THE PHILOSOPHY OF STRICT FINITISM

Ernest J. WELTI

ABSTRACT

The philosophy of strict finitism is a research programme containing developmental theory and mathematics as its main branches. The first branch is concerned with the ontogenetic and historical development of various concepts of infinity. The framework is Jean Piaget's genetic epistemology. Based upon these developmental studies, the mathematical branch introduces a new concept of infinity into mathematics. Cantor propagated the actual infinite, Brouwer and the constructivists the potential infinite. Still more radical is strict finitism, favoring the natural infinite, i.e. the phenomena of the unsurveyable, unfeasible, unreachable. There exist by this time strict finitistic reconstructions for arithmetic, geometry, calculus, and even for infinitistic set theory.

1. Developmental Investigations of Strict Finitism

1.1. Ontogenesis

The framework of the developmental studies of strict finitism is Jean Piaget's genetic epistemology. The school of Piaget has elaborated mathematical models for several cognitive activities of natural thought and its developmental factors. The essential developmental factor is called the *mechanism of majorizing equilibration*. At the beginning of any development are states of cognitive instability, manifesting themselves by the absence of cognitive instruments that were needed to solve a certain problem, or by unsufficient differentiation of available instruments. States of cognitive instability start the mechanism of majorizing equilibration that directs by the agency of typical interim stages to a better cognitive equilibrium.

Concerning various concepts of mechanics, geometry and algebra,

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Piaget and his disciples have worked out developmental stages and mechanisms that occur as well in ontogenesis as in historiogenesis. Such parallelisms are not instances of strict law, but rather of a heuristic principle, comparable to Haeckel's biogenetic principle. Following the biogenetic principle, the biological ontogenesis can be regarded as a recapitulation of phylogenesis. Analogically we speak of the *psychohistorical principle*: cognitive ontogenesis recapitulates in many instances the history of science. In the case of the development of the concepts of infinity, the psychohistorical principle is crowned with success.

From the standpoint of cognitive development, the role of science may be seen in the reconstruction of the various stage-specific equilibria of the ontogenetic and historiogenetic development. This reconstruction starts at the finally achieved stage and goes the backwards. Science therefore has not only a progressive trend, but also a trend which is enantiodromic to the ontogenetic and historiogenetic development. We speak of the *principle of cognitive regression*: scientific progress is in many cases combined with a cognitive regression, i.e. it takes us back from the ontogenetically and historiocally later stages to ever earling stages. The development of the concepts of infinity gives us again an instance to this principle.

1.2 Historiogenesis

Our particular purpose is to work out development processes and to understand them as majorizing equilibrations. The historical development of the concepts of infinity, in accordance with the psychohistorical principle, runs largely in parallel with ontogenetic development. Three major stages may be discerned:

(1) The stage of the *natural infinite*: At this stage, wich begins in the Old Stone Age and continues to Presocratics, infinity is perceived as something unsurveyable, unfeasible, unreachable.

(2) The stage of the *potential infinite*: This stage begins in Classical Greece. Characteristic is the newly developed concept of limit which may be found in the mathematical works of Eudoxus and plays a central role in the Aristotelian theory of infinity.

(3) The stage of the *actual infinite*: This stage commences in the Middle Ages. Various scholastics have regarded infinite sets

as accessible to the all-embracing power of God and anticipated thereby certain concepts and theorems of modern set theory.

Of interest, however, are not only the steps from one major stage to the next but also the many small achievements within each stage. At the stage of the natural infinite, the transition fron ancient oriental concepts of infinity to Anaximander's apeiron is of particular importance.

At the stage of the potencial infinite, the development can be described which commences with the discovery of the irrational and the Zenonian paradoxes, passing through interim stages to end ultimately in Classical Greek mathematics.

At the stage of the actual infinite, the development is produced and promoted by various set-theoretical paradoxes. While the scholastics did not ultimately achieve a stable cognitive equilibrium, the initially confused relation of subset and relation of injective embedding, for exemple, have begun to crystallize.

2. Mathematical Investigations of Strict Finitism

2.1 The Research Programme

Formal methods allow us to reconstruct historically emerging concepts of infinity. True to the principle of cognitive regression, this reconstruction takes us back from the historically later stages to ever earling stages. Consequently, the actual infinite is first reconstructed, in the form of axiomatic set theory, followed soon after, within the framework of constructivism, by potential infinite. The third stage then tackles *the reconstruction of the natural infinite.*

This third stage of cognitive regression has so far been implemented only hesitantly and in scattered instances. In the book of the author [Welti 1986], these instances are introduced in detail, conceptually coordinated and united in the common research programme of strict finitism. This research programme may be characterized by following four points:

(1) Formal reconstruction of phenomena and concepts of the natural infinite.

(2) Building thereon, the reconstruction of the concepts and methods of classical mathematics.

(3) Complete dispense with the axiom of infinity. All sets used should be finite. Yet, within the finite, a distinction can be made in a suitable way between 'small' and 'large'.

(4) Investigation to what extent sequences of theorems relating to the large finite can converge to infinitistic theorems.

The findings on the fourth point can be interpreted as indicating that infinitistic theories may be perceived as an approximative treatment on the large finite (or natural infinite). This approximation necessarily veils certain details which enter the picture only on finitistic approach. Infinitistic limit-theories or limit-models are less informative than those sequences of finitistic theories or models which converge towards them.

Concepts of infinity can therefore be adduced on evolutionary lines from finite cognitive activities, i.e. may be explained as the result of a limit process, not only in ontogenetic and historical development but even in the field of formal mathematics itself.

2.2. Examples and Results

Concerning technical details of the following mathematical theories, see the work of the author [Welti 1986] and literature there cited. The present article aspires to give an intuitive understanding of strict finitism and the phenomena of the natural infinite. The phenomena of the natural infinite fall essentially into two groups, namely (1) phenomena of open horizon and (2) phenomena of unreachability. We begin by *phenomena of open horizon.*

A sequence of many (say 100'000) equidistant trees of an avenue is phisically finite, but appears to our view as an infinite sequence. There is a first tree, and to each tree we see a successor tree. The iteration of an object, produced by two suitably arranged mirrors, is another phenomenon of open horizon. We see an endless repetition of the object, but physically the number of the visible images is bounded by the absorption of the light in the glass and by the resolving power of the eye. As a further example, P. Vopěnka has given the ape story. We consider the linearly ordered finite D such that the first element of D is an ape (say a ramapithecus) and the last element is Darwin.

Each non-first element is a son of the immediately preceding element of D. The subclass A of all apes belonging to D has a first element and is closed by successor operation, because sons of apes are apes. Therefore the subclass A, while physically finite, behaves like an infinite set of Cantorian set theory.

Historically famous examples are the heap paradox and the bald man paradox, both produced in antiquity by Stoic Eubulides. Let us regard the predicate G(x) meaning 'a man with x hairs on his head is bals'. The following three propositions are correct:

$$G(0),\ G(x) \rightarrow G(x+1),\ \neg G(10^6).$$

Of course, a man without any hair is bald. If a bald man gets an additional hair, he is still bald. A man with 10^6 hairs, however, is not bald.

The mentioned examples cannot be treated within the framework of classical arithmetic and set theory. In all these examples we have a situation in which the eyents of an infinite process are to be identified with the parts of a finite object. Following A.S. Yessenin-Volpin, we speak of Zenonian situations. Such situations illustrate how finite beings (e.g. homines sapientes) living in a finite world can catch the idea of the infinite: by the treatment of large, unsurveyable sets: The hypothesis of a transcendent realm of eternel ideas is superfluous. The concepts of the actual infinite and the potential infinite can be explained as approximative attempts to reconstruct scientifically such large sets. Strict finitism, as it were, strives for a refined theory of the infinite that remains close to the phenomena of the natural infinite.

A strict finitistic reconstruction of the above mentioned examples is realized e.g. by fuzzy set theory or by Vopěnka's alternative set theory. Still another approach is given by Parikh. His 'quasiconsistent arithmetics' are formal systems in which large numbers are treated as classically infinite numbers. All proofs up to some proof complexity ċ result in clasically correct sentences. A later epoch disposing of a greater proof complexity c' will unmask some of the infinite numbers at issue as finite, but on the other side will treat some very large numbers still as infinite.

We now turn to the second group of phenomena of the natural infinite, namely the *phenomena of unreachability*. Of central importance is their relatedness to the cognitive abilities of operating subjects. By improvement of concept formation or deductive competence, the hitherto unreachable may get indeed reachable. We illustrate this point by the following example that was given first by D. van Dantzig.

It is possible that always when a mathematician A uses the term 'transfinite number', a more competent mathematician B interprets it as 'finite number' without ever coming to an inconsistency. This will be the case if B disposes of a method of defining far larger numbers than A does. B can then construct some finite number (is his sense) surpassing by far all those which A can reach with his methods. If then A speaks of the transfinite numbers $\omega, \omega+1, \omega^2$ and so on, B interprets them as finite numbers n, n+1, n^2 and so on. The essential point comes now. Even the mathematician C who is the acting world champion of defining large numbers faces the possibility of loosening his championship. Therefore C will not surrender to the illusion that he gazes at the eyes of the actual infinite. More humbly, he will use the term 'transfinite' or transfinite symbols only in the strict finitistic sense of 'numbers surpassing everything I can ever obtain'. In fact, the new world champion D will unmask them as finite.

In Engeler's theory SFT, there is a whole sequence M_i of mathematicians disposing of ever greater competence. These mathematicians M_i perform thought experiments that are governed by fixed and communicable programs. The mathematicians M_i operate in a strict finitistic way: (1) The sets considered are always (hereditarily) finite and (2) each mathematician M_i has his frustration constant i, i.e. he thinks only during a restricted period of time, and has only restricted imagination. Surprisingly, the more intelligent M_i are able to come to a consensus about all axioms of Zermelo-Fraenkel set theory, including the axiom of infinity. The theory SFT explains why strict finitistic beings, operating of finite objects, can perceive so-called infinite sets and prove properties of them.

In this respect, Engeler's theory SFT formalizes some ontogenetic and historical developments: sets that originally are believed to be infinite are later known to be infinite. For examples, see author's book [Welti 1986, 1.4, 2.4, 6.4]. True to the principle of cognitive regression, the limit-theory of this development was first worked out (in classical set theory), and only now Engeler has reconstructed the sequence of finitistic theories converging towards the infinitistic limit-theory.

In a similar approach, Kaluza has shown that transfinite cardinal theory can be perceived as an approximative treatment of very large finite integers. For instance, the often mystified sentences like $\omega + 1 \sim \omega$ transform into sentences like $10^{12}+1 \sim 10^{12}$ used by scientists without awe. While there is only one relation of equivalence $\sim$ in the infinitistic theory, finitism has a whole sequence $\sim_i$ of them, opening a more subtle and differentiate proceeding. Exact conditions can be given under which a number x is to be identified with its sucessor x+1, with its double 2x, with its square x^2, and so on. When i and x are growing, more and more so-called transfinite sentences can be verified.

Calculus, too, can be handled in a strict finitistic way. Mycielski has given a formal theory FIN of analysis which has the important property that each finite subset of the axioms of FIN has a finite model. It is therefore always possible to work within a suitable finite model. If necessary, one cas pass over into a larger finite model by adding more and more axioms. Instead of the real line it suffices to take rational numbers whose denominators are bounded by some very large constant. We have no need to complete these rational numbers by Dedekind cuts or something else, because rationals with large denominators behave in FIN like classically irrationals. The supremum of a number-set is the maximum, the infimum is the minimum. Integrals are large (but finite) sums. FIN is strong enough to develop that part of calculus which is used or has potential applications in natural science.

Even more radical are the finite geometries FG_p that were studied in the fifties by Kustaanheimo. The prime p is the number of points on a line. Here one starts by finite Galois fields and then goes on to build a comprehensive, logically finite cosmology. By this cosmology, there exist only a finite numbers of objects and a finite number of relations between these objects.

Even space and time are strictly finitized. The spatiotemporal structure depends upon the largest prime p of the universe and certain properties. of primes. A finite geometry FG_p has a Euclidean kernel inside of which the Euclidean axioms hold. This Euclidean kernel models our everyday experiences. But if we pass over to very small (atomic) distances or very large (astronomical) distances, we leave the Euclidean kernel. Therefore the usual properties of order and congruence break down. The largest prime q of the Euclidean kernel that is quadratic residue modulo p can be deduced by quantum physical and astronomical observations and seems at present to be about 10^{80}. It is hence numerically identical with the nucleonic number, i.e. the number of nucleons in the universe. Arithmetical operations beyond the nucleonic number leave the Euclidean kernel, and hence do not conserve the usual order properties of classical arithmetic.

As we see from the discussed examples, strict finitism is able to reconstruct some important parts of classical mathmatics. Moreover, strict finitism performs everywhere more subtle and differentiate results than infinitistic mathematics. But this is only the beginning. Our hope is that man will wake up from his vain dreams of actual infinity and will henceforth cultivate his finite garden on earth.

BIBLIOGRAPHY

WELTI, E. (1986), *Die Philosophie des Strikten Finitismus.* Entwicklungstheoretische und mathematische Untersuchungen über Unendlichkeitsbegriffe in Ideengeschichte und heutiger Mathematik (Europäische Hochschulschriften, Reihe XX, Band 201). Bern, Frankfurt am Main, New York: Peter Lang Verlag.

Eidgenössische Technische Hochschule. Mathematik
Zürich. Schweiz.