A COMMON AXIOM FOR CLASSICAL DIVISION RULES FOR CLAIMS PROBLEMS

M. J. ALBIZURI*, J. C. SANTOS

Abstract. In this paper we introduce a new axiom, denoted claims separability, that is satisfied by several classical division rules defined for claims problems. We characterize axiomatically the entire family of division rules that satisfy this new axiom. In addition, employing claims separability, we characterize the minimal overlap rule, given by O'Neill (1982), Piniles' rule and the rules in the TAL-family, introduced by Moreno-Ternero and Villar (2006), which includes the uniform gains rule, the uniform losses rule and the Talmud rule.

JEL Classification: C71.

Key words: claims problems, claims separability.

1. Introduction

Consider the problem in which there is a resource to be divided among several agents who have claims on it and cannot be all fulfilled. This is an issue studied not only in the economic literature or the game theory literature but also in the 2000 year old document Talmud or in the twelfth century by Maimonides, Rabad and Ibn Ezra. It is formally denoted as claims problem or bankrutcy problem.

O'Neill (1982) originated the application of cooperative game theory in solving claims problem. That is, in finding division rules, which specify allocations of the resource among the agents. Good surveys of the related literature are due to Thomson (2003, 2013a, 2013b) and Moulin (2002). Concepts, as the Lorenz ranking (Thomson, 2012) have also been studied for that problem. When studying claims problems there is an axiomatic approach: fair properties for division rules are considered and division rules satisfying fair properties are seek or identified. This is the approach of this paper.

Date: February 25, 2015

University of the Basque Country. Faculty of Economics and Business. Department of Applied Economics IV. Bilbao.

*Corresponding author. Phone number:+34946013805

E-mail addresses: mj.albizuri@ehu.es, carlos.santos@ehu.es.

The following classical division rules for claims problems can be mentioned. The uniform gains rule and the uniform losses rule, both of them referred by Maimonides; the Talmud rule, present in the Talmud; the rule by Piniles from the nineteenth century; the minimal overlap rule, which is the generalization by O'Neill (1982) of a rule by Rabad and Ibn Ezra (defined only when the endowment is not greater than the smallest claim); and the well known proportional rule. It can mentioned the axiomatic characterizations of the uniform gains rule and uniform losses rule by Herrero and Villar (2001), that of the Talmud rule by Aumann and Mashler (1985), and that of the minimal overlap rule by Alcalde et al. (2008) .

In this paper we propose a new axiom for claims problems, named claims separability. It is satisfied by the uniform gains rule, the uniform losses rule, the Talmud rule, Piniles' rule , the minimal overlap rule and the proportional rule. We would like to point out that so far there is only one axiom in the literature fulfilled by all of them: order preserving, introduced by Auman and Maschler (1985). This new axiom is also satisfied by the rules in the TAL-family defined by Moreno-Ternero and Villar (2006), and the alternative extension of the Ibn Ezra rule introduced by Bergantiños and Mendez-Naya (2001) and characterized by Alcalde et al. (2005). Claims separability follows from the fact that if agent j claims more than agent i, then the claim of agent j is formed by the claim of agent i plus the remaining claim of agent j. Claims separability requires the allocation of agent j to be equal to the allocation of agent i plus the allocation of agent j in a remaining claims problem. This remaining problem will be explained in detail in Section 3. Roughly writing, it has as claims the remaining claims and as endowment the remaining one.

We determine all the rules that satisfy claims separability, which turns to form a family of serial like rules. We also give characterizations for the uniform gains rule and the uniform losses rule related to the characterizations given by Herrero and Villar (2001), in which the consistency axiom employed by these authors is substituted by claims separability and independence of null demands. We give an axiomatic characterization for the Talmud rule, related to a characterization given by Aumann and Mashler (1985), employing our axiom and self-duality. Moreover, if instead of self-duality, we consider a weaker axiom than the composition axiom introduced by Young (1988), Piniles' rule is characterized. Finally, we provide a characterization for the minimal overlap rule by means of three axioms: claims separability, invariance of claims truncation, and a new one related to the composition axiom introduced by Young (1988).

There are well known rules which do not satisfy claims separability. For example, the random arrival rule given by O'Neill (1982).

The rest of the paper is organized as follows: preliminaries are presented in Section 2. Claims separability and the family characterized by that axiom are in

Section 3. In Section 4 are provided axiomatic characterizations of the uniform gains rule, the uniform losses rule, the Talmud rule and the minimal overlap rule.

2. Preliminaries

Let N be a finite set of nonnegative integers. For $q \in \mathbb{R}^N$ and $S \subseteq N$ we denote $q(S) = \sum_{j \in S} q_j$ and $s = |S|$. The zero vector is denoted by $0 = (0, \ldots, 0)$. Given two vectors x, $y \in \mathbb{R}^N$, $x \leq y$ means that $x_j \leq y_j$, for all $j \in N$. The set of all nonnegative N-dimensional real vectors is denoted by $\mathbb{R}^N_+ = \left\{x \in \mathbb{R}^N : x \geqslant 0\right\}$ and $X^N = \left\{x \in \mathbb{R}^N_+ : x_j \geqslant x_i \text{ if } j > i\right\}$. If $|N| = 1$, it is written $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$. The set of natural numbers is denoted by $\mathbb{N} = \{1, 2, ...\}$. For notational convenience and without loss of generality, it can be assumed that $N = \{1, ..., n\}$.

A *claims problem* (or bankruptcy problem) with set of claimants N is an ordered pair (c, E) where $c = (c_1, \ldots, c_n)$, $0 \leq c_1 \leq \cdots \leq c_n$, specifies for each agent i a claim c_i, and $0 \leq E \leq c_1 + \cdots + c_n$ represents the amount to be divided.

The space of all claims problems is denoted by C, and by C^N the set of all claims problems with set of claimants N.

A *division rule* (or bankruptcy rule) is a function that associates with each claims problem $(c, E) \in C^N$ a vector $\varphi(c, E) \in \mathbb{R}^n_+$ specifying an award for each agent i such that $0 \leqslant \varphi(c, E) \leqslant c$ and $\varphi_1(c, E) + \cdots + \varphi_n(c, E) = E$.

N can be fixed or not. Throughout the paper if it is noted in which case we are.

Now we present some classical bankruptcy rules. First, the following three. The uniform gains rule, which shares the endowment equally without giving anyone more than his/her claim; the uniform losses rule, which allocates losses equally without giving anyone a negative amount; and the Talmud rule, which allocates the endowment equally to agents, so that no-one receives more than half of his/her claim, and, if the endowment is greater than the sum of half of the claims, allocates losses equally. Those three rules can be formalized as follows.

The *uniform gains rule*, UG, shares the following amount for each $(c, E) \in C^N$ and each $i \in N$:

$$\min\{c_i, \lambda\},$$

where $\lambda \geq 0$ satisfies $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

The *uniform losses rule*, UL, allocates the following amount for each $(c, E) \in C^N$ and each $i \in N$:

$$\max\{0, c_i - \lambda\},$$

where $\lambda \geq 0$ satisfies $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

The *Talmud rule*, TAL, provides each $(c,E)\in C^N$ and each $i\in N$ with the following quantity:

$$\min\left\{\frac{c_i}{2},\lambda\right\},$$

if $E\le\frac{c(N)}{2}$, and

$$\max\left\{\frac{c_i}{2},c_i-\mu\right\},$$

if $E\ge\frac{c(N)}{2}$, where λ and μ are such that $\sum_{i\in N}\mathrm{TAL}_i(c,E)=E$.

The *TAL-family*, which comprises the above three rules, was introduced by Moreno-Ternero and Villar (2006). Each rule in the TAL-family is associated with a parameter $\theta\in[0,1]$ and is denoted by R^θ. It allocates the endowment equally until each of the agents receives no more than the fraction θ of his/her claim, and if the endowment is greater than the fraction θ of the total claim then losses are shared equally. Therefore, if $\theta=0$ the uniform losses rule results, if $\theta=1/2$ the Talmud rule and if $\theta=1$ the uniform gains rule.

That is, R^θ provides each $(c,E)\in C^N$ and each $i\in N$ with the following quantity:

$$\min\left\{\theta c_i,\lambda\right\},$$

if $E\le\theta c(N)$, and

$$\max\left\{\theta c_i,c_i-\mu\right\},$$

if $E\ge\theta c(N)$, where λ and μ are such that $\sum_{i\in N}R_i^\theta(c,E)=E$.

The *minimal overlap rule* allocates each agent the sum of the partial awards from the various units on which he/she lays claims, where for each unit equal division prevails among all the agents claiming it and claims are arranged on specific parts of the amount available, called units, so that the number of units claimed by exactly one claimant is maximized, and for each $k=2,...,n-1$ successively, the number of units claimed by exactly k claimants is maximized subject to the $k-1$ maximization exercises being solved. Alcalde et al. (2008) formalize the minimal overlap rule, denoted by φ^{mo}, as follows.

For each (c,E) and each $i\in N$,
(a) if $E\ge c_n$,

$$\varphi_i^{mo}(c,E)=\sum_{j=1}^{i}\frac{\min\left\{c_j,t\right\}-\min\left\{c_{j-1},t\right\}}{n-j+1}+\max\left\{c_i-t,0\right\},$$

where $c_0=0$, and t is the unique solution for the equation

$$\sum_{k=1}^{n}\max\left\{c_k-t,0\right\}=E-t,$$

or

(b) if $E < c_n$,

$$\varphi_i^{mo}(c,E) = \sum_{j=1}^{i} \frac{\min\{c_j, E\} - \min\{c_{j-1}, E\}}{n-j+1}.$$

Albizuri and Santos (2014) associates with each $(c,E) \in C^N$ and $i \in N$ a subproblem $(c^{-i}, E^{-i}) \in C^N$. They write that a division rule satisfies balanced contributions associated with $(c^{-i}, E^{-i}) \in C^N$ if

$$\varphi_j(c,E) - \varphi_i(c,E) = \varphi_j(c^{-i}, E^{-i}) - \varphi_i(c^{-j}, E^{-j})$$

for all $(c,E) \in C^N$, and $i,j \in N$. Albizuri and Santos (2014) take different possibilities for c^{-i} and E^{-i}. Among them, those presented in this paper:

$$c^{-i} = (\max\{c_k - c_i, 0\})_{k \in N}$$

and the following two expressions for E^{-i}.

a) $E^{-i} = \min\left\{\sum_{k=i+1}^{n} c_k^{-i}, \max\{E - \theta(c_1 + \cdots + c_i + (n-i)c_i), 0\}\right\}$, where $\theta \in [0,1]$. The resulting (c^{-i}, E^{-i}) is referred as the θ-TAL-subproblem.

b) $E^{-i} = \min\left\{\sum_{k=i+1}^{n} c_k^{-i}, \max\{E - c_i, 0\}\right\}$. The resulting (c^{-i}, E^{-i}) is referred as the MO-subproblem.

Albizuri and Santos (2014) prove the following theorem.

Theorem 1. *A division rule φ satisfies the balanced contributions property associated with the θ-TAL-subproblem (MO-subproblem) if and only if φ coincides with R^θ (the minimal overlap rule).*

3. Claims separability and the associated family

Before writing the main axiom let us consider the following, which is introduced by Aumann and Maschler (1985). We denote by φ a division rule with domain C^N.

Order preserving. For all claims problem $(c,E) \in C^N$,

$$0 \le \varphi_1(c,E) \le \ldots \le \varphi_n(c,E) \text{ and } 0 \le c_1 - \varphi_1(c,E) \le \ldots \le c_n - \varphi_n(c,E).$$

It requires gains and losses not to be smaller for anyone who claims more. As pointed by Aumann and Maschler (1985), the uniform gains rule, the uniform losses rule, the Talmud rule, the minimal overlap rule[1] and the proportional rule satisfy order preserving. It is clear also that all the rules in the TAL-family satisfy order preserving.

[1] In fact, they referred to Ibn Ezra or Rabad's rule, that is, to the specif case in which $E \le c_n$.

Claims separability. If $(c, E) \in C^N$ and $i \in \{1, \ldots, n-1\}$, then $(c^{-i}, E^i) \in C^N$ and

$$\varphi_j(c, E) = \varphi_i(c, E) + \varphi_j(c^{-i}, E^i) \tag{1}$$

for all $j > i$, where

$$E^i = E - \sum_{h=1}^{i} \varphi_h(c, E) - (n-i)\,\varphi_i(c, E). \tag{2}$$

The axiom we propose relates the allocations of two claims problems. Given a claimant j and any claimant i who asks no more than him/her, the claim of agent j is formed by two quantities: the claim of agent i and the rest. Suppose that all the agents who claim no less than i are given for the former quantity of their claims the allocation corresponding to agent i. Those who claim less or equal than i are given simply their allocations. As a result of that, the endowment reduces by those allocations, that is, it becomes E^i. And agents still have the rest of their claims to be asked for, that is, c_h^{-i}. Claims separability requires those remaining claims and the reduced endowment to form a claims problem, and the allocation of claimant j who asks no less than i to be the allocation of agent i plus the allocation of agent j in the remaining claims problem, that is, in (c^{-i}, E^i).

In the consistency axiom for claims problems, the claims of the agents are not divided into two pieces. Instead, a group of agents leave the scene with their allocations, and those who continue remain with their entire claims. Therefore, the endowment is reduced only by the allocations of the agents who leave. Consistency requires the allocation of the agents who remain to coincide with their allocations in the remaining problem. If we compare our axiom and consistency, we can say that our axiom can be seen as an additive consistency axiom. It requires the division rule to be consistent when claims are divided according to lower claims.

With respect to the notation, we employ E^i to denote the remaining endowment to make easier the writing of the paper, although this endowment depends on the division rule, the agent and the claims.

The requirement $(c^{-i}, E^i) \in C^N$ in claims separability is not so strange or demanding. We show that if a division rule satisfies order preserving, then (c^{-i}, E^i) is a claims problem.

Lemma 1. *For each claims problem $(c, E) \in C^N$ and each $i = 1, \ldots, n-1$, if a division rule satisfies order preserving then $(c^{-i}, E^i) \in C^N$.*

Proof. It is sufficient to prove that $\sum\limits_{j=i+1}^{n} (c_j - c_i) \geq E^i$.

It holds

$$E^i = E - \sum_{h=1}^{i} \varphi_h(c,E) - (n-i)\,\varphi_i(c,E) = \sum_{j=i+1}^{n} \left(\varphi_j(c,E) - \varphi_i(c,E)\right),$$

and order preserving implies that for each $j > i$

$$\varphi_j(c,E) - \varphi_i(c,E) \leq c_j - c_i.$$

Therefore,

$$\sum_{j=i+1}^{n} (c_j - c_i) \geq \sum_{j=i+1}^{n} \left(\varphi_j(c,E) - \varphi_i(c,E)\right) = E^i,$$

and the proof is complete. □

Remark 1. *We would like to note the following. Let φ be a division rule. If (c^{-i}, E^i) is a claims problem for each $(c,E) \in C^N$ and $i = 1, ..., n-1$, and φ satisfies equality* (1)*, then φ satisfies order preserving. Indeed, if $j > i$, then $\varphi_j(c,E) = \varphi_i(c,E) + \varphi_j(c^{-i}, E^i) \geq \varphi_i(c,E)$ and $\varphi_j(c,E) - \varphi_i(c,E) = \varphi_j(c^{-i}, E^i) \leq c_j - c_i$, where the two inequalities hold since φ is a division rule. Therefore, φ satisfies order preserving. As a result of that, claims separability is equivalent to requiring order preserving and equality* (1)*.*

Proposition 1. *The rules in the TAL-family, the minimal overlap rule and the proportional rule satisfy claims separability.*

Proof. Let us prove that R^θ satisfies claims separability. Let $j > i$. Since R^θ satisfies the balanced contributions property associated with the θ-TAL-subproblem, it holds

$$R_j^\theta(c,E) - R_i^\theta(c,E) = R_j^\theta(c^{-i}, E^{-i}) - R_i^\theta(c^{-j}, E^{-j}),$$

where

$$E^{-h} = \min\left\{\sum\nolimits_{k=h+1}^{n} c_k^{-h}, \max\left\{E - \theta\left(c_1 + \cdots + c_h + (n-h)\,c_h\right), 0\right\}\right\},$$

if $h = i, j$. Since $j > i$, then the above equality turns into

$$R_j^\theta(c,E) - R_i^\theta(c,E) = R_j^\theta(c^{-i}, E^{-i}).$$

Taking into account this equality, that R^θ is a division rule and equality $R_j^\theta(c^{-i}, E^{-i}) = 0$ if $j \leq i$, the following holds:

$$E^{-i} = \sum_{j=1}^{n} R_j^\theta(c^{-i}, E^{-i}) = \sum_{j=i+1}^{n} \left(R_j^\theta(c,E) - R_i^\theta(c,E)\right) = E^i.$$

Thus, $(c^{-i}, E^i) \in C^N$ and

$$R_j^\theta(c,E) - R_i^\theta(c,E) = R_j^\theta(c^{-i}, E^i),$$

that is, R^θ satisfies claims separability.

The proof for the minimal overlap rule can be done in the same way, taking the corresponding E^{-h}, if $h = i, j$.

As for the proportional rule, it can be easily proved that it satisfies the balanced contributions property with c^{-i} and $E^{-i} = \sum_{k=i+1}^{n} \frac{c_k - c_i}{c(N)} E$. Therefore, the proof for the proportional rule is as above considering that E^{-i}. □

Observe that a division rule satisfies claims separability if and only if it satisfies the balanced contributions property associated with a claims problem (c^{-i}, E^{-i}), where E^{-i} is any non-negative number not depending on j (smaller or equal than $c^{-i}(N)$).

Notice that claims separability does not imply consistency. The minimal overlap rule is claims separable but it is not consistent. And the other way round, consistency does not imply claims separability. Cassel's parametric rule (see Thomson, 2003) is a consistent rule, which is not claims separable.

On the other hand, the random arrival rule (O'Neill, 1982) is not either consistent or claims separable. If φ^{ra} denotes the random arrival rule, for each (c, E) and each $i \in N$,

$$\varphi_i^{ra}(c, E) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \min \left\{ c_i, \max \left\{ E - \sum_{j \in N, \pi(j) < \pi(i)} c_j, 0 \right\} \right\},$$

where Π^N denotes the class of bijections from N into itself. Take $(c, E) = ((2, 3, 5, 6), 6)$ and $i = 2$. Then, $\varphi^{ra}(c, E) = (9/12, 13/12, 23/12, 27/12), (c^{-i}, E^i) = ((0, 0, 2, 3), 2)$ and $\varphi^{ra}((0, 0, 2, 3), 2) = (0, 0, 1, 1)$. Observe that

$$\varphi_3^{ra}(c, E) = \frac{23}{12} \neq \frac{13}{12} + 1 = \varphi_2^{ra}(c, E) + \varphi_3^{ra}(c^{-2}, E^2).$$

We have seen that several classical division rules satisfy claims separability. Now, the question is: what are the rules satisfying this axiom like? We show that they form a family determined by several functions satisfying some properties.

Let a list of functions $\mathcal{F} = \left\{F^k\right\}_{k=1}^{n-1}$, $F^k : X^{n-k+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ be such that:

(C1) $F^k(x; y) \geq \max \left\{ 0, y - \sum_{j=2}^{n-k+1} (x_j - x_1) \right\}$ if $y \leq x_1 + \cdots x_{n-k+1}$.

(C2) $F^k(x; y) \leq \min \left\{ (n - k + 1) x_1, y \right\}$ if $y \leq x_1 + \cdots x_{n-k+1}$.

(C3) $F^k(x; y) = 0$ if $y > x_1 + \cdots + x_{n-k+1}$.

Conditions C1 and C2 fixe an upper bound and a lower bound for the values of the functions F^k when $y \leq x_1 + \cdots x_{n-k+1}$. Condition C3 is a thecnical requirement in order to F^k be a function. In fact, the values of F^k when $y > x_1 + \cdots x_{n-k+1}$ are not needed to define the division rules in our family, and therefore we can choose any number to be the value.

Definition 1. *The division rule associated with a list of functions* $\mathcal{F} = \left\{F^k\right\}_{k=1}^{n-1}$ *which satisfy C1, C2 and C3 is defined for each claims problem* $(c, E) \in C^N$ *as follows:*

If $i < n$,

$$\varphi_i^{\mathcal{F}}(c, E) = \sum_{j=1}^{i} \frac{F^j(c^{j-1}; E^{j-1})}{n-j+1},$$

and if $i = n$,

$$\varphi_n^{\mathcal{F}}(c, E) = E - \sum_{j=1}^{n-1} \varphi_j^{\mathcal{F}}(c, E),$$

where $c^0 = c$ *and* $E^0 = E$*; and for each* $j = 1, \ldots, n-2$*, the vector* $c^j \in \mathbb{R}^{n-j}$ *satisfies* $c_h^j = c_{j+h}^{-j}$ *for each* $h = 1, \ldots, n-j$*, and the real number* E^j *satisfies expression* (2)*, that is,* $E^j = E - \sum\limits_{h=1}^{j} \varphi_h^{\mathcal{F}}(c, E) - (n-j)\, \varphi_j^{\mathcal{F}}(c, E)$.

The division rule $\varphi^{\mathcal{F}}$ in the above definition can be seen as a serial like rule. Notice that c^j is composed by the last $n-j$ terms of c^{-j}. The endowment E is given in n steps. In the first step the same quantity is allocated to all the agents. This quantity can be seen as the total allocation corresponding to the amount c_1 claimed by all the agents. The number $F^1(c^0; E^0)$ tells us the total amount given to the agents at this step. The amount $F^1(c^0; E^0)$ is not bigger than n times the claim of agent 1, that is, the total claim corresponding to this step, and it is not bigger than the endowment. That is precisely the requirement C2. Also, since in the following steps the claims are reduced by c_1, the total amount $F^1(c^0; E^0)$ has to be big enough to allocate the endowment E in all the steps. And that is the requirement C1. After sharing $F^1(c^0; E^0)$ agent 1 leaves the scene, and we proceed in the second step in the same way with the new endowment E^1 and the claims reduced by c_1. The same quantity is allocated to agents $2, ..., n$, the corresponding to the amount $c_1^1 = c_2 - c_1$ claimed by those agents. The number $F^2(c^1; E^1)$ is the total amount given in that step and it has as before a lower bound and an upper bound determined by the amount c_1^1. The process continues until the end. At the final step, the rest is given to agent n.

In the proofs below we take into account the following equalities clearly implied by the previous definition[2].

$$\varphi_i^{\mathcal{F}}(c, E) = \varphi_{i-1}^{\mathcal{F}}(c, E) + \frac{F^i(c^{i-1}; E^{i-1})}{n-i+1}, \qquad \text{if } i \neq n, \tag{3}$$

and

$$\varphi_n^{\mathcal{F}}(c, E) = \varphi_{n-1}^{\mathcal{F}}(c, E) + E^{n-2} - F^{n-1}(c^{n-2}; E^{n-2}). \tag{4}$$

[2] We assume $\varphi_0^{\mathcal{F}}(c, E) = 0$.

It is necessary that the division rule $\varphi^{\mathcal{F}}$ is well defined.

Lemma 2. *The mapping $\varphi^{\mathcal{F}}$ is a division rule.*

Proof. We show that $\varphi_1^{\mathcal{F}}(c,E)+\cdots+\varphi_n^{\mathcal{F}}(c,E)=E$ and $\theta \leqslant \varphi^{\mathcal{F}}(c,E) \leqslant c$. The former equality is true by construction of $\varphi^{\mathcal{F}}$.

By definition, $\varphi_i^{\mathcal{F}}(c,E) \geq 0$ if $i<n$, and taking into account (4) it holds

$$\varphi_n^{\mathcal{F}}(c,E)=\varphi_{n-1}^{\mathcal{F}}(c,E)+E^{n-2}-F^{n-1}(c^{n-2};E^{n-2}) \geq \varphi_{n-1}^{\mathcal{F}}(c,E) \geq 0,$$

where in the before last inequality C2 is applied.

Now we show that $\varphi_i^{\mathcal{F}}(c,E) \leqslant c_i$ for all $i \in N$.

If $i \neq n$, by definition and C2,

$$\begin{aligned}\varphi_i^{\mathcal{F}}(c,E) &= \frac{F^1(c^0;E^0)}{n}+\cdots+\frac{F^i(c^{i-1};E^{i-1})}{n-i+1}\\ &\leq \frac{nc_1^0}{n}+\cdots+\frac{(n-i+1)c_1^{i-1}}{n-i+1}=c_1+\cdots+(c_i-c_{i-1})=c_i.\end{aligned}$$

And if $i=n$, taking into account (4) ,

$$\begin{aligned}\varphi_n^{\mathcal{F}}(c,E) &= \varphi_{n-1}^{\mathcal{F}}(c,E)+(E^{n-2}-F^{n-1}(c^{n-2};E^{n-2}))\\ &\leq c_{n-1}+\left(E^{n-2}-F^{n-1}(c^{n-2};E^{n-2})\right)\\ &\leq c_{n-1}+E^{n-2}-\left(E^{n-2}-((c_{n-1}-c_{n-2})-(c_n-c_{n-2}))\right)=c_n,\end{aligned}$$

where in the first inequality we have considered the case $i=n-1$ and in the second one C1. □

Lemma 3. *The mapping $\varphi^{\mathcal{F}}$ satisfies claims separability.*

Proof. To prove that $(c^{-i},E^i) \in C^N$, by Lemma 1 it is sufficient to prove that $\varphi^{\mathcal{F}}$ satisfies order preserving. Let $(c,E) \in C^N$. Taking into account that C2 implies that $E^{n-2}-F^{n-1}(c^{n-2};E^{n-2}) \geq 0$, by (3) and (4) it is clear that $0 \leq \varphi_1^{\mathcal{F}}(c,E) \leq \cdots \leq \varphi_n^{\mathcal{F}}(c,E)$.

To show that $0 \leq c_1-\varphi_1^{\mathcal{F}}(c,E) \leq \cdots \leq c_n-\varphi_n^{\mathcal{F}}(c,E)$ it is sufficient to prove that $\varphi_{i+1}^{\mathcal{F}}(c,E)-\varphi_i^{\mathcal{F}}(c,E) \leq c_{i+1}-c_i$ when $i=1,...,n-1$.

If $i \neq n-1$, it holds

$$\begin{aligned}\varphi_{i+1}^{\mathcal{F}}(c,E)-\varphi_i^{\mathcal{F}}(c,E) &= \frac{F^{i+1}(c^i;E^i)}{n-i}\\ &\leq \frac{\min\left\{(c_{i+1}-c_i)(n-i),E^i\right\}}{n-i} \leq c_{i+1}-c_i,\end{aligned}$$

where the first equality is implied by (3) and the first inequality by C2.

If $i=n-1$, we get

$$\begin{aligned}\varphi_n^{\mathcal{F}}(c,E)-\varphi_{n-1}^{\mathcal{F}}(c,E) &= E^{n-2}-F^{n-1}(c^{n-2};E^{n-2})\\ &\leq E^{n-2}-\left(E^{n-2}-((c_{n-1}-c_{n-2})-(c_n-c_{n-2}))\right)=c_n-c_{n-1},\end{aligned}$$

applying (4) in the first equality and C1 in the inequality.

Now we prove equality (1). Let $(c,E) \in C^N$ and $i \in \{1,...,n-1\}$. We make the proof by induction on $j > i$.

Observe that if $h \leq i$ then

$$\varphi_h^{\mathcal{F}}(c^{-i}, E^i) = 0, \tag{5}$$

since $0 \leq \varphi_h^{\mathcal{F}}(c^{-i}, E^i) \leq c_h^{-i} = 0$.

The first step of the induction: if $j = i+1$, by definition,

$$\begin{aligned}
\varphi_j^{\mathcal{F}}(c^{-i}, E^i) &= \frac{F^1(\left(c^{-i}\right)^0; \left(E^i\right)^0)}{n} + \cdots + \frac{F^{i+1}(\left(c^{-i}\right)^i; \left(E^i\right)^i)}{n-i} \\
&= \varphi_i^{\mathcal{F}}(c^{-i}, E^i) + \frac{F^{i+1}(\left(c^{-i}\right)^i; \left(E^i\right)^i)}{n-i} \\
&= \frac{F^{i+1}(\left(c^{-i}\right)^i; \left(E^i\right)^i)}{n-i},
\end{aligned}$$

where in the third equality we take into account expression (5).

Moreover, if $h = 1, \ldots n-i$, then

$$\left(c^{-i}\right)_h^i = c_{h+i}^{-i} - c_i^{-i} = (c_{h+i} - c_i) - (c_i - c_i) = c_{h+i} - c_i = c_h^i,$$

and

$$\begin{aligned}
\left(E^i\right)^i &= \left(E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c,E) - (n-i)\,\varphi_i^{\mathcal{F}}(c,E)\right)^i \\
&= E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c,E) - (n-i)\,\varphi_i^{\mathcal{F}}(c,E) \\
&- \left(\sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c^{-i},E^i) - (n-i)\,\varphi_i^{\mathcal{F}}(c^{-i},E^i)\right) = E^i,
\end{aligned}$$

where expression (5) has been applied in the last equality.

Hence,

$$\varphi_{i+1}^{\mathcal{F}}(c^{-i}, E^i) = \frac{F^{i+1}(c^i; E^i)}{n-i},$$

and since by (3) we have that

$$\varphi_{i+1}^{\mathcal{F}}(c,E) - \varphi_i^{\mathcal{F}}(c,E) = \frac{F^{i+1}(c^i; E^i)}{n-i},$$

then it holds

$$\varphi_{i+1}^{\mathcal{F}}(c,E) = \varphi_i^{\mathcal{F}}(c,E) + \varphi_{i+1}^{\mathcal{F}}(c^{-i}, E^i),$$

as was to be shown.

Assume that $\varphi_k^{\mathcal{F}}(c,E) = \varphi_i^{\mathcal{F}}(c,E) + \varphi_k^{\mathcal{F}}(c^{-i}, E^i)$ if $k < j$, and let us prove the equality for j. Distinguish two cases.

(a) If $i+2 \leq j < n$, by definition and expression (5),

$$\varphi_j^{\mathcal{F}}(c^{-i}, E^i) = \frac{F^1\left(\left(c^{-i}\right)^0; \left(E^i\right)^0\right)}{n} + \cdots + \frac{F^j\left(\left(c^{-i}\right)^{j-1}; \left(E^i\right)^{j-1}\right)}{n-j+1}$$

$$= \frac{F^{i+1}\left(\left(c^{-i}\right)^i; \left(E^i\right)^i\right)}{n-i} + \cdots + \frac{F^j\left(\left(c^{-i}\right)^{j-1}; \left(E^i\right)^{j-1}\right)}{n-j+1}.$$

For $k = i, \ldots, j-1$, and $h = 1, \ldots n-k$,

$$\left(c^{-i}\right)_h^k = c_{h+k}^{-i} - c_k^{-i} = (c_{h+k} - c_i) - (c_k - c_i) = c_{h+k} - c_k = c_h^k,$$

and

$$\left(E^i\right)^k = \left(E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c, E) - (n-i)\,\varphi_i^{\mathcal{F}}(c, E)\right)^k$$

$$= E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c, E) - (n-i)\,\varphi_i^{\mathcal{F}}(c, E)$$

$$- \left(\sum_{h=1}^{k} \varphi_h^{\mathcal{F}}(c^{-i}, E^i) + (n-k)\,\varphi_k^{\mathcal{F}}(c^{-i}, E^i)\right)$$

$$= E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c, E) - (n-i)\,\varphi_i^{\mathcal{F}}(c, E)$$

$$- \left(\sum_{h=i+1}^{k} \varphi_h^{\mathcal{F}}(c^{-i}, E^i) + (n-k)\,\varphi_k^{\mathcal{F}}(c^{-i}, E^i)\right)$$

$$= E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c, E) - \left((k-i)\,\varphi_i^{\mathcal{F}}(c, E) + \sum_{h=i+1}^{k} \varphi_h^{\mathcal{F}}(c^{-i}, E^i)\right)$$

$$- \left((n-k)\,\varphi_i^{\mathcal{F}}(c, E) + (n-k)\,\varphi_k^{\mathcal{F}}(c^{-i}, E^i)\right)$$

By induction, this expression turns into

$$E - \sum_{h=1}^{k} \varphi_h^{\mathcal{F}}(c, E) - (n-k)\,\varphi_k^{\mathcal{F}}(c, E) = E^k.$$

And hence the initial expression can be rewritten as follows:

$$\varphi_j^{\mathcal{F}}(c^{-i}, E^i) = \frac{F^{i+1}(c^i; E^i)}{n-i} + \cdots + \frac{F^j(c^{j-1}; E^{j-1})}{n-j+1}.$$

Since by definition,

$$\varphi_j^{\mathcal{F}}(c, E) - \varphi_i^{\mathcal{F}}(c, E) = \frac{F^{i+1}(c^i; E^i)}{n-i} + \cdots + \frac{F^j(c^{j-1}; E^{j-1})}{n-j+1},$$

it holds

$$\varphi_j^{\mathcal{F}}(c, E) = \varphi_i^{\mathcal{F}}(c, E) + \varphi_j^{\mathcal{F}}(c^{-i}, E^i).$$

(b) If $j = n$, we have

$$\varphi_i^{\mathcal{F}}(c,E) + \varphi_n^{\mathcal{F}}(c^{-i},E^i) = \varphi_i^{\mathcal{F}}(c,E) + E^i - \sum_{j=1}^{n-1} \varphi_j^{\mathcal{F}}(c^{-i},E^i)$$

$$= \varphi_i^{\mathcal{F}}(c,E) + E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c,E) - (n-i)\,\varphi_i^{\mathcal{F}}(c,E) - \sum_{j=i+1}^{n-1} \varphi_j^{\mathcal{F}}(c^{-i},E^i),$$

where the first equality holds because $\varphi^{\mathcal{F}}$ is a division rule and the second by definition of E^i. By induction and case (a), the last expression becomes

$$E - \sum_{h=1}^{i} \varphi_h^{\mathcal{F}}(c,E) - (n-i-1)\,\varphi_i^{\mathcal{F}}(c,E) - \sum_{j=i+1}^{n-1} \left(\varphi_j^{\mathcal{F}}(c,E) - \varphi_i^{\mathcal{F}}(c,E)\right)$$

$$= E - \sum_{j=1}^{n-1} \varphi_j^{\mathcal{F}}(c,E)$$

$$= \varphi_n^{\mathcal{F}}(c,E),$$

where in the last equality it is taken into account the definition of φ. □

Theorem 2. *A division rule satisfies claims separability if and only if there exists a list of functions* $\mathcal{F} = \left\{F^k\right\}_{k=1}^{n-1}$ *which satisfy C1, C2 and C3 such that the division rule is the one associated with* $\mathcal{F} = \left\{F^k\right\}_{k=1}^{n-1}$.

Proof. Existence is proved in the previous Lemma, so now we prove unicity. Let φ be a division rule satisfying claims separability. Define the list of functions $\mathcal{F} = \left\{F^k\right\}_{k=1}^{n-1}$, F^k from $X^{n-k+1} \times \mathbb{R}_+$ to $\mathbb{R}_+$ as follows:

$$F^k(x;y) = \begin{cases} (n-k+1)\varphi_k((\overbrace{0,\dots,0}^{k-1},x),y) & \text{if } y \le x_1 + \dots + x_{n-k+1}, \\ 0 & \text{if } y > x_1 + \dots + x_{n-k+1}. \end{cases}$$

We prove that these functions satisfy C1 and C2. First we show condition C1, that is, $F^k(x;y) \ge \max\left\{0, y - \sum_{j=2}^{n-k+1}(x_j - x_1)\right\}$ when $y \le x_1 + \cdots x_{n-k+1}$.

As φ is a division rule, then $\varphi_k(c,E) \ge 0$ for all $(c,E) \in C^N$, and therefore, $F^k(x;y) \ge 0$ for all $(x;y) \in X^{n-k+1} \times \mathbb{R}_+$.

On the other hand, since φ satisfies order preserving (Remark 1), then

$$\sum_{j=k}^{n}(\varphi_j((\overbrace{0,\dots,0}^{k-1},x),y) - \varphi_k((\overbrace{0,\dots,0}^{k-1},x),y))$$

$$\le \sum_{j=k}^{n}(x_{j-k+1} - x_1) = \sum_{j=2}^{n-k+1}(x_j - x_1).$$

And then

$$F^k(x;y) = (n-k+1)\varphi_k((\overbrace{0,\dots,0}^{k-1},x),y)$$

$$\geq \sum_{j=k}^{n} \varphi_j((\overbrace{0,\ldots,0}^{k-1},x),y) - \sum_{j=2}^{n-k+1}(x_j - x_1) = y - \sum_{j=2}^{n-k+1}(x_j - x_1),$$

where in the last equality it is taken into account that φ is a division rule.

Now we prove condition C2, that is, $F^k(x;y) \leq \min\{(n-k+1)\,x_1, y\}$ when $y \leq x_1 + \cdots x_{n-k+1}$.

Again by order preserving,

$$F^k(x;y) = (n-k+1)\varphi_k((\overbrace{0,\ldots,0}^{k-1},x),y) \leq y.$$

And finally, as φ is a division rule,

$$F^k(x;y) = (n-k+1)\varphi_k((\overbrace{0,\ldots,0}^{k-1},x),y) \leq (n-k+1)\,x_1,$$

and the proof is complete. □

Remark 2. *Notice that if in Theorem 2 the domain of division rules is C instead of C^N, then for each $n \in \mathbb{N}$ there exists a list of functions $\mathcal{F} = \{F^k\}_{k=1}^{n-1}$, and all of them determine the division rule.*

Theorem 2 allows to show that Piniles' rule and the alternative extension of the Ibn Ezra solution introduced by Bergantiños and Mendez-Naya (2001) and characterized by Alcalde et al. (2005), satisfy claims separability.

Piniles' rule, denoted by φ^{Pi}, provides each $(c,E) \in C^N$ with the following quantities:

$$UG(\frac{c}{2},E),$$

if $E \leq \frac{c(N)}{2}$, and

$$\frac{c}{2} + UG(\frac{c}{2}, E - \frac{c(N)}{2}),$$

if $E \geq \frac{c(N)}{2}$.

The alternative extension of the Ibn Ezra solution, denoted by φ^{GIE}, is defined as follows. Let $(c,E) \in C^N$. If $c_n \geq E$, then $\varphi^{GIE}(c,E) = \varphi^{IE}(c,E)$, where φ^{IE} denotes the Ibn Ezra solution. If $c_n < E$, a new problem $(c',E') \in C^N$ is considered, where $E' = E - c_n$ and $c' = c - \varphi^{IE}(c,c_n)$. If $c'_n \geq E'$ the process ends and each agent i is given $\varphi_i^{IE}(c,c_n)+\varphi_i^{IE}(c',E')$. If $c'_n < E'$, a new problem $(c'',E'') \in C^N$ is defined in the same way as $(c',E') \in C^N$ is defined with respect to (c,E). If $c''_n \geq E''$ the process ends and each agent i is given $\varphi_i^{IE}(c,c_n) + \varphi_i^{IE}(c',c'_n) + \varphi_i^{IE}(c'',E'')$, and if $c''_n < E''$ a new problem $(c''',E''') \in C^N$ is calculated and so on. It has been proved that in a finite number of steps the process ends, and therefore, each agent is given the corresponding sum of allocations.

Proposition 2. *φ^{Pi} and φ^{GIE} satisfy claims separability.*

Proof. To prove that φ^{Pi} satisfies claims separability, let $\mathcal{F} = \left\{F^k\right\}_{k=1}^{n-1}$, F^k from $X^{n-k+1} \times \mathbb{R}_+$ to $\mathbb{R}_+$ defined as follows:

$$F^k(x;y) = \begin{cases} (n-k+1)\varphi_k^{Pi}((\overbrace{0,\ldots,0}^{k-1},x),y) & \text{if } y \leq x_1 + \cdots + x_{n-k+1}, \\ 0 & \text{if } y > x_1 + \cdots + x_{n-k+1}. \end{cases}$$

By definition of φ^{Pi}, this division rule satisfies the equalities in Definition 1 with those F^k. Moreover, it can be easily proved that conditions C1 and C2 are satisfied. Therefore, by Theorem 2, φ^{Pi} satisfies claims separability. Similarly, if we take φ^{GIE} instead of φ^{Pi} and consider the corresponding F^k, by definition of φ^{GIE} the equalities in Definition 1 are also satisfied, as well as conditions C1 and C2. □

Remark 3. *As written before, the random arrival rule does not satisfy claims separability. However, it satisfies a variation of that axiom. Let $j \in N$. Consider the following definition for the subproblem $(c^{-i}, E^i) \in C^N$: $c_j^{-i} = c_j - c_i$ and $c_k^{-i} = c_k$ if $k \neq j$, and $E^i = E - \varphi_i(c,E)$. That is, only the claim of agent j is reduced by c_i and the other claims are not affected. Therefore, the endowment changes only by the allocation given to j by his/her portion c_i. The random arrival rule satisfies this variation and the proof is as in Proposition 1, taking into account that it satisfies the balanced contribution property associated with the subproblem determined by that c^{-i} and endowment $E - \varphi_i(c,E)$ (Albizuri and Santos, 2014).*

4. Some characterizations

Now we characterize the uniform gains rule, the uniform losses rule, the Talmud rule, Piniles' rule and the minimal overlap rule employing claims separability. In the first, claims separability and independence of null demands substitutes consistency in a characterization provided by Herrero and Villar (2001). The other two axioms employed by these authors are the following ones. It is denoted by φ a division rule with domain C.[3]

Path independence (Moulin, 1987). For all claims problems $(c,E) \in C^N$ and all $E' > E$ we have

$$\varphi(c,E) = \varphi(\varphi(c,E'),E).$$

According to it, if a problem is resolved with an endowment E', and after that the actual endowment, denoted by E, is even lower than the expected one, if the interin allocations are considered as claims, the corresponding allocations with E coincide with the allocations with E and the original claims.

[3]In the proofs of this section the set of agents varies, so we ask for the entire domain of claims problems.

Exemption. For all claims problems $(c, E) \in C^N$, if $c_i \leq \frac{E}{n}$ then $\varphi_i(c, E) = c_i$.

This axiom requires the small claimants not to be held responsible for the shortage. Observe that if the claims of all agents are lower than E/n there would be no bankruptcy.

In our case, we have to consider the well known axiom independence of null demands. Given a claims vector $c \in \mathbb{R}^n$ and $j \in \mathbb{N}$, we denote as before by $c^j \in \mathbb{R}^{n-j}$, the vector which satisfies $c^j_h = c^{-j}_{j+h}$ for each $h = 1, \ldots, n-j$.

Independence of null demands. Let $(c, E) \in C^N$ be such that $c_k = 0$ if $k \leq i$. Then

$$\varphi_j(c, E) = \varphi_j(c^i, E)$$

for all $j > i$.

This axiom states that the allocations are the same when null demands are not taken into account.

Theorem 3. *The uniform gains rule is the only rule that satisfies independence of null demands, claims separability, path independence and exemption.*

Proof. It is clear that the uniform gains rules satisfies independence of null demands. By Proposition 1 it satisfies claims separability, and by Herrero and Villar (2001) the other two axioms.

Now let us prove unicity. Let φ be a division rule which satisfies the four axioms. By independence of null demands we can assume that the claims are not null. We proceed by induction on n. If $n = 1$, since φ is a division rule, it holds $\varphi(c, E) = UG(c, E)$ for all $(c, E) \in C^N$. Assume that the equality is true when the number of agents is smaller than n, and we prove it for n. Let $(c, E) \in C^N$. Distinguish two cases.

(a) If $E/n \geq c_1$, exemption implies

$$\varphi_1(c, E) = c_1. \tag{6}$$

If $j > 1$, by claims separability,

$$\varphi_j(c, E) = \varphi_1(c, E) + \varphi_j(c^{-1}, E - n\varphi_1(c, E)).$$

Applying equality (6) , we get

$$\varphi_j(c, E) = c_1 + \varphi_j(c^{-1}, E - nc_1).$$

By independence of null demands and induction $\varphi_j(c^{-1}, E - nc_1)$ is uniquely determined.

(b) If $E/n \leq c_1$, consider $E' = nc_1 \geq E$. Taking into account claims separability,

$$\varphi_j(c, E') = \varphi_1(c, E') + \varphi_j(c^{-1}, E' - n\varphi_1(c, E'))$$

for all $j > 1$. Exemption implies $\varphi_1(c, E') = c_1$, and substituting in the above equality,

$$\varphi_j(c, E') = c_1 + \varphi_j(c^{-1}, E' - nc_1) = c_1 + \varphi_j(c^{-1}, 0) = c_1,$$

where in the last equality we take into account that φ is a division rule. By path independence, we have $\varphi_j(c, E) = \varphi_j(\varphi(c, E'), E)$ for all $j \in N$. Since we have proved that $\varphi_i(c, E') = \varphi_j(c, E')$ for all $i, j \in N$, claims separability implies $\varphi_i(\varphi(c, E'), E) = \varphi_j(\varphi(c, E'), E)$ for all $i, j \in N$, and hence $\varphi_j(\varphi(c, E'), E) = E/n$ for all $j \in N$. Therefore, $\varphi_j(c, E) = E/n$ and the proof is complete. □

We cannot drop out independence of null demands from the above characterization. The following rule differs from the uniform gains rule and it satisfies all the axioms except that axiom (it is straightforward but cumbersome to prove it, so we omit it). We write

$$(c', E') = ((c_1, c_1 + a, c_1 + a + b), 3c_1 + d),$$

where $a, b \geq 0$ and $0 \leq d \leq 2a + b$.

$$\varphi(c, E) = \begin{cases} UG(c, E), & \text{if } n \neq 3, \\ UG(c, E), & \text{if } n = 3 \text{ and } 3c_1 \geq E, \\ & \text{or } (c, E) = (c', E') \text{ and } d \geq 3a, \\ \left(c_1, c_1 + \frac{d}{3}, c_1 + \frac{2d}{3}\right), & \text{if } (c, E) = (c', E'), \\ & d \leq 3a \text{ and } d \leq 3b, \\ \left(c_1, c_1 + \frac{d-b}{2}, c_1 + b + \frac{d-b}{2}\right), & \text{if } (c, E) = (c', E'), \\ & d \leq 3a \text{ and } d \geq 3b. \end{cases}$$

We can give an alternative characterization for the uniform gains rule, considering the following axiom instead of path independence.

Equal allocations. Let $(c, E) \in C^N$. If for all $i \in N$ it holds $c_i \geq \frac{E}{n}$, then $\varphi_i(c, E) = \frac{E}{n}$ for all $i \in N$.

According to this axiom, if everybody claims at least the egalitarian allocation of the endowment, then this allocation has to be satisfied. Observe that instead of $\varphi_i(c, E) = E/n$ for all $i \in N$, we can write alternatively, $\varphi_i(c, E) = \varphi_j(c, E)$ for all $i, j \in N$.

Theorem 4. *The uniform gains rule is the only rule that satisfies independence of null demands, claims separability, equal allocations and exemption.*

Proof. It is is immediate that the uniform gains rules satisfies equal allocations. Unicity is proved with the first step of the above proof while in the second step equal allocations is applied. Indeed, if $E/n \leq c_1$ then $E/n \leq c_i$ for all $i \in N$. Hence, equal allocations implies $\varphi_i(c, E) = E/n$ and the proof is complete. □

Now we characterize the uniform losses rule. For that, as Herrero and Villar (2001), we take into account that the uniform losses rule is the dual rule of the uniform gains rule. Given a division rule φ, the *dual rule* of φ, denoted by φ^*, is defined by Aumann and Maschler (1985) as follows:

$$\varphi^*(c, E) = c - \varphi(c, L),$$

where $L = \left(\sum_{i \in N} c_i\right) - E$. We simply have to consider the dual properties of the above theorems. It is said that $\mathcal{P}^*$ is the *dual property* of $\mathcal{P}$ if for each division rule φ,

$$\varphi \text{ satisfies } \mathcal{P} \text{ if and only if } \varphi^* \text{ satisfies } \mathcal{P}^*.$$

It can be easily proved that the dual properties of claims separability and independence of null demands are themselves. The dual properties of path independence and exemption are, respectively, composition and exclusion, which have also been employed by Herrero and Villar (2001) to characterize the uniform losses rule.

Composition. For all claims problems $(c, E) \in C^N$ and all $E_1, E_2 \in \mathbb{R}_+ \setminus \{0\}$ such that $E_1 + E_2 = E$,

$$\varphi(c, E) = \varphi(c, E_1) + \varphi(c - \varphi(c, E_1), E_2).$$

Exclusion. For all claims problems $(c, E) \in C^N$, if $c_i \leq \frac{L}{n}$ then $\varphi_i(c, E) = 0$.

Theorem 5. *The uniform losses rule is the only rule that satisfies independence of null demands, claims separability, composition and exclusion.*

On the other hand, the dual property of equal allocations is this axiom.

Equal losses. Let $(c, E) \in C^N$. If for all $i \in N$ it holds $c_i \geq \frac{L}{n}$, then $c_i - \varphi_i(c, E) = c_j - \varphi_j(c, E)$ for all $i, j \in N$.

That is, if everybody claims more than an egalitarian share of losses, then losses are equal. In this case, we can also write alternatively $\varphi_i(c, E) = c_i - \frac{L}{n}$ for all $i \in N$.

Therefore, we have the following:

Theorem 6. *The uniform losses rule is the only rule that satisfies independence of null demands, claims separability, equal losses and exclusion.*

Moulin (2000) proves that the uniform gains rule, the uniform losses rule and the proportional rule are the only rules that satisfy composition, path independence, equal treatment of equals, scale invariance and consistency when the set of agents is smaller than a fixed set of agents with cardinality at least three. We notice that in this result consistency cannot be either substituted by claims separability and independence of null demands. Assume that the cardinality of the fixed set of agents is three. Consider the division rule φ which satisfies independence of null demands and

$$\varphi_j((c_1, c_2, c_3), E) = UG_1((c_1, c_2, c_3), E)$$
$$+P_j((0, c_2 - c_1, c_3 - c_1), E - 3UG_1((c_1, c_2, c_3), E)),$$

where $j = 1, 2, 3$, and P denotes the proportional rule. This division rule different from the uniform gains rule, uniform losses rule and the proportional rule satisfies all the axioms except consistency and satisfies claims separability.

We can also give a characterization of the Talmud rule employing claims separability. For that, we recall the following concept, which is also employed by Aumann and Maschler (1985) to characterize that rule.

A division rule φ is *self-dual* if $\varphi^* = \varphi$.

That is, the division rule treats gains and losses in the same way. We also require the following two properties, which are variations of the ones that characterize the uniform gains rule.

Half claims exemption. Let $(c, E) \in C^N$ be such that $E \leq \frac{(\sum_{i \in N} c_i)}{2}$. If $\frac{c_i}{2} \leq \frac{E}{n}$, then $\varphi_i(c, E) = \frac{c_i}{2}$.

Half claims equal allocations. Let $(c, E) \in C^N$ be such that $E \leq \frac{(\sum_{i \in N} c_i)}{2}$. If for all $i \in N$ it holds $\frac{c_i}{2} \geq \frac{E}{n}$, then $\varphi_i(c, E) = \frac{E}{n}$ for all $i \in N$.

In both axioms, the endowment does not reach half of the claims. If we assume that, if it is possible, everybody has right to obtain half of his/her claim, in both cases the allocations will be less or equal than half of the claims. In the first case, any agent obtains half of his/her half claim, if, sharing the endowment equally, he/she is able to obtain that half claim. In the second case, half claims of all the agents are not smaller than the egalitarian division of the endowment. Hence, everybody is given that egalitarian share.

Theorem 7. *The Talmud rule is the only rule that satisfies self-duality, claims separability, half claims equal allocations and half claims exemption.*

Proof. It is clear that the Talmud rule satisfies half claims equal allocations and half claims exemption. We have proved that it satisfies claims separability and as Aumann and Mashler (1985) point out it is a self-dual rule. To prove unicity, by self-duality we only have to consider the case $E \leq \left(\sum_{i\in N} c_i\right)/2$. If $c_1/2 \leq E/n$ then we can reason as in (a) of Theorem 4 and if $c_1/2 \geq E/n$ as in (b). □

Notice that Piniles' rule satisfies the four axioms in Theorem 7 but self-duality. It turns out that if we consider instead a weaker axiom than composition, it can be characterized. Clearly, Piniles' rule does not satisfy composition, but the equality in that axiom does hold in the particular case in which the endowment is divided by half of the claims. So, if we write $E^{HC} = \min\{c(N)/2, E\}$, the following axiom is satisfied by that division rule.

Half claims composition. For all claims problems $(c, E) \in C^N$,

$$\varphi(c, E) = \varphi(c, E^{HC}) + \varphi(c - \varphi(c, E^{HC}), E - E^{HC}).$$

Theorem 8. *Piniles' rule is the only rule that satisfies half claims composition, claims separability, half claims equal allocations and half claims exemption.*

Proof. It is clear that Piniles' rule satisfies the four axioms. For unicity, half claims composition implies that we only have to consider the case $E \leq \left(\sum_{i\in N} c_i\right)/2$. The rest follows as in Theorem 7. □

We would like also to point out that consistency cannot be substituted by claims separability (and independence of null demands if needed) in all the cases. For example, Young (1988) proves that the proportional rule is the only rule that satisfies equal treatment of equals, composition and self-duality. Not only the proportional rule but also the Talmud rule satisfies equal treatment of equals, claims separability, self-duality and independence of null demands.

For the characterization of the minimal overlap rule, we introduce a composition axiom. If $c \in \mathbb{R}^N_+$ and $x \in \mathbb{R}_+$ we denote by c^{-x} the vector defined by

$$c^{-x} = \left(\max\{c_j - x, 0\}\right)_{j\in N}.$$

Composition.* For all claims problems $(c, E) \in C^N$, and all $E_1, E_2 \in \mathbb{R}_+$ such that $E_1 + E_2 = E$ and $(c^{-E_1}, E_2) \in C^N$,

$$\varphi(c, E) = \varphi(c, E_1) + \varphi(c^{-E_1}, E_2).$$

As in composition, the state is provided in two steps. First, E_1 is given with the original claims. After sharing E_1, each claim is reduced by the quantity already given and then the rest of the state is allocated. The difference between composition and composition* is that in composition each claim is reduced by what the claimer has received instead of the entire quantity already allocated.

Proposition 3. *The minimal overlap rule satisfies composition*.*

Proof. Let $(c, E) \in C^N$. Distinguish two cases.

(a) If $E \leq c_n$, let $E_1, E_2 \in \mathbb{R}_+$ be such that $E_1 + E_2 = E$. Then $E_1 \leq c_n$ and $E_2 \leq c_n - E_1 = \left(c^{-E_1}\right)_n$. The latter inequality implies $(c^{-E_1}, E_2) \in C^N$. Let $i \in N$. Distinguish two subcases.

(i) If $c_i \leq E_1$, then

$$\begin{aligned}\varphi_i^{mo}(c, E) = \sum_{j=1}^{i} \frac{\min\{c_j, E\} - \min\{c_{j-1}, E\}}{n - j + 1} &= \sum_{j=1}^{i} \frac{c_j - c_{j-1}}{n - j + 1} \\ = \varphi_i^{mo}(c, E_1) = \varphi_i^{mo}(c, E_1) &+ \varphi_i^{mo}(c^{-E_1}, E_2),\end{aligned}$$

where the last equality is true because $\left(c^{-E_1}\right)_i = 0$.

(ii) If $c_i > E_1$, let $k = \min\{j \in N : c_j \geq E_1\}$. It holds

$$\begin{aligned}\varphi_i^{mo}(c, E) &= \sum_{j=1}^{i} \frac{\min\{c_j, E\} - \min\{c_{j-1}, E\}}{n - j + 1} \\ = \sum_{j=1}^{k} \frac{\min\{c_j, E_1\} - \min\{c_{j-1}, E_1\}}{n - j + 1} &+ \frac{\min\{c_k - E_1, E - E_1\}}{n - k + 1} \\ + \sum_{j=k+1}^{i} &\frac{\min\{c_j - E_1, E - E_1\} - \min\{c_{j-1} - E_1, E - E_1\}}{n - j + 1} \\ &= \varphi_i^{mo}(c, E_1) + \varphi_i^{mo}(c^{-E_1}, E_2),\end{aligned}$$

where the last equality is true because the first term on the left coincides with $\varphi_i^{mo}(c, E_1)$ and the the sum of the other two terms on the left coincides with $\varphi_i^{mo}(c^{-E_1}, E_2)$.

(b) If $E > c_n$, let $E_1, E_2 \in \mathbb{R}_+$ be such that $E_1 + E_2 = E$ and $(c^{-E_1}, E_2) \in C^N$. Let $t \leq c_n$ be the unique solution for the equation

$$\sum_{j=1}^{n} \max\{c_j - t, 0\} = E - t.$$

Now let us prove that $(c^{-E_1}, E_2) \in C^N$ if and only if $E_1 \leq t$. We have $(c^{-E_1}, E_2) \in C^N$ if and only if

$$\sum_{j=1}^{n} \left(c^{-E_1}\right)_j = \sum_{j=1}^{n} \max\{c_j - E_1, 0\} \geq E_2.$$

Consider the continuous function

$$f(x) = \left(\sum_{j=1}^{n} \max\{c_j - x, 0\} \right) - (E - x),$$

which has the unique zero t. If $x < t$ then $f(x) > 0$, and if $x > t$ then $f(x) < 0$. Therefore, if $E_1 \leq t$, then

$$\sum_{j=1}^{n} \max\{c_j - E_1, 0\} \geq E - E_1 = E_2.$$

And if $E_1 > t$,

$$\sum_{j=1}^{n} \max\{c_j - E_1, 0\} < E - E_1 = E_2.$$

Now distinguish two subcases.

(i) If $c_i \leq E_1$, then $c_i \leq t$ and

$$\varphi_i^{mo}(c, E) = \sum_{j=1}^{i} \frac{\min\{c_j, t\} - \min\{c_{j-1}, t\}}{n - j + 1} = \sum_{j=1}^{i} \frac{c_j - c_{j-1}}{n - j + 1}$$

$$= \varphi_i^{mo}(c, E_1) = \varphi_i^{mo}(c, E_1) + \varphi_i^{mo}(c^{-E_1}, E_2),$$

where as in case (a) the last equality is true because $\left(c^{-E_1}\right)_i = 0$.

(ii) If $c_i > E_1$, as in case (a), let $k = \min\{j \in N : c_j \geq E_1\}$. Then

$$\varphi_i^{mo}(c, E) = \sum_{j=1}^{i} \frac{\min\{c_j, t\} - \min\{c_{j-1}, t\}}{n - j + 1} + \max\{c_i - t, 0\}$$

$$= \sum_{j=1}^{k} \frac{\min\{c_j, E_1\} - \min\{c_{j-1}, E_1\}}{n - j + 1} + \frac{\min\{c_k - E_1, t - E_1\}}{n - k + 1}$$

$$+ \sum_{j=k+1}^{i} \frac{\min\{c_j - E_1, t - E_1\} - \min\{c_{j-1} - E_1, t - E_1\}}{n - j + 1}$$

$$+ \max\{(c_i - E_1) - (t - E_1), 0\}.$$

$$= \varphi_i^{mo}(c, E_1) + \frac{\min\{c_k - E_1, t - E_1\}}{n - k + 1}$$

$$+ \sum_{j=k+1}^{i} \frac{\min\{c_j - E_1, t - E_1\} - \min\{c_{j-1} - E_1, t - E_1\}}{n - j + 1}$$

$$+ \max\{(c_i - E_1) - (t - E_1), 0\}.$$

Since the unique solution for the equation

$$\sum_{j=1}^{n} \max\{(c_j - E_1) - t', 0\} = E_2 - t'$$

is $t' = t - E_1$, the expression above turns into $\varphi_i^{mo}(c, E_1) + \varphi_i^{mo}(c^{-E_1}, E_2)$ and the proof is complete. □

We also require invariance under claims truncation, which already appears in Curiel et al. (1987), and Dagan and Volij (1993), that states that what is claimed over the state does not change the allocations.

Invariance under claims truncation. For all claims problems $(c, E) \in C^N$,

$$\varphi(c, E) = \varphi((\min\{c_j, E\})_{j \in N}, E).$$

Theorem 9. *The minimal overlap rule is the only rule that satisfies invariance under claims truncation, claims separability and composition*.*

Proof. We have proved in the above proposition that the minimal overlap rule satisfies composition* and we have already proved that it satisfies claims separability. It is clear that it satisfies invariance under claims truncation. So let us prove unicity.

By claims separability, we only need to know the share for the first non null claim, if it exists. If all the claims are null, then the division rule gives zero to all the agents, as the minimal overlap rule does. So, let j be the first agent whose claim $c_j \neq 0$. Distinguish two cases.

(a) If $E \leq c_n$, consider two subcases.

(i) If $E \leq c_j$, let $c' \in \mathbb{R}_+^N$ be such that $c'_k = 0$ if $k = 1, ..., j-1$, and $c'_k = E$ if $k = j, ..., n$. By invariance under claims truncation,

$$\varphi_j(c, E) = \varphi_j(c', E).$$

Claims separability implies $\varphi_j(c', E) = \varphi_k(c', E)$ if $k = j, ..., n$. Thus, taking into account that φ is a division rule,

$$\varphi_j(c', E) = \frac{E}{n - j + 1},$$

and hence,

$$\varphi_j(c, E) = \frac{E}{n - j + 1}.$$

(ii) If $E > c_j$, consider composition* with $E_1 = c_j$. We have

$$\varphi_j(c, E) = \varphi_j(c, c_j) + \varphi_j(c^{-c_j}, E_2) = \varphi_j(c, c_j),$$

where the last equality is true because $(c^{-c_j})_j = 0$. Let $c^* \in \mathbb{R}_+^N$ be such that $c_k^* = 0$ if $k = 1, ..., j-1$, and $c_k^* = c_j$ if $k = j, ..., n$. Invariance under claims truncation implies

$$\varphi_j(c, c_j) = \varphi_j(c^*, c_j) = \frac{c_j}{n - j + 1},$$

where, as above, the last equality follows since φ is a division rule and satisfies claims separability. Hence,

$$\varphi_j(c, E) = \frac{c_j}{n - j + 1}.$$

(b) If $E > c_n$ take in composition* $E_1 = t$ where t is the unique solution for the equation

$$\sum_{k=1}^{n} \max\{c_k - t, 0\} = E - t.$$

Then,

$$\varphi_j(c, E) = \varphi_j(c, t) + \varphi_j(c^{-t}, E - t).$$

Since φ is a division rule it holds $\varphi_j(c^{-t}, E - t) = \max\{c_j - t, 0\}$, and substituting in the above equality,

$$\varphi_j(c, E) = \varphi_j(c, t) + \max\{c_j - t, 0\},$$

and the expression on the right is determined because (c, t) belongs to case (a) since $c_n \geq t$ by definition of t. $\square$

If we compare the characterization of the minimal overlap provided in Theorem 8 and the charaterization due to Alcalde et al. (2005) for the alternative extension of the Ibn Ezras's rule, we see that they employ a form of composition axiom and dummy axioms, joint with anonymity. Claims separability would be in that case a stronger requirement than anonymity and independent of the other two axioms.

Acknowledgement

This research has been partially supported by the University of the Basque Country (GIU13/31) and the Spanish MCINN (project ECO2012-33618).

5. References

Albizuri, M.J., Santos, J.C. (2014) A potential approach to claims problems. Working paper.

Alcalde, J., Marco, M., Silva, J. A. (2005) Bankruptcy problems and the Ibn Ezra's proposal, Econ. Theor. 26, 103-114.

Alcalde, J., Marco, M.C., Silva, J.A., (2008) The minimal overlap rule revisited. Soc. Choice Welfare 31,109-128.

Aumann, R.J., Maschler, M. (1985) Game-theoretic analysis of a bankruptcy problem from the Talmud. J Econ Theory 36, 195-213.

Bergantiños, G. and Méndez-Naya, L. (2001) Additivity in bankruptcy problems and in allocation problems, Spanish Econ. Rev. 3, 223-229

Curiel, I., Maschler, M., Tijs, S.H., (1987) Bankruptcy games. Zeitschrift für Operations Research 31, A143-A159.

Dagan, N., Volij, O. (1993) The bankruptcy problem: a cooperative bargaining approach, Math. Soc. Sci, 26, 287-297.

Herrero, C. and Villar, A. (2001) The three musketeers: Four classical solutions to bankruptcy problems, Math. Soc. Sci. 39, 307-328.

Moreno-Ternero, J.D., Villar, A., (2006) The TAL-family of rules for bankruptcy problems. Soc. Choice Welfare 27, 231-249.

Moulin, H., (1987). Equal or proportional division of a surplus, and other methods. Int. Jour. of Game Theory 16, 161-186.

Moulin, H., (2000) Priority rules and other asymmetric rationing methods. Econometrica 68, 643-684.

Moulin, H., (2002) Axiomatic cost and surplus-sharing. In: Arrow, K.J., Sen, A.K., Suzumura, K.(Eds.), Handbook of Social Choice and Welfare, vol. 1, North Holland, Elsevier, Amsterdam, pp. 289-357.

O'Neill, B., (1982) A problem of rights arbitration from the Talmud. Math. Soc. Sci. 2, 345-371.

Thomson, W., (2003) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math. Soc. Sci. 45, 249-297.

Thomson, W., (2012) Lorenz rankings of rules for the adjudication of conflicting claims. Econ. Theory 50, 547-569.

Thomson, W., (2013a) Game-theoretic analysis of bankruptcy and taxation problems: recent advances. Int. Game Theory Rev., 1-14.

Thomson, W., (2013b) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. Working Paper 578, University of Rochester.

Young, P., (1988) Distributive justice in taxation. Jour. of Econ. Theory 43, 321-335.