminus S))$$

Proof. By Lemma 8 (S, v_S^x) is SD-relevant. We denote by f_S^x the analog of function f for the game (S, v_S^x) .

By definition of $f_S^x(i, T)$ it holds that

$$\begin{aligned} f_S^x(i, T) &= \min_{i \in U \subset T} F^{(S, v_S^x)}(U) = \min_{i \in U \subset T} \min_{R \subset N \setminus S} F(U \cup R) = \\ &= \min_{i \in M \subset T \cup (N \setminus S)} F(M) = f(i, T \cup (N \setminus S)). \end{aligned}$$

Therefore

$$\begin{aligned} v_S^x(T) &= x(T) - \sum_{i \in T} f_S^x(i, T) = x(T) - \sum_{i \in T} f(i, T \cup (N \setminus S)) = \\ &= \sum_{i \in T} z_i(T \cup (N \setminus S)) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} z_i(T \cup (N \setminus S)). \end{aligned}$$

Here we use the fact that the coalition $T \cup (N \setminus S)$ is relevant in the game (N, v) . ■

The corollary below presents a simple formula for computing some SD-reduced games. This result is used in the proof of the main theorem.

Corollary 11 *Let (N, v) be an SD-relevant TU game, $x = SD(N, v)$ and $S \in \mathcal{B}(x)$. Consider the SD-reduced game $(N \setminus S, v^x)$ and assume that coalition $T \subset N \setminus S$ is relevant in game $(N \setminus S, v^x)$ at x . Then*

$$v_S^x(T) = v(T \cup S) - \sum_{i \in S} z_i(T \cup S) = v(T \cup S) - v(S).$$

Proof. Since $S \in \mathcal{B}(x)$ it holds that $f(i, S) = \frac{x(S) - v(S)}{|S|}$. Since (N, v) is SD-relevant, for any T such that $S \subset T$ it holds that $f(i, T) = f(i, S)$. Therefore,

$$\begin{aligned} \sum_{i \in S} z_i(T \cup S) &= \sum_{i \in S} x_i(T \cup S) - \sum_{i \in S} f(i, T \cup S) = \\ \sum_{i \in S} (x_i(S) - f(i, S)) &= x(S) - |S| \frac{x(S) - v(S)}{|S|} = v(S). \end{aligned}$$

■

The notion of SD-equivalent games is also needed in the proof of the main results of the paper. We say that two TU games are SD-equivalents if the two games have in the set of coalitions of minimal satisfaction with respect to the SD-prenucleolus the same antipartition.

Definition 12 *Let $x = SD(N, v)$ and let $y = SD(N, w)$. We say that TU games (N, v) and (N, w) are SD-equivalent if there exists an antipartition Q such that $Q \subseteq \mathcal{B}(x)$ and $Q \subseteq \mathcal{B}(y)$.*

Next lemma allows us to consider only SD-relevant and SD-equivalent game while analyzing the monotonicity of the SD-prenucleolus in the class of SD-relevant games.

Lemma 13 *For some $S \subset N$ and any $\gamma \in [0, \alpha]$, let $(N, v + \gamma u_S)$ be an SD-relevant TU game. Then, there exists β , $0 < \beta \leq \alpha$ such that:*

- 1 (N, v) and $(N, v + \beta u_S)$ are SD-equivalent TU games.
- 2 $(N, v + \beta u_S)$ and $(N, v + \alpha u_S)$ are SD-equivalent TU games.

Proof. Let $y = SD(N, v + \alpha u_S) = SD(N, w)$ and $x = SD(N, v)$. Let \mathcal{Q} be an antipartition contained in $\mathcal{B}(x)$. If \mathcal{Q} is contained in $\mathcal{B}(y)$ then it is evident that $\alpha = \beta$. If \mathcal{Q} is not contained in $\mathcal{B}(y)$ then it must be the case that $S \in \mathcal{B}(y)$ and $F(\mathcal{Q}, v) > F(S, y, w)$ and any antipartition in $\mathcal{B}(y)$ must include S . Let \mathcal{M} be an antipartition in $\mathcal{B}(y)$. If \mathcal{M} is an antipartition in $\mathcal{B}(y)$ the proof is completed and $\alpha = \beta$. If not, it is clear that $F(\mathcal{M}, v) > F(\mathcal{Q}, v) > F(\mathcal{Q}, w) > F(\mathcal{M}, w)$. Therefore by decreasing α we can find a new game $(N, q) = (N, v + \beta u_S)$ such that

$$F(\mathcal{M}, q) = F(\mathcal{Q}, v) = F(\mathcal{Q}, q) > F(\mathcal{M}, w)$$

Therefore with the game (N, q) the statement of the lemma is proved for this last case. ■

Now we are in a position to prove the main theorem of this section.

Theorem 14 *For some $S \subset N$ and any $\gamma \in [0, \alpha]$, let $(N, v + \gamma u_S)$ be an SD-relevant TU game. Then $SD_i(N, v + \alpha u_S) \geq SD_i(N, v)$ for any $i \in S$.*

Proof. The fact can be proved by induction for $|N|$. If $|N| \leq 2$ then the monotonicity holds. Assume that it holds for all games with no more than k players. Now we show that it also holds for each game with $k + 1$ players.

Consider the game (N, v) with $|N| = k + 1$ and a game $(N, w) \equiv (N, v + \alpha u_S)$ for $S \subset N$ and $\alpha > 0$. We will show that for each $i \in S$ it holds that $SD_i(N, v) \leq SD_i(N, w)$. Assume that (N, w) and (N, v) are SD-equivalent.

From Lemma 9 for the two games there is an antipartition, \mathcal{Q} , in the set of coalitions with minimal satisfaction. Consider a coalition T of this antipartition \mathcal{Q} . We seek to compare the SD-prenucleolus of the two SD-reduced games $(N \setminus T, v^{SD(v)})$ and $(N \setminus T, w^{SD(w)})$. We distinguish 3 cases:

1. $S \notin \mathcal{Q}$ and T is not a subset of S . By applying Corollary 11, the two SD-reduced games must coincide. Therefore players in $S \cap N \setminus T$ receive the same payoff in both games. Since the SD-prenucleolus satisfies the SD-reduced game property it must be the case that in games (N, v) and (N, w) players in $S \cap N \setminus T$ also must receive the same payoff.

2. $S \in \mathcal{Q}$. Note that this implies that S must be in the same antipartition with T since otherwise the two games cannot be SD-equivalent.

If $|N \setminus T| = 1$ then it is clear that $SD(T, v) = SD(T, w)$ and consequently $SD_i(N, v) = SD_i(N, w)$ for $i \in N \setminus T$. Therefore we only consider the case $|N \setminus T| > 1$.

Let $F(S, SD(N, v), v) = F_1$ and $F(S, SD(N, w), w) = F_2$. Since S is in the same antipartition in both games by applying Theorem 5 it is quite immediately apparent that $F_2 = F_1 - \alpha \frac{1}{|N|(|Q|-1)}$ and

$$SD(S, w) = w(S) + |S| F_2 = v(S) + \alpha + |S| \left(F_1 - \alpha \frac{1}{|N|(|Q|-1)} \right) =$$

$$SD(S, v) + \alpha \left(1 - \frac{|S|}{|N|(|Q|-1)} \right) > SD(S, v).$$

In this case the characteristic function w^S for relevant coalitions in the reduced game (S, w^{SD}) with respect to the SD-prenucleolus of the game (N, w) results

$$w^{SD}(U) = \begin{cases} v^{SD}(S) + \alpha \left(1 - \frac{|S|}{|N|(|Q|-1)} \right) & U = N \setminus T \\ v^{SD}(U) & \text{otherwise} \end{cases}$$

This means that (by strong aggregate monotonicity of the SD-prenucleolus) for each $i \in S \cap (N \setminus T)$ it holds that

$$SD_i(N \setminus T, w^{SD(w)}) > SD_i(N \setminus T, v^{SD(v)}) \Leftrightarrow SD_i(N, w) > SD_i(N, v).$$

$\exists S \notin Q$ and $T \subset S$.

In this case by applying Corollary 11,

$$w^{SD}(U) = \begin{cases} v^{SD}(S) + \alpha & U = S \setminus T \\ v^{SD}(U) & \text{otherwise.} \end{cases}$$

In this case the TU game $(N \setminus T, w^{SD(w)})$ can be written as $(N \setminus T, v^{SD(v)} + au_{S \setminus T})$. From Lemma 8 the SD-reduced games $(N \setminus T, v^{SD(v)})$ and $(N \setminus T, v^{SD(v)} + au_{S \setminus T})$ are SD-relevant. Note also that for any $\gamma \in [0, \alpha]$ it also holds that $(N \setminus T, v^{SD(v)} + \gamma u_{S \setminus T})$ is an SD-relevant game⁷. We distinguish two cases;

⁷This is so because $(N \setminus T, v^{SD(v)} + \gamma u_{S \setminus T})$ is the SD-reduced game of $(N, v + \gamma u_S)$ with respect to the SD-prenucleolus of $(N, v + \gamma u_S)$. Recall that $(N, v + \gamma u_S)$ is by assumption SD-relevant.

3a) $(N \setminus T, v^{SD(v)})$ and $(N \setminus T, v^{SD(v)} + au_{S \setminus T})$ are SD-equivalent. The analysis can be repeated for these two TU games. If the analysis ends in case 1 or 2 the proof is complete. Otherwise the analysis is repeated for the resulting two new SD-reduced games. In this last case the new TU games have fewer players. Since the result is true when the number of players is 2 it can be asserted that at some stage the procedure will end in case 1 or 2.

3b) $(N \setminus T, v^{SD(v)})$ and $(N \setminus T, v^{SD(v)} + au_{S \setminus T})$ are not SD-equivalent. By Lemma 13 there exists $\beta, \beta < \alpha$, such that $(N \setminus T, v^{SD(v)})$ and $(N \setminus T, v^{SD(v)} + \beta u_{S \setminus T})$ are SD-equivalent and $(N \setminus T, v^{SD(v)} + \beta u_{S \setminus T})$ and $(N \setminus T, v^{SD(v)} + \alpha u_{S \setminus T})$ are SD-equivalent and SD-relevant. The analysis can be repeated for these two pairs of TU games. If the analysis ends in case 1 or 2 the proof is complete. Otherwise the analysis is repeated for the resulting two new SD-reduced games. In this last case the new TU games have fewer players. Since the result is true when the number of players is 2 it can be asserted that at some stage the procedure will end in case 1 or 2. It is thus proved that for any player i in $S \cap (N \setminus T)$ it holds that $SD_i(N, w) \geq SD_i(N, v)$. Since this is true for any coalition T in the antipartition Q it must be concluded that for any player i in S it holds that $SD_i(N, w) \geq SD_i(N, v)$.

Assume that (N, w) and (N, v) are not SD-equivalent. By Lemma 13 there exists $\beta, \beta < \alpha$, such that (N, v) and $(N, v + \beta u_S)$ are SD-equivalent and (N, w) and $(N, v + \beta u_S)$ are SD-equivalent. Therefore the above arguments can be used to conclude that for any player i in S it holds that $SD_i(N, v) \leq SD_i(N, v + \beta u_S)$. Similarly, it must be concluded that for any player i in S it holds that $SD_i(N, w) \geq SD_i(N, v + \beta u_S)$. ■

Therefore, SD-relevant games and monotonicity of the SD-prenucleolus are strongly related.

5 Convex games

5.1 The SD-prenucleolus

In the class of convex games (Shapley, 1971) core stability and coalitional monotonicity are compatible. In convex games, the Shapley value satisfies

the two properties. In general, the Shapley value is not a core concept, even in the class of veto balanced games. Therefore the issue of whether a core concept satisfying monotonicity in the class of convex games and in the class of veto balanced games exists has been an open question⁸. The following theorem answers the question in the affirmative.

Theorem 15 *In the class of convex games the SD-prenucleolus satisfies coalitional monotonicity.*

The proof of this theorem results immediately from the facts that convex games are SD-relevant games (see lemma below) and the fact that if (N, v) and $(N, v + \alpha u_S)$ are convex then $(N, v + \gamma u_S)$ is convex for any $\gamma \in [0, \alpha]$.

Lemma 16 *Let (N, v) be a convex game and x be an allocation. Then all coalitions are relevant with respect to x . Therefore convex games are SD-relevant games.*

Proof. The lemma is obviously true for coalitions with minimal satisfaction, coalitions in $\mathcal{B}(x)$. We seek to prove that given any two relevant coalitions, S and T , $S \cup T$ is relevant.

Assume that S and T are relevant coalitions, $S \cup T \neq N$ and $S \cup T$ is non relevant. By convexity

$$x(S \cup T) - v(S \cup T) + x(S \cap T) - v(S \cap T) \leq x(S) - v(S) + x(T) - v(T).$$

Since S and T are relevant:

$$x(S) - v(S) = \sum_{i \in S} f_{\mathcal{H}, F}(i, S)$$

$$x(T) - v(T) = \sum_{i \in T} f_{\mathcal{H}, F}(i, T).$$

Since $S \cup T$ is non relevant:

$$x(S \cup T) - v(S \cup T) = \sum_{i \in S \cup T} f_{\mathcal{H}, F}(i, S \cup T) + (F(x, S \cup T) - \max_{i \in S} f_{\mathcal{H}, F}(i, S \cup T)).$$

⁸See Arin (2012) for a discussion on this issue.

We consider two cases.

a) There is no relevant coalition $Q \subset S \cup T$ such that $Q \not\subseteq S$, $Q \not\subseteq T$ and $F(Q, x) < \max(F(S, x), F(T, x))$.

In this case it holds that

$$\begin{aligned} & \sum_{i \in S \cup T} f_{\mathcal{H}, F}(i, S \cup T) = \\ = & \sum_{i \in S \setminus T} f_{\mathcal{H}, F}(i, S) + \sum_{i \in T \setminus S} f_{\mathcal{H}, F}(i, T) + \sum_{i \in T \cap S} \min(f_{\mathcal{H}, F}(i, T), f_{\mathcal{H}, F}(i, S)) + \alpha \end{aligned}$$

where $\alpha > 0$ since $S \cup T$ is non relevant. Therefore

$$\alpha + x(S \cap T) - v(S \cap T) \leq \sum_{i \in T \cap S} \max(f_{\mathcal{H}, F}(i, T), f_{\mathcal{H}, F}(i, S))$$

or (since $\alpha > 0$)

$$x(S \cap T) - v(S \cap T) < \sum_{i \in T \cap S} \max(f_{\mathcal{H}, F}(i, T), f_{\mathcal{H}, F}(i, S))$$

or (assuming $S \cap T$ is relevant⁹)

$$\sum_{i \in T \cap S} f_{\mathcal{H}, F}(i, T \cap S) < \sum_{i \in T \cap S} \max(f_{\mathcal{H}, F}(i, T), f_{\mathcal{H}, F}(i, S))$$

Since $(S \cap T) \subset S$ for any $i \in S \cap T$ it holds that

$$f_{\mathcal{H}, F}(i, S) \leq f_{\mathcal{H}, F}(i, T \cap S)$$

and similarly, since $(S \cap T) \subset T$ for any $i \in S \cap T$ it holds that

$$f_{\mathcal{H}, F}(i, T) \leq f_{\mathcal{H}, F}(i, T \cap S).$$

Consequently, for any $i \in S \cap T$ it holds that

$$f_{\mathcal{H}, F}(i, T \cap S) \geq \max(f_{\mathcal{H}, F}(i, T), f_{\mathcal{H}, F}(i, S))$$

⁹If it is non relevant the proof is identical: a strictly positive number α just needs to be added on the right-hand side of the inequality. Since α is positive the arguments do not change.

which contradicts the fact that

$$x(S \cap T) - v(S \cap T) = \sum_{i \in T \cap S} f_{\mathcal{H}, F}(i, T \cap S) < \sum_{i \in T \cap S} \max(f_{\mathcal{H}, F}(i, T), f_{\mathcal{H}, F}(i, S)).$$

b) There is a relevant coalition $Q \subset S \cup T$ such that $Q \not\subseteq S$, $Q \not\subseteq T$ and $F(x, Q) < \max(F(S, x), F(T, x))$. Among the relevant coalitions satisfying these conditions Q has the minimal satisfaction.

Consider the following coalitions S^1 and T^1 defined as follows:

$$S^1 = \begin{cases} S & \text{if } F(Q, x) \geq F(S, x) \\ S \cup Q & \text{if } F(Q, x) < F(S, x) \end{cases} \text{ and}$$

$$T^1 = \begin{cases} T & \text{if } F(Q, x) \geq F(T, x) \\ T \cup Q & \text{if } F(Q, x) < F(T, x) \end{cases} .$$

We consider two cases:

b1) Coalitions S^1 and T^1 are relevant.

Using coalitions S^1 and T^1 , repeat the arguments used for coalitions S and T . Note that since Q has been chosen with minimal satisfaction then for these two coalitions (S^1 and T^1) case b) does not occur and it is concluded that $S^1 \cup T^1 = S \cup T$ is relevant.

b2) Assume, without loss of generality, that S^1 is non relevant. Note that $S^1 \subset S \cup T$ and the set of players is finite. Repeat the proof with coalitions S and Q . This ends up either in a contradiction (cases a) and b1)) or in case b2) with two coalitions S and P (or Q and P) such that $S \cup P$ (or $Q \cup P$) is non relevant. Repeat the proof again for coalitions S and P (or Q and P). If the proof ends in case a) or b1) the contradiction is found. If not, repeat the proof with another two coalitions. Note that at the end two coalitions need to be found for which case b2) does not occur since the number of players is finite and the size of the coalitions is reduced at each step whenever the proof ends in case b2). ■

In general, it is not necessarily true that if a game (N, v) is SD-relevant then for a given allocation x , $x \neq SD(N, v)$, all coalitions are relevant. Con-

sider the following 4-person balanced game:

$$v(S) = \begin{cases} 4 & \text{if } S \in \{\{1, 2\}, \{1, 3\}, \{1, 4\}\} \\ 7 & \text{if } S \in \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\} \\ 9 & \text{if } S = \{2, 3, 4\} \\ 18 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

It can be checked that $SD(N, v) = (3, 5, 5, 5)$ and the game is SD-relevant. Considering the allocation $x = (0, 6, 6, 6)$ it can be checked that $F(\{1, 2, 3\}, x) > F(\{1, 2\}, x) = F(\{1, 3\}, x)$ and therefore coalition $\{1, 2, 3\}$ is not relevant at x .

5.2 The SD-prekernel

In what follows we prove that in the class of convex games the SD-prekernel (Arin and Katsev, 2011) and the SD-prenucleolus coincide.

Given a TU game (N, v) and an allocation $x \in X(N, v)$ the complaint of player i against player j is defined as follows:

$$s_{ij}(x) = \min_{S: i \in S, j \notin S} F(S, x).$$

The SD-prekernel of a TU game (N, v) is:

$$SDPK(N, v) = \{x \in X(N, v) : s_{ij}(x) = s_{ji}(x) \text{ for all } i \neq j\}$$

The SD-prenucleolus belongs to the SD-prekernel but, in general, do not coincide. Consider a 4-person game (N, v) defined as follows:

$$v(S) = \begin{cases} 4 & \text{if } S = N \\ 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1, 2\}, \{3, 4\}\} \\ 2 & \text{otherwise.} \end{cases}$$

It is immediately apparent that $(2, 2, 0, 0)$ is an element of the SD-prekernel of the game and that the SD-prenucleolus of (N, v) is $(1, 1, 1, 1)$.

Theorem 17 *Let (N, v) be a convex game. Then $SDPK(N, v) = \{SD(N, v)\}$.*

Proof. *Let $x = SD(N, v)$ and assume that there exists y , $y \neq x$, such that $y \in SDPK(N, v)$. First of all we seek to prove that $\mathcal{B}(y)$ contains an antipartition. Let S be a maximal coalition in $\mathcal{B}(y)$, that is, there is no coalition T in $\mathcal{B}(y)$ such that $S \subset T$. $\mathcal{B}(y)$ must contain for each $i \in S$ a coalition, T^i , such that $i \notin T^i$. Since (N, v) is convex (and therefore all coalitions are relevant for any allocation) the maximality of S implies that $N \setminus S \subset T^i$. Let $\{T^i : i \in S\}$ be the set of maximal coalitions for each i in S (maybe the case that for two players i, j it holds that $T^i = T^j$). Then $\{T^i : i \in N \setminus S\} \cup \{S\}$ is an antipartition. It is immediately apparent that $(N \setminus T^i) \cap (N \setminus T^j)$ is empty. If for some $i, j \in S$ it holds that $(N \setminus T^i) \cap (N \setminus T^j)$ is nonempty it is clear that $T^i \cup T^j \neq N$ and that contradicts the maximality of T^i and T^j since the fact that (N, v) is convex implies that $T^i \cup T^j$ is an element of the set $\mathcal{B}(y)$. Since the satisfaction of an antipartition only depends on the characteristic function $\mathcal{B}(x)$ and $\mathcal{B}(y)$ should contain the same antipartition. And for any coalition S in the antipartition it must hold that $y(S) = x(S)$. Finally, the SD-reduced games (S, v^y) and (S, v^x) are identical and since the SD-prekernel satisfies SD-reduced game property it must be concluded that $x_i = y_i$ for all $i \in S$. ■*

6 Concluding remarks

This paper follows up the research started by Arin and Katsev in 2011. Considering the results included in the two papers the SD-prenucleolus stands out as the only known core concept that satisfies monotonicity in the class of convex games and in the class of veto balanced games. Convex games¹⁰ and games with veto players have been widely used to model many different economic situations. In both classes the compatibility between core stability and monotonicity was known. However the existence of a continuous core concept satisfying monotonicity in those two classes was an open question that has been answered with the study of the SD-prenucleolus.

¹⁰This research opens up several questions concerning the study of the SD-prenucleolus for subclasses of convex games such as bankruptcy games and airport games.

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