Finite graphs, free groups and Stallings' foldings

Final Degree Dissertation
Degree in Mathematics

Jone Lopez de Gamiz Zearra

Supervisors:
Montserrat Casals Ruiz
Javier Gutiérrez García

Leioa, 20 June 2017

Contents

Introduction

The main goal of the work is to study some basic properties of free groups, by using group theory and topology. For this, graphs play a main role, since we will prove that the fundamental group of a connected graph is a free group.

The matter is, as far as possible, self-contained, although the reader must have an elementary background on topology and group theory. Since an introduction to algebraic topology is not given until the last year and the course is optional, a brief summary of the fundamental group is given in the Appendix B. However, even if one uses the topological theory of coverings some results for graphs, such as those presented in Section 4.3, could be obtained more directly, we have tried to make the proofs using exclusively the theory developed in the dissertation. When this has not been possible, as in Proposition 4.3.5, a reference of the topological proof has been given.

The first three chapters are drawn mainly from [6]. With the aim of completing the theory and facilitating the understanding, the books [9], [8] and [2] have also been used. The last chapter, however, is based on [10]. Even so, all the examples of the last chapter, some examples of the previous chapters, some proofs of Section 2.2, Section 4.3 and Section 4.4, and the details of some proofs that are briefly given have been carried out by the author.

In the beginning, it may seem that the chapters have no relation. The reason is that *Ampliación de Topología* is taught during the second term, so I began to study the subject from [6] at the beginning of the academic year. When I started to write the work, in order not to exceed in length, I removed some chapters. Thus, in the first three chapters I have only written what it is not studied in the degree.

In Chapter 1 we give the definitions and some famous examples of graphs, simplicial complexes and cell complexes, which are three different ways of constructing topological spaces.

Chapter 2 talks about free groups and their most basic properties. It is also proved that the fundamental group of a connected graph is a free group, which is one of the main results of the work.

In Chapter 3, by using free groups, we will recall group presentations. In the ending, in Section 3.4, some useful applications of the famous Seifert-Van Kampen Theorem are given by means of group presentations.

The solved exercises of the first three chapters have been presented in Appendix A. The exercises of the last chapter, however, are put as examples with the theory, in order to make it lighter.

The second part of the work is developed in the last chapter.

Section 4.1 is dedicated to the category of graphs, with special emphasis in pushouts and pullbacks, due to their later importance.

Section 4.2, Section 4.3 and Section 4.4 are mainly focused on defining and giving the properties of the type of maps that will be essential when representing subgroups of free groups. Throughout those sections, Algorithm 1 stands out because of its numerous applications later on. Theorem 4.3.8 and Theorem 4.4.6 are also of great importance, because they show a way of representing the join and the intersection of free groups.

Section 4.5 is exclusively about Marshall Hall's Theorem.

Finally, the last section is only introduced in order to close Algorithm 1, and we will discuss whether a finitely generated subgroup of a free group is of finite index.

Before starting with the dissertation, it would be interesting to mention some of the properties of free groups which make them interesting, due to the fact that they do not always hold when working with other groups.

Firstly, we will prove that any subgroup of a finitely generated free group is free. When working with direct products, for example, we cannot state that any subgroup of a direct product is a direct product, as this example illustrates.

Secondly, after introducing group presentations, it will be trivial that a finitely generated subgroup of a free group is finitely presented. Here is an example of a group where such property does not hold.

Thirdly, as Howson's Theorem states, the intersection of finitely generated subgroups of a free group is again finitely generated. Nevertheless, if we do not work with free groups, we find counterexamples.

Finally, by using that the fundamental group of a connected graph is a free group, we will be able to find easily a free basis of any finitely generated subgroup of a free group. However, in well-known groups, such as the direct product of two free groups of rank 2, there are finately generated subgroups with unknown structure.

Notation

In general, if we talk about a topological space, since the endowed topology is going to be the natural one, it is not going to be mentioned. Therefore, instead of talking about a topological space (X, τ_X), we will only talk about the topological space X. Furthermore, since we are going to work always with topological spaces, sometimes the notion topological space will be replaced simply by space.

In relation to this, we will assume that the unit interval $[0, 1]$ is endowed with the usual topology.

In addition, Γ, Δ and Θ will always denote a graph. Moreover, V and E are going to be used for the vertex set and the edge set of a graph. However, if such graph has to be emphasized, V_Γ and E_Γ may also be used, if we are working with the graph Γ.

Chapter 1

Constructing spaces

The main purpose of the chapter is to give three different ways of building topological spaces, by using only simple building blocks.

Firstly, we will describe graphs, where the building blocks are vertices and edges. Although we will start defining them combinatorially, we will finish giving them a topological structure. We will also introduce Cayley graphs, which are a way of constructing graphs by using groups. Secondly, we will generalise this notion and we will introduce simplicial complexes, where the building blocks are simplices. Finally, due to the fact that simplicial complexes are often unwieldy, we will present cell complexes, which are often a much more efficient way of building a topological space.

1.1 Graphs

We are going to start with the definition of a graph. Intuitively, this is a countable collection of points, known as vertices, joined by a countable collection of arcs, known as edges. The formal definition is as follows.

Definition 1.1.1. A *graph* Γ consists of two sets E and V, where V and E are countable, and two maps $^{-}\colon E \longrightarrow E$ and $\iota\colon E \longrightarrow V$. Moreover, two rules must be satisfied for all $e \in E$:

$$\bar{\bar{e}} = e \quad \text{and} \quad \bar{e} \neq e.$$

An $e \in E$ is a *directed edge of* Γ, and $\bar{e} \in E$ is the *reverse of* e. As told before, the elements of V are called *vertices of* Γ, and $\iota(e)$ and $\tau(e) \coloneqq \iota(\bar{e})$ are the *initial vertex* and *terminal vertex of* e, respectively.

An *orientation of* Γ consists of a choice of exactly one edge in each pair $\{e, \bar{e}\}$. Alternatively, defining $\phi\colon \mathbb{Z}_2 \times E \longrightarrow E$,

$$\phi(g, e) = \begin{cases} e, & \text{if } g = \bar{0}, \\ \bar{e}, & \text{if } g = \bar{1}, \end{cases} \qquad g \in \mathbb{Z}_2,\ e \in E,$$

it is routine that the group $(\mathbb{Z}_2, +)$ acts freely on E, and an orientation is a choice of a representative in each orbit.

In practice, a graph is often represented by a diagram, using the following convention: a point marked on the diagram corresponds to a vertex of the graph, and a line joining two marked points corresponds to a set of edges of the form $\{e, \bar{e}\}$.
For example, a graph having two vertices v_1 and v_2 and two edges e and $\bar{e}$ with $\iota(e) = v_1$ and $\tau(e) = v_2$ can be represented in these two ways:

Let Γ be a graph and let V and E be the vertex and edge set, respectively. We form the topological space T which is the disjoint union of $E \times [0, 1]$ and V, where V and E are provided with the discrete topology. Let $\sim$ be the finest equivalence relation on T for which

$$(e, t) \sim (\bar{e}, 1 - t), \quad (e, 0) \sim \iota(e) \quad \text{and} \quad (e, 1) \sim \tau(e),$$

for $e \in E$ and $t \in [0, 1]$.
The quotient space $T/\sim$ is called the *realisation of the graph* Γ. From now on, graphs will also refer to the respective topological realisation.

Example 1.1.1. Let us consider the previous graph, where the vertex set is $V = \{v_1, v_2\}$, the edge set is $E = \{e, \bar{e}\}$ and $\iota(e) = v_1$, $\tau(e) = v_2$. Then, $\sim$ is the finest equivalence relation on $T = V \cup E \times [0, 1]$ such that

$$v_1 \sim (e, 0), \quad v_2 \sim (e, 1) \quad \text{and} \quad (\bar{e}, t) \sim (e, 1 - t), \quad \text{for all} \quad t \in [0, 1].$$

Hence, $T/\sim = \{[x] \mid x \in V \cup E \times [0, 1]\} = \{[(e, t)] \mid t \in [0, 1]\}$, which is homeomorphic to the unit interval, via the homeomorphism that sends each element $[(e, t)]$ to t.

Let us introduce a tool that describes many properties of a group in a topological way.

Definition 1.1.2. Let G be a group and let S be a set of generators for G. The associated *Cayley graph* is a graph with vertex set G and edge set $G \times S$, such that the initial and terminal vertices of a general edge (g, s) are respectively g and gs.

Remark 1.1.1. Notice that the Cayley graph of a group depends on the choice of the generators.

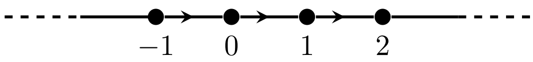


Figure 1.1: Cayley graph of $\mathbb{Z}$ with respect to $\{1\}$.

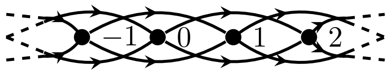


Figure 1.2: Cayley graph of $\mathbb{Z}$ with respect to $\{2, 3\}$.

If S is a generating set of G, each element g can be written as $s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$, where s_i is an element of S and $\epsilon_i \in \{1, -1\}$, for all $i \in \{1, \ldots, n\}$, $n \in \mathbb{N} \cup \{0\}$. This specifies a path, starting at the identity vertex, and running in an olderly way along the edges labelled $(s_1^{\epsilon_1} \cdots s_{i-1}^{\epsilon_{i-1}}, s_i)$ in the forwards direction if $\epsilon_i = 1$, backwards if $\epsilon_i = -1$, for $i \in \{1, \ldots, n\}$. Conversely, if we pick any path from the identity vertex to the vertex g, then this specifies a way of expressing g as a product of the chosen generators and their inverses. Therefore, we conclude that the equality $s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} = e$ holds in the group G if and only if the corresponding path starting at the identity vertex is a closed path.

1.2 Simplicial complexes

Firstly, we are going to define simplices, which are the standard pieces to build simplicial complexes.

Definition 1.2.1. The *standard n-simplex* is the set

$$\Delta^n = \Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \forall i \in \{0, \ldots, n\} \text{ and } \sum_{i=0}^{n} x_i = 1\Big\}.$$

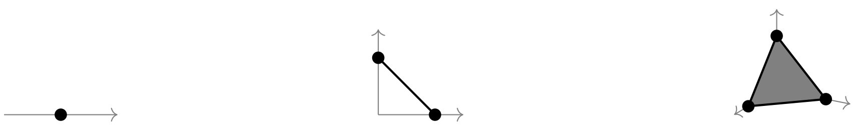
Its *vertices*, denoted by $V(\Delta^n)$, are those points $(x_0, \ldots, x_n)$ in Δ^n where $x_i = 1$ for some i (and hence $x_j = 0$ for the other $j \neq i$).
For each non-empty subset A of $\{1, \ldots, n\}$ there is a corresponding *face of* Δ^n, which is

$$\big\{(x_0, \ldots, x_n) \in \Delta^n \mid x_i = 0, \forall i \notin A\big\}.$$

To finish with the definitions, the *interior of* Δ^n is

$$\operatorname{int}(\Delta^n) = \big\{(x_0, \ldots, x_n) \in \Delta^n \mid x_i > 0, \forall i \in \{0, \ldots, n\}\big\}.$$

Note that $V(\Delta^n)$ determines a basis of the vector space $\mathbb{R}^{n+1}$. Hence, any map $f\colon V(\Delta^n) \longrightarrow \mathbb{R}^m$ extends to a unique linear map $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^m$. The restriction of this map to Δ^n is termed the *affine extension of* f.



(a) Δ^0 (b) Δ^1 (c) Δ^2

Figure 1.3: Standard simplicial complexes.

Definition 1.2.2. A *face inclusion of a standard m-simplex into a standard n-simplex* (where $m < n$) is the affine extension of an injection from $V(\Delta^m)$ to $V(\Delta^n)$.

At this stage, we are able to start constructing our new spaces.

Definition 1.2.3. An *abstract simplicial complex* is a pair (V, Σ) where V is a set, whose elements are called *vertices*, and Σ is a finite set of non-empty finite subsets of V, called *simplices*, such that

(i) for each $v \in V$, $\{v\}$ is in Σ,

(ii) if σ is an element of Σ, so is any non-empty subset of σ.

(V, Σ) is said to be *finite* if V is a finite set.

Definition 1.2.4. The *topological realisation* $|K|$ of an abstract simplicial complex $K = (V, \Sigma)$ is the space obtained by the following procedure:

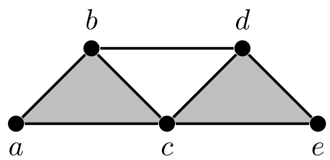
(i) For each $\sigma \in \Sigma$, take a copy of the standard n-simplex, where $n+1$ is the number of elements of σ. Denote this simplex by Δ_σ and label its vertices with the elements of σ.

(ii) Whenever σ is contained in $\tau \in \Sigma$, identify Δ_σ with a subset of Δ_τ, via the face inclusion which sends the elements of σ to the corresponding elements of τ.

Example 1.2.1. Let us take the abstract simplicial complex $K = (V, \Sigma)$, where $V = \{a, b, c, d, e\}$, and

$$\Sigma = \big\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\},\\ \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}\big\}.$$

If we take the elements $\{a\}$ and $\{a, b\}$ of Σ, we first have to take a copy of a standard 0-simplex, named $\Delta_{\{a\}}$, and a copy of a standard 1-simplex, $\Delta_{\{a,b\}}$, and secondly, we must identify $\Delta_{\{a\}}$ with a subset of $\Delta_{\{a,b\}}$. Performing this operation with all the possible cases, we obtain



Note that $|K|$ is the union of the interiors of the simplices. In addition, clearly the interiors of any two simplices are disjoint. Thus, any element $x \in |K|$ is expressed as

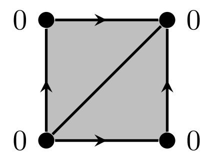
$$x = \sum_{i=0}^{n} \lambda_i v_i,$$

for a unique simplex with vertices $\{v_0, \dots, v_n\}$ and unique positive real numbers $\lambda_0, \dots, \lambda_n$ which sum to one.

The formal definition of a triangulation of a space, which we have used plenty of times, is based on this notion.

Definition 1.2.5. A *triangulation* of a space X is a simplicial complex K together with a homeomorphism $|K| \longrightarrow X$.

Example 1.2.2. Let us take a square with a diagonal line and identify opposite sides of the square.

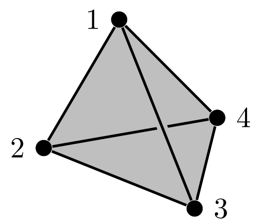


Note that this is not a triangulation of the torus, because we are identifying two vertices of the same 1-simplex, so the corresponding face inclusion is not well-defined; we are not working with a simplicial complex.

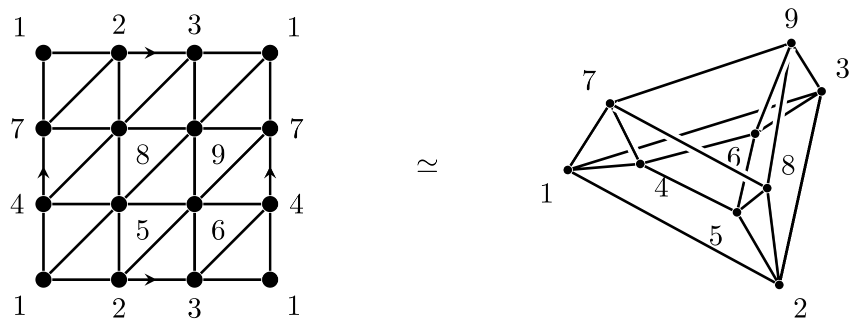
Example 1.2.3. A possible triangulation of the sphere is as follows. Let $K = (V, \Sigma)$ be an abstract simplicial complex such that $V = \{1, 2, 3, 4\}$, and

$$\Sigma = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{2,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3,4\}\big\}.$$

Thus, its topological realisation is homeomorphic to the sphere:



Example 1.2.4. The torus has a triangulation using nine vertices.



Definition 1.2.6. A *subcomplex* of a simplicial complex (V, Σ) is a simplicial complex (V', Σ') such that $V' \subseteq V$ and $\Sigma' \subseteq \Sigma$.

Definition 1.2.7. A *simplicial map* between abstract simplicial complexes (V_1, Σ_1) and (V_2, Σ_2) is a map $f\colon V_1 \longrightarrow V_2$ such that for all $\sigma_1 \in \Sigma_1$, $f(\sigma_1) \in \Sigma_2$.

Example 1.2.5. Let $V_1 = V_2 = \{1,2,3\}$ and $\Sigma_1 = \Sigma_2 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. Then, any $f\colon V_1 \longrightarrow V_2$ is a simplicial map, whereas the same is not true for $\Sigma_1 = \Sigma_2 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$.

A simplicial map f between abstract simplicial complexes K_1 and K_2 induces a map $|f|$ using the unique affine extension, with the following properties:

(i) $|f|$ is defined in $V(|K_1|)$ according to f,

(ii) if σ is a simplex of K_1 with vertices $v_0, \ldots, v_n$, then $f(v_0), \ldots, f(v_n)$ span a simplex of K_2,

(iii) if $x = \sum_{i=0}^{n} \lambda_i v_i$ for unique positive numbers $\lambda_0, \ldots, \lambda_n$ which sum to one and for a unique simplex with vertices $v_0, \ldots, v_n$, then $|f|(x) = \sum_{i=0}^{n} \lambda_i f(v_i)$.

Remark 1.2.1. $|K|$ can be seen as a quotient space, as well as a subspace of $\mathbb{R}^n$. However, the topology in both cases is the same (see [8]).

1.3 Cell complexes

In this final section, we introduce a useful generalisation of simplicial complexes, which are more efficient when building topological spaces.

Definition 1.3.1. Let X be a space, and let $f\colon S^{n-1} \longrightarrow X$ be a map. Then, the space obtained by *attaching an n-cell to X along f* is defined to be the quotient of the disjoint union $X \cup D^n$, such that for each point $x \in X$, $f^{-1}(x)$ and x are identified to a point. The space is denoted by $X \cup_f D^n$.

Definition 1.3.2. A finite cell complex is a space X descomposed as

$$K^0 \subset K^1 \subset \cdots \subset K^n = X,$$

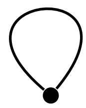
where

(i) K^0 is a finite set of points,

(ii) K^i is obtained from K^{i-1} by attaching a finite collection of i-cells.

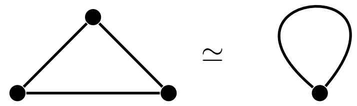
Example 1.3.1. Any simplicial complex is a finite cell complex by letting each n-simplex to be an n-cell.

Example 1.3.2. If we take a vertex and we attach the unit interval identifying the endpoints and that vertex (that is, we obtain a loop), we do not obtain a simplicial complex. The reason is that we are identifying the 0-simplex with a subset of the 1-simplex via the face inclusion; but if we identify it with two points, we will not obtain a well-defined map.



However, it is clearly a cell complex.

Nevertheless, we can always subdivide a loop to get a simplicial complex by adding other two vertices.



Remark 1.3.1. It can be proved that the realisation of a graph is a cell complex formed with 0 and 1-cells (see [9]). Therefore, by Example 1.3.2, topologically it can be seen as a simplicial complex.

Example 1.3.3. The construction of the torus by identifying opposite sides of the square may be viewed as a 2-dimensional cell structure with one 0-cell (the vertex 0), two 1-cells (the edges a and b) and one 2-cell (the square).

(a) Construction of the torus.

(b) Cell structure of the torus.

Chapter 2

Free groups

In this chapter, we will talk about free groups: for any set S, we will define a group, known as the free group on S. Their importance in this dissertation derives from the fact that the fundamental group of a connected graph is a free group. In order to prove it, we will present not only some properties of free groups, but also some terminology and basic results from graph theory. Finally, the chapter will end up with such proof and some easy examples.

2.1 Main definitions and properties

Informally speaking, we should view S as an alphabet, and the elements in $F(S)$ as words in this alphabet. The group operation is concatenation; in order to compose two words in $F(S)$, we simply write one down and then follow it by the other. For example, if $S = \{a, b\}$, then ab and ba are elements in $F(S)$, and their concatenation is $abba$. However, groups have inverses, so whenever a is an element of S, a^{-1} may also appear in a word. But then, $aa^{-1}b$ and b should represent the same element in the group, so instead of working with words, we will define an equivalence relation on the set of words on the alphabet S.

Let S be a set, known as the *alphabet*. From this set, we create a new set S^{-1} which is a copy of S, but for $x \in S$, we denote the corresponding element of S^{-1} by x^{-1}. Moreover, $S \cap S^{-1} = \varnothing$ and if $x^{-1} \in S^{-1}$, then $(x^{-1})^{-1} = x$. The elements of $S \cup S^{-1}$ are the *letters* of the alphabet.

Definition 2.1.1. A *word w on the alphabet S* is a finite sequence $x_1 \cdots x_m$ where $m \in \mathbb{N} \cup \{0\}$ and $x_i \in S \cup S^{-1}$ for $i \in \{1, \ldots, m\}$. Note that the empty sequence, where $m = 0$, is allowed as a word, and we denote it by $\varnothing$.

Definition 2.1.2. The *concatenation* of the words $x_1 \cdots x_m$ and $y_1 \cdots y_n$ where x_i, $y_j \in S \cup S^{-1}$, for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, is the word $x_1 \cdots x_m y_1 \cdots y_n$.

Definition 2.1.3. A word w' is an *elementary contraction* of a word w, written $w \searrow w'$, if $w = y_1xx^{-1}y_2$ and $w' = y_1y_2$, for words y_1 and y_2, and some $x \in S \cup S^{-1}$. We also say that w is an *elementary expansion* of w', written $w' \nearrow w$.

For example, if $S = \{a, b\}$, then $aa^{-1}b$ is an elementary expansion of b, and b is an elementary contraction of $aa^{-1}b$.

Definition 2.1.4. Two words w' and w on the alphabet S are *equivalent*, written $w \sim w'$, if there are words $w_1, \dots, w_n$ such that $w = w_1$ and $w' = w_n$, and for each $i \in \{1, \dots, n-1\}$, $w_i \nearrow w_{i+1}$ or $w_i \searrow w_{i+1}$.
This is, in fact, an equivalence relation defined on the set of words on the alphabet S.

With the previous tools, we are now in condition to define free groups.

Definition 2.1.5. The *free group on the set S*, denoted by $F(S)$, is the set of equivalence classes of words on the alphabet S.

Theorem 2.1.1. *Free groups are, in fact, groups, where the operation is defined as follows: if w and w' are words on S, then $[w] \cdot [w'] = [ww']$. The identity element is $[\varnothing]$ and the inverse of $[x_1 \cdots x_n]$ is $[x_n^{-1} \cdots x_1^{-1}]$.*

Definition 2.1.6. If a group G is isomorphic to $F(S)$, for some set S, the copy of S in G is known as a *free generating set.*

The next step is to analyse each equivalence class and to determine particular representatives.

Definition 2.1.7. A word is *reduced* if it does not admit an elementary contraction.

Lemma 2.1.2. *Let w_1, w_2 and w_3 be words on the alphabet S such that $w_1 \nearrow w_2 \searrow w_3$. Then, either there is a word w_2' such that $w_1 \searrow w_2' \nearrow w_3$ or $w_1 = w_3$.*

Proof. Since $w_1 \nearrow w_2$, it follows that $w_1 = ab$ and $w_2 = axx^{-1}b$, for some $x \in S \cup S^{-1}$ and some words a and b on S. Similarly, $w_2 \searrow w_3$, so w_3 is obtained from w_2 by deleting yy^{-1}, for some $y \in S \cup S^{-1}$.
Let us consider three cases:

(i) If $y \neq x$ and $y \neq x^{-1}$, it is possible to remove yy^{-1} from w_1 before inserting xx^{-1}. Hence, if we denote by w_2' the word obtained by removing yy^{-1} from w_1, we are in the first case: $w_1 \searrow w_2' \nearrow w_3$.

(ii) If $y = x$, all we have done is to insert and remove a pair of letters, so $w_1 = w_3$.

(iii) If $y = x^{-1}$, then either $w_2 = \tilde{a}x^{-1}xx^{-1}b = \tilde{a}yy^{-1}yb$ (where $a = \tilde{a}y$) or $w_2 = axx^{-1}x\tilde{b} = ay^{-1}yy^{-1}\tilde{b}$ (where $b = y^{-1}\tilde{b}$). In the first case, $w_1 = ab = ay^{-1}\tilde{b} = w_3$ and in the second case, $w_1 = ab = \tilde{a}yb = w_3$. □

Proposition 2.1.3. *Any element in the free group $F(S)$ is represented by a unique reduced word.*

Proof. Let us define the length of a word as the number of letters of such word. Note that an elementary contraction reduces the length by two. Hence, a shortest representative for an element in $F(S)$ must be reduced. Then, we only have to check that this shortest representative is unique.
Suppose that, on the contrary, there are distinct reduced words w and w' that are equivalent. Thus, there are words $w_1, \dots, w_n$ where $w = w_1$, $w' = w_n$ and for each $i \in \{1, \dots, n-1\}$, $w_i \nearrow w_{i+1}$ or $w_i \searrow w_{i+1}$. Consider a sequence of this type where $w_i \neq w_j$, for any $i \neq j$ (if w_i did equal w_j, we could miss out all the words in the sequence between them, creating a shorter sequence of words joining w and w').
Now, if, at some point, $w_i \nearrow w_{i+1} \searrow w_{i+2}$, then by Lemma 2.1.2, there is another word w'_{i+1} such that $w_i \searrow w'_{i+1} \nearrow w_{i+2}$. In this way, we may perform all elementary contractions before the elementary expansions.
Therefore, the sequence starts with $w = w_1 \searrow w_2$ or ends with $w_{n-1} \nearrow w_n = w'$, but this is a contradiction because w and w' were reduced words of the equivalence class. $\square$

One of the main characteristics of free groups is that they satisfy the following universal property:

Theorem 2.1.4. *Given any set S, any group G and any map $f \colon S \longrightarrow G$, there is a unique homomorphism $\phi \colon F(S) \longrightarrow G$ such that the following diagram commutes:*

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F(S) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \phi} \\ & & G \end{array}$$

where $\iota \colon S \longrightarrow F(S)$ denotes the canonical inclusion sending each element of S to the corresponding generator in $F(S)$.

Proof. We first show the existence of ϕ.
Consider a word $w = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ where $x_i \in S$ and $\epsilon_i \in \{1, -1\}$, for $i \in \{1, \dots, n\}$. We define

$$\phi(w) = f(x_1)^{\epsilon_1} \cdots f(x_n)^{\epsilon_n}.$$

In order to check that ϕ does not depend on the selected representative, it suffices to prove it when w' is an elementary contraction of w; that is, when $w = w_1 x x^{-1} w_2$ or $w = w_1 x^{-1} x w_2$ and $w' = w_1 w_2$ for words w_1 and w_2 on S and $x \in S$. In the first case,

$$\phi(w) = \phi(w_1) f(x) f(x)^{-1} \phi(w_2) = \phi(w_1)\phi(w_2) = \phi(w').$$

The other case is analogous. Therefore, ϕ is well-defined and it is clearly a homomorphism.
Finally, we have to check that ϕ is the unique homomorphism for which the diagram commutes, but this is routine because a homomorphism of groups is determined by what it does to a set of generators. □

2.2 The fundamental group of a graph

Our goal in this section is to prove that the fundamental group of a connected graph is a free group.
In order to obtain that result, we need some terminology and basic results from graph theory.

Definition 2.2.1. Let Γ be a graph with vertex set V, edge set E and maps $^{-}\colon E \to E$ and $\iota\colon E \to V$. A *subgraph* of Γ is a graph with vertex set $V' \subseteq V$, edge set $E' \subseteq E$ and the maps being the restrictions of the previous ones. For this to be defined, $\overline{e'}$ and $\iota(e')$ must be in E' and V', respectively, for each $e' \in E'$. Observe that under such conditions, $\tau(e')$ will also be an element of V', for each $e' \in E'$.

Definition 2.2.2. A *path* p in Γ of length $|p| = n$ with initial vertex u and terminal vertex v is an n-tuple of edges of Γ, $p = e_1 \cdots e_n$, such that for $i \in \{1, \ldots, n-1\}$ we have $\tau(e_i) = \iota(e_{i+1})$, $u = \iota(e_1)$ and $v = \tau(e_n)$.
For $n = 0$, given any vertex v, there is a unique path A_v of length 0 whose initial and terminal vertices coincide and are equal to v.
A path p is called a *circuit* if its initial and terminal vertices coincide.

If p and q are paths in Γ and the terminal vertex of p equals the initial vertex of q, then they may be concatenated to form a path pq such that $|pq| = |p| + |q|$, and whose initial vertex is that of p and whose terminal vertex is that of q.

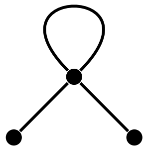
Definition 2.2.3. A *round-trip* is a path of the form $e\bar{e}$.

If a path p contains two adjacent edges forming a round-trip, then by deleting that round-trip we get a path p' with the same initial and terminal vertices as p, and $|p'| = |p| - 2$. In this case, we say that p' is an *elementary reduction of* p (and p an *elementary expansion of* p') and we write $p \searrow p'$.

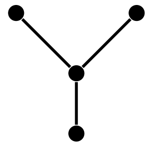
Definition 2.2.4. A *reduced path* in a graph Γ is a path containing no round-trip.

Definition 2.2.5. A graph is *connected* if any pair of vertices can be joined by some path.

Definition 2.2.6. A graph is a *forest* if the only reduced circuits have length 0, and if such graph is connected, it is called a *tree*.



(a) It is not a tree.



(b) It is a tree.

At this stage, it is pretty clear that there is a relationship between the notions of free groups and those we have just defined from graph theory. However, we still need to prove a few more properties.

Lemma 2.2.1. *In a tree, there is a unique reduced path between any two vertices.*

Proof. By connectedness, any two vertices are connected by a path. The shortest such path is reduced. Then, we only need to show that it is unique. By contradiction, suppose that there exist two distinct reduced paths between distinct vertices,

$$p = e_1 \cdots e_n \quad \text{and} \quad p' = e'_1 \cdots e'_m.$$

Thus, $e_1 \cdots e_n \overline{e'_m} \cdots \overline{e'_1}$ is a circuit, and since the graph is a forest, it must have length 0. That is, $e_n = e'_m$, $e_{n-1} = e'_{m-1}$ and so on. In this way, we conclude that $n = m$ and $e_i = e'_i$, for all $i \in \{1, \dots, n\}$. □

Definition 2.2.7. A *maximal tree* in a connected graph Γ is a subgraph T that is a tree, but where the addition of any edge of $E_\Gamma \setminus E_T$ to T gives a graph which is not a tree.

Lemma 2.2.2. *Let Γ be a connected graph and let T be a subgraph that is a tree. Then, the following are equivalent:*

(i) $V_T = V_\Gamma$,

(ii) T is maximal.

Proof. (i) $\Longrightarrow$ (ii). Let e be an edge of $E_\Gamma \setminus E_T$. If the initial and terminal vertices are the same, then adding e to T results in a subgraph that is not a tree. On the other hand, if the endpoints of e are distinct, then by hypothesis they lie in T and by Lemma 2.2.1, they are connected by a reduced path p in T. Thus, $p\bar{e}$ is a reduced circuit, so it is not a tree.

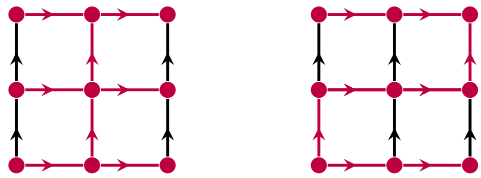
(ii) $\Longrightarrow$ (i). Suppose that there is a vertex v of Γ which is not in V_T. Pick a shortest path from T to v. Note that the first edge of the path starts in V_T, but it cannot end in V_T (otherwise, it would not be the shortest path), so we can add it to T and create a larger tree. □

We finish the main results of graph theory with the existence of maximal trees.

Lemma 2.2.3. *Any connected graph Γ contains a maximal tree.*

Proof. Since the vertex set of a graph is countable, we may pick a total ordering on V_Γ. Γ is connected, so we may assume that for each $i \geq 2$, the i-th vertex shares an edge with one of the earlier vertices.
Set T_1 to be the first vertex and for $i \geq 2$, let T_i be the subgraph such that the vertices are the first i-th ones, and the edges are the ones of T_{i-1} and the edge joining the i-th vertex and one of the previous vertices. Then, T_i is a tree, for all $i \geq 1$.
In conclusion, $T = \bigcup_{i\geq 1} T_i$ is also a tree, and since it contains all the vertices of Γ, by Lemma 2.2.2, T is maximal. □

Remark 2.2.1. The previous lemma guarantees the existence of maximal trees. However, they may not be unique. For example, in the following graph two different maximal trees are drawn in purple:



We are now in condition to prove the main theorem of this chapter.

Theorem 2.2.4. *The fundamental group of a connected graph is a free group.*

Proof. Note that loops in graphs may be thought as circuits, and two circuits are homotopic relative to $\{0, 1\}$ if and only if one can be obtained from the other one by a finite number of elementary reductions and expansions.
Let T be a maximal tree of Γ (Lemma 2.2.3) and b a vertex of Γ; in particular, it is also a vertex of the maximal tree, by Lemma 2.2.2.
Firstly, if a loop based at b lies in T, it is homotopic relative to $\{0, 1\}$ to the trivial circuit. Therefore, we only have to take care of loops that do not lie in T.
For any vertex v of Γ, let $\theta(v)$ be the unique reduced path (Lemma 2.2.1) from b to v in T. Then, if $p = e_1 \cdots e_n$ is a circuit which does not lie in T, p is homotopic relative to $\{0, 1\}$ to a path

$$p' = e'_1 \cdots e'_n,$$

where $e'_i = \theta(\iota(e_i))\, e_i\, \overline{\theta(\tau(e_i))}$, for all $i \in \{1, \ldots, n\}$. Now, if we set

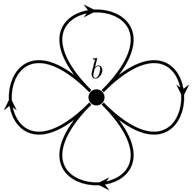
$$S = \left\{\theta(\iota(e))\, e\, \theta(\tau(e))^{-1} \mid e \in E_\Gamma \setminus E_T\right\},$$

and we define $f \colon S \longrightarrow \pi_1(\Gamma, b)$ as follows

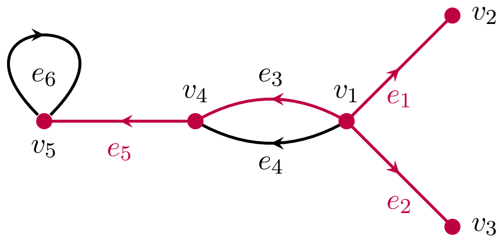
$$\theta(\iota(e))\, e\, \theta(\tau(e))^{-1} \longmapsto \left[(\iota(e))\, e\, \theta(\tau(e))^{-1}\right],$$

it is routine to prove that the homomorphism of Theorem 2.1.4 is an isomorphism. □

Example 2.2.1. Let Γ be the graph with a single vertex b and four edges. Hence, a maximal tree T consists of just the vertex. Thus, $\pi_1(\Gamma, b)$ is the free group on four generators.



Example 2.2.2. Let us consider the following graph.



A maximal tree is drawn in purple. Then, if we take as basepoint the vertex v_1, the fundamental group of such graph is the free group on 2 generators, where the generators are

$$\left\{e_4\, e_3^{-1}, e_3\, e_5\, e_6\, e_5^{-1}\, e_3^{-1}\right\}.$$

Chapter 3

Group presentations

In this chapter, we are going to define group presentations, and free groups will be the main tool for that.

3.1 Generators and relations

Let us start recalling some properties of normal subgroups.

Definition 3.1.1. Let B be a subset of a group G. The *normal subgroup generated by* B is the intersection of all normal subgroups of G that contain B, and we denote it by $\langle\langle B\rangle\rangle$.

Remark 3.1.1. The intersection of a family of normal subgroups is again a normal subgroup. Therefore, $\langle\langle B\rangle\rangle$ is the smallest normal subgroup containing B.

Proposition 3.1.1. *Let G be a group and $B \subseteq G$. The subgroup $\langle\langle B\rangle\rangle$ consists of all expressions of the form*

$$\prod_{i=1}^{n} g_i b_i^{\epsilon_i} g_i^{-1},$$

where $n \in \mathbb{N} \cup \{0\}$, $g_i \in G$, $b_i \in B$ and $\epsilon_i \in \{1, -1\}$, for all $i \in \{1, \ldots, n\}$.

We can now specify what it means to define a group via generators and relations.

Definition 3.1.2. Let X be a set and let R be a collection of elements in $F(X)$. The group with *presentation* $\langle X \mid R\rangle$ is defined to be $F(X)/\langle\langle R\rangle\rangle$.

Example 3.1.1. The dihedral group D_{2n} has the following presentation:

$$\langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma\rangle.$$

That is, the alphabet of the free group is $\{\sigma, \tau\}$, and the normal subgroup is generated by $R = \{\sigma^n, \tau^2, \tau\sigma\tau\sigma\}$.

The word $\tau\sigma^n\tau$ represents the identity element in D_{2n}. To see this, note that in $F(\{\sigma,\tau\})$

$$\tau\sigma^n\tau = (\tau\sigma^n\tau^{-1})\tau^2,$$

which lies in $\langle\langle R\rangle\rangle$ by Proposition 3.1.1.

Remark 3.1.2. Note that two words w and w' on the alphabet X represent the same element in $\langle X \mid R\rangle$ precisely when there is an element y in $\langle\langle R\rangle\rangle$ such that $w' = wy$, where the equality holds in the free group $F(X)$.

An alternative way to decide whether w and w' represent the same element in $\langle X \mid R\rangle$ is the following.

Proposition 3.1.2. *Let $G = \langle X \mid R\rangle$. Then, two words w and w' on the alphabet X represent the same element in G if and only if they differ by a finite sequence of the following moves:*

(i) perform an elementary contraction or expansion,

(ii) insert somewhere into the word one of the relations in R or its inverse.

Proof. The sufficient condition is trivial.
In order to prove the other implication, it suffices to show that if two words w and w' represent the same element in G, then they differ by a finite sequence of moves (i) and (ii). By Remark 3.1.2, we know that in this case, $w' = wy$ where $y \in \langle\langle R\rangle\rangle$. So, by Proposition 3.1.1,

$$w' = w\prod_{i=1}^{n} g_i r_i^{\epsilon_i} g_i^{-1},$$

for $n \in \mathbb{N}\cup\{0\}$, $g_i \in F(X)$, $r_i \in R$ and $\epsilon_i \in \{1,-1\}$, for all $i \in \{1,\dots,n\}$. Thus, we can obtain $wg_1g_1^{-1}$ from w by move (i), and then obtain $wg_1r_1^{\epsilon_1}g_1^{-1}$ from this by move (ii). Proceeding in this way, we obtain w' using moves (i) and (ii). □

Example 3.1.2. By Example 3.1.1, the word $\tau\sigma^n\tau$ represents the identity element in the dihedral group D_{2n}. Let us use the previous proposition to verify it.

$$\tau\sigma^n\tau \xrightarrow{\text{(ii)}} \tau\sigma^n\sigma^{-n}\tau \xrightarrow{\text{(i)}} \tau^2 \xrightarrow{\text{(ii)}} \tau^2\tau^{-2} \xrightarrow{\text{(i)}} e.$$

Remark 3.1.3. It is important to distinguish the product of the group and the concatenation of the free group.
Let $F(G)$ be the free group on the group G. Hence, if x_1 and x_2 are non-trivial elements in G and $x_3 = x_1x_2$ in G, x_3 and x_1x_2 represent distinct elements in $F(G)$, because they are non-equivalent words on the alphabet G.

By Theorem 2.1.4, there is a canonical homomorphism sending each generator in $F(G)$ to the corresponding element in G, which is clearly surjective. Let $R(G)$ be the kernel of this homomorphism; for example, in the previous case, $x_3x_2^{-1}x_1^{-1}$ lies in $R(G)$. Then, G is isomorphic to $F(G)/R(G)$, so G has presentation $\langle G \mid R(G)\rangle$, and it is called the *canonical presentation for* G.

Definition 3.1.3. A presentation $\langle X \mid R\rangle$ is *finite* if X and R are both finite sets. Moreover, a group is *finitely presented* if it admits a finite presentation.

The following result allows us to check whether a map from a group $\langle X \mid R\rangle$ to another group is a homomorphism.

Lemma 3.1.3. *Let $\langle X \mid R\rangle$ and G be groups. Let a map $f\colon X \longrightarrow G$ induce a homomorphism $\phi\colon F(X) \longrightarrow G$. This descends to a homomorphism $\varphi\colon \langle X \mid R\rangle \longrightarrow G$ if and only if $\phi(r) = e$ for all $r \in R$.*

Proof. The condition $\phi(r) = e$ for each $r \in R$ is necessary for φ to give a well-defined homomorphism.
Conversely, by Proposition 3.1.1, any element w in $\langle\langle R\rangle\rangle$ can be written as

$$\prod_{i=1}^{n} w_i r_i^{\epsilon_i} w_i^{-1},$$

where $n \in \mathbb{N} \cup \{0\}$, $w_i \in F(X)$, $r_i \in R$ and $\epsilon_i \in \{1, -1\}$, for $i \in \{1, \dots, n\}$. Since $\phi(r) = e$ for all $r \in R$, and taking into account that ϕ is a homomorphism, $\phi(w) = e$. Hence,

$$\begin{aligned} \varphi\colon F(X)/\langle\langle R\rangle\rangle &\longrightarrow H \\ x\langle\langle R\rangle\rangle &\longmapsto \phi(x) \end{aligned}$$

is well-defined. □

3.2 Tietze transformations

We have previously said that G has a presentation $\langle X \mid R\rangle$ if G is isomorphic to $F(X)/\langle\langle R\rangle\rangle$. In this section, however, we are going to use an equivalent definition: If G is isomorphic to $F(X)/\langle\langle R\rangle\rangle$, we can construct an epimorphism from $F(X)$ to G, such that the kernel is $\langle\langle R\rangle\rangle$. Thus, we obtain the following definition.

Definition 3.2.1. The group G has presentation $\langle X \mid R\rangle$ if there exists an epimorphism $\varphi\colon F(X) \longrightarrow G$ such that $\ker\varphi = \langle\langle R\rangle\rangle$. In this case we denote it by $\langle X \mid R\rangle^{\varphi}$.

A group G can have many presentations. For instance, the trivial group has presentation $\langle x \mid x\rangle$ and also $\langle x, y \mid x, xy\rangle$. We now look at how different presentations of the same group compare with each other.

Let $\langle X \mid R\rangle^{\varphi}$ be a presentation of G. Then so is $\langle X \mid R \cup S\rangle^{\varphi}$ for any set S contained in $\langle\langle R\rangle\rangle$. In this case, we say that $\langle X \mid R \cup S\rangle^{\varphi}$ comes from $\langle X \mid R\rangle^{\varphi}$ by a *general Tietze transformation of type I*, and that $\langle X \mid R\rangle^{\varphi}$ comes from $\langle X \mid R \cup S\rangle^{\varphi}$ by a *general Tietze transformation of type I′*. If $|S| = 1$, we refer to *simple Tietze transformations.*

Let Y be a set such that $X \cap Y = \varnothing$, and let u_y be an element in $F(X)$ for each $y \in Y$. Then, let us check that $\langle X \cup Y \mid R \cup \{yu_y^{-1} \mid y \in Y\}\rangle^{\psi}$ also presents G, where $\psi(x) = \varphi(x)$ for all $x \in X$ and $\psi(y) = \varphi(u_y)$ for all $y \in Y$.
Let N be the normal subgroup of $F(X\cup Y)$ generated by $R\cup\{yu_y^{-1} \mid y \in Y\}$. Since $N \subseteq \ker\psi$, ψ induces an epimorphism

$$\pi\colon F(X \cup Y)/N \longrightarrow G.$$

But, by Lemma 3.1.3, there is also a homomorphism

$$\theta\colon G \longrightarrow F(X \cup Y)/N,$$

with $\theta(\varphi(x)) = xN$, that it is in fact an epimorphism because $F(X \cup Y)/N$ is generated with words on the alphabet X. Note that $\pi \circ \theta$ is the identity map. Moreover, since $\theta(\pi(yN)) = \theta(\psi(y)) = \theta(\varphi(u_y)) = u_yN = yN$, $\theta \circ \pi$ is also the identity.
We say that $\langle X \cup Y \mid R \cup \{yu_y^{-1} \mid y \in Y\}\rangle^{\psi}$ comes from $\langle X \mid R\rangle^{\varphi}$ by a *general Tietze transformation of type II*, and that $\langle X \mid R\rangle^{\varphi}$ comes from $\langle X \cup Y \mid R \cup \{yu_y^{-1} \mid y \in Y\}\rangle^{\psi}$ by a *general Tietze transformation of type II′*. If $|Y| = 1$, we refer to *simple Tietze transformations.*

Theorem 3.2.1. *Any two presentations of the same group can be obtained from each other by a sequence of general Tietze transformations. If both presentations are finite, then each can be obtained from the other by a sequence of simple Tietze transformations.*

Proof. Let $\langle X \mid R\rangle^{\varphi}$ and $\langle Y \mid S\rangle^{\psi}$ both present a group G. Assume that $X \cap Y = \varnothing$. For each $y \in Y$ choose $u_y \in F(X)$ with $\psi(y) = \varphi(u_y)$, and for each $x \in X$ choose $v_x \in F(Y)$ with $\varphi(x) = \psi(v_x)$.
Defining θ by $\theta(x) = \varphi(x)$ and $\theta(y) = \varphi(u_y)$, we get a presentation

$$\left\langle X \cup Y \mid R \cup \{yu_y^{-1} \mid y \in Y\}\right\rangle^{\theta}$$

of G, obtained from the presentation $\langle X \mid R\rangle^{\varphi}$ by a general Tietze transformation of type II.
Now, $\theta(y) = \varphi(u_y)$, and this equals $\psi(y)$ by definition. It follows that

$\theta(w) = \psi(w)$ for any $w \in F(Y)$. In particular, $\theta(s) = \psi(s) = 1$ and $\theta(v_x) = \psi(v_x)$ which, by definition, equals $\varphi(x) = \theta(x)$.
We find from this that the presentation

$$\left\langle X \cup Y \mid R \cup S \cup \{yu_y^{-1} \mid y \in Y\} \cup \{xv_x^{-1} \mid x \in X\} \right\rangle^{\theta}$$

comes from the previous presentation by a general Tietze transformation of type I.
By simmetry, this presentation also comes from $\langle Y \mid S\rangle^{\psi}$ by a general Tietze transformation of type II followed by one of type I.
Finally, if X, Y, R and S are all finite, each general Tietze transformation used can be replaced by a finite sequence of simple Tietze transformations.
□

Example 3.2.1. Let $\langle x, y \mid x^3, y^2, (xy)^2\rangle$ be a presentation of the symmetric group of degree 3. Through Tietze transformations this presentation can be converted to $\langle y, z \mid (yz)^3, y^2, z^2\rangle$.

$$\begin{aligned}
\langle x, y \mid x^3, y^2, (xy)^2\rangle &\xrightarrow{\text{II}} \langle x, y, z \mid x^3, y^2, (xy)^2, zy^{-1}x^{-1}\rangle \\
&\longrightarrow \langle x, y, z \mid x^3, y^2, (xy)^2, zyx^{-1}\rangle \\
&\longrightarrow \langle x, y, z \mid x^3, y^2, (xy)^2, xy^{-1}z^{-1}\rangle \\
&\xrightarrow{\text{II}'} \langle y, z \mid (zy)^3, y^2, z^2\rangle.
\end{aligned}$$

3.3 Push-outs

In this section, we use presentations to define a construction which is important in group theory and it will allow us to introduce the Seifert-Van Kampen Theorem.

Definition 3.3.1. Let G_0, G_1 and G_2 be groups, and let $\phi_1 \colon G_0 \longrightarrow G_1$ and $\phi_2 \colon G_0 \longrightarrow G_2$ be homomorphisms. Let $\langle X_1 \mid R_1\rangle$ and $\langle X_2 \mid R_2\rangle$ be the canonical presentations of G_1 and G_2, where $X_1 \cap X_2 = \varnothing$. Then, the *push-out* $G_1 *_{G_0} G_2$ *of*

$$G_1 \xleftarrow{\phi_1} G_0 \xrightarrow{\phi_2} G_2$$

is the group

$$\left\langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(g) = \phi_2(g) \mid g \in G_0\} \right\rangle.$$

Remark 3.3.1. It can be proved that one may substitute other presentations for G_1 and G_2 in the definition and obtain the same group.

Remark 3.3.2. The inclusions

$$X_i \longrightarrow \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(g) = \phi_2(g) \mid g \in G_0\}\rangle,$$

for $i \in \{1,2\}$, induce canonical homomorphisms $\alpha_1 : G_1 \longrightarrow G_1 *_{G_0} G_2$ and $\alpha_2 : G_2 \longrightarrow G_1 *_{G_0} G_2$, by Lemma 3.1.3. Moreover, since $\phi_1(g) = \phi_2(g)$ holds in $G_1 *_{G_0} G_2$ for each $g \in G_0$, the following diagram commutes.

$$\begin{array}{ccc} G_0 & \xrightarrow{\phi_1} & G_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ G_2 & \xrightarrow{\alpha_2} & G_1 *_{G_0} G_2 \end{array}$$

Definition 3.3.2. When G_0 is the trivial group, the push-out $G_1 *_{G_0} G_2$ depends only on G_1 and G_2. It is known as the *free product* $G_1 * G_2$.

Definition 3.3.3. When $\phi_1 : G_0 \longrightarrow G_1$ and $\phi_2 : G_0 \longrightarrow G_2$ are injective, the push-out $G_1 *_{G_0} G_2$ is known as the *amalgamated free product of* G_1 *and* G_2 *along* G_0.

Example 3.3.1. The free product $\mathbb{Z} * \mathbb{Z}$ is isomorphic to the free group on two generators, because by Remark 3.3.1 we can take the presentations $\langle x \mid \varnothing \rangle$ and $\langle y \mid \varnothing \rangle$ for $\mathbb{Z}$.

3.4 Topological applications of the Seifert-Van Kampen Theorem

First of all, let us recall two different formulations of this theorem.

Theorem 3.4.1. *Let K be a space, which is a union of two path-connected open sets K_1 and K_2, where $K_1 \cap K_2$ is also path-connected. Let b be a point in $K_1 \cap K_2$, and let $\iota_1 : K_1 \cap K_2 \longrightarrow K_1$ and $\iota_2 : K_1 \cap K_2 \longrightarrow K_2$ be the inclusion maps. Then, $\pi_1(K, b)$ is isomorphic to the push-out of*

$$\pi_1(K_1, b) \xleftarrow{\iota_{1*}} \pi_1(K_1 \cap K_2, b) \xrightarrow{\iota_{2*}} \pi_1(K_2, b).$$

The second alternative formulation is as follows:

Theorem 3.4.2. *Let K, K_1, K_2, ι_1 and ι_2 be as in the previous theorem. Let $\langle X_1 \mid R_1 \rangle$ and $\langle X_2 \mid R_2 \rangle$ be the presentations for $\pi_1(K_1, b)$ and $\pi_1(K_2, b)$, with $X_1 \cap X_2 = \varnothing$. Then, a presentation of $\pi_1(K, b)$ is given by*

$$\big\langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\iota_{1_*}(g) = \iota_{2_*}(g) \mid g \in \pi_1(K_1 \cap K_2, b)\}\big\rangle.$$

The Seifert-Van Kampen Theorem can be used in order to prove that the fundamental group of a bouquet of n-circles is the free group on n generators.

Another important application is that it allows us to compute the fundamental group of cell complexes.

Theorem 3.4.3. *Let K be a path-connected cell complex, and let $l_i \colon S^1 \longrightarrow K^1$ be the attaching maps of the 2-cells, where $1 \leq i \leq n$. Let b be a basepoint in K^0. Let $[l'_i]$ be the conjugacy class of the loop l_i in $\pi_1(K^1, b)$. Then, $\pi_1(K, b)$ is isomorphic to $\pi_1(K^1, b)/\langle\langle [l'_1], \ldots, [l'_n] \rangle\rangle$.*

Remark 3.4.1. Note that a map from S^1 to a topological space may be seen as a loop in such topological space.

Remark 3.4.2. The loops l_i are not necessarily based on b, and hence they do not give well-defined elements in $\pi_1(K^1, b)$. However, they do give well-defined conjugacy classes: let w_i be a path from b to $l_i(0) = l_i(1)$, and let l'_i be $w_i\, l_i\, w_i^{-1}$. Then, $[l'_i] \in \pi_1(K^1, b)$.

Remark 3.4.3. Since a cell complex that consists of 0 and 1-cells can be seen as a graph, $\pi_1(K^1, b)$ is free.

Proof. We are only going to give an outline of the proof. We will describe how the fundamental group behaves when an n-cell is attached to a space, when $n \geq 2$.

Let X be a path-connected space, and let $f \colon S^{n-1} \longrightarrow X$ be the attaching map of an n-cell. Decompose $Y = X \cup_f D^n$ into the open and path-connected sets

$$K_1 = \big\{ z \in D^n \mid |z| < 2/3 \big\} / \sim \quad \text{and} \quad K_2 = \big\{ z \in D^n \mid |z| > 1/3 \big\} \cup X / \sim .$$

Then, K_1 is homeomorphic to an open n-ball, $K_1 \cap K_2$ is homeomorphic to $S^{n-1} \times (1/3, 2/3)$ (which is homotopy equivalent to S^{n-1}) and K_2 is homotopy equivalent to X.
Applying the Seifert-Van Kampen Theorem, when $n > 2$, $\pi_1(K_1 \cap K_2)$ and $\pi_1(K_1)$ are both trivial, so attaching an n-cell has no effect on the fundamental group. When $n = 2$, $\pi_1(K_1 \cap K_2)$ is isomorphic to $\mathbb{Z}$, so attaching a 2-cell has the effect of adding a relation to $\pi_1(X)$. □

Corollary 3.4.4. *Any finitely presented group can be realised as the fundamental group of a finite connected cell complex.*

Proof. Let $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$ be a finite presentation of a group. Let K^0 be a single point, and let K^1 be a bouquet of m-circles. Then, $\pi_1(K^1)$ is a free group on m generators, where each generator consists of a loop that goes round one of the circles.
Now attach 2-cells along the words r_j for $j \in \{1, \ldots, n\}$. By Theorem 3.4.3, the resulting space has the required fundamental group. □

Chapter 4

Stallings' foldings

4.1 The category of Graphs

One of the main goals of the chapter is to establish a relation between operations in the category of graphs, namely the pullback and the pushout, and group theoretic operations between subgroups of free groups, namely the intersection and the join. Let us begin recalling some basics of graph theory which were introduced in Chapter 1.
In this chapter we are going to consider only oriented and finite graphs.

Recall that a *graph* Γ consists of two sets E and V (as said before, in this chapter they are finite), and two maps $^{-}\colon E \longrightarrow E$ and $\iota\colon E \longrightarrow V$ such that

$$\bar{\bar{e}} = e \quad \text{and} \quad \bar{e} \neq e.$$

An *orientation* $\mathcal{O}$ of Γ consists of a choice of exactly one edge in each pair $\{e, \bar{e}\}$.
Finally, we can construct the realisation of the graph, which is in fact topologically equivalent to a simplicial complex.

Definition 4.1.1. A *map of graphs* $f\colon \Gamma \longrightarrow \Delta$ consists of a pair of maps which brings edges to edges, vertices to vertices and preserves the structure; that is,

$$f(\iota(e)) = \iota(f(e)), \quad f(\tau(e)) = \tau(f(e)) \quad \text{and} \quad f(\bar{e}) = \overline{f(e)},$$

for all $e \in E$.

Thus, oriented graphs and maps of graphs form a category denoted by **Grph**.
In addition, there are two functors, named *edges* and *vertices* from **Grph** to **Set** defined trivially.

The next step is to analyse pullbacks and pushouts.

Let us check that *pullbacks* in the category of graphs always exist:

Let $f_1\colon \Gamma_1 \longrightarrow \Delta$ and $f_2\colon \Gamma_2 \longrightarrow \Delta$ be maps of graphs. Define the graph Γ_3 to have vertex set

$$V_{\Gamma_3} = \{(u_1, u_2) \mid u_1 \text{ is a vertex of } \Gamma_1,\ u_2 \text{ is a vertex of } \Gamma_2 \text{ and } f_1(u_1) = f_2(u_2)\},$$

edge set

$$E_{\Gamma_3} = \{(e_1, e_2) \mid e_1 \text{ is an edge of } \Gamma_1,\ e_2 \text{ an edge of } \Gamma_2 \text{ and } f_1(e_1) = f_2(e_2)\},$$

and maps

$$\begin{array}{rcl} \iota\colon E_{\Gamma_3} & \longrightarrow & V_{\Gamma_3} \\ (e_1, e_2) & \longmapsto & \big(\iota_1(e_1), \iota_2(e_2)\big), \end{array} \qquad \begin{array}{rcl} {}^{-}\colon E_{\Gamma_3} & \longrightarrow & E_{\Gamma_3} \\ (e_1, e_2) & \longmapsto & \big(\overline{e_1}^{1}, \overline{e_2}^{2}\big), \end{array}$$

where ι_i and ${}^{-i}$ are the maps of the graph Γ_i, for $i \in \{1, 2\}$.
Define also p_1 and p_2 to be the projection maps from Γ_3 to Γ_1 and from Γ_3 to Γ_2, respectively. Then it is easy to check that:

(PB1) $f_1 \circ p_1 = f_2 \circ p_2$,

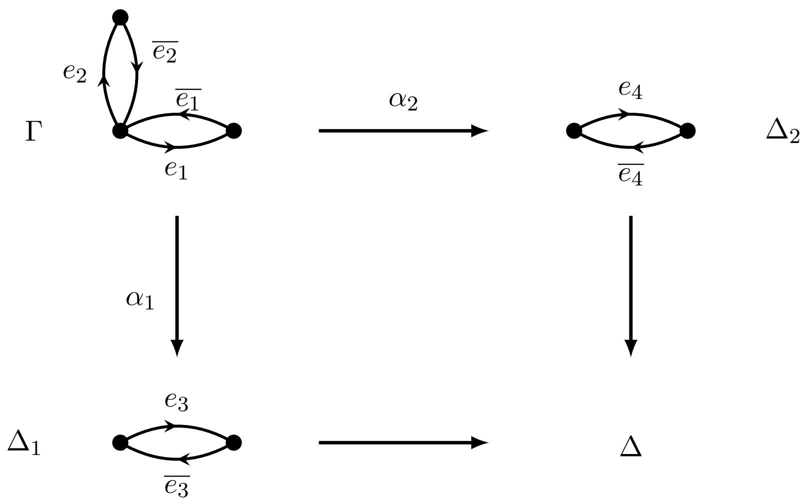
(PB2) If (Γ, q_1, q_2) is a further pair such that Γ is a graph and $q_i\colon \Gamma \longrightarrow \Gamma_i$ $(i \in \{1, 2\})$ are maps of graphs with $f_1 \circ q_1 = f_2 \circ q_2$, then there exists a unique map of graphs $f\colon \Gamma \longrightarrow \Gamma_3$ satisfying the conditions $q_i = p_i \circ f$ $(i \in \{1, 2\})$. Indeed, f is given by

$$f(u) = (q_1(u), q_2(u)) \quad \text{and} \quad f(e) = (q_1(e), q_2(e)),$$

for each $u \in V_\Gamma$ and $e \in E_\Gamma$.
In conclusion, (Γ_3, p_1, p_2) is the pullback of the maps of graphs f_1 and f_2.

Pushouts, however, do not always exist. Let us give an example.

Example 4.1.1. Let us consider the following diagram,



where $\alpha_1 \colon \Gamma \longrightarrow \Delta_1$ and $\alpha_2 \colon \Gamma \longrightarrow \Delta_2$ are defined in the following way:

$$\alpha_1(e_1) = \alpha_1(e_2) = e_3 \quad \text{and} \quad \alpha_2(e_1) = e_4,\ \alpha_2(e_1) = \overline{e_4}.$$

If the given diagram were a pushout diagram, the images of e_3 and e_4 in Δ should be equal, but also the images of e_3 and $\overline{e_4}$. In particular, the images of e_4 and $\overline{e_4}$ should be the same, which is not possible if Δ is a graph.

Even if in Chapter 2 we introduced paths, this time we are going to go in depth with them.

Recall that a *path* p in Γ, of length $n = |p|$, with initial vertex u and terminal vertex v, is an n-tuple of edges of Γ, $p = e_1 \cdots e_n$, such that for $i \in \{1, \ldots, n-1\}$, we have

$$\tau(e_i) = \iota(e_{i+1}), \quad u = \iota(e_1) \quad \text{and} \quad v = \tau(e_n).$$

For $n = 0$, given any vertex v, there is a unique path A_v of length 0 whose initial and terminal vertices coincide and are equal to v. Another way of defining paths is as follows: the standard arc of length n, A_n, can be described as the interval $[0, n]$ subdivided at the integer points; then, our path p is a map of graphs $p \colon A_n \longrightarrow \Gamma$ such that $p(0) = u$ and $p(n) = v$. Finally, paths are called *circuits* if the initial and terminal vertices coincide.
We also have an operation between compatible paths, the *concatenation*, which consists of joining both paths.

Hence, we can construct the category $\mathbf{P(\Gamma)}$, where the objects are the vertices of the graph Γ and the morphisms between two vertices are the paths joining them. Finally, the composition of the morphisms is the concatenation of paths, where the identity morphisms are the paths of length 0 with the necessary initial and terminal vertices.
Moreover, a map of graphs $f \colon \Gamma \longrightarrow \Delta$ induces a length-preserving functor denoted by the same symbol, $f \colon P(\Gamma) \longrightarrow P(\Delta)$.

Using that category, we can define the fundamental group from the point of view of category theory.
Recall that a *round-trip* is a path of the form $e\bar{e}$. If a path p contains two adjacent edges forming a round-trip, then by deleting them we get a path p' with the same initial and terminal vertices as p, and with $|p'| = |p| - 2$. In this case, p' is an elementary reduction of p, and we write $p \searrow p'$.
A *reduced path* is a path containing no round-trip.
The equivalence relation on $P(\Gamma)$ generated by $\searrow$ is denoted by $\sim$ and it is called *homotopy*. Every path is clearly homotopic to a reduced path. Concatenation of paths is compatible with homotopy, and thus the set of $\sim$-classes of $P(\Gamma)$ forms a small category denoted by $\boldsymbol{\pi}(\boldsymbol{\Gamma})$.
Each element in $\boldsymbol{\pi}(\boldsymbol{\Gamma})$ has an inverse: If A_v is a path of length 0, then $[A_v]^{-1} = A_v$. If $p = e_1 \cdots e_n$, then $[p]^{-1} = [\overline{e_n} \cdots \overline{e_1}]$.

To sum up, if we consider the set of elements in $\boldsymbol{\pi}(\boldsymbol{\Gamma})$ starting and ending at a fixed vertex v, we obtain a group $\pi_1(\Gamma, v)$: the *fundamental group of* Γ *based at* v. In this case too, given a map of graphs $f\colon \Gamma \longrightarrow \Delta$ there is a homomorphism denoted by the same symbol, $f\colon \pi_1(\Gamma, v) \longrightarrow \pi_1(\Delta, f(v))$, such that

$$f([p]) = [f(p)].$$

4.2 Stars

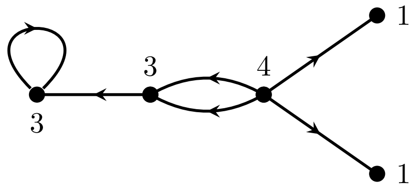
The goal of this section is to introduce special maps between graphs, called immersions and coverings, which are essential to study subgroups of a free group.

Definition 4.2.1. If v is a vertex of the graph Γ, the *star of* v *in* Γ is the set of edges of Γ:

$$\mathrm{St}(v, \Gamma) = \{e \in E \mid \iota(e) = v\}.$$

The cardinality of $\mathrm{St}(v, \Gamma)$ is called the *valence of* v in Γ.

Example 4.2.1. The valences of the vertices of the graph are written near each vertex.

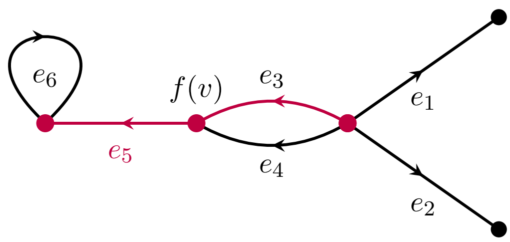


A map of graphs $f\colon \Gamma \longrightarrow \Delta$, yields, for each vertex v of Γ, a map

$$f_v\colon \mathrm{St}(v, \Gamma) \longrightarrow \mathrm{St}(f(v), \Delta)$$
$$e \longmapsto f(e).$$

Definition 4.2.2. If, for each vertex v of Γ the map f_v is injective, we call f an *immersion*. If f_v is surjective for all the vertices, we say that f is *locally surjective*. Finally, if f_v is bijective for each vertex, we call f a *covering*.

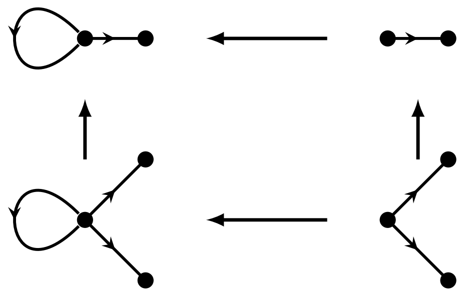
Example 4.2.2. A reduced path of length n in Γ is exactly the same as an immersion from the standard arc of length n to Γ. However, it may not be locally surjective.



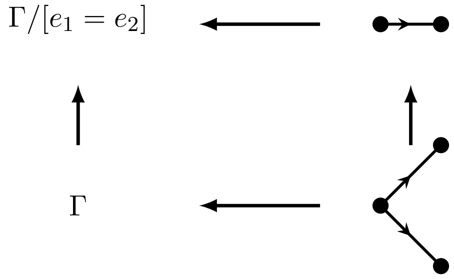
If we consider the path drawn in purple, $\overline{e_4}$ is an element of the star of $f(v)$ in the graph, but it is not the image of an edge of the standard arc of length 2.

Definition 4.2.3. A pair of edges (e_1, e_2) of Γ is said to be *admissible* if $\iota(e_1) = \iota(e_2)$ and $e_1 \neq \overline{e_2}$. In this case, we can identify $\tau(e_1)$ to $\tau(e_2)$, e_1 to e_2 and $\overline{e_1}$ to $\overline{e_2}$ to obtain a graph denoted by $\Gamma/[e_1 = e_2]$, which we call the result of *folding* (e_1, e_2) *in* Γ.

Example 4.2.3. Let us present an easy example of this construction.



In general, folding (e_1, e_2) in Γ is a particularly simple instance of the following pushout construction.



Let $f\colon \Gamma \longrightarrow \Delta$ be a map of graphs which is not an immersion. Then, there exists a vertex v of Γ such that

$$\begin{aligned} f_v\colon \mathrm{St}(v, \Gamma) &\longrightarrow \mathrm{St}(f(v), \Delta) \\ e &\longmapsto f(e) \end{aligned}$$

is not injective. Thus, there are two distinct edges e_1, e_2 of Γ with initial vertex v and $f(e_1) = f(e_2)$, so f folds the admissible pair (e_1, e_2) non-trivially (note that if $\overline{e_2} = e_1$, then $\overline{f(e_1)} = f(e_1)$, which is not possible).

Thus, if $f\colon \Gamma \longrightarrow \Delta$ is a map of graphs, we can find a finite sequence of foldings: $\Gamma = \Gamma_0 \longrightarrow \Gamma_1 \longrightarrow \Gamma_2 \longrightarrow \ldots \longrightarrow \Gamma_n$ and an immersion $\Gamma_n \longrightarrow \Delta$, so that the composition of the immersion and the sequence of foldings is equal to f.

As we mentioned before, coverings and immersions are used to represent subgroups of free groups. Therefore, we will analyse them separately. Let us start with coverings.

4.3 Coverings

The theory of coverings of graphs is a particular case of the topological theory of covering spaces. Let us prove some properties.

Proposition 4.3.1 (Path-lifting)**.** *If $f\colon \Gamma \longrightarrow \Delta$ is a covering, v a vertex of Γ and p a path in Δ with initial vertex $f(v)$, then there exists a unique path $\tilde{p}$ in Γ with initial vertex v such that $f(\tilde{p}) = p$.*

Proof. Suppose that p is equal to $e_1 \cdots e_n$ where $\iota(e_1) = f(v)$ and e_i is an edge of Δ, for all $i \in \{1, \dots, n\}$. Since f is a covering, f_v is bijective, so there exists a unique edge $\widetilde{e_1}$ of Γ with initial vertex v such that $f(\widetilde{e_1}) = e_1$. If we denote $\tau(\widetilde{e_1})$ by w, then $f(w) = \tau(f(\widetilde{e_1})) = \tau(e_1) = \iota(e_2)$, and we can continue as in the previous case until we obtain a path

$$\tilde{p} = \widetilde{e_1} \cdots \widetilde{e_n},$$

where $\widetilde{e_i}$ is an edge of Γ, for all $i \in \{1, \dots, n\}$ and $\iota(\widetilde{e_1}) = v$.
Finally, those edges are unique, because f is a covering. Therefore, $\tilde{p}$ is unique. □

Proposition 4.3.2 (Homotopy-lifting)**.** *In the notation of Proposition 4.3.1, if p is a round-trip, then $\tilde{p}$ is a round-trip. Hence, if $p \sim q$, then $\tilde{p} \sim \tilde{q}$.*

Proof. If $p = e\bar{e}$ for some edge e of Δ and initial vertex $f(v)$, following the same procedure as above, there exists a unique edge $\tilde{e}$ of Γ such that $f(\tilde{e}) = e$ and $\iota(\tilde{e}) = v$. Then, $\bar{e} = \overline{f(\tilde{e})} = f(\bar{\tilde{e}})$, so by uniqueness, $\tilde{p} = \tilde{e}\bar{\tilde{e}}$. □

Proposition 4.3.3 (General lifting)**.** *Let $f\colon \Gamma \longrightarrow \Delta$ be a covering and $g\colon \Theta \longrightarrow \Delta$ be a map of graphs with Θ connected. Further, let u and v be vertices of Γ and Θ such that $f(u) = g(v)$. Then, there exists $\tilde{g}\colon \Theta \longrightarrow \Gamma$ such that $\tilde{g}(v) = u$ and $f \circ \tilde{g} = g$ if and only if $g(\pi_1(\Theta, v)) \subseteq f(\pi_1(\Gamma, u))$. Moreover, if $\tilde{g}$ exists, it is unique.*

$$\begin{array}{ccc} & & \Gamma \\ & \overset{\tilde{g}}{\nearrow} & \downarrow f \\ \Theta & \xrightarrow{\;g\;} & \Delta \end{array}$$

Proof. The right implication is trivial. In order to prove the left implication, first of all let us construct $\tilde{g}$.
We fix $\tilde{g}(v)$ to be u. Now, let w be a vertex of Θ. Since Θ is connected, there exists a path p in Θ with initial vertex v and terminal vertex w. Thus, $g(p)$ starts in $g(v) = f(u)$ and finishes in $g(w)$. By Proposition 4.3.1, there exists a unique path $\tilde{p}$ in Γ with initial vertex u such that $f(\tilde{p}) = g(p)$. Then, we define $\tilde{g}(w)$ to be the terminal vertex of $\tilde{p}$. We can define $\tilde{g}$ in the edges of Θ in a similar way.
It remains to check that $\tilde{p}$ is well-defined; that is, that it does not depend on the selected path. Suppose that there are two paths in Θ joining v and w, p_1 and p_2. Thus, $g(p_1\overline{p_2}) \in f(\pi_1(\Gamma, u))$. This implies that

$$g(p_1\overline{p_2}) \sim f(p_3),$$

for some circuit p_3 in Γ based at u, and equivalently,

$$g(p_1) \sim f(p_3)g(p_2).$$

Finally, $\widetilde{p_1}$ and $p_3\widetilde{p_2}$ are the unique paths in Γ with initial vertex u that project by f into the previous paths, so by Proposition 4.3.2, $\widetilde{p_1} \sim p_3\widetilde{p_2}$. Therefore, both of them have the same final vertex.
The uniqueness of the paths of Proposition 4.3.1 imply the uniqueness of $\tilde{g}$. □

Proposition 4.3.4. *If $f\colon \Gamma \longrightarrow \Delta$ is a covering and u a vertex of Γ, then*

$$f\colon \pi_1(\Gamma, u) \longrightarrow \pi_1(\Delta, f(u))$$

is injective.

Proof. Let $[p_1]$ and $[p_2]$ be elements in $\pi_1(\Gamma, u)$ such that $f([p_1]) = f([p_2])$; that is, $f(p_1) \sim f(p_2)$. Thus, the uniqueness and Proposition 4.3.2 directly imply that $p_1 \sim p_2$. □

Proposition 4.3.5 (Existence of coverings)**.** *If Δ is connected, v a vertex of Δ, and $H \subseteq \pi_1(\Delta, v)$ a subgroup, then there exists a covering $f\colon \Gamma \longrightarrow \Delta$ where Γ is connected, with vertex u, such that $f(u) = v$ and $f(\pi_1(\Gamma, u)) = H$. The index of H in $\pi_1(\Delta, v)$ is the cardinality of $f^{-1}(v)$.*

Proof. It follows from the general theory of covering spaces. See, for instance, [4]. □

There is an interesting consequence of the previous theorem, which is called the Nielsen-Schreier Theorem.

Theorem 4.3.6 (Nielsen-Schreier)**.** *Any subgroup of a finitely generated free group is free.*

Proof. Let F be a free group on n generators. Then, $F \cong \pi_1(\Delta, v)$, where Δ is the bouquet of n-circles and v is the central vertex.
Let H be any subgroup of F. By Proposition 4.3.5, there exists a covering $f\colon \Gamma \longrightarrow \Delta$, where Γ is connected, with vertex u, such that $f(u) = v$ and $f(\pi_1(\Gamma, u)) = H$. Moreover, by Proposition 4.3.4, f is injective, so

$$H \cong \pi_1(\Gamma, u).$$

Finally, we know that $\pi_1(\Gamma, u)$ is a free group, so H is too. □

Another way of constructing graphs is using actions of groups on sets. Recall that a group G acts on a set M if for each $g \in G$ and $m \in M$, an element $g \cdot m \in M$ is defined such that $g_2 \cdot (g_1 \cdot m) = (g_2 g_1) \cdot m$ and $1 \cdot m = m$ for all $m \in M$, g_1, $g_2 \in G$.

Definition 4.3.1. A *group G acts on a graph* Γ if the actions of G on the set of edges and vertices satisfy that

$$g \cdot \iota(e) = \iota(g \cdot e), \quad g \cdot \bar{e} = \overline{g \cdot e} \quad \text{and} \quad g \cdot e \neq \bar{e},$$

for all $g \in G$ and all the edges of Γ.

Let a group G act on a graph Γ. For every edge and vertex x we denote by $\mathcal{O}(x)$ the orbit of x with respect to this action,

$$\mathcal{O}(x) = \{g \cdot x \mid g \in G\}.$$

In this case, we can define the *factor graph* as the graph with vertices $\mathcal{O}(v)$, for each vertex v of Γ, and edges $\mathcal{O}(e)$, for each edge e of Γ, with the following conditions:

(1) $\mathcal{O}(v)$ is the initial vertex of $\mathcal{O}(e)$ if there exists $g \in G$ such that $g \cdot v$ is the initial vertex of e (note that it does not depend on the representative of the orbit),

(2) the inverse of the edge $\mathcal{O}(e)$ is the edge $\mathcal{O}(\bar{e})$ (and so $\overline{\overline{\mathcal{O}(e)}} = \mathcal{O}(e)$).

Notice that the edges $\mathcal{O}(e)$ and $\mathcal{O}(\bar{e})$ do not coincide since G acts on Γ without inversion of edges ($g \cdot e \neq \bar{e}$). Therefore, the factor graph, denoted by Γ/G is in fact a graph, and the projection map,

$$\begin{aligned} p\colon \Gamma &\longrightarrow \Gamma/G \\ x &\longmapsto \mathcal{O}(x), \end{aligned}$$

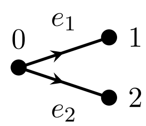
not only is a map of graphs, but it is also locally surjective.

Proof. Let v be a vertex of Γ and $\mathcal{O}(e)$ be an element of the star of $\mathcal{O}(v)$ in Γ/G. By definition, $g \cdot v$ is the initial vertex of e, for some $g \in G$, so

$$\iota(g^{-1} \cdot e) = g^{-1} \cdot (\iota(e)) = g^{-1} \cdot (g \cdot v) = (g^{-1} g) \cdot v = v.$$

Thus, $g^{-1} \cdot e$ is an element of the star of v in Γ which projects by p_v on $\mathcal{O}(g^{-1} \cdot e) = \mathcal{O}(e)$. $\square$

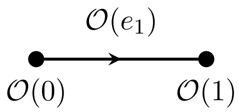
Example 4.3.1. Let us consider the following graph.



If we consider the group $G = \langle (1,2) \rangle \leq \Sigma_3$ and the actions on the vertex and edge set

$$\begin{array}{rcl} G \times V & \longrightarrow & V \\ (\sigma, i) & \longmapsto & \sigma(i), \end{array} \qquad \begin{array}{rcl} G \times E & \longrightarrow & E \\ (\sigma, e_i) & \longmapsto & e_{\sigma(i)}, \\ (\sigma, \overline{e_i}) & \longmapsto & \overline{e_{\sigma(i)}}, \end{array}$$

we obtain the same graph as folding the pair (e_1, e_2).



Proposition 4.3.7. *If G acts freely on Γ, p is a covering.*

Proof. We only need to check that p is an immersion.
Let v be a vertex of the graph and e_1, e_2 be edges of Γ with initial vertex v such that $\mathcal{O}(e_1) = \mathcal{O}(e_2)$. Hence, $e_2 = g \cdot e_1$ for some $g \in G$, so $v = \iota(e_2) = g \cdot \iota(e_1) = g \cdot v$. Thus, by hypothesis, $g = 1$. $\square$

At this stage, we have all the required tools to establish a relation between the pushout of graphs and the join of subgroups.

Theorem 4.3.8 (Pushout represents join)**.** *Suppose that*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\alpha_1} & \Delta_1 \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle \beta_1} \\ \Delta_2 & \xrightarrow{\beta_2} & \Theta \end{array}$$

is a pushout diagram, where Γ, Δ_1, Δ_2 and Θ are connected graphs and β_1 is onto. Let z be a vertex of Γ; call the images of z in Δ_1, Δ_2 and Θ respectively v_1, v_2 and w. Then,

$$\pi_1(\Theta, w) = \beta_1(\pi_1(\Delta_1, v_1)) \vee \beta_2(\pi_1(\Delta_2, v_2)).$$

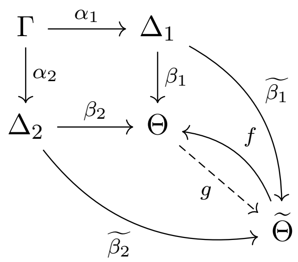
Proof. If we denote by H the subgroup of the right hand side, taking into account that Θ is connected and w is a vertex of the graph, by Proposition 4.3.5, there exists a covering $f\colon \widetilde{\Theta} \longrightarrow \Theta$ where $\widetilde{\Theta}$ is connected, with vertex u, such that $f(u) = w$ and

$$f(\pi_1(\widetilde{\Theta}, u)) = H.$$

Moreover, by Proposition 4.3.3, there exist

$$\widetilde{\beta_1}\colon \Delta_1 \longrightarrow \widetilde{\Theta} \text{ and } \widetilde{\beta_2}\colon \Delta_2 \longrightarrow \widetilde{\Theta},$$

such that $\widetilde{\beta_1}(v_1) = \widetilde{\beta_2}(v_2) = u$, $f(\widetilde{\beta_1}) = \beta_1$ and $f(\widetilde{\beta_2}) = \beta_2$. Then, $\widetilde{\beta_1} \circ \alpha_1$ and $\widetilde{\beta_2} \circ \alpha_2$ are liftings of $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$, so by uniqueness, $\widetilde{\beta_1} \circ \alpha_1 = \widetilde{\beta_2} \circ \alpha_2$.



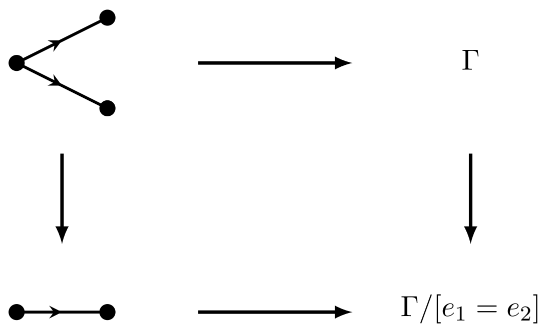
Using the fact that our diagram is a pushout diagram, there is a map of graphs $g\colon \Theta \longrightarrow \widetilde{\Theta}$ such that $g(w) = g(\beta_1(v_1)) = \widetilde{\beta_1}(v_1) = u$. Let us show that f is surjective.
If x is an element of the graph Θ, since β_1 is surjective, there exists y_1 in Δ_1 such that $\beta_1(y_1) = x$. Hence, $g(x) = \widetilde{\beta_1}(y_1)$, and $f(g(x)) = \beta_1(y_1) = x$. In conclusion, $f \circ g = \mathrm{id}$. Thus, considering

$$f\colon \pi_1(\widetilde{\Theta}, u) \longrightarrow \pi_1(\Theta, w),$$

by Proposition 4.3.4 it is injective, and taking into account that $f \circ g = \mathrm{id}$, it is routine to check that it is in fact an isomorphism. Hence, we achieve the required equality. □

Corollary 4.3.9. *If (e_1, e_2) is an admissible pair of edges in a connected graph Γ, then the folding map $\Gamma \longrightarrow \Gamma/[e_1 = e_2]$ is surjective on fundamental groups.*

Proof. As we have mentioned before, folding (e_1, e_2) is a particular example of the pushout construction



Since the hypothesis of the previous theorem hold and the lower left corner has trivial fundamental group, we immediately obtain the conclusion. □

4.4 Immersions

Immersions have some of the properties of coverings, and they do represent subgroups more efficiently than do coverings. We now start with some basic properties.

Proposition 4.4.1 (Preservation of reduced paths). *If $f\colon \Gamma \longrightarrow \Delta$ is an immersion of graphs, and p is a reduced path in Γ, then $f(p)$ is a reduced path in Δ.*

Proof. Let us prove, first of all, that the composition of two immersions is again an immersion.
Let $f\colon \Gamma \longrightarrow \Delta$ and $g\colon \Delta \longrightarrow \Theta$ be immersions and v be a vertex of Γ.

$$\begin{aligned}(g \circ f)_v \colon \mathrm{St}(v, \Gamma) &\longrightarrow \mathrm{St}((g \circ f)(v), \Theta)\\ e &\longmapsto (g \circ f)(e).\end{aligned}$$

If e_1 and e_2 are edges of Γ with initial vertex v and $g(f(e_1)) = g(f(e_2))$, since g is an immersion and $f(e_1)$ and $f(e_2)$ lie in the star of $f(v)$ in Δ, then $f(e_1) = f(e_2)$. Finally, f is an immersion and e_1 and e_2 are elements of the same star, so $e_1 = e_2$.
To sum up, since a reduced path in Γ is exactly the same as an immersion from a standard arc to Γ, $f(p)$ is an immersion in Δ. □

Proposition 4.4.2 (Uniqueness of path-lifting). *If $f\colon \Gamma \longrightarrow \Delta$ is an immersion, p and q are paths in Γ having the same initial vertex, and $f(p)=f(q)$, then $p = q$.*

Proof. It is easy to prove it by induction on $|p|$. □

Proposition 4.4.3. *If $f\colon \Gamma \longrightarrow \Delta$ is an immersion, Θ a connected graph and g_1, $g_2\colon \Theta \longrightarrow \Gamma$ are maps of graphs such that $f \circ g_1 = f \circ g_2$ and $g_1(v) = g_2(v)$ for some vertex v of Θ, then $g_1 = g_2$.*

Proof. We are going to show that $g_1(v) = g_2(v)$ for all $v \in V_\Theta$. In the case of the edges, the argument is similar.
Let w be a vertex of Θ. Since Θ is connected, there is a path from v to w in Θ, p. Then, $g_1(p)$ is a path in Γ from $g_1(v)$ to $g_1(w)$ and $g_2(p)$ is a path in Γ from $g_2(v)$ to $g_2(w)$. By hypothesis, $f \circ g_1 = f \circ g_2$, so $(f \circ g_1)(p)$ and $(f \circ g_2)(p)$ are equal. Moreover, $g_1(v) = g_2(v)$, so by Proposition 4.4.2, $g_1(p) = g_2(p)$. Thus, the terminal vertices must be the same. □

Proposition 4.4.4. *If p and q are reduced, homotopic paths in Γ, then $p = q$.*

Proof. It is a consequence of the fact that the fundamental group of a graph is a free group, and in each equivalence class of a free group there is a unique reduced word. □

As in the case of coverings, we also have the following property.

Proposition 4.4.5 (Injectivity of π_1)**.** *If $f\colon \Gamma \longrightarrow \Delta$ is an immersion and v is a vertex of Γ, then*

$$f\colon \pi_1(\Gamma, v) \longrightarrow \pi_1(\Delta, f(v))$$

is injective.

Proof. Let α be a non-trivial element in $\pi_1(\Gamma, v)$. Then, α is represented by a circuit p based at v with p reduced and $|p| \geq 1$. By Proposition 4.4.1, $f(p)$ is reduced, and f is length-preserving, so $|f(p)| \geq 1$. Thus, by Proposition 4.4.4 it is not homotopic to a path of length 0, so $f \circ \alpha$ is non-trivial. □

We are now going to introduce a useful way of representing certain subgroups of free groups by immersions.

Algorithm 1. Given a finite set of elements $\{\alpha_1, \ldots, \alpha_n\} \subseteq \pi_1(\Delta, u)$ there is an algorithm that represents the subgroup H generated by $\{\alpha_1, \ldots, \alpha_n\}$ by an immersion $f\colon \Gamma \longrightarrow \Delta$, as follows:

Represent α_i by a circuit p_i based at u. Let Γ_2 be a wedge of n-circles, where the i-th circle is subdivided in $|p_i|$ pieces, and $f_2\colon \Gamma_2 \longrightarrow \Delta$ maps the i-th circle to p_i.
Then, $f_2(\pi_1(\Gamma_2, v)) = \langle \alpha_1, \ldots, \alpha_n \rangle = H$, where v is the vertex of the wedge. f_2 can be factored through a series of folds and an immersion:

$$\Gamma_2 \longrightarrow \Gamma_3 \longrightarrow \ldots \longrightarrow \Gamma_k \xrightarrow{f_k} \Delta,$$

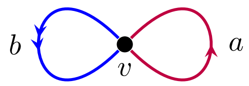
where the last arrow is an immersion and the other ones are foldings. By Corollary 4.3.9, each fold is surjective on π_1, and so, letting w be the image of v in Γ_k,

$$f_k(\pi_1(\Gamma_k, w)) = H.$$

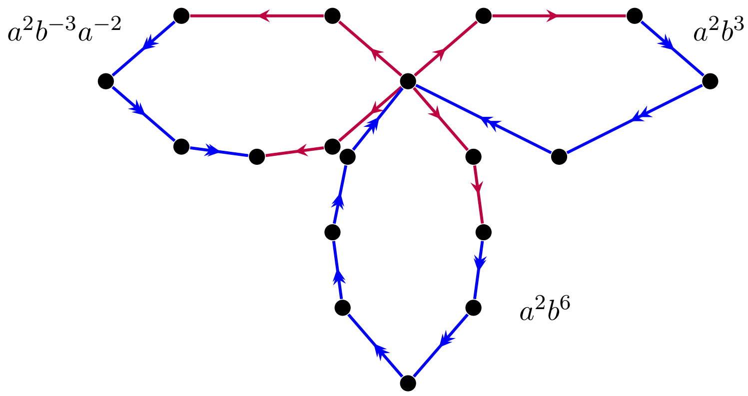
Thus, f_k is the desired immersion. Now, we know how to find a free basis of $\pi_1(\Gamma_k, w)$, which by Proposition 4.4.5 yields a free basis of H.

Example 4.4.1. Let H be the subgroup $\langle a^2b^3, a^2b^6, a^2b^{-3}a^{-2}\rangle$ of the free group $F(\{a, b\})$. Let us find a basis of H using the previous algorithm.

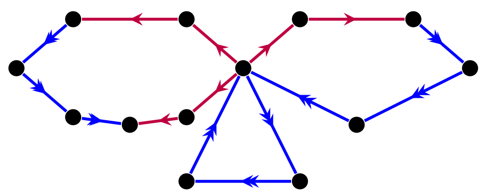
We identify the free group of rank two with the bouquet of 2-circles,



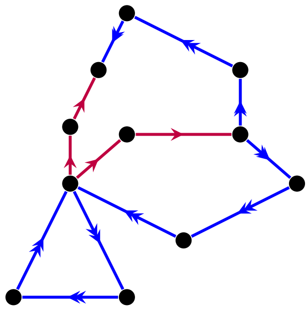
Firstly, we represent the subgroup by a wedge of 3 subdivided circles and a map that sends each circle to the corresponding element. In the figure, each section is drawn with its image under the map.



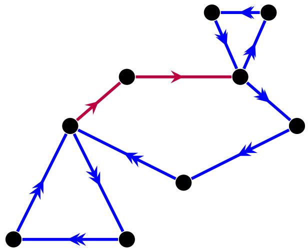
Secondly, note that we may fold all the edges of the above right circle with some edges of the below circle.



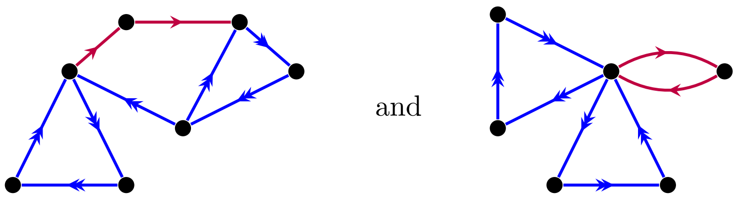
Thirdly, we can fold two edges of the above left circle with two edges of the above right circle.



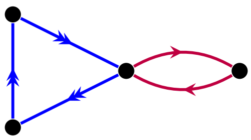
Fourthly, we fold the edges drawn in purple.



Fifthly, we fold the edges drawn in blue of the circle of the top with the ones of the big circle (it is done in two steps).



Finally, we fold the circles which are drawn in blue.



In this way, we have obtained that $\{a^2, b^3\}$ is a free basis of H.

We are now ready to establish the correspondence between pullbacks of graphs and intersections of free groups. In this result, with the intention of not confusing the notation of maps and paths, the projections defined when constructing the pullback are named g_1 and g_2.

Theorem 4.4.6 (Pullback of immersions represents intersection)**.** *Let*

$$\begin{array}{ccc} \Gamma_3 & \xrightarrow{g_1} & \Gamma_1 \\ {\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle f_1} \\ \Gamma_2 & \xrightarrow{f_2} & \Delta \end{array}$$

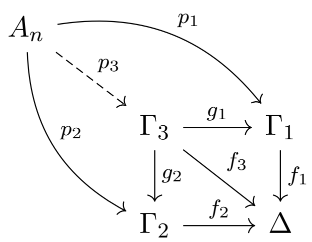
be a pullback diagram of graphs. Suppose that f_1 and f_2 are immersions. Let v_1, v_2 be vertices in Γ_1, Γ_2 such that $f_1(v_1) = f_2(v_2) = w$; let v_3 be the corresponding vertex of Γ_3. Define $f_3 = f_1 \circ g_1 = f_2 \circ g_2 \colon \Gamma_3 \longrightarrow \Delta$, and

$$H_i = f_i(\pi_1(\Gamma_i, v_i)) \quad \textit{for} \quad i \in \{1, 2, 3\}.$$

Then,

$$H_3 = H_1 \cap H_2.$$

Proof. It is clear from $f_1 \circ g_1 = f_2 \circ g_2$ that H_3 is contained in $H_1 \cap H_2$. In order to show the reverse inclusion, let $[\alpha] \in H_1 \cap H_2$. Then, there are reduced circuits p_1 and p_2 in Γ_1 and Γ_2 based at v_1 and v_2, respectively, such that $f_1(p_1)$ and $f_2(p_2)$ belong to the homotopy class of α. By Proposition 4.4.1, $f_1(p_1)$ and $f_2(p_2)$ are reduced equivalent paths, so by Proposition 4.4.4, $f_1(p_1) = f_2(p_2)$. If we denote by n the length of both paths, we are in the following conditions.



By the pullback property, there exists a path p_3 in Γ_3 such that $p_1 = g_1(p_3)$ and $p_2 = g_2(p_3)$. Having on mind the definitions of g_1 and g_2 and taking into account that p_1 and p_2 are both circuits, we obtain that p_3 is a circuit based at v_3. Thus, $f_3(p_3)$ represents an element of H_3, and it represents also α. □

Corollary 4.4.7 (Howson's theorem)**.** *If H_1 and H_2 are finitely generated subgroups of a free group F, then $H_1 \cap H_2$ is finitely generated (and a free basis of $H_1 \cap H_2$ can be determined algorithmically).*

Proof. Represent F as $\pi_1(\Delta)$, where Δ is a graph with one vertex. Since H_1 and H_2 are finitely generated subgroups of $\pi_1(\Delta)$, using Algorithm 1, we can represent them by immersions

$$f_1 \colon \Gamma_1 \longrightarrow \Delta \quad \text{and} \quad f_2 \colon \Gamma_2 \longrightarrow \Delta,$$

where Γ_1 and Γ_2 are connected graphs, $H_1 = f_1(\pi_1(\Gamma_1, v_1))$ and $H_2 = f_2(\pi_1(\Gamma_2, v_2))$.
Note that it is compulsory for $f_1(v_1)$ and $f_2(v_2)$ to be equal to v, so after constructing the pullback Γ_3, by Theorem 4.4.6,

$$H_1 \cap H_2 = f_3(\pi_1(\Gamma_3, v_3)),$$

where $f_3 = f_1 \circ g_1 = f_2 \circ g_2$ and v_3 is the corresponding vertex of Γ_3.
Let us now check that f_3 is an immersion. If w is a vertex of Γ_3, it is equal to (u, v), where u is a vertex of Γ_1, v is a vertex of Γ_2 and $f_1(u) = f_2(v)$.

$$\begin{aligned}(f_3)_w \colon \operatorname{St}(w, \Gamma_3) &\longrightarrow \operatorname{St}(f_3(w), \Delta)\\ (e_1, e_2) &\longmapsto f_3((e_1, e_2)) = f_1(e_1) = f_2(e_2).\end{aligned}$$

Since f_1 and f_2 are both immersions, $(f_3)_w$ is injective.
Summarizing, by Proposition 4.4.5, $H_1 \cap H_2$ is isomorphic to $\pi_1(\Gamma_3, v_3)$, which is finitely generated. Moreover, f_3 is an immersion, so applying f_3 to a free generating set of $\pi_1(\Gamma_3, v_3)$, we achieve a free generating set of $H_1 \cap H_2$. □

4.5 Marshall Hall's Theorem

The goal of this section is to prove Marshall Hall's Theorem, which states that for any non-trivial element of a free group, there exists a finite index subgroup which does not contain it.

Theorem 4.5.1. *Let $f : \Gamma \longrightarrow \Delta$ be an immersion of graphs. Suppose that Δ has only one vertex. Then there exists a graph Γ' containing Γ, such that $\Gamma' \setminus \Gamma$ consists only of edges, and there exists a map $f' \colon \Gamma' \longrightarrow \Delta$ extending f, such that f' is a covering.*

Proof. Let V_Γ and E_Γ be the sets of vertices and edges of Γ and $\mathcal{O}$ the orientation of Δ. For each $e \in \mathcal{O}$, define

$$\begin{aligned}R_e \colon \big\{\iota(e_1) \mid e_1 \in E_\Gamma \text{ and } f(e_1) = e\big\} &\longrightarrow \big\{\tau(e_1) \mid e_1 \in E_\Gamma \text{ and } f(e_1) = e\big\}\\ \iota(e_1) &\longmapsto \tau(e_1).\end{aligned}$$

If we check that R_e is well-defined, it is clear that R_e is bijective.
Suppose that $\iota(e_1) = \iota(e_2)$ where e_1, $e_2 \in E_\Gamma$ and $f(e_1) = f(e_2) = e$. Since f is an immersion, we conclude that $e_1 = e_2$, so $\tau(e_1) = \tau(e_2)$.

Since V_Γ is finite, we may extend R_e to all V_Γ, for each $e \in \mathcal{O}$:

$$S_e \colon V_\Gamma \longrightarrow V_\Gamma.$$

The next step is to construct the graph Γ'. In order to do it, we define the set of vertices and the set of edges of Γ' in the following way:

$$V_{\Gamma'} = V_\Gamma,$$

$$E_{\Gamma'} = \big\{(u, v, e) \mid u, v \in V_\Gamma,\ e \text{ edge of } \Delta;\ \text{if } e \in \mathcal{O}, v = S_e(u);\ \text{if } \bar{e} \in \mathcal{O}, u = S_{\bar{e}}(v)\big\}.$$

For $\epsilon = (u, v, e) \in E_{\Gamma'}$, if we define $\bar{\epsilon}$ to be $(v, u, \bar{e})$ and $\iota(\epsilon) = u$, it is routine to prove that Γ' is a graph.

We also have to construct a covering $f' \colon \Gamma' \longrightarrow \Delta$. Define it by taking each vertex v to the unique vertex of Δ, and $f'(\epsilon) = e$, for each $\epsilon = (u, v, e) \in E_{\Gamma'}$. Now, let us show that it is, in fact, a covering.
Let w be a vertex of Γ'.

$$\begin{aligned} (f')_w \colon \mathrm{St}(w, \Gamma') &\longrightarrow \mathrm{St}(f'(w), \Delta) \\ (u, v, e) &\longmapsto e. \end{aligned}$$

Suppose that there are elements (u_1, v_1, e_1) and (u_2, v_2, e_2) in the star of w in Γ' such that $e_1 = e_2$. Moreover, in order those elements to be in such star, $u_1 = u_2 = w$. Then, if $e_1 = e_2 \in \mathcal{O}$, $v_1 = S_e(u_1) = S_e(w)$ and $v_2 = S_e(u_2) = S_e(w)$, but S_e is a bijection, so $v_1 = v_2$. If $\overline{e_1} = \overline{e_2} \in \mathcal{O}$, the argument is similar. Thus, f' is an immersion.
In order to check that f' is locally surjective, for $e \in \mathrm{St}(f'(w), \Delta)$ and $e \in \mathcal{O}$, taking $(w, S_e(w), e) \in \mathrm{St}(w, \Gamma')$ we are done, and if $\bar{e} \in \mathcal{O}$, $(w, S_{\bar{e}}^{-1}(w), e) \in \mathrm{St}(w, \Gamma')$.

Finally, embed Γ into Γ' by a map $a \colon \Gamma \longrightarrow \Gamma'$ where for a vertex $v \in V_\Gamma$, $a(v) = v$ and for an edge $e \in E_\Gamma$, $a(e) = (\iota(e), \tau(e), f(e))$. It is injective because f is an immersion, and $f' \circ a = f$. □

Corollary 4.5.2 (Marshall Hall's Theorem). *Let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$ be elements in a free group F. Let H be the subgroup of F generated by $\{\alpha_1, \ldots, \alpha_k\}$. Suppose that $\beta_i \notin H$, for $i \in \{1, \ldots, l\}$. Then there exists a subgroup H' of finite index in F, such that $H \subseteq H'$, $\beta_i \notin H'$ for $i \in \{1, \ldots, l\}$, and there exists a free basis of H' having a subset which is a free basis of H.*

Proof. Represent F as $\pi_1(\Delta)$, where Δ has only a vertex.
Let Γ_1 be a wedge of circles and arcs, subdivided appropriately, and f_1 a map from Γ_1 to Δ, so that the circles in Γ_1, under f_1, represent α_j and the arcs in Γ_1 represent β_i. Thus, with appropriate basepoint v_1,

$$f_1(\pi_1(\Gamma_1, v_1)) = H.$$

Since Γ_1 is a finite connected graph, f_1 can be factored through a series of folds and an immersion $f \colon \Gamma \longrightarrow \Delta$, and as we argued in Algorithm 1,

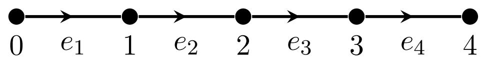
$$f(\pi_1(\Gamma, v)) = H,$$

with appropriate vertex v. Since $\beta_i \notin H$, the image of the i-th arc of Γ_1 in Γ is not a circuit.

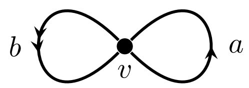
Now apply Theorem 4.5.1 to f, extending it to a covering $f'\colon \Gamma' \longrightarrow \Delta$, without adding new vertices. We define H' to be $f'(\pi_1(\Gamma', v))$.
The index of H' in F is the number of vertices of Γ, by Proposition 4.3.5, which is finite. Firstly, it is clear that $H \subseteq H'$. Secondly, the paths in Γ which represent β_i (the images of the arcs of Γ_1) are not circuits; therefore, taking into account that since f' is a covering the induced homomorphism is injective, $\beta_i \notin H'$. Finally, a maximal tree T of Γ is also a maximal tree of Γ', by Lemma 2.2.2. Using this, we find a free basis of $\pi_1(\Gamma', v)$ of which a subset is a free basis of $\pi_1(\Gamma, v)$. □

Example 4.5.1. Let $F(\{a, b\})$ be a free group of rank 2 and $g = abaa \in F(\{a, b\})$. We are going to find a subgroup H of finite index such that $g \notin H$.

Let us follow the steps of the previous proof.
We take H to be the trivial subgroup, Γ the standard arc of length 4,



and Δ the bouquet of 2-circles,



Now, we have to construct the graph Γ' and the covering f' of Theorem 4.5.1.
We can choose the maps S_a and S_b in the following way:

$$\begin{array}{rcl} S_a\colon \{0,1,2,3,4\} & \longrightarrow & \{0,1,2,3,4\} \\ 0 & \longmapsto & 1, \\ 1 & \longmapsto & 0, \\ 2 & \longmapsto & 3, \\ 3 & \longmapsto & 4, \\ 4 & \longmapsto & 2, \end{array} \qquad \begin{array}{rcl} S_b\colon \{0,1,2,3,4\} & \longrightarrow & \{0,1,2,3,4\} \\ 0 & \longmapsto & 0, \\ 1 & \longmapsto & 2, \\ 2 & \longmapsto & 1, \\ 3 & \longmapsto & 3, \\ 4 & \longmapsto & 4. \end{array}$$

Then, the vertex set of Γ' is $\{0, 1, 2, 3, 4\}$, and the edge set, $E_{\Gamma'}$, is

$$\{\epsilon_i, \overline{\epsilon_i} \mid i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\},$$

such that

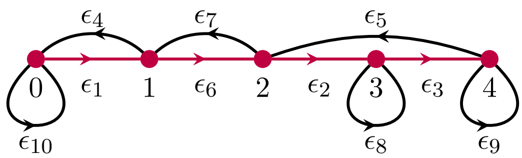
$$\epsilon_1 = (0, 1, a), \epsilon_2 = (2, 3, a), \epsilon_3 = (3, 4, a), \epsilon_4 = (1, 0, a), \epsilon_5 = (4, 2, a),$$

$$\epsilon_6 = (1, 2, b), \epsilon_7 = (2, 1, b), \epsilon_8 = (3, 3, b), \epsilon_9 = (4, 4, b), \epsilon_{10} = (0, 0, b).$$

The covering f' is defined trivially in the vertex set, and in the edge set it is defined in the following way:

$$f'\colon E_{\Gamma'} \longrightarrow E_{\Delta}$$
$$\epsilon_i \longmapsto f'(\epsilon_i) = \begin{cases} a, & \text{if } i \in \{1,2,3,4,5\}, \\ b, & \text{if } i \in \{6,7,8,9,10\}. \end{cases}$$

Finally, Γ' can be drawn as follows.



A maximal tree of such graph is drawn in purple. Therefore, its fundamental group is the free group of rank 6, where the generating set is this:

$$\big\{\epsilon_{10}, \epsilon_1\epsilon_4, \epsilon_1\epsilon_6\epsilon_7\epsilon_1^{-1}, \epsilon_1\epsilon_6\epsilon_2\epsilon_8\epsilon_2^{-1}\epsilon_6^{-1}\epsilon_1^{-1},$$
$$\epsilon_1\epsilon_6\epsilon_2\epsilon_3\epsilon_9\epsilon_3^{-1}\epsilon_2^{-1}\epsilon_6^{-1}\epsilon_1^{-1}, \epsilon_1\epsilon_6\epsilon_2\epsilon_3\epsilon_5\epsilon_6^{-1}\epsilon_1^{-1}\big\}.$$

To sum up, applying f', we obtain the free generating set of H',

$$\big\{b, a^2, ab^2a^{-1}, ababa^{-1}b^{-1}a^{-1}, aba^2ba^{-2}b^{-1}a^{-1}, aba^3b^{-1}a^{-1}\big\}.$$

4.6 Core graphs

The goal of this section is to describe an algorithm that decides whether or not a finitely generated subgroup of a free group is of finite index.

Definition 4.6.1. A *cyclically reduced circuit* in a graph Γ is a circuit, $p = e_1e_2\cdots e_n$, which is reduced as a path, and for which $e_1 \neq \overline{e_n}$.

Definition 4.6.2. A graph Γ is said to be a *core-graph* if Γ is connected, has at least one edge, and every edge belongs to at least one cyclically reduced circuit.

Remark 4.6.1. Every cyclic permutation of a cyclically reduced circuit is again a cyclically reduced circuit. Thus, we may assume that if a graph is a core-graph, each edge is the first element of a cyclically reduced circuit.

(a) It is a core-graph. (b) It is not a core-graph.

If Γ is a connected graph with non-trivial fundamental group, an *essential edge of* Γ is an edge belonging to some cyclically reduced circuit. Then, we define the *core of* Γ as all essential edges of Γ and all initial vertices of essential edges.
For example, in the previous case, the core-graph is drawn in purple.

Definition 4.6.3. If H is a subgroup of a group G, we say that H satisfies the *Burnside condition* when, for every $g \in G$, there exists some positive integer n such that $g^n \in H$.

Lemma 4.6.1. *If H satisfies the Burnside condition in G, then so does any conjugate subgroup.*

Proof. Suppose that H is a subgroup of G which satisfies the Burnside condition and let H^h be a conjugate subgroup, with $h \in G$.
Let $g \in G$. Since $G^h = G$, $g = g_1^h$ for some $g_1 \in G$. H satisfies the Burnside condition, so there exists a positive integer m such that $g_1^m \in H$. Thus, $(g_1^h)^m = (g_1^m)^h \in H^h$. □

Proposition 4.6.2. *Let $f\colon \Gamma \longrightarrow \Delta$ be a finite-sheeted covering of connected graphs (that is, the cardinality of the fibres is finite) and v a vertex of Γ. Then, $f(\pi_1(\Gamma, v)) \subseteq \pi_1(\Delta, f(v))$ satisfies the Burnside condition.*

Proof. Let us prove a more general property: Subgroups of finite index satisfy the Burnside condition.
Let H be a finite index subgroup of G and

$$G = g_1 H \,\dot{\cup} \cdots \dot{\cup}\, g_n H,$$

for some elements $g_i \in G$, $i \in \{1, \ldots, n\}$.
Let $g \in G$. Then, $g \in g_i H$, for some $i \in \{1, \ldots, n\}$, so $gH = g_i H$. Similarly, $g^2 H = g_j H$, for some $j \in \{1, \ldots, n\}$.
Finally, since $\{g_1 H, \ldots, g_n H\}$ is finite, there exist $n_0, n_1 \in \mathbb{N}$, $n_0 > n_1$ such that

$$g^{n_0} H = g^{n_1} H,$$

so $g^{n_0 - n_1} \in H$.
Returning again to our hypothesis, note that the index of $f(\pi_1(\Gamma, v))$ in $\pi_1(\Delta, f(v))$ is the cardinality of $f^{-1}(f(v))$, which is finite by hypothesis. □

Conversely:

Proposition 4.6.3. *Let $f\colon \Gamma \longrightarrow \Delta$ be an immersion of connected graphs. Suppose that Δ is a core-graph, v a vertex of Γ and $f(\pi_1(\Gamma, v)) \subseteq \pi_1(\Delta, f(v))$ satisfies the Burnside condition. Then, f is a covering.*

Proof. We only need to show that f is locally surjective. Let w be a vertex of Γ and e be an edge of Δ such that $\iota(e) = f(w)$. $f(\pi_1(\Gamma, v))$ satisfies the Burnside condition, and $f(\pi_1(\Gamma, w))$ is a conjugate subgroup of the previous one (Γ is connected), so it also satisfies it (Lemma 4.6.1).
The graph Δ is a core-graph, so by Remark 4.6.1 we may assume that there is a cyclically reduced circuit p in Δ whose first term is e. Therefore, $[p] \in \pi_1(\Delta, f(w))$. By the Burnside condition, there exists $n \in \mathbb{N}$ such that the homotopy class of p^n belongs to $f(\pi_1(\Gamma, w))$. That is, there is a reduced circuit q in Γ based at w such that $f(q) \sim p^n$.
But p is cyclically reduced, so p^n is reduced. By Proposition 4.4.1, $f(q)$ is also reduced, so $f(q) = p^n$ (Proposition 4.4.4). Then, the first term of q is an edge e_1 of Γ with $\iota(e_1) = w$ and $f(e_1) = e$. □

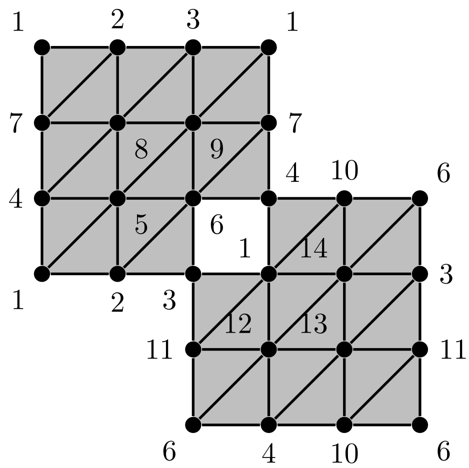
Remark 4.6.2. The previous proposition shows whether a given finitely generated subgroup of a free group is of finite index.
By Algorithm 1, we can represent the subgroup by an immersion $f\colon \Gamma \longrightarrow \Delta$, where Γ is finite and Δ is a one-vertex graph (therefore, a core-graph); if f is a covering, by Proposition 4.3.5, the subgroup is of finite index equal to the number of vertices of Γ. If f is not a covering, then Proposition 4.6.3 shows that the subgroup does not satisfy the Burnside condition, so it is of infinite index.

Appendix A

Solved exercises

Exercise 1. Give a triangulation of the 2-genus surface.

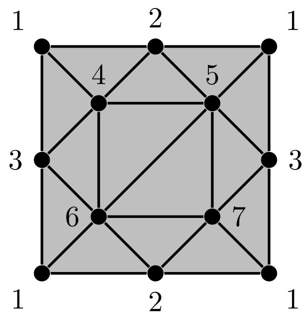
Solution. If we base on the triangulation of the torus that was given in Example 1.2.4, we obtain the following triangulation of the 2-genus surface.



□

Exercise 2. Give a triangulation of the torus with 14 triangles and 7 vertices.

Solution. Let us draw such triangulation.



□

Exercise 3. Given a word w on the alphabet $A = \{a, b, c, d\}$, give an algorithm that determines whether or not the word w is trivial in $F(A)$. Do the same in the case of a free abelian group on the alphabet A.

Solution. Let us give a code done in Mathematica in order to decide whether a word is trivial or not in such free group. In the code, a^{-1}, b^{-1}, c^{-1} and d^{-1} are denoted by e,f g and h. If the code returns a 1, the word is trivial; if the answer is -1, it is not trivial.

First of all, let us summarise the idea behind the program:

(i) In order a word w to be trivial in $F(A)$, the number of times a letter a, b, c or d appears in the word must be equal to the number of times the letter e, f, g or h appears, respectively. If this does not hold, the word is not trivial.

(ii) We have to check if there exists a pair of letters together such that one is the inverse of the other one. If it happens, we can erase them and we obtain an equivalent word. Proceeding in this way, if the word is trivial, at some point we will obtain an equivalent word of length 0. If not, the word is not trivial.

```
istrivialfreegroup[t_] :=
 Module[{char, l, num, i, list, p1, p2, p3, p4, p5, p6,
  p7, p8, j, sort},
  list = {"a", "b", "c", "d", "e", "f", "g", "h"};
  char = Characters[t];
  sort = 0;
  l = Length[char];
  If[Mod[l, 2] == 0,
   num = {};
   p1 = 0;
   p2 = 0;
```

```
  p3 = 0;
  p4 = 0;
  p5 = 0;
  p6 = 0;
  p7 = 0;
  p8 = 0;
  For[i = 1, i <= l, i++,
   If[char[[i]] == "a", AppendTo[num, 1]; p1 = p1 + 1;];
   If[char[[i]] == "b", AppendTo[num, 2]; p2 = p2 + 1;];
   If[char[[i]] == "c", AppendTo[num, 3]; p3 = p3 + 1;];
   If[char[[i]] == "d", AppendTo[num, 4]; p4 = p4 + 1;];
   If[char[[i]] == "e", AppendTo[num, -1]; p5 = p5 + 1;];
   If[char[[i]] == "f", AppendTo[num, -2]; p6 = p6 + 1;];
   If[char[[i]] == "g", AppendTo[num, -3]; p7 = p7 + 1;];
   If[char[[i]] == "h", AppendTo[num, -4]; p8 = p8 + 1;];
   ];
  If[p1 == p5 && p2 == p6 && p3 == p7 && p4 == p8,
   For[j = 1, j <= l/2, j++,
     If[erase[num][[1]] > 0,
       num = erase[num][[2]]; ,
       sort = -1; Break[];
       ];
     ];, sort = -1;];, sort = -1];
sort
]

erase[num_] := Module[{g, l, i, numb, s},
s = {};
numb = num;
g = 0;
l = Length[num];
For[i = 1, i <= l - 1, i++,
 If[numb[[i]] == -numb[[i + 1]],
   numb = Join[numb[[1 ;; i - 1]], numb[[i + 2 ;; l]]];
   g = g + 1;
   Break[];
   ];
 ];
AppendTo[s, g];
AppendTo[s, numb];
s
]
```

In the case of the free abelian group on the alphabet A, we only need to verify whether the number of times each letter appears in the word is equal to the number of times its inverse appears. Hence, we obtain the following code.

```
istrivialabelian[t_] :=
 Module[{list, char, l, sort, p1, p2, p3, p4, p5, p6,
  p7, p8, i},
  list = {"a", "b", "c", "d", "e", "f", "g", "h"};
  char = Characters[t];
  l = Length[char];
  If[Mod[l, 2] == 0,
   p1 = 0;
   p2 = 0;
   p3 = 0;
   p4 = 0;
   p5 = 0;
   p6 = 0;
   p7 = 0;
   p8 = 0;
   For[i = 1, i <= l, i++,
    If[char[[i]] == "a", p1 = p1 + 1;];
    If[char[[i]] == "b", p2 = p2 + 1;];
    If[char[[i]] == "c", p3 = p3 + 1;];
    If[char[[i]] == "d", p4 = p4 + 1;];
    If[char[[i]] == "e", p5 = p5 + 1;];
    If[char[[i]] == "f", p6 = p6 + 1;];
    If[char[[i]] == "g", p7 = p7 + 1;];
    If[char[[i]] == "h", p8 = p8 + 1;];
    ];
   If[p1 == p5 && p2 == p6 && p3 == p7 && p4 == p8,
    sort = 0, sort = -1];, sort = -1];
  sort]
```

□

Exercise 4. Recall that two elements g and g' in a group G are conjugate if there exists h in G such that $g' = g^h = h^{-1}gh$. Explain when two elements in a free group are conjugated.

Solution. Let $F(X)$ be a free group on the alphabet X and $w \in F(X)$. Then, $w = x_{i_1} \cdots x_{i_n}$ for $x_{i_j} \in X \cup X^{-1}$, $j \in \{1, \ldots, n\}$. In addition, let us suppose that w is a reduced word.

If $x_{i_n} = x_{i_1}^{-1}$,

$$x = x_{i_n}^{-1}(x_{i_2} \cdots x_{i_{n-1}})x_{i_n}.$$

Similarly, if $x_{i_2} = x_{i_{n-1}}^{-1}$,

$$x = (x_{i_{n-1}} x_{i_n})^{-1} (x_{i_3} \cdots x_{i_{n-2}})(x_{i_{n-1}} x_{i_n}).$$

Proceeding in this way, we obtain that

$$w = w_1^{-1}(x_{j_1} \cdots x_{j_m}) w_1,$$

where w_1 is a reduced word on the alphabet X and $x_{j_1} \cdots x_{j_m}$ is a cyclically reduced word on X.

Let g, $g' \in F(X)$. Then,

$$g = w_1^{-1} \bar{w} w_1, \quad g' = w_2^{-1} \tilde{w} w_2,$$

where w_1, w_2 are reduced words and $\bar{w}$, $\tilde{w}$ are cyclically reduced words on X.
If g and g' are conjugate, then $g = h^{-1} g' h$ for some element $h \in F(X)$, so

$$\bar{w} = (w_1 h^{-1} w_2^{-1}) \tilde{w} (w_2 h w_1^{-1}) = (w_2 h w_1^{-1})^{-1} \tilde{w} (w_2 h w_1^{-1}).$$

Therefore, we claim that g and g' are conjugate if and only if $\bar{w}$ and $\tilde{w}$ are.

Proof. $\Rightarrow$). We have already proved it.
$\Leftarrow$). If $\bar{w} = k^{-1} \tilde{w} k$ for some $k \in F(X)$, then

$$g = w_1^{-1} \bar{w} w_1 = w_1^{-1} k^{-1} \tilde{w} k w_1 = (w_2^{-1} k w_1)^{-1} g' (w_2^{-1} k w_1).$$

□

In conclusion, we only need to check when two cyclically reduced words $\bar{w}$ and $\tilde{w}$ are conjugate.

Let us show that $\bar{w}$ and $\tilde{w}$ are conjugate if and only if $\bar{w}$ and $\tilde{w}$ are cyclic permutations of each other.

Proof. $\Leftarrow$). If $\tilde{w} = s_1 \cdots s_t$ and $\bar{w} = s_i s_{i+1} \cdots s_t s_1 \cdots s_{i-1}$ for letters on the alphabet X and $i \in \{2, \dots, t\}$, then

$$\bar{w} = (s_1 \cdots s_t) \tilde{w} (s_i \cdots s_t)^{-1}.$$

$\Rightarrow$). Take any conjugate $h^{-1} \bar{w} h$ of $\bar{w}$ which is cyclically reduced. Thus, the first element of h must be either $\bar{w}_1$ or ${\bar{w}_n}^{-1}$, where $\bar{w}_1$ and $\bar{w}_n$ are the first and last letters of $\bar{w}$, respectively. Hence,

$$h^{-1} \bar{w} h = v^{-1} \bar{\bar{w}} v,$$

and the length of $\bar{\bar{w}}$ is smaller than the length of $\bar{w}$.
Claim: $\bar{\bar{w}}$ is a cyclic permutation of $\bar{w}$.

Proof. If the first letter of h is $\bar{w_1}$, let us suppose that $h = \bar{w_1} h_2 \cdots h_s$. Then,

$$h^{-1} \bar{w} h = h_s^{-1} \cdots h_2^{-1} \bar{w}_2 \cdots \bar{w_n} \, \bar{w_1} h_2 \cdots h_s,$$

and $\bar{\bar{w}} = \overline{w_2} \cdots \overline{w_n} \, \overline{w_1}$.
If the first letter of h is $\bar{w_n}^{-1}$, the argument is analogous. □

We have to continue with the same procedure until the length of v is 0. □

□

Exercise 5. Does the following presentation define the trivial group?

$$G = \langle a, b, c \mid aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2 \rangle.$$

Solution. If we show that $a = b = c = 1$, then we will have obtained that G is the trivial group.
Using the first relation, $b^{-2} = ab^{-1}a^{-1}$, so $b^{-1} = bab^{-1}a^{-1}$ and

$$b^{-1}a = bab^{-1}.$$

By symmetry, $c^{-1}b = cbc^{-1}$. Firstly,

$$c^2bc^{-2} = ccbc^{-1}c^{-1} = cc^{-1}bc^{-1} = bc^{-1} = c^{-2}b,$$

so $c^{-2}bc^2 = c^2b$.
Secondly,

$$bcac^{-1}b^{-1} = c^2bab^{-1}c^{-2} = c^2b^{-1}ac^{-2} = c^2b^{-1}c^{-2}c^2ac^{-2} = b^{-1}c^2a^4,$$

and in each equality we have used the following relations:
(i) In the first one, $bc = c^2b$,
(ii) In the second one, $bab^{-1} = b^{-1}b$,
(iii) In the fourth one, $c^2b^{-1}c^{-2} = b^{-1}c^2$ and $c^2ac^{-2} = ccac^{-1}c^{-1} = ca^2c^{-1} = cac^{-1}cac^{-1} = a^4$.
Thirdly,

$$ba^2b^{-1} = bab^{-1}bab^{-1} = b^{-1}ab^{-1}a = b^{-1}b^{-2}aa = b^{-3}a^2,$$

and in the third equality we have used that $ab^{-1}a^{-1} = b^{-2}$.
Fourthly,

$$b^{-1}c^2a^4 = bcac^{-1}b^{-1} = ba^2b^{-1} = b^{-3}a^2,$$

so $c^2 = b^{-2}a^{-2}$.

Finally, we obtain that

$$c^{-2}bc^2 = a^2b^2bb^{-2}a^{-2} = a^2ba^{-2} = b^4,$$

and the last equality is achieved because of the first relation.

In conclusion, $c^2b = b^4$, so $c^2 = b^3$. Therefore, $bcb^{-1} = b^3$, so $c = b^3 = c^2$. That is, $c = 1$. If we now return to the relations of G, we obtain that $a = b = c = 1$.

Remark A.0.1. This group is known as the Higman's group of rank 3. Although this one is trivial, the Higman's group of rank 4,

$$G = \langle a, b, c, d \mid aba^{-2} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle,$$

is an infinite group.

□

Appendix B

The fundamental group

The goal of this appendix is to recall the definitions and the properties related to homotopies and fundamental groups that are needed in order to understand the dissertation. Therefore, all the proofs are avoided.

We will start with the definitions and basic properties corcerning to homotopy theory.

Definition B.0.1. Let X and Y be topological spaces. A *homotopy* between two continuous maps $f, g\colon X \longrightarrow Y$ is a continuous map $H\colon X \times [0,1] \longrightarrow Y$ such that

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x) \quad \text{for all} \quad x \in X.$$

We then say that f and g are *homotopic*, and write $f \simeq g$.

Example B.0.1. Suppose that X is a topological space and Y is a convex subset of $\mathbb{R}^n$. Then, any two continuous maps $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Y$ are homotopic, via the homotopy

$$\begin{aligned} H\colon X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto (1-t)f(x) + tg(x). \end{aligned}$$

This is known as the *straight-line homotopy.*

Lemma B.0.1. *For any two spaces X and Y, homotopy is an equivalence relation defined on the set of continuous maps from X to Y.*

Definition B.0.2. Two spaces X and Y are *homotopy equivalent*, written $X \simeq Y$, if there exist maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that

$$g \circ f \simeq \mathrm{id}_X \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y.$$

Lemma B.0.2. *Homotopy equivalence is an equivalence relation defined on the set of topological spaces.*

Definition B.0.3. A topological space X is *contractible* if it is homotopy equivalent to a space with a unique point.

Definition B.0.4. Let X and Y be topological spaces and let A be a subspace of X. Then, two continuous maps $f, g\colon X \longrightarrow Y$ are *homotopic relative to* A if there exists a homotopy H such that

$$H(x,t) = f(x) = g(x) \quad \text{for all} \quad x \in A \quad \text{and} \quad t \in [0,1].$$

At this stage, we can introduce the concept of fundamental group.

Definition B.0.5. Let X be a topological space and let u and v be paths in X such that $u(1) = v(0)$. The *composite path* $u.v$ is given by

$$\begin{aligned} u.v\colon [0,1] &\longrightarrow X \\ t &\longmapsto \begin{cases} u(2t), & \text{if } 0 \leq t \leq 1/2, \\ v(2t-1), & \text{if } 1/2 \leq t \leq 1. \end{cases} \end{aligned}$$

Definition B.0.6. A *loop in the space* X *based at a point* $b \in X$ is a path l such that $l(0) = l(1) = b$.

Definition B.0.7. The *fundamental group* of the space X and the basepoint b is the set of homotopy classes relative to $\{0,1\}$ of loops in X based at b, and it is denoted by $\pi_1(X,b)$.

Theorem B.0.3. *If X is a topological space and $b \in X$, $\pi_1(X,b)$ is, in fact, a group, where the operation is defined as follows: if l and l' are loops in X based at b, and $[l]$ and $[l']$ are their homotopy classes relative to $\{0,1\}$, then $[l] \cdot [l']$ in the group is defined to be $[l.l']$. Moreover, $[c_b]$ is the identity element of the group, where c_b sends all the elements of $[0,1]$ to b. Finally, the inverse element of $[l]$ is $[\bar{l}]$, where $\bar{l}(t) = l(1-t)$, for all $t \in [0,1]$.*

Proposition B.0.4. *Let X and Y be path-connected spaces such that $X \simeq Y$. Then, $\pi_1(X,b) \cong \pi_1(Y,b')$, for all $b \in X$ and $b' \in Y$.*

Bibliography

[1] O. Bogopolski, Introduction to Group Theory, European Mathematical Society, Zürich, 2008.

[2] D. Cohen, Combinatorial Group Theory: a topological approach, Cambridge University Press, London, 1989.

[3] L. Cote, Free groups and graphs: the Hanna Neumann Theorem, University of Chicago, 2008.

[4] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2001.

[5] G. Higman, A finitely generated infinite simple group, Journal of the London Mathematical Society. Second Series 26, 1951, 61–64.

[6] M. Lackenby, Topology and Groups, University of Oxford, 2008.

[7] Mathematics Stack Exchange. Stack Exchange inc. https://math.stackexchange.com/

[8] C.R.F. Maunder, Algebraic Topology, Van Nostrand Reinhold, London, 1970.

[9] J-P. Serre, Trees, Springer-Verlag, Berlin-Heiderlberg-New York, 1980.

[10] J.R. Stallings, Topology of finite graphs, Inventiones mathematicae 71, 1983, 551–565.