Early numerical cognition and mathematical processes*

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ABSTRACT: In this paper I study the development of arithmetical cognition with the focus on metaphorical thinking. In an approach developing on Lakoff and Núñez (2000), I propose one particular conceptual metaphor, the Process → Object Metaphor (POM), as a key element in understanding the development of mathematical thinking.

Keywords: Philosophy of mathematics, arithmetic, conceptual metaphor theory, processes, number cognition.

RESUMEN: En este artículo estudio el desarrollo de la cognición aritmética enfocado en el pensamiento metafórico. En una propuesta que se desarrolla desde la de Lakoff y Núñez (2000), propongo una metáfora particular conceptual, la Metáfora Proceso → Objeto (POM), como un elemento clave para comprender el desarrollo del pensamiento matemático.

Palabras clave: Filosofía de las matemáticas, aritmética, teoría conceptual de la metáfora, procesos, cognición numérica.

Introduction

This paper is motivated by the great potential in two approaches to explaining the nature of arithmetical knowledge. First is the empirical research conducted in the field of early numerical cognition, which has gone through enormous growth in recent years both in terms of quantity and quality. Second is the focus on conceptual metaphors in mathematical thinking, as presented in Lakoff and Núñez (2000). At the same time, however, this paper is also motivated by a concern. It seems that in both approaches there are important gaps when we try to understand the development of arithmetical knowledge. In the explanations based on early numerical cognition, whether the focus is on subitizing (Carey 2009) or approximate number sense (Dehaene 1997/2011) as the basis for learning number concepts, there remains a major problem in explaining the development from an early ability with numerosities to grasping the general concept of natural number. As for the theory based on metaphors, it is hard to see where the conceptual metaphors come to mathematics from. In general, conceptual metaphors seem to be too developed and too specific to fit together with the research on early numerical cognition.

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In short, there is a gap. The research on early numerical cognition seems to provide a good explanation why we come to have shared quantity concepts in the first place. The metaphor theory seems to provide a plausible account of how new abstract mathematical concepts are developed. But the early numerosity concepts are limited and/or approximate, while abstract mathematical concepts are precise and general. I see this conceptual leap as the key question in explaining the nature of arithmetical knowledge based on early numerical cognition.

In this paper I suggest an answer. Following Lakoff and Núñez, I argue that metaphors provide a fruitful framework for explaining the emergence of new abstract mathematical concepts. But I argue further that metaphors help us explain why we have abstract mathematical concepts in the first place. The key difference to the account Lakoff and Núñez is that I do not propose metaphors specific to mathematically developed concepts, such as infinity. Instead, I suggest that the key to metaphorical thinking in mathematics can be found in the general way we treat end products of processes metaphorically as objects. Using access to infinite objects as an example, I show that this approach, what I call the Process → Object Metaphor (POM), can do everything that the infinity-specific conceptual metaphor of Lakoff and Núñez does. Furthermore, I argue that POM can explain how we acquire abstract number concepts, thus combining the research on early numerical cognition with metaphorical thinking in mathematics.

1. *What is numerical cognition?*

The contemporary literature on what is called "number cognition" or "numerical cognition" has an unfortunate tendency to conflate the meaning of terms like "number", "natural number", "numerosity" and "quantity". Some of the same vocabulary that is used for formal mathematics is often also used to describe primitive, core cognitive ability for treating quantities. When reading about "arithmetic", it is thus often difficult to determine what the relevant concept is. The use that mathematicians and philosophers have for the term —usually referring to a formal system such as Peano arithmetic— can be remarkably different from the use of cognitive scientists and psychologists. In the line of research following Wynn (1992), for example, it is commonplace to use the term "infant arithmetic". Equally common is the use of the term "arithmetic" in describing animal ability with quantities. To cite just two examples, there are recent papers with the titles "Arithmetic in newborn chicks" (Rugani *et al.* 2009), and "Numerical and arithmetical abilities in non-primate species" (Agrillo 2014).

While perhaps understandable in informal talk about infants and animals and their ability with quantities, in philosophy —as well as in the empirical study of numerical cognition— this use of arithmetical terminology can cause serious problems. While no author would claim that "infant arithmetic" or "chick arithmetic" is arithmetic in the formal sense, the discourse often uses the same mathematical terms. Wynn, for example, called her paper "Addition and subtraction by human infants". Already by using those arithmetical terms, there is at least an implicit commitment to the infant behaviour being arithmetical in some relevant sense. In the abstract of the paper, Wynn writes:

> Here I show that 5-month-old infants can calculate the results of simple arithmetical operations on small numbers of items. This indicates that infants possess true numerical concepts, and suggests that humans are innately endowed with arithmetical abilities. (Wynn 1992)

However, as ground-breaking as Wynn's work was, there seems to be no reason to accept that the infants were calculating arithmetical operations. By reacting to the unnatural situation where one doll and one doll equal one doll, for example, the children may have merely kept track of one quantity, perhaps through a completely different cognitive mechanism. Similarly, there is no reason to believe that infants possess true numerical concepts. The experiment shows that infants can individuate objects, but certainly that ability can be explained without evoking numerical concepts.

When we try to establish the relevance of these kinds of empirical studies to the philosophy of mathematics, such differences are absolutely crucial. If we describe infant or animal behaviour in terms of developed arithmetic, we are in constant danger of distorting the explanations concerning the cognitive processes. In the worst case scenario, the terminology we use can even determine the kind of cognitive abilities we assume infants and animals to possess. At the very least, the conceptual coherence of the research is greatly compromised. Consider the following quotation:

> Humans possess two nonverbal systems capable of representing numbers, both limited in their representational power: the first one represents numbers in an approximate fashion, and the second one conveys information about small numbers only. Conception of exact large numbers has therefore been thought to arise from the manipulation of exact numerical symbols. (Izard *et al.* 2008, 491)

What are the nonverbal systems representing in that account? *Numbers*. But are "small numbers" numbers? How about "exact large numbers"? The terminology is confusing but even more worryingly, by the use of that terminology, the quotation seems to conflate the *explanans* with the *explanandum*. If we are trying to figure out what the concept of number is, we cannot simply assume that the nonverbal systems are representing it.

That is why our first order of business in explaining number cognition is to make clear just what we are explaining. For that purpose, we should start by defining the relevant concepts. This is not a trivial matter. As was mentioned above, in the philosophy of mathematics it is commonplace, for example, to use the term "arithmetic" to refer to a formal system of mathematics. From a developmental perspective, however, this seems needlessly narrow. In the discussion about arithmetical knowledge, it would be odd to limit the question to knowing formal arithmetic. Children can be highly proficient with arithmetical operations without having any knowledge of axiomatic systems or formal languages. From the developmental perspective, this kind of knowledge should count as arithmetical.

For this reason, I suggest a definition of arithmetic that captures its essential qualities, without demanding the understanding of formal systems. When we are concerned with the development of arithmetical knowledge, the key stage is to understand how natural number concepts are formed and how arithmetical operations are grasped. Thus arithmetic is understood in this paper as a *sufficiently rich discrete system of explicit number words or symbols with specified rules of operations*. What it means for a system to be sufficiently rich is not easily determined: it is impossible to state, for example, how many number words the system must include. Instead of providing any strict criteria for the richness of both numbers and the operations, the idea of the above definition is that for a system to be arithmetical, it has to sufficiently follow the structure of the omega progression, i.e., the standard ordering of the set of natural numbers. In this manner, if a child is able to correctly add and multiply numbers greater than, say, 100, it seems likely that she has grasped something essential about the system. That kind of knowledge cannot be acquired through rote memorization of multiplication tables which, while seemingly arithmetical, may involve little understanding of the numeral structure. Instead, for a number system to count as arithmetical, it has to follow a distinct recursive structure that the child has grasped. Usually, at this point of development children realize that the same operations work for larger and larger numbers, often also developing an early conception of infinity (Tirosh 1999; Monaghan 2001).

That kind of a system, while not the kind of formal axiomatic system that philosophers and mathematicians are after, has a key developmental difference to the primitive, core cognitive systems for quantities. While the former is arithmetical, the latter can clearly be merely *proto*-arithmetical. As we will see, it can contain the seed for arithmetical knowledge by determining the conceptual content children later obtain for numbers, but at that stage of development, the system is simply too weak and fuzzy to be called arithmetical. Correspondingly, the objects of a proto-arithmetical system cannot be called natural numbers without causing conceptual confusion. Rather, it is better to talk about *numerosities*.

In the literature on number cognition, a commonly accepted theory is that the developed natural number concepts, and the arithmetical knowledge about them, is based on our proto-arithmetical ability with numerosities. Philosophically, this can be understood in two different ways. First, we may accept that the cognitive core systems play a role in our growing understanding of natural numbers, yet we may refuse that the proto-arithmetical abilities have anything to do with what *makes* arithmetical sentences true. This was basically already the view of Frege (1884), according to which we must always separate the psychological processes in the context of discovery from the logical context of justification. The second, stronger, position is that the cognitive core systems also play a role as truthmakers of arithmetical sentences. According to such theories, a sentence like $3 + 2 = 5$ is true (at least partly) because the meaning of the number concepts is determined by the proto-arithmetical ability with numerosities.

In this paper, my focus is not on this debate, which essentially comes down to the old philosophical problem of Platonism or some form of constructivism and nominalism as the basic philosophical theory of mathematics. Instead of going into the traditional discussion on ontological and epistemological issues, I want to approach the question purely from the context of discovery. How can the proto-arithmetical ability develop into arithmetic proper? What other, e.g. linguistic, tools are required? In what follows, I identify one particular element central to arithmetical cognition: the importance of *processes*. I will argue that a cognitively informed epistemological theory of arithmetic does not need to involve problematic means of epistemic access to mathematical objects. Instead, mathematical objects can be reached by simple metaphorical thinking. When we think of mathematical objects in terms of processes, the connection to cognitive origins becomes clearer. Starting from the proto-arithmetical origins, I will try to show that the development of mathematical thinking is intimately tied to observing and executing processes in our physical environment.

As we will see in the final section, these questions are not independent of the ontological issues. Indeed, while my theory is perfectly compatible with both Platonist and nominalist ontology, one of its great strengths is providing an ontologically economical alternative to Platonism. Natural numbers may or may not have mind-independent existence, but we will see that one conclusion of the present theory is that arithmetical knowledge can be plausibly explained without making that assumption.

2. *Metaphors and mathematics*

Although there is no wider consensus about the details, it is now commonly accepted that human infants and many nonhuman animals process observations in terms of quantities. The systems for representing numerosities in this non-symbolic manner are called *core cognitive* (Spelke 2000). Two such systems have been presented. First is a system for parallel individuation (or object tracking) that allows determining the amount of objects in the field of vision without counting (Starkey & Cooper 1980; Spelke 2000). Second is the ability to estimate the numerosity of a group of objects and determine the difference in group sizes. The first ability is called *subitizing*, and the second system is referred to —with some possible variation— as either the Approximate Number System (ANS), the Analogue Magnitude System, or Number Sense (Dehaene 1997/2011). While both abilities deal with numerosities, they have important limitations and cannot be reasonably considered to be arithmetical. Subitizing, while dealing with discrete numerosities, only works for small quantities; usually up to four objects. The ANS works for larger groups, but it is an estimation system that becomes increasingly inaccurate as the collections become larger, and thus is generally not considered to be discrete.[1] Both systems are therefore at best proto-arithmetical, and as explained in the previous section, should not be characterized in arithmetical terminology.

Based on many experiments, most importantly concerning the lack of ratio effect in treating small quantities (Feigenson *et al.* 2004), it is commonplace to accept that subitizing and the ANS are indeed two separate abilities for treating quantities (Agrillo 2015). However, there are others who suggest that a single estimation mechanism can sufficiently explain the data (see e.g. Beran *et al.* 2006). Here I will follow the more widely accepted position that subitizing and the ANS are two different mechanisms, but ultimately the arguments in this paper do not depend on this choice. The important part is that both abilities can indeed be considered to be proto-arithmetical, i.e., there is a relevant sense in which both abilities contribute to the development of arithmetical understanding. This is not something we can trivially assume, as there is much work to be done in the empirical research, as well as the philosophical analysis of the results, before we can conclusively establish such development. In Pantsar (2014, 2015, 2016) I have made an effort to show that there indeed is a strong case for believing that subitizing and the ANS both play a part in the development of arithmetical knowledge. While there remain important open questions, the empirical data and philosophical foundation certainly seem to be strong enough to pursue this line of thinking.

Following that work, at this point it would be commonplace to propose an explanation of the way our proto-arithmetical ability determines the content of our developed arithmetical number concepts. Here, however, I would like to temporarily reverse the standard order of exposition. In explaining the cognitive basis for natural numbers, at some point we need to ask an important question: how do we get to know that there is an infinite amount of them? In the proto-arithmetical ability, there is nothing to suggest that. Neither subitizing nor the ANS provide a framework for infinity. Indeed, any theory that explains arith-

[1] Although it should be noted that there are also models of ANS in which it is described in terms of discrete whole numbers. See e.g. Zorzi & Butterworth 1999.

metical knowledge in terms of biological primitives, needs to be able to explain an interesting conceptual leap. Starting from handling small collections of objects, or larger ones in an approximate manner, the idea for an infinite amount of discrete natural numbers somehow emerges.

It needs to be noted that there is nothing inevitable about such conceptual leap and it certainly does not follow automatically from the manipulation of small collections of objects. Many cultures have not developed the idea of an infinity of numbers. These include tribes such as Pirahã and Munduruku, whose ability with numerosities does not rise much above the proto-arithmetical origins (Gordon 2004, Pica *et al.* 2004), but it does not appear that even the Mayans —for all their great ability in calculations with natural numbers— had the concept of infinity, at least not in the form we have it (Ifrah 1998). This move from finite to infinite collections is crucial for mathematics, and any well-informed cognitive account of the development of mathematical thinking should be able to provide a plausible explanation of it.

Perhaps the best-known such effort is by Lakoff and Núñez (2000), who proposed that all cases of actual infinity in mathematics are applications of one conceptual metaphor, the *Basic Metaphor of Infinity* (BMI). The general idea of BMI is that processes that can be continued indefinitely are thought to have a metaphorical ultimate result. Just like we speak of the final resultant state of a completed iterative process (such as counting to ten), with the help of BMI we can speak of the metaphorical "final resultant state" of iterative processes that go on and on. In the simplest case, the definition that the successor of each natural number is also a natural number gives us an indefinite iterative process. When one realizes that the process goes on and on, she no longer expects a final resultant state. Instead, the metaphorical "final resultant state" becomes the concept of actual infinity we use in mathematics. Like the end process of a completed iterative process, the "final resultant state" is unique and follows every non-final state of the indefinite iterative process (2000, 158-59).[2]

In Pantsar (2015) I have reviewed some of the arguments against BMI, as well as providing my own criticism of it. One critical point I emphasized was that the final resultant states of completed iterative processes and "final resultant states" of unending processes do not behave similarly. In terms of Conceptual Metaphor Theory (Lakoff & Johnson 1980), which provides the general framework for Lakoff and Núñez, metaphors consist of three main components. First, there is a source domain, which is the conceptual domain of the metaphorical expressions. Second is the target domain, the conceptual domain we use the metaphors to explain. Third, there needs to be a mapping between the two domains. This mapping does not need to be bijective, or even a function. The important point is that it retains enough of the structure of the source domain in the target domain, thus making the metaphor useful. To give a standard example, in the phrase "love is war", *war* is part of the source domain, *love* is part of the target domain, and the metaphor

[2] Here we must make the Aristotelean distinction between *apeiron dunamei* (the *potential* infinity) and *apeiron hos aphorismenon* (the *actual* infinity) (*Physics* 208a6). The central idea behind the theory of Lakoff and Núñez is that we need a cognitive account for actual infinity in mathematics, in order to be able to talk about infinite *things*: infinite sets, points at infinity, transfinite numbers etc. (Lakoff and Núñez 2000, 155).

is successful if what we say about war retains enough of the structure of what we believe to be the case about love.[3]

In the account in Lakoff and Núñez (2000), one problem was that we could not be sure that the BMI retains enough of the structure. Completed iterative processes have actual final resultant states, which do not behave in the same way as the metaphorical "final resultant states". In a simple case, counting to ten and eleven are different processes with different final resultant states. But, for example, the series 1, 2, 3, … , n, …. and the series 1, 4, 9, … , n^2, … are two different processes that give the *same* "final resultant state". If BMI is the metaphor we use, how do we know that we cannot use our knowledge of completed iterative processes in this way to concern endless iterative processes? Núñez (2005) has later developed the account so that the BMI is a double conceptual blend, getting input both from the completed and endless iterative processes. This, what he called now the Basic *Mapping* of Infinity, seems to provide a better picture of what is happening cognitively. By drawing from two kinds of elements, the completed and endless iterative processes, with BMI one is then able to use the knowledge of those processes to introduce the concept of actual infinity to mathematics.

While the Basic Mapping of Infinity better elucidates the conceptual elements involved, I believe it remains to be a flawed concept even in this improved form. Most importantly, there remains too big a cognitive gap between an actual final resultant state and a metaphorical "final resultant state". In the source domain, understanding final resultant states can be cognitively a much simpler process, as it can involve mere mechanical following of a simple algorithm. Understanding that a process is unending, however, requires grasping something essential about the process. In the improved version of the BMI, Núñez suggests that input from endless iterative processes also plays a role, which addresses this issue. Clearly this is a step forward, but the important question concerns the details of *how* that happens. In particular, what role does BMI play and what other means of drawing knowledge of different kinds of processes are used?

However, while the particulars of the theory may need further clarification, there are clear merits for the metaphorical account of Lakoff and Núñez. The above problems notwithstanding, they seem to reveal something essential about cognitive access to actual infinity in their account. Even if we were extremely capable with finite collections, the leap to infinite collections is a qualitative one that does not necessarily occur. In explaining this qualitative leap, a metaphorical account has great potential.[4] From our experience with fi-

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[3] It should be noted here that Conceptual Metaphor Theory remains to be somewhat controversial among cognitive scientists. However, the present account does not rely in any way on accepting that particular theory of metaphors. It could be presented in the framework of other theories of metaphors, such as Way (1991) or Kövecses (2002), but it is also compatible with the "Quinean bootstrapping" account of acquiring concepts used by Carey (2009). The reason for focusing on Conceptual Metaphor Theory here is to compare my account with the metaphorical account of Lakoff and Núñez, for which it provides a natural framework. The theoretical versatility of the present approach will be made clear in the final section in which I will clarify what is meant by metaphors in the present cognitive setting.

[4] This is the case also if we accept, for example, the bootstrapping theory of Carey (2009) in which grasping the cardinality principle ultimately leads to having the concept of infinity. As transfinite cardinals do not behave like finite cardinals, a qualitative leap from the finite to the infinite must be made also in that account.

nite collections and completed processes, we somehow successfully move the discourse to unending processes and infinite collections. For cognitively informed epistemology of mathematics, metaphorical thinking provides an appealing explanation.

So the problem is in the proposed metaphor, not generally in metaphorical thinking. Above we have seen some of the problems with the Basic Metaphor (or Mapping) of Infinity. However, I do not see any of the problems as fatal. The metaphor perhaps needs more explanation, but ultimately it seems possible to formulate it in a way that mirrors the cognitive processes in making the leap to infinite objects in mathematics.

Yet there appears to be one problem with BMI that is more general and harder to solve. Infinity is a theoretical concept that is not universally developed even in arithmetically quite advanced cultures. Could it really be the case that our most basic cognitive toolbox for mathematical thinking includes such a metaphor?[5] Or is it more likely that BMI is not the kind of primitive cognitive tool we are looking for? In what follows, I will argue that there is indeed a more general, more basic mode of metaphorical thinking which is ubiquitous in mathematics —and which explains everything that BMI does.

3. *Process → Object Metaphor*

In Pantsar (2015), I suggested an alternative to the Basic Metaphor of Infinity: the Process → Object Metaphor (POM). Let's take a simple mathematical concept such as the Fibonacci sequence as an example how the two metaphors work. The Fibonacci sequence, of course, is the number sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

and it is defined recursively by the function

$$F(0) = 0, F(1) = 1 \text{ and for all } n : F(n)=F(n-1) + \mathrm{F}(n-2).$$

Clearly the sequence is unending and thus the process of calculating new members in the sequence has no final resultant state. However, according to the BMI, it has the metaphorical "final resultant state" of being infinite. By grasping how the sequence works, one is able to see that it is unending, and with the use of metaphorical thinking we realize its infinity. I do not want to claim that there is necessarily anything wrong in an explanation like that. However, I believe that we can reach the same inference by the use of a more general, more basic conceptual metaphor.

We talk about the Fibonacci sequence, but already that term is showing considerable mathematical sophistication. For the uninitiated, it may seem strange to talk of a sequence of numbers as an object. Instead, the easier cognitive access to learning about the sequence is provided by learning how one can calculate the next number (Rubio-Sánchez & Hernán-

[5] There is of course an inevitable vagueness in the term "basic cognitive toolbox". The purpose of this paper is not to pursue the question which cognitive abilities should be included in that. Instead, I will merely argue that BMI is unlikely to be included since there is a more basic conceptual metaphor which can account for everything that is achieved with the help of BMI.

Losada 2007). In other words, before children can understand what the sequence is, they need to learn the *process* behind the sequence. By understanding the process, it becomes obvious that no finite part of the sequence will ever give us the infinite series. Indeed, based on the well-known "Kripkenstein argument" (Kripke 1982), it will neither unassailably give us the rule that the sequence follows.

But of course the recursive definition we gave above *does* give us the Fibonacci sequence unequivocally. What it does not give us, however, is the series as a *thing*. It gives us the series as a process, which is clearly not finite. Thus, the Fibonacci sequence gives us a paradigmatic case: when we talk about infinite things in mathematics, we are treating the things defined by potentially endless processes as objects of mathematics. This is what I mean by the Process → Object Metaphor. We take (an unending) process and by applying the POM we can talk about it as an (infinite) object. It should be clear that like the BMI, POM is a metaphor. Unending processes are not objects —at least not like any other objects we know— and we do not need to understand things defined by such processes as literally existing.[6]

What is the essential difference between BMI and POM? Both would seem to be able to explain how we move from unending mathematical processes to infinite mathematical objects. However, there is one important difference: in POM there are no problematic concepts like the "final resultant state". The metaphor is not used because we realize that the Fibonacci sequence is unending. Instead, POM is used because it can *always* be used for mathematical processes, regardless of whether they are completed or unending. If we took a finite subset of the Fibonacci sequence, the exact same metaphorical thinking could be used, this time using a completed process to give a finite sequence as an object. This is the great strength of POM over BMI: it is in no way *infinity-specific.* Thus it has potential to be exactly the kind of more basic and more general metaphor that we were looking for.

Let us see just how basic and how general POM is. In mathematics education, the importance of processes has been acknowledged for a long time (e.g. Thompson (1984) and Hiebert (2003)). This can be seen in the dynamic language used to elucidate mathematical concepts, starting from such simple cases as two and three *make* five. That kind of language has been shown to be ubiquitous in teaching mathematics on every level. One good example is the teaching of functions. Functions are taught as *picking* one element from the image for each element of the domain. Another common way is to show the elements of domain as the *input* and the elements of the image as the *output*. Both of these ways treat functions as processes, and it is only much later that the students learn to treat functions as sets of ordered pairs, i.e., objects.[7]

This also goes for university-level mathematics. In analysis, the process-based language is fundamental to learning. A sequence is said to *converge* and numbers are said to *approach*

[6] Thus POM also avoids the Kripkenstein problem. We cannot unassailably determine the process of new Fibonacci numbers if all we have are finite parts of the sequence. But since the process is actually *primary* to the sequence, we can establish the generality of it.

[7] There have been numerous works recently studying the use of dynamic language in mathematical practice. Núñez (2006), for example, has analyzed the use of motion-based terminology in mathematics, in an account that fits well not only with BMI, but also with POM.

the limit of the sequence. While methods are used which treat the functions as static objects —e.g. the (ε,δ)-definition for the limit and convergence of a function— in explanatory remarks they are usually amended by dynamic process-based expressions. The language of mathematical objects has clear advantages, but the (ε,δ)-definition, for example, is notoriously hard for students to learn. By treating the function as a process —by drawing a graph, for example— students usually find it much easier to understand the nature of convergence (Dawkins 2014).

None of this, of course, is in any way surprising. Indeed, the use of process-based language and teaching methods is so wide that remarks like the above may seem like merely pointing out the obvious. However, the interesting part here is not how mathematics is taught so often by focusing on processes. What should be clarified is *why* this seems so natural to us. In the Process → Object Metaphor we would appear to have a simple answer: process-based language is so easy to grasp because in (especially early) mathematics education we so often learn primarily about processes. Objects only come later, through metaphorical thinking.

How can we evaluate POM as a hypothesis in explaining mathematical cognition? This is not an easy question. Clearly POM —or any other conceptual metaphor— is rarely, if ever, explicitly used in mathematics. Rather, by its characteristics, POM is something that is used implicitly throughout the development of mathematical thinking. If POM is indeed the kind of an important tool I have claimed it is, we should expect to find evidence for it in at least the following ways:

— POM can explain key abstraction steps in mathematics.
— The informal language of mathematics includes regular references to processes.
— Process-based teaching methods are beneficial in introducing new mathematical concepts.
— Processes play an important role already at the initial stages of developing mathematical thinking.

We will return soon to the fourth point, but we have already seen that the first three predictions are fulfilled. Because of that, it seems clear that processes play a key role in the development of mathematical thinking, both historically as a subject and in individual development. Those three predictions, however, leave open one plausible scenario: while implicit metaphorical thinking clearly plays an important role, is POM the correct metaphor? In the previous section I have criticized BMI on the grounds that we can find a more general, more basic metaphor to account for actual infinity. Can we be sure that POM does not run into the same danger?

It should first be made clear that POM as a conceptual metaphor is meant to elucidate the role that processes play in mathematics and how objects work as their counterpart. I do not want to claim that each time we make the jump from processes to objects, we specifically employ a single conceptual metaphor, whether explicitly or implicitly. There may, for instance, be different forms of POM based on the kind of processes and objects we are dealing with. In that way, it could be better to talk about a family of metaphors. But as of writing this, there is still very little empirical knowledge of how conceptual metaphors work on the level of the brain. Hence, any account that would make an effort to be too specific would eventually run into trouble with the details of the neural architecture involved. Instead of providing a detailed explanation, POM is meant to be understood as a more general description of the way we in mathematics move from one domain of discourse (processes) to another (objects).[8] This does not imply that POM is not an empirically testable hypothesis, or that it could not provide new empirically testable hypotheses concerning the possible lower-level cognitive processes that it is meant to characterize. Indeed, it is my contention that theoretical concepts like POM can be highly useful in designing experiments and interpreting empirical data. At this point, however, we cannot expect to explicate conceptual metaphors in terms of our cognitive architecture.

Even with this understanding of POM, however, there remains the important question whether there is a better explanation of metaphorical thinking in mathematics. Obviously I do not want to reject this possibility. As we learn more about metaphorical thinking, it may indeed be possible to break POM down into more primitive parts of conceptual processing. However, as I have explained above, such efforts are not feasible within the current state of the art in cognitive science. As of now, there does not seem to be a better alternative than speaking about metaphors like BMI and POM as simple conceptual metaphors, and hence we should assess the strengths of POM correspondingly.

In this context, there are two possibilities which limit the applicability of POM. First, there can be a level of mathematical thinking high enough that does not involve any metaphorical thinking, or the metaphorical thinking is not process-based. Second, there can be a more primitive level of mathematical thinking on which POM is not used. The first possibility seems quite possible. Mathematicians may indeed become so proficient in the discourse on objects that they will no longer have any need to grasp mathematics in terms of processes. It is similarly possible that other metaphors are used in higher-level mathematical thinking. I do not want to dispute either possibility, but neither seems to be in any conflict with the importance I have argued for POM. Obviously I do not claim that all mathematical thinking is metaphorical, or that POM covers all metaphorical thinking in mathematics. What I do argue for is the position that POM can be fruitful in explaining the development of mathematical understanding, from early origins to high levels of mathematics.[9]

This way, the second possible limit to the applicability of POM seems much more relevant. Is there a primitive level of mathematical thinking that does not apply POM? If there

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[8] The account here is certainly not the only one to emphasize the importance of both processes and objects to mathematical cognition. Gray & Tall (1994) have written about *procepts,* the amalgams of mathematical processes and concepts. In their theory, a "procept" refers to the way we use mathematical symbols to refer to both processes and concepts. Thus a symbol such as "5" refers to the object five, but also to the various processes by which we can reach 5 (counting to five, 3 + 2, 10/2, etc.). The procept account is cognitively plausible and compatible with POM. However, if mathematical objects are seen as metaphorical counterparts of processes —as in the present account— the role they play in procepts becomes secondary. Thus, interpreting procepts in terms of POM makes one part of the amalgam redundant.

[9] Another way to see the role of POM is through the psychological concept of *internalization* as presented by Vygotsky (1978). Tallying, finger counting, as well as more sophisticated processes, are internalized as one develops her mathematical skills and knowledge. In higher mathematics, these processes are no longer enacted, but they remain in the background even when the discourse moves primarily to objects. I thank Regina Fabry for pointing out this connection.

is, it would be a potentially serious problem for the present theory. In particular, the question would arise whether there is some other metaphor, or perhaps a form of non-metaphorical thinking, behind POM.

4. Metaphors and the roots of arithmetic

In Section 2, I presented the viewpoint that arithmetical knowledge is based on our cognitive core systems. There are different ways of approaching the matter, as well as different explanations about the roles that the different core systems play, but a widely shared view among empirical researchers (if not necessarily philosophers) is that in some way the limited ability we have with quantities already as infants determines how arithmetical understanding develops. In Section 3, I have proposed that metaphorical thinking captured by the POM provides a highly useful platform for explaining that development. The question we need to ask in this section is how the two theories fit together.

Based on the considerations in the previous section, it appears that in order to be a viable hypothesis for explaining arithmetical thinking, POM needs to be present already at the early stages of arithmetical thinking. I will argue here that not only is this the case, but POM can already explain the *proto*-arithmetical processing of numerosities. When it comes to neuronal representations of numerosities and possible metaphorical elements in them, we currently have no reliable way of gathering data. There is, however, interesting data on the neuronal representation of numerosities. Distinct groups of neurons in the parietal and frontal lobes appear to be associated with different numerosities, which can be interpreted as the stimulus for observing a collection of objects of a given size. These neurons are independent of modality, i.e., the same group activates regardless of whether we, say, see three objects or hear three tones (Nieder *et al.* 2002; Nieder & Dehaene 2009; Nieder 2012; Nieder 2013). How the processing of numerosities in the brain continues after that is largely unknown. Based on neuronal data it is difficult to say, for example, at what stage we form exact number concepts and how this process is carried out.

Another problematic issue is that that the proto-arithmetical ability is not independent of the developed ability of processing natural number concepts. While we appear to share subitizing and ANS with many non-human animals, it is clear that we develop conceptual representations that go beyond those proto-arithmetical origins. However, studies show that training in arithmetic also develops the acuity of our estimation ability. College students show the same brain patterns as monkeys in estimation tasks, thus suggesting that the ANS is an independent ability, but studies also show that better mathematical skills are associated with better acuity in estimation tasks (Cantlon *et al.* 2006; Brannon & Merritt 2011). This two-way connection between the proto-arithmetical ability and arithmetical thinking is also seen in the way that developmental dyscalculia and damage to the proto-arithmetically important areas in the brain are connected to lower mathematical skill-levels (Dehaene *et al.* 1999; Butterworth 2010). Clearly there is a way in which the proto-arithmetical ability is closely connected with the way we acquire number concepts, and this connection remains even after we have acquired exact number concepts.

This prompts the question how subitizing and ANS are related to the Process → Object Metaphor, i.e., how do we come to use our proto-arithmetical ability in the kind of way that enables us to have sufficiently well-formed processes whose end products we can then conceive of metaphorically as objects? There exist various frameworks of how the ANS can give us numerosity concepts, the most famous of which is perhaps the Dehaene-Changeux (2011) model. It is not possible here to go deeper into the topic of neuronal modelling, but it should be noted that in the literature there is no consensus of how the proto-arithmetical ability gives us number concepts. Some (like Carey 2009) believe that subitizing and a resulting "bootstrapping" process are the keys to explaining how we get the concept of cardinality. Others (like Dehaene 1997/2011) believe that the answer can be found in the ANS.

How does POM fit within all this? Simply put, with the current state of the art in neural modelling, we don't know how exactly discrete numerosity concepts are formed. For the POM-based account, however, this is only a minor setback. Whether by bootstrapping or the number sense of Dehaene (or by some other process), we know that we *do* acquire discrete numerosity concepts. The main aim of the present account is then to show how these concepts are developed in terms of processes. Thus the approach here is compatible with various proposed explanations of the initial acquisition of numerical conceptual representations, including those by Carey and Dehaene.

In this way, the best place to start from is the earliest stage we can be certain that there are conceptual representations for numerosities, i.e., when we learn to use number words or symbols to refer to numerosities.[10] One of the earliest known ways of doing this were systems of tallying. By making notches onto a piece of wood or bone, or by some similar means, human beings have for a long time been able to keep track of discrete quantities. The famous Ishango bone is believed to show such a system, and it has been dated to be more than 20,000 years old (Bogoshi *et al.* 1987). Probably at least equally old are different systems of keeping track of quantities with the help of body parts. Indigenous tribes all over the world have developed such systems and they can be used for quantities up to at least thirty (Butterworth 1999).

It is clear that representing discrete quantities is a practice much older than our arithmetical theories. As such, the different methods of keeping track of quantities provide an interesting basis to study the cognitive basis of natural numbers. Tallying and counting with fingers are such basic processes that it is easy to dismiss their philosophical importance. Yet there is one basic question that we need to ask: why do systems of tallying and using body parts *work*? How are we able to interpret a group of notches or fingers as representing a quantity? I propose that the simple explanation is that they are early cases of using the Process → Object Metaphor. Both tallying and extending (or folding, or pointing to —the particular method differs greatly) are clearly processes. More specifically, they are processes based on the "plus one" (successor) operation. In both systems, information can be communicated without having number words. By showing a certain body part (or configuration of different body parts), one is able to determine which numerosity is referred

[10] It is important to note that this is not the stage when children learn to recite part of the natural number sequence. It takes longer for children to learn to use the sequence properly to refer to numerosities (Fuson 1988; Wynn 1990; Davidson *et al.* 2012).

to. Similarly, in tallying one can match objects one-to-one with the strokes and thus determine the correct amount.[11]

However, as clear as it is that such methods are among the earliest to keep track of and communicate numerosities, it is equally clear that the end product of neither process *is* a numerosity. Indigenous people do not get confused that, say, three fingers *are* the numerosity three. Instead, they use tallying and body parts as methods of representing the numerosity. However, different cultures may have quite different ideas of what numerosities or numbers are. For many cultures, such questions are not familiar and in others (like ours), they get radically different answers. Often there is quite little in common in the way numerosities are used, referred to and developed.

However, one thing remains constant: numerosities are treated in terms of processes. Three extended fingers at the end of a counting process represents the end product of the process of extending a finger three times. When we are familiar with the process, we only need to see the three fingers in a context involving numerosities and we immediately understand what they signify. We also understand that the fingers of one person refer to the same numerosity as those of another. The particular fingers do not matter, just as the particular means of tallying does not. What matters is the *process* that has led to it.

In Japan, for example, counting with fingers begins from five extended fingers on one hand and with each number, one finger is folded inwards. If a European person not familiar with this custom only saw a Japanese person's hand at the end of the process, she would most likely get the expressed numerosity wrong. However, if she saw the process, there is little chance she would be mistaken. There might be variation in the particular methods, but the overall idea is the same: a step-by-step process is used and its end product is referred to in an object-like fashion.

This implicit use of POM continues as we learn more about numbers. Addition, for example, is often first taught with the help of either the body part method or tallying. In this way, process-based metaphorical thinking clearly plays an important role in the development of early arithmetical thinking. It generally does not take long for children to dispense with the processes in early arithmetic. As they get more adept at manipulating numbers as objects, there is less need for metaphorical thinking. But as we have seen, learning new mathematical concepts via processes is a pursuit that continues all the way to university education. This makes both the basic character and the generality of POM apparent. Right from the proto-arithmetical beginnings, we are able to communicate ideas about numerosities because of shared understanding of processes. As seen in mathematics education —as well as in mathematical practice— this is an enduring idea that retains its importance on all levels of mathematical communication. With POM, I believe we have a viable general hypothesis concerning the cognitive activity involved.

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[11] Parsons (2007) and Jeshion (2014) argue that mathematical intuition is responsible for grasping that members of stroke-systems of tallying (e.g. |||) represent members of the ω —sequence (e.g. the natural number three). Their account has important connections with the present theory, but mathematical intuition is a tricky concept that has some quite problematic connotations. In this way, the metaphorical account would seem to provide a better framework for basic mathematical cognition, while reserving a role for mathematical intuition as a highly useful concept in explaining more sophisticated mathematical thinking.

5. *What are metaphors in mathematics?*

Above we have seen how POM appears to capture something essential about the way arithmetical cognition develops, as well as about the general cognitive structure behind mathematical knowledge. However, what we still need to ask is whether there are stages in the development of arithmetical cognition in which POM does *not* play a role. This is an important question for several reasons. As explained in the first section, there is a significant qualitative leap from proto-arithmetic to arithmetic. However, this is not the only qualitative leap involved. For example, the ability to match the number word sequence to increasing quantities is clearly a vital stage in developing knowledge about natural numbers. Many other similar stages can be identified; such is the complexity of the development of arithmetical knowledge.

It is not possible here to go into all the levels involved in that question. Instead of proposing a detailed picture of the development of arithmetical knowledge, akin to Lakoff and Núñez, I will focus here on the early stages of numerical cognition, with the aim of explaining an essential difference to more developed stages. With this explanation, I hope to also shed light on what mathematical metaphors actually are.

It seems clear that the core cognitive systems of subitizing and ANS are non-metaphorical. Whatever our understanding of mathematical metaphors may be, the idea that the limited and/or inexact core cognition based means of treating quantities would involve conceptual metaphors seems necessarily flawed. As mentioned above, in terms of conceptual metaphor theory a metaphor needs to include three things: a source domain, a target domain and a structure-preserving mapping between the two. It is hard to see how the Approximate Number System, for example, could be explained by the conceptual metaphor theory. Thus the early origins of arithmetical cognition do not involve POM, or other metaphorical thinking.[12]

However, it is possible to see already at the stage of tallying or using body parts in counting, that POM plays a crucial role. Although not present initially, during the development of arithmetical cognition metaphors clearly enter the picture early on. What is the key cognitive step involved? Here I want to suggest a straight-forward answer: the core cognitive systems do not involve the use of conceptual metaphors simply because, on that level, there are no conceptual *representations* of numerosities. Let me explain that a bit more. To be sure, *some* kind of representations in the brain are made already at the level of core cognition.[13] For example, it has been observed that there are quantity-specific neurons in the brain. When test subjects' observations deal with quantities, regardless of the modality of the observation, (partly) the same group of neurons activate in different subjects (Nieder 2012, 2013, 2016).[14] So when a subject A sees three things and subject B hears three sounds, in addition to many other neurons firing due to the particular circumstances, there

[12] This point can be made, *mutatis mutandis*, with regard to other theories of metaphor.

[13] If we indeed accept that there are representations in the human mind. For the present theory it is enough that we assume that there are explanatorily relevant neural representations of numerosities, we don't need to assume the existence of content-bearing mental representations.

[14] The experiments here were conducted on monkeys, but based on the similarity in the brain regions that monkeys and humans use for subitizing and estimation, it is plausible that the human proto-arithmetic ability shares the same characteristics (Nieder 2012, 2013, 2016; Piazza *et al.* 2007).

is also a common group of neurons —the so-called *number neurons* in the parietal and frontal lobes— activating because the subjects observe the quantity three. For two objects, this group would be (partly) different.

However, whatever these core conceptual representations may be, they are in important ways different from the conceptual representations that later emerge. Subitizing usually only works for up to three or four objects and the ANS gets increasingly inaccurate as the quantities get larger. Moreover, the development of arithmetical knowledge is not a straight-forward process in which the core cognitive ability in clearly specified steps develops into arithmetic. Many cultures never develop numeral systems beyond a few number words and even those words are not always used consistently (Gordon 2004). There is nothing inevitable in developing arithmetical thinking. While the core cognitive resources are universal, it takes specific cultural conditions to develop them further. Menary (2015) has called this process *enculturation*. Following Dehaene (2009), he argues that in learning arithmetic, we employ brain circuits evolutionarily developed for other purposes. Thus the development of arithmetical thinking in an individual is tied to the redeployment of the proper brain areas, but also to cultural input. An important part of the cultural resources is to have number words in the language. But as was noted above, even when children learn number words, the process is not a simple matching of words to quantities. Even after learning number words, for a surprisingly long time, children cannot match the sequence to quantities (Davidson *et al.* 2012).

I believe that from all this, we get a potential explanation of what the POM actually is. Why can children at certain age recite the sequence of number words but not match them with the proper quantities? The answer must be that there is more to understanding numbers than merely being able to recite a list of words in an order. Indeed, it is quite clear what the missing element is: in the mere reciting of the number word list, the words do not necessarily *represent* anything. This, I argue, is the key to developing numerical cognition beyond the primitive origins —and, as such, the key to the cognitive basis of the Process → Object Metaphor. The important cognitive leap toward arithmetic is made when we start using conceptual representations for the number words in the counting sequence. When a child learns to match the number words with numerosities, it is because he understands that the words represent quantities. This enables him to count, i.e., use the end product of a process to represent a numerosity.[15] Already in this early indispensable step in learning arithmetic, POM can help us provide an explanation.

From the above considerations, we can get a better idea of the nature of the Process → Object Metaphor. It is a way of moving from executing or observing a process —e.g. counting— to having a conceptual representation of its end product —e.g. a discrete quantity. The same thing happens all through the development of mathematical knowledge, getting increasingly theoretical forms. In sets, sequences, functions and other mathematical objects, we find ways of treating the end products of mathematical processes as objects. When the processes are unending, the objects are infinite. But all through this development, the central idea remains the same. We acquire conceptual representations of mathematical objects, not because we have special epistemic access to them, but because the objects them-

[15] This is called being *cardinality principle* knower in the literature (LeCorre & Carey 2007; Sarnecka & Wright 2013).

selves are metaphorical counterparts of well-defined —and well-understood— mathematical processes.

Finally, we should tackle one concern. Throughout the paper we have been talking about mathematical processes as being epistemologically less problematic than their metaphorical counterparts, abstract mathematical objects. But while the epistemic problems of mathematical objects are well known, just how epistemologically unproblematic are mathematical processes? Indeed, aren't mathematical processes also abstract, thus suggesting that they would share the same problems as mathematical objects?

While that is a legitimate concern, it fails to appreciate an important difference between processes and objects. Abstract mathematical objects, Platonically understood, have the same general problem of epistemic access regardless of their complexity. It seems clear that the epistemic access to small natural numbers, for example, is simpler than that to complex numbers. By observing collections of four objects, one can have access to the natural number four in the kind of unmediated way that is not possible for complex numbers. But even though conceptually simpler, natural numbers face the same famous epistemological problem of Benacerraf (1973) as complex numbers do: how can we as physical subjects have epistemic access to non-physical objects?

With mathematical processes, the matter is not similar. While abstract mathematical processes may indeed share Benacerraf's epistemological problem, as we have seen, we can track mathematical processes down to concrete processes, such as finger counting and tallying, as well as more developed embodied methods such as writing down symbols.[16] Of course these processes are not completely unproblematic epistemologically, either. Someone from a different culture may struggle to understand the significance of tallying, for example, especially in cultures with few or no numerals like the Pirahã. Moreover, we do not know which exact cognitive processes are involved in grasping processes like tallying and how they relate to processes involved in higher mathematical thinking.

Nevertheless, there is little doubt that concrete processes can generally facilitate grasping abstract objects. Gelman and Butterworth (2005), for example, observe how people in the Oksapmin tribe, who have an extensive body-part counting system, were quick to learn counting rules and verbal numeral systems when introduced to them in a new culture. This same realization of the importance of concrete processes is also at the basis of the universal method of teaching basic arithmetic to children with the help of physical objects. Many studies have shown that children who manipulate concrete objects have an advantage in learning basic arithmetic (see e.g. Sowell 1989). Interestingly, Clements and McMillen (1996) noted that computer software that allows students to manipulate objects can have the same beneficial effect in learning. Thus the important part does not appear to be that the manipulated objects are physical. What is important is the manipulation itself, i.e., the process.

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[16] It could be argued that the subitizing ability gives similar access to natural numbers via concrete objects. While I agree that subitizing is likely to play a role in giving rise to natural numbers, I find postulating this kind of direct access to natural numbers problematic. In any case, even if we accept that by subitizing numerosities of collections of concrete objects we can form natural number concepts, this ability is limited to the first three or four numbers.

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