EXHAUSTIVELY AXIOMATIZING RMO$_\rightarrow$ WITH AN APPROPIATE EXTENSION OF ANDERSON AND BELNAP'S "STRONG AND NATURAL LIST OF VALID ENTAILMENTS"

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ABSTRACT

RMO$_\rightarrow$ is the result of adding the 'mingle principle' (viz. $A \rightarrow (A \rightarrow A)$) to Anderson and Belnap's implicative logic of relevance R$_\rightarrow$. The aim of this paper is to provide all possible axiomatizations with independent axioms of RMO$_\rightarrow$ formulable with Anderson and Belnap's list extended with three characteristic minglish principles.

INTRODUCTION

Let R$_\rightarrow$ be the implication fragment of Anderson and Belnap's Logic of Relevance R. Then, RMO$_\rightarrow$ is the result of adding the axiom ('mingle') $A \rightarrow (A \rightarrow A)$ to R$_\rightarrow$ (see [1], §8.15). We have shown in [3] how to extend RMO$_\rightarrow$ with conjuction, disjunction and negation in order to define the logic of relevance RMO. This logic is suggested as an alternative to Anderson and Belnap's R; therefore, RMO$_\rightarrow$ has to be understood as an alternative to the pure logic of relevance R$_\rightarrow$. These results contradict Anderson and Belnap's conclusive remark: "*relevance and mingle are incompatible when truth-functions are added*"([1], p. 97).

We have shown in [2] how to "exhaustively" axiomatize R$_\rightarrow$ with

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Anderson and Belnap's *"strong and natural list of valid entailments"* ([1], p. 26). The aim of this paper is to exhaustively axiomatize $RMO_{\rightarrow}$ relative to an extension of this list with three characteristic minglish principles.

1. EXTENDING ANDERSON AND BELNAP'S LIST

Anderson and Belnap's list is the following

1. $A \rightarrow A$
2. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
3. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
4. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
5. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
6. $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$
7. $(D \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow (D \rightarrow C)))$
8. $(C \rightarrow D) \rightarrow ((A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow D)))$
9. $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow ((D \rightarrow B) \rightarrow (D \rightarrow C)))$
10. $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow ((C \rightarrow D) \rightarrow (B \rightarrow D)))$
11. $(A \rightarrow ((B \rightarrow C) \rightarrow D)) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow D))$
12. $(B \rightarrow C) \rightarrow ((A \rightarrow ((B \rightarrow C) \rightarrow D)) \rightarrow (A \rightarrow D))$
13. $(A \rightarrow B) \rightarrow ((A \rightarrow B) \rightarrow C) \rightarrow C)$
14. $(((A \rightarrow A) \rightarrow B) \rightarrow B)$

For R we add

15. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
16. $B \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
17. $A \rightarrow ((A \rightarrow B) \rightarrow B)$

and for RMO

18. $A \rightarrow (A \rightarrow A)$
19. $(A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))$
20. $(A \rightarrow B) \rightarrow (B \rightarrow (A \rightarrow B))$

We note that **18**, **19** and **20** are equivalent in the presence of **1**, **2**, and **3**.

2. MATRICES

We provide five matrices to be used in the independence proofs of section **3.** We record the theses in the list **falsified** by each matrix (theses omitted are verified). Designated values are starred.

MATRIX I. Falsifies **1**, **14**, and **18**

$\rightarrow$	0	1	2
0	2	2	2
1	0	0	2
*2	0	0	2

MATRIX II. Falsifies **15**, **16**, and **17**

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
*2	0	0	2

MATRIX III. Falsifies **4**, **5**, and **6**

$\rightarrow$	0	1	2
0	2	2	2
1	1	2	2
*2	0	1	2

MATRIX IV. Falsifies **2**, **3**, **7**, **8**, **9**, and **10**

→	0	1	2	3	4
0	3	3	3	3	3
1	0	4	2	3	4
*2	0	1	2	3	4
*3	0	0	2	3	0
*4	0	1	2	3	4

MATRIX V. Falsifies **18**, **19**, and **20**

→	0	1	2	3
0	3	3	3	3
1	0	2	0	3
*2	0	1	2	3
*3	0	0	0	3

3. EXHAUSTIVELY AXIOMATIZING RMO$_{\rightarrow}$

We prove

THEOREM. RMO$_{\rightarrow}$ may be axiomatized (with modus ponens) using any selection that includes one (and only one) thesis from each of the groups in (a) and (b) below:

(a) {**1**, **14**}, {**2**, **3**, **7**, **8**, **9**, **10**}, {**4**, **5**, **6**}, {**15**, **16**, **17**}, {**19**, **20**}.

(b) {**18**}, {**2**, **3**, **7**, **8**, **9**, **10**}, {**4**, **5**, **6**}, {**15**, **16**, **17**}.

The 270 resulting selections are the only axiomatizations of RMO$_{\rightarrow}$ with independent axioms formulable with the twenty theses in the list.

To prove that any selection in the Theorem is an axiomatization of RMO$_{\rightarrow}$, the reader may use the following

LEMMA (i) Each one of the 108 selections in {**1**, **14**}, {**2**, **3**, **7**, **8**, **9**, **10**}, {**4**, **5**, **6**}, {**15**, **16**, **17**} is an axiomatization of R$_{\rightarrow}$.

(ii) **1** is derivable from **4** and **18.**

(iii) **1** is derivable from **5**, **18** and **17** (**15**, **16**).

(iv) **1** is derivable from **6**, **18** and **17** (**15**, **16**).

Proof of the Lemma. (i) : Cfr. [2]; (ii), (iii) and (iv) are left to the reader.

By (i) of the Lemma, it is obvious that any selection in (a) is an axiomatization of RMO$_{\rightarrow}$; by (ii), (iii), (iv) and (a) it is evident that the same holds for (b).

To prove that all selections in (a) and (b) have independent axioms, use the matrix in section **2.** as follows:

- Independence of **1**, **14** and **18** : MATRIX I.
- Independence of **2**, **3**, **7**, **8**, **9**, and **10** : MATRIX IV.
- Independence of **4**, **5**, and **6** : MATRIX III.
- Independence of **15**, **16** and **17** : MATRIX II.
- Independence of **19** and **20** : MATRIX V.

Finally, it suffices to inspect the matrices in § 2 to see that all axiomatizations (with independent axioms) of RMO$_{\rightarrow}$ formulable with **1-20** are exactly those of (a) and (b) in the above Theorem (Note that **11**, **12** and **13**, restricted versions of **15**, **16** and **17**, have been not employed).

REFERENCES

1 ANDERSON, ALAN R. and BELNAP, NUEL D. Jr. **Entailment,** vol. 1, Princeton University Press, Princeton, N. J. , 1975.

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