NOTES ON TYPES, SETS AND LOGICISM, 1930-1950[1]

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ABSTRACT: The present paper is a contribution to the history of logic and its philosophy toward the mid-20th century. It examines the interplay between logic, type theory and set theory during the 1930s and 40s, before the reign of first-order logic, and the closely connected issue of the fate of logicism. After a brief presentation of the emergence of logicism, set theory, and type theory (with particular attention to Carnap and Tarski), Quine's work is our central concern, since he was seemingly the most outstanding logicist around 1940, though he would shortly abandon that viewpoint and promote first-order logic as all of logic. Quine's class-theoretic systems NF and ML, and his farewell to logicism, are examined. The last section attempts to summarize the motives why set theory was preferred to other systems, and first-order logic won its position as the paradigm logic system after the great War.

Keywords: Mathematical logic, logicism, set theory, simple type theory, first-order logic, NF, ML. Philosophy of logic, Platonism, constructivism, nominalism, 'bankruptcy theory', paradoxes, analyticity, existential assumptions, Gödel's incompleteness theorem. Ramsey, Carnap, Tarski, Gödel, Church, Quine, Rosser, Wang.

> (...) to say that mathematics is logic is merely to replace one undefined term by another. When we go back of the word 'logic' to its definition in the logistic systems, we find that they run the gamut from extreme Platonism to pure formalism. (Haskell B. Curry, 1939)

Contrary to the customary assumption of historians until 15 years ago, it is now well known that first-order logic evolved from the higher-order calculuses of Frege, Peano and Russell, and that it took a long process for logicians to focus on first-order, even as an important subsystem (van Heijenoort 1967) (Goldfarb 1979) (Moore 1980, 1988) (Shapiro 1991, chap. 7). The first time that a language with first-order variables only was studied can be found in the work of Löwenheim, in 1915;[2] starting in 1922, Skolem would claim that first-order logic is all there is. During the 1920s and 30s the ground was laid for the subsequent concentration on first-order systems, but it is frequently forgotten that the theory of types was still the paradigm logical system during the 1930s. Several examples will be given below, but we may mention at this point that Hilbert's 'erweiterter Funktionenkalkül' (expanded functional calculus; Hilbert & Ackermann 1928), although peculiar, was essentially a form of type theory.[3] The theory of types reigned from the 1920s, and it took one more chapter in the history of logic to evolve to the now customary distribution of tasks between first-order logic and set theory. Thus, the interplay between logic, type theory and set theory may be seen as one of the main motives behind the evolution of mathematical logic from 1930 to 1950.

Many different issues coalesced around the question of the right logic system. It seems to this writer that the evolution of logic up to 1950 cannot be properly understood unless we pay attention to issues in the philosophy of mathematics, since this was by then the main reference point for logic. To use Shapiro's terminology (1991, 25), the 1920s and 30s were the heyday of *foundationalism*, the view that it is highly desirable, and *must* be possible, to reconstruct the whole of mathematics on a completely secure basis, "one that is maximally immune to rational doubt". A single symbolic system providing a framework for all of mathematics was being sought (see Church 1939). The threats of the paradoxes for logic and set theory, and of intuitionism for classical mathematics generally, were felt as powerful motives for such an enterprise. The desire for intellectual security commended recourse to perfectly specifiable, controlable, and well-behaved formal systems. First-order logic increasingly became the indispensable basic system after 1940, but it is a misconception, too frequent among philosophers, that it became *all* of logic; it could be argued that it has never been.

The motives I have sketched merged in many ways with philosophical questions concerning the nature of logic, of its relations to mathematics, and of mathematical knowledge itself. After 1930, attempts were made to reconcile the famous triad of opposing foundational views, logicism, intuitionism and formalism, to locate points of agreement and to overcome the divisions. Among other things, this implied the need to clarify the relations between logic and mathematics. As is well known, logicists believed that logic is *the* foundation of mathematics, while intuitionists held the converse, and formalists tried a middle course, according to the conception that mathematics and logic are two different fields, neither is reducible to the other, but both need each other for a rigorous formal development (see, e.g., Benacerraf & Putnam 1983). Thus, the panorama in the philosophy of mathematics demanded the establishing of clear borderlines between logic and mathematics, a need that philosophical minds must have felt strongly.

In the present paper particular attention will be paid to the classical alternative of logicism, still widely influential during the 1930s, but soon to lose its importance. We will consider the views of some key figures that can be related to logicism and its fate, especially the young Willard Van

Quine, though also Rudolf Carnap and other logicians such as Alonzo Church, Alfred Tarski and Kurt Gödel. An exploration of the work of these logicians will help clarify the way in which logicism was slowly abandoned, and will shed some light on how the theory of types (hereafter TT) came to be replaced by the combination of first-order logic and Zermelo-Fraenkel set theory (ZFC). In what follows, we assume that the reader has a basic knowledge of ZFC.

1. Logicism, types, and sets

1.1. Origins of logicism

In brief outline, it can be said that logicism emerged as an important viewpoint as a consequence of the success obtained, during the 19th century, in rigorizing mathematics by means of so-called *arithmetization*.[4] Generally speaking, and disregarding the case of Kronecker, arithmetization meant reduction of all notions of classical mathematics, particularly analysis, to natural numbers and sets. A key example is the notion of continuity, crucial for real analysis. It is well known that Cauchy and Bolzano defined the continuity of a function in arithmetic terms, but they were unable to establish full proofs of the most basic theorems of analysis. This would only be possible after the continuity of the real number system had been clarified. Thus it was that around 1870 a number of authors, including Mèray, Weierstrass, Cantor and Dedekind, came to propose 'definitions' of the real numbers in terms of series or sequences of rationals, or cuts in the rationals. Such definitions clarified the notion of real number and the continuity of the set $\mathbb{R}$, making it possible to prove theorems such as the existence of a limit to each monotonically increasing and bounded sequence of real numbers (see, e.g., Dedekind 1872). Thus, the most important remnant of geometrical thought within analysis, the notion of continuity, was satisfactorily arithmetized, in the above sense.

Once the real numbers had been reduced to the rationals, it was easy to see that all other traditional kinds of numbers could be similarly reduced to natural numbers via sets. The most difficult step was reducing the natural numbers themselves to sets or equivalent 'devices'; this was done, independently, by Frege (1884; 1893/1903) and Dedekind (1888). Since traditional conceptions made it natural to think of the notion of set as an integral part of logic (see Ferreirós 1996), this meant that the program of arithmetization had been taken one step further, and the notions of arithmetic and analysis had been reduced to logical notions. Logicism was thus born.[5]

As important as noting the essential coincidences between Gottlob Frege (1848-1925) and Richard Dedekind (1831-1916), is remembering the important differences between them. The latter was only interested in an informal axiomatic presentation of the theory of sets and mappings, which he regarded as a part of logic and established as the foundation of arithmetic, algebra and analysis. The former was mostly interested in the strict formalization of all linguistic devices employed in mathematical deduction, and in using it for a stringent definition of the natural and real numbers. Frege started the mature phase of logicism, in which the use of the symbolic or logistic method was crucial. This led to formal systems of propositional and higher-order predicate logic.

A second phase in the history of logicism was precipitated by the discovery of the paradoxes, notably Russell's paradox, which was particularly striking because of its formulation in terms of the most basic 'logical' notions-class or set (property), negation, and membership (predication). Frege, having heavily relied on the free definition of sets by means of concepts or propositional functions (presupposed in his Basic Law V), quickly came to the conclusion that logicism was untenable. Dedekind had had a similar confidence in the comprehension principle, but an abstract and purely extensional presentation of set theory made it almost invisible in his (1888). Nevertheless, his whole construction depended upon the famous (and, by the way, frequently misunderstood) theorem establishing the existence of infinite sets on the basis of a universal domain. Thus his theory was also shattered.

Two alternative courses were taken in response to this situation, leading to type theory (TT) and set theory (ZFC). Ernst Zermelo (1871-1953) was not so worried about the paradoxes as about constructivist criticism of his proof of the well-ordering theorem, based on the axiom of choice (Moore 1982). He decided to take "set theory as it is historically given" and build an informal axiomatic foundation for it (Zermelo 1908), just as Hilbert had built an axiom system for geometry. He was following Cantor in taking set theory as a branch of mathematics, but under the influence of Dedekind and others he kept saying that it deals with "the logical foundations of all of arithmetic and analysis" (Zermelo 1908, 200). Zermelo presented seven axioms: extensionality, elementary sets, separation (his answer to the paradoxes), power set, union, choice, and infinity.

Meanwhile, Bertrand Russell (1872-1970) tried to save as much as possible of the Fregean approach: he clung to the principle of comprehension but restricted membership by the strict conditions of type theory. Thus, any "propositional function" (open sentence, predicate) determines a class, but its elements must always belong to a fixed type-a set of type n may only have objects of type $n-1$ as its elements, type 0 being a collection of individuals, type 1 a collection of predicates (classes) of individuals, etc. (Russell 1908). Russell was trying to keep Frege's logicist ideal alive, and after the First World War the logicist program was known mainly in the classic version of Whitehead and Russell's *Principia Mathematica* (1910-13).[6] The horns of the dilemma-higher-order logic vs. branch of mathematics, TT vs. ZFC-had been stated, and controversy followed.

1.2. Developments in the twenties

The controversy was further complicated by the contributions of Brouwer and Poincaré. The latter opposed logicism and the new logic by criticizing impredicative definitions, and tried to show that mathematical induction has to be taken as the quintessential, primitive mathematical notion, as a synthetic *a priori* truth (see Goldfarb 1988). Brouwer started an assault on classical mathematical thinking, based upon his intuitionist philosophy and the search for constructive proofs. Other authors had already criticized the axiom of choice because of its purely existencial, i.e., strictly non-constructive, character (see Moore 1982). The combined effect of all these controversies was a long debate, which reached its high point in the 1920s, and a generalized feeling of insecurity among authors interested in logic and the foundations of mathematics. By this time, the question of the paradoxes had attained mythical dimensions, and it continued to be felt as pressing up to 1940, promulgating a suspicious attitude toward the notion of set and all its derivatives or relatives.

The attack of the intuitionists and constructivists was not successful in obtaining many converts, but it was successful in affecting the way in which logic and the foundations of mathematics were studied. Attention to constructive procedures increased greatly, and most logicians working in the 1930s preferred them. After all, the goal of pursuing the symbolic or logistic method, i.e., establishing a formal deductive system that might work mechanically, was the basic idea behind mathematical logic. In the late 1920s, David Hilbert (1862-1943) became the most influential logician; the way in which he studied logic was framed by his goal of a finitistic metamathematical proof of the consistency of formal systems codifying mathematical theories. It is not coincidental that almost all of the great contributions of the 1930s had a strong constructivist flavor, the main exception being Tarski's seminal contributions to metamathematics and semantics (Tarski 1956).

As has been said, ZFC and TT were taken to be the main alternatives for the reform of logical theory called forth by the paradoxes (see e.g. Gödel 1931, 144-145; Tarski 1986, vol. 1, 236 and passim; Church 1939, 69-70; Carnap 1963, 33). Both systems received substantial improvements during the 1920s. Fraenkel and Skolem introduced the axiom of replacement in 1922, and both proposed modifications of Zermelo's axiom of separation; the most successful was Skolem's proposal to formulate it within the language of first-order set theory (Skolem 1922). Indeed, Thoralf Skolem (1887-1963) was the main defender of first-order logic as all of logic; he showed the paradoxical consequences of the Löwenheim-Skolem theorem as applied to first-order set theory (the so-called Skolem paradox). Von Neumann also presented several improvements and an alternative axiom system, based on functions taken as primitives, in 1925. One further important step was Zermelo's (1930) proposal of the axiom of foundation, which had the effect of splitting the domain of sets into levels-the cumulative hierarchy; this was the first presentation of the intuitive semantical picture underlying set theory (frequently called the iterative conception of sets).[7]

The theory of types also underwent substantial improvement, this time simplification. Russell's system avoided so-called impredicative definitions, which he-like Poincaré-regarded as responsible for the paradoxes. A class not only had a characteristic type, depending on that of its elements, but also a characteristic *order*, which depended on the complexity of the expression that defined the class. This yielded an extremely complex version of TT, that came to be called *ramified* type theory.[8] The introduction of orders (i.e., the requirement of predicativity) brought unbearable complications to the theoretical development, so Whitehead and Russell came to introduce the infamous *axiom of reducibility*. It would have been easier to simply forget about orders and predicativity, since this axiom in fact amounted to the un-ramification of the theory.

Frank Ramsey (1903-1930) found arguments in favor of impredicative definitions and introduced the simple theory of types, which avoided all reference to orders and to "the blemish" of the axiom of reducibility. First, he presented the famous distinction between "logical antinomies", the ones that make crucial use of the notion of class (such as Russell's or Burali-Forti's paradoxes), and those "antinomies" which depend upon peculiar linguistic or "psychological" devices, particularly the notion of meaning (such as Richard's paradox, etc.) (Ramsey 1926, 171-172). It was only the logical ones that affected a formal development of the theory of classes, since within a formally specified language it is impossible to formulate the semantic antinomies. This eliminated many of Russell's arguments against impredicativity.

Second, Ramsey argued that impredicative definitions are not meant to define objects in the sense of *creating* them, but only to single out some objects. Thus, they are perfectly admissible, at the price of accepting that existential and universal quantifiers involve reference to classes and relations that are not explicitly definable within the formal language (1926, 173-174, 188). (Although Carnap (1931, 50) criticized this argument as based upon an unacceptable form of platonism, it seems to work well in the context of a Zermelo-style theory that secures the existence of sets by means of further axioms.) Finally, Ramsey observed that in order to attain a reduction of mathematics to logic, it was necessary to make a radical move from Russell's propositional functions to a logic based on "propositional functions in extension", i.e., extensional classes (Ramsey 1926, 201-203). This yielded a version of TT strongly reminiscent of set theory.[9]

The wide influence exerted by *Principia Mathematica* was one of the reasons why higher-order logic, in the form of TT (*viz* ω-order logic), served as the main logic system in the 1930s. Also important is the fact that, before the crucial contributions of Zermelo (1930) and Gödel (1940), ZFC was still perceived by many as an artificial system, more dangerous than TT. Being more restrictive, TT conveyed a feeling of security against the antinomies, and it was more in line with a logical conception of classes as dependent upon concepts or propositional functions (the idea that Frege and Russell relied on). Thus, simple TT became the preferred system for European logicians. It can be found in the crucial contributions of Gödel (1931) and Tarski (1936) (written in 1931), in a compact formulation that externally is strongly reminiscent of today's first-order logic. Quine would talk about the "neo-classical or Tarski-Gödel theory of types".

Let us take Kurt Gödel's (1906-1978) version of simple TT and 'quine' it, which means that we render '$u(v)$' as '$v \in u$' (compare Gödel 1931, 150-155). The set of logical constants includes '$\neg$', '$\vee$', '$\forall$', '$\in$'. We have indexed variables 'x_i', 'y_i', 'z_i',... for all natural numbers i, where i is the type of the objects referred to by the variable. Elementary formulas are of the form '$x_i \in x_{i+1}$', and we have the usual definitions for '$\neg(\alpha)$', '$(\alpha \vee \beta)$', '$\forall x(\alpha)$'. Then come four axioms for propositional logic and two for quantification, which Gödel formulated using schematic letters. And finally we find the TT axioms, the *axiom of comprehension*, any instance of

$\exists u_{n+1} \forall v_n (v_n \in u_{n+1} \leftrightarrow \alpha)$ where 'u_{n+1}' is not free in the arbitrary formula 'α';

and the *axiom of extensionality*, any instance of

$\forall x_{i-1} [(x_{i-1} \in x_i \leftrightarrow x_{i-1} \in y_i) \rightarrow x_i = y_i)]$ for any natural number i.

The system is formally very simple. It keeps the axiom of comprehension, which was responsible for the appearance of paradoxes in Frege's system, but these are avoided because of the type restrictions that we have imposed, by allowing only formulas of the form '$x_i \in x_{i+1}$' (a class of type i+1 can only have classes of type i as its elements).

1.3. Logicism in the thirties

So long as *Principia Mathematica* was the main reference work in the field of logic, logicism was quite widespread. Logicians of the 1930s, most of whom learnt their discipline through that work, give plenty of evidence for it. Rudolf Carnap (1891-1970) was the most important logicist in the early 1930s. In 1929 he published his *Abriss der Logistik* (Carnap 1929) as the first volume in a collection of writings supporting the conceptions of the Vienna Circle. The book summarized the logical system of *Principia Mathematica*, simplified according to Ramsey's proposal of simple TT, and showed how such system was useful for diverse applications in the sciences; among them was, of course, the reduction of mathematics to pure logic.

By this time, Carnap had already experienced the strong influence of Ludwig Wittgenstein's *Tractatus*, which had also been decisive for Ramsey (see his 1926, 152-156). The work of Frege had prepared Carnap for the idea that "knowledge in mathematics is analytic in the general sense that it has essentially the same nature as knowledge in logic", and Wittgenstein made this idea "more radical and precise" (Carnap 1963, 12, 25; see Wittgenstein 1921, 6.1, 6.11). Thus, logic and mathematics are a mass of tautologies, say nothing about the world, have no factual content. This doctrine fit the positivist credo of the Circle, and complemented it nicely by giving a more satisfactory account of mathematics than Mill's thoroughgoing empiricism. The ill-explained analiticity of logic and mathematics became Carnap's main dogma, a fixed point in all his future philosophical contributions. It gave logicism a very clear epistemological import (see section 3).

In 1931, Carnap organized a series of talks and a roundtable discussion on the foundations of mathematics (see Dawson 1984), with contributions by Heyting on intuitionism, and von Neumann on formalism; Carnap defended the logicist viewpoint (Carnap 1931). As in his *Abriss*, Carnap proposed a type-theoretic construction of arithmetic, set theory and higher mathematics. He tried to find common ground with formalism, and analyzed the main problems facing logicism-that of justifying the logical character of the axioms of infinity and choice (Carnap 1931, 44-45), and the problem of predicativity. Three years later, in his classic *Logische Syntax der Sprache* (translated as Carnap 1937), he was able to propose a solution to all these problems.

According to the 1931 paper, "the *most difficult problem* confronting contemporary studies in the foundations of mathematics" was how to develop logic in such a way that a definition of the real numbers becomes possible, while at the same time impredicative definitions are avoided (Carnap 1931, 49). Carnap's concern with impredicative definitions was in line with Russell's work, but also shows the strong effects of constructivism upon him. Carnap rejected Ramsey's solution, appealing to Frege's *dictum* that, in mathematics, only that may be taken to exist whose existence has been proved in finitely many steps. He said he agreed with intuitionists that "the finiteness of every logico-mathematical operation, proof and definition (...) is required by the very nature of the subject" (Carnap 1931, 50).[10] But his attempted solution (along the lines of interpreting the universal quantifier not extensionally, but hypothetically) was not satisfactory. For in such a case, we would only have the properties that can be explicitly defined within our language; this gives denumerably many properties, so we will be unable to define the real numbers. In 1931, Carnap was finding serious difficulties in attempting to escape from constructivism.

Logische Syntax der Sprache was his way out of those problems, and of the new one posed by Gödel's incompleteness theorem. Rejecting Wittgenstein's stern distinction between saying and showing,[11] *Logische Syntax* was an important contribution to metamathematics-which Carnap began to call 'syntax' (see Carnap 1937, 153-275). At this point in his career, he was convinced that all of logic, the foundations of mathematics and science, and philosophy, could and had to be expressed in syntactical form, as the logical syntax of languages.[12]

Carnap's main new argument was couched in terms of the famous Principle of Tolerance. According to him, any language (formal system) deserves study for its own sake, and can be employed whenever it is convenient for the purpose at hand. Choices of language are a simple matter of convenience, and imply no further presupposition, in particular no extra-linguistic (e.g., ontological) assumption:

> *It is not our business to set up prohibitions, but to arrive at conventions. (...) In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. (Carnap 1937, 51-52).

As Sarkar has remarked, this was "Carnap's response to the foundational disputes: by tolerance and convention they are defined out of existence" (Sarkar 1992, 195).

Carnap exemplified his Principle of Tolerance by studying two formal systems. For the first time he considered a form of first-order logic, which he had earlier associated with constructivism, probably having Skolem in mind (see Carnap 1931, 52). It was his "language I", which included primitive recursive arithmetic, and thus was apt to define its own formal syntax by means of Gödel's procedure of arithmetization (see Gödel 1931). He also considered "language II", a form of type theory built on top of language I, suited to the needs of classical mathematics and set theory (see Sarkar 1992). Since the Principle of Tolerance is in action, language II does not commit us to platonism, but is simply a linguistic convention. Such was Carnap's solution.

The Principle of Tolerance was perfectly coherent with Carnap's faith in the purely analytic character of logic, mathematics, and all linguistic conventions, and we may grant that something of its kind seems sensible as a pragmatic rule for logicians. But it is difficult to take it seriously as the basis of a philosophy of mathematics. Once more, it is hard to see how it could be reconciled with anything other than constructivism. If the formal system has to pose no presupposition, it ought to be a mere matter of playing with symbols, and of course we ought not accept a platonistic system (language II) that works as if unnamable sets, or unnamable numbers, exist. As regards logicism, with Carnap's Principle it seems to reduce to the void thesis that logic can be conventionally defined as a system strong enough to deduce mathematics. As Sarkar writes, "The trouble with that principle is that it almost seems to make a mockery out of the thesis of logicism" (Sarkar 1992, 226). Although the unsatisfactory character of Carnap's position was repeatedly emphasized by Quine and Tarski (see below), Carnap remained immovable in his absolute faith in the analytic character of logic, set theory, mathematics, and everything linguistic. He was an 'analytical' logicist to the end.

One even wonders whether the Principle of Tolerance did not help promoting a shift away from logicism. One of the earliest expressions of such a shift can be found in the -then unpublished- paper by Haskell B. Curry (1900-1982) that was quoted at the top of this article (see above). Given the intrinsic vagueness and elusiveness of logicism, Curry was led to remark that "we do not have here a third view of mathematics parallel with" intuitionism and formalism (Curry 1939, 206). The scope of logic, and its epistemological interpretation, were seen by Curry essentially as a matter of convention.

Another important logician that was attracted by logicism was Alonzo Church (1903-1995). Without entering into a detailed discussion of his views, which he seldom presented explicitly, it can be said that he seems to have been a defender of 'weak logicism' all the while. He was a consistent defender of higher-order systems, but also used to emphasize the role of the axiom of infinity as a kind of frontier between logic and mathematics. Thus, for instance, in a review of Carnap's *Foundations of Logic and Mathematics*, he criticized "an oversimplification of the relation between logic and arithmetic, partly due to the failure to make explicit mention of the axiom of infinity, (...) an unfortunately misleading impression is given".[13]

A more surprising example is that of Alfred Tarski (1901-1983) himself, who presented logicist ideas in his 1941 elementary textbook *Introduction to Logic and the Methodology of Deductive Sciences*. Interestingly enough, the book devotes almost no space to quantification theory, concentrating instead on the propositional calculus, the theory of identity, and the theory of classes and relations. This is consistent with the goal

> to show that the concepts of logic permeate the whole of mathematics, that they comprehend all specifically mathematical concepts as special cases, and that logical laws are constantly applied -be it consciously or unconsciously- in mathematical reasonings (Tarski 1941, preface to the original 1936 ed., p. xvii-xviii).

Particularly relevant is section 26 on cardinal numbers, finite and infinite classes, and "arithmetic as a part of logic" (ibid. 79). After presenting the Frege-Russell definition of cardinal numbers as classes of equipollent classes, Tarski claims it has been shown that "the notion of number itself, and likewise all other arithmetical concepts are definable within the field of logic" -"a most interesting consequence of far-reaching importance" (ibid., 81; see 130). Slightly below, in reference to the works of Frege and Whitehead & Russell, he stresses that it has been possible to develop all of arithmetic, "algebra, analysis, and so on", as parts of pure logic.

These comments are only qualified by the need to accept the axiom of infinity, "a statement which is intuitively less evident than the others" (Tarski 1941, 81), in order to derive all theorems of arithmetic, and by "a certain logical difficulty" in connection with the definition of finite and infinite classes (ibid., sec. 33, p. 105; see 80). (Tarski is referring to the need for the axiom of choice in order to show that the set-theoretic definition is equivalent to the traditional one, a topic that had greatly interested him (see Tarski 1986, vol.1).)

It thus seems that Tarski was, at least verbally, a logicist as of 1941. This is understandable if we consider that, until 1935, he used TT as the basic logic system, as can be seen in his famous book *Logic, Semantics, Metamathematics*, containing papers from the 1920s and 30s.[14] Particularly relevant in this connection is the famous paper on the 'Wahrheitsbegriff', which gives a precise statement of simple TT based on axioms of comprehension (called "pseudo-definitions", following Leśniewski), extension and infinity (Tarski 1956, 241-243).[15] A simplified version of the theory of classes, based on a distinction of levels ("orders") which is explicitly acknowledged as akin to Russell's logical types, can be found in Tarski's *Introduction to Logic* (1941, secs. 21 & 23, p. 68, 73-74). Later on, however, he moved from TT to first-order ZFC, as is recorded in the 1935 *Postscript* to the German edition of the paper on truth (see 1956, 271n). Thus, even before the Second World War Tarski had overcome his previous misgivings about first-order ZFC as a convenient system of logic.

But one should also note that in (1941, sec. 36, p. 120) he advances the view that the deductive method is the only essential distinguishing feature of mathematics, and mentions that, according to this, "deductive logic" is one of the mathematical disciplines (see also ibid., xvii on the origins of mathematical logic). This shows that logicism may be an elusive position -Tarski's logicism may not have meant a reduction of mathematics to something else, but its reduction to set theory. In a footnote within the new bibliographic guide appended to *Introduction to Logic* for its 1964 edition, Tarski qualified his earlier statements accordingly: they had been made under the assumption that logic is not restricted to its elementary parts, the propositional calculus and the theory of quantifiers, but embraces also the theory of classes and relations. Now he stated the idea in an equivalent form: arithmetic, analysis and algebra can be developed within set theory, or within an expanded logic system including classes and relations.

The elusiveness of logicism becomes even clearer when we consider Tarski's disagreement with the positivists concerning the issue of analyticity. As early as 1934, at the International Congress of Philosophy, Tarski criticized the use of the term "tautological" among logical positivists, recommending that it be restricted to sentences of the propositional calculus. He denied rules such as *modus ponens* any tautological character, since "either (they are) propositions about our action (Verfahren) or parts of a syntactic definition of the concept of consequence" (1986, vol. 4, 694). Two years later he made public his disagreement with the conception that logic and mathematics are tautological; the concept 'analytical', meaning logically true, was regarded as relative to a division of terms into logical and extra-logical (Tarski 1956, 419-420). As late as 1958 he would still remark that the vagueness in the demarcation of logical symbols admits for treating '$\in$' as one of them, which of course would shortcut Skolem's paradox (see Tarski 1986, vol. 4, 723).

Even after most mathematicians and logicians had abandoned the logicist viewpoint, it persisted as a popular account of mathematics among philosophers. Logical positivists were strong adherents to logicism, which was the viewpoint of some of their main inspirations: Frege, Russell, Carnap, and -less clearly- the early Wittgenstein. Thus, as late as 1945 we find Carl G. Hempel publishing an elementary exposition of the viewpoint in the *American Mathematical Monthly*.[16] With the fall of positivism, we can find its former dauphins giving arguments against logicism, as is the case of Hilary Putnam in 'The Thesis that Mathematics Is Logic' (Putnam 1967), ironically a paper published in a volume commemorating Russell as "Philosopher of the Century".

2. Quine's class-theoretic systems

In some ways, Willard Van O. Quine (1908-) may be regarded as a second Russell. His early concentration upon logic and the foundations of mathematics, and subsequent devotion to more philosophical topics, remind one of the author of *Principia Mathematica*. Moreover, Quine's systems NF ('New Foundations' (1937)) and ML (*Mathematical Logic* (1940)) are important developments of Russell's line of thought in taking classes as determined by open sentences. I propose that both systems could be called 'class-theoretic', since on the one hand they are very far from a Zermelo-style theory based on some iterative notion of "set of" (as Gödel (1947) said), and on the other they dispense with any kind of type condition. Therefore, they come closer than any other system to rescuing as much as possible from the old, naive idea of class -in particular, the universal class is retained. Furthermore, Quine has always shown a preference for the word "class" over "set".

2.1. Early work and viewpoints

Quine started his career with a dissertation on "The Logic of Sequences: A Generalization of *Principia Mathematica*", defended in 1932, which summarized a great part of the huge *Principia*.[17] Quine's generalization was based on an ingenious set of new primitive operations, allowing for simultaneous treatment of classes and n-ary relations, but, according to the author, "the touted generalization is unimpressive":

> What is rather to be commended in the dissertation is its cleaning up of *Principia*. It resolved confusions of use and mention of expressions. Propositional functions, in the sense of attributes, ceased to be confused either with open sentences or with names of attributes. One effect of sorting these matters out was the recognition of *predication* as a primitive operation of the system. Another effect was justification of Ramsey's proposal that ramified types and the axiom of reducibility be skipped. (see Quine 1941) (...) My system was extensional. Predication was membership. (Quine 1986, 10)

Extensional preferences were obscured by the fact that Quine was still talking about propositional functions, not about classes. However, the extensional viewpoint was to become one of the characteristics of Quine's philosophy of logic. He argued that extensional classes enjoy clear identity conditions, while co-extensional attributes may still be regarded as different.[18]

Quine's first system was rather complex in several ways. It employed six basic operations, including "an ugly binary operation called superplexion" (Quine 1986, 11), which was really complex: it served to define inclusion, the conditional, negation and quantification. Dreben writes: "Thus even at the very beginning of his career, Quine's drive for system building, for formal simplicity and elegance holds sway" (Dreben 1990, 87). We may doubt whether, in this particular case, we really find formal simplicity and elegance, but it is certainly characteristic of Quine to search endlessly for economy and simplicity of the system by means of ingenious, or even tricky, notational devices. As Quine humbly acknowledges, by that time Tarki and Gödel were publishing their "now classical thumbnail formulation" of TT, based simply upon truth functions, quantification, and membership (or predication; see above).

It took Quine several more years to "settle on a sanest comprehensive system of logic -or, as I would now say, logic and set theory" (Quine 1986, 17). But before we talk about his later contributions, it will be worthwhile to discuss several traits of Quine's position, that were already clear by the mid-thirties. We have already seen his rejection of Russell's propositional functions on behalf of extensional classes, which would become a crucial part of his philosophy of logic. Related to this was a preference for set theories over second-order logic, which he would regard -in one of his famous phrases- as "set theory in disguise" or "set theory in sheep's clothing". His earliest argument against higher-order logic, as used in *Principia*, was that it amounts to a leap from quantification theory to a (platonistic) theory of attributes (Quine 1941, 145). Of course, we may interpret predicate and relation symbols as referring to sets, but then what we have is simply set theory -whether we write $\phi \in \psi$ or $\psi(\phi)$ is immaterial, simply a notational variant. Thus, Quine's rejection of second-order logic is a consequence of his theory that quantified variables refer ("to be is to be the value of a variable"), a doctrine that emerged during the 1930s (see Quine 1986, 13, 14, 19).[19]

A third, most important point is what has been called the "bankruptcy theory". Quine has always been convinced that the paradoxes showed commonsense logic (or the commonsense notion of class) to be contradictory (Quine 1940, §29, 166). Since common sense is bankrupt, no way out is 'natural' -all set theories are equally artificial, and there is no intuitive reason to prefer one of them. Quine has thus opposed the viewpoint that became customary after World War II, namely that Zermelo's cumulative hierarchy (*viz* the iterative notion of set, or the notion of "set of") affords a convenient intuitive picture justifying the ZFC system (see Gödel 1947).

The point is best expressed in the introduction to *Set Theory and its Logic*. The notion of class is so fundamental to thought that we cannot hope to define it in more fundamental terms. And the "natural attitude" on the question what classes exist is that any open sentence determines a class; here Quine reveals himself as heir to Frege and Russell, not to Cantor, Dedekind or Zermelo (see Parsons 1986, 383-386; Quine agrees that his view was Fregean on p. 403). Since the natural attitude is discredited by the paradoxes, intuition is not to be trusted, and we are faced by a variety of alternative axiom systems-those of Russell, Zermelo, von Neumann, Quine. They are largely incompatible with one another, and none of them clearly deserves to be singled out as standard. Quine warns: "it would be imprudent at the present day to immerse ourselves in just one system to the point of retraining our intuition to it" (Quine 1963, 5; also 1, viii).

Zermelo's set theory is less compatible with the idea that open sentences determine sets. This appears only in the axiom of separation as a way of singling out subsets, while the main process of set formation depends on the rest of the axioms, which are existential. Thus, the process of set generation becomes laborious and insecure (Quine 1940, §29, 164). Moreover, the class of everything is not a set in Zermelo's sense, and none of Zermelo's sets embraces more than an infinitesimal portion of the totality of entities (Quine 1953, 97). Although completely natural when viewed from an iterative standpoint, these traits were felt by Quine as important shortcomings. He also complained that no complement of a set is itself a set (1940, §29); obviously, he was not restricting Boolean algebra to the status of the algebra of subsets of any given domain, but requiring that it apply to classes in general (see Quine 1937, 92). This is equivalent to saying that he was assuming the universal domain, the class of everything (which we have already seen).

Russell's theory of types is based on the idea that open sentences determine sets, though Russell conflated this with the idea that abstract properties determine sets. Quine has always been very critical of Russell's thought that his theory was a "no-class" theory, since all that it afforded was the elimination of one kind of abstract objects, classes, on behalf of less clear abstract objects-properties. "Such definition rests the clearer on the obscurer, and the more economical on the less" (Quine 1941, 148). Another flaw in Russell's theory was the way it "imputed meaninglessness": ($\alpha \in \beta$) is meaningless whenever the value of β is not of next higher type than that of α. Quine would have liked a construction of the theory that rendered the relevant formulas meaningless "in the straightforward sense of not getting defined" (Quine 1986, 14). One further flaw was "the distasteful type ontology" (ibid. 17): the universal class and the null class give way to an infinite series of quasi-universal (and null) classes, one for each type; even arithmetic is subject to the same reduplication, with new natural numbers within each type.

> Not only are all these cleavages and reduplications intuitively repugnant, but they call continually for more or less elaborate technical maneuvers by way of restoring the severed connections. (Quine 1937, 91-92)

Such were the main viewpoints, complaints and preferences that informed Quine's search for a "sanest comprehensive system of logic".

2.2. New foundations

Thus, the main alternative systems that had been proposed as a way out of the paradoxes (as Quine viewed it) were both unsatisfactory. Quine's efforts to remedy it led to 'New Foundations for Mathematical Logic', an invited address at a meeting of the Mathematical Association of America, on December 1936, that was felt by him as real and important progress. This is confirmed by the words with which he presented the draft to Carnap on Nov. 15, 1936:

> I am anxious to have you look this over as soon as possible, to see whether you have reason to suppose the system contradictory: for it looks dangerous, and on the other hand if consistent it looks important. (Creath 1990, 281)

In the name of clarity, the system NF used three primitives, corresponding to the "neoclassical" notation employed by Tarski and Gödel (Quine 1986, 17) and to what Quine had come to see as the three main parts of logic: truth-function theory, quantification theory, class theory. The primitives were Sheffer's stroke '|' ("not both... and..."), the universal quantifier, and membership. But the important point was Quine's alternative to type theory, which avoided the uncomfortable traits of Russell's system outlined above. Instead of adopting the hierarchy of types, to which type-indexed variables correspond, and banishing all formulas that are not of the form $(x_i \in y_{i+1})$ as meaningless, Quine's idea was the following. We may leave the language alone, using universal variables and countenancing all formulas, while we build restrictions essentially equivalent to Russell's into conditions for the formulas that may occur in instances of the axiom of comprehension.

The crucial notion is that of *stratified formula* (Quine 1937, 91). A formula in expanded form-i.e., eliminating all defined terms-is called *stratified* if there is an assignment of numerals to its variables, such that all instances of '$\in$' occur in contexts of the form '$n \in n+1$'. For instance, '$(x \in x)$' and '$(x \in y \wedge y \in x)$' are not stratified. Russell accepted only stratified formulas, Quine admits both stratified and unstratified ones. Now, the axiom of comprehension (axiom R3' in Quine's paper) is any instance of

$\exists x\, \forall y\, [(y \in x) \leftrightarrow \phi]$ where ϕ is stratified and does not contain x (ibid. 92).

The resulting system is more convenient than Russell's in having unique universal and null classes, and unique natural numbers (under the Frege-Russell definition); its classes behave in accordance with Boolean algebra. Moreover, it is more powerful than *Principia*. First, R3' covers all instances of the axiom of comprehension in simple TT (see above), so all proofs in simple TT are proofs in NF; second, in some cases it can be shown indirectly that there are classes corresponding to unstratified formulas.

Quine was particularly pleased to find that the axiom of infinity, which has to be postulated in TT, seems to become provable in NF. If we take the stratified formula '(x = x)' as ϕ in R3', the existence of the universal class V is proven. Likewise we can prove the existence of infinitely many members of that class, different from each other, namely $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, etc. (Quine 1937, 93-94). Thus, Quine's paper ended on a-cautiously stated-triumphal note, since the "striking instance" of the axiom of infinity was within the deductive power of NF.

As we see, early in 1937, Quine could think he had solved most of the main problems with TT and *Principia*, including its inconveniences, unnatural consequences (see above), and the problems with the axiom of reducibility (already irrelevant for simple TT) and the axiom of infinity. Both of these axioms had seemed to take away from the logical character of Russell's system. It should not be surprising to find that the paper begins with a confident assertion of logicism:

> In Whitehead and Russell's *Principia Mathematica* we have good evidence that all mathematics is translatable into logic. (...) It must be admitted that the logic that generates all this is a more powerful engine than the one provided by Aristotle. (...) The primitive notions in terms of which these calculi are ultimately expressed are not standard notions of traditional logic; still they are of a kind which one would not hesitate to classify as logical. (...) All logic in the sense of *Principia*, and hence all mathematics as well, can be translated into a (far more meager) language which consists only of an infinity of variables (...) and (...) three modes of notational composition. (Quine 1937, 80-81)

Similar statements appear both in earlier and later work of this period, in a tentative but confident way in (Quine 1936, 337, 354), and later in the assertive statement "mathematics reduces to logic" (1940, introd., 23-24). It seems that Quine envisioned the final triumph of Russell's program with his new systems, first NF and later ML*.

However, that was a deceptive appearance, and it was not late before the strange characteristics of NF began to emerge. First, Paul Bernays (1888-1978) was quick to point out problems with the axiom of infinity in his review of the paper (Bernays 1937). It is impossible in NF to prove the existence of a class answering to the ZF axiom of infinity (i.e., a class containing $\varnothing$, and $\{c\}$ for each element c), although intuitively V seems to be such a class. Thus, apparently we have nothing to play the role of the set of natural numbers. Bernays concluded:

> So it is still doubtful whether the proposed system-its consistency assumed-will be adequate for mathematics. As a first test might be taken the problem of proving that, under a suitable definition of infinity, the universal class is infinite. (Bernays 1937, 87)

Curiously, until 1953 it seemed impossible for NF to pass the test, and when it did, it was due to the failure of the axiom of choice (see below).

Another queer feature was pointed out by Quine himself, later in that year. The existence of the universal class V, however natural it seemed to Quine, entails strange consequences. The most obvious is that Cantor's theorem, that to every set there is a cardinally greater one (its power set), must fail for V and thus for NF generally. Quine devoted a paper to discussing the strange situation of that theorem within NF.[20] Cantor's original proof depended upon an unstratified condition, that is, the subclass of V that we would need, in order to show that V cannot be mapped one-to-one to its subclasses, is defined by an unstratified condition, so R3' does not prove its existence. Otherwise, the theorem might have been used to derive a contradiction from the existence of V, but as it is, the theorem -under the form "the subclasses of a class cannot be correlated with the members"- is refutable in NF. However, under the form used in *Principia*, "the subclasses of a class outnumber the unit classes of the members", the theorem is indeed provable. This means that, for NF to be consistent, V must contain no universal mapping "between objects and their unit classes" (ibid., 124). This is not contradictory in itself, but certainly is a strange feature for a system allowing for proper classes.

We may certainly ask, what about other problematic aspects? What about the axiom of choice, and Gödel's incompleteness theorem? Surprisingly enough, the question of the axiom of choice does not come up either in 'New Foundations' or in the book *Mathematical Logic*; but we will encounter it below. It seems that Quine, also following Russell here, was willing to eliminate important parts of classical mathematics and set theory in the name of logical sense. At any rate, his lack of interest in the axiom of choice, which was then a crucial point for mathematicians interested in foundations, is an indication that Quine was behaving more as a philosopher than a mathematician.

As for Gödel's theorem, Quine was aware of its import, namely that the "totality of principles" of logic and mathematics can never be completely reproduced by the theorems of a formal system (Quine 1936, 345; 1937, 89). Later on he would emphasize its importance by speaking of "the second great modern crisis in the foundations of mathematics -precipitated in 1931 by Gödel's proof" (Quine 1948, 19). Chapter 7 of *Mathematical Logic* contained an original proof of Gödel's theorem based upon what Quine -reminiscing Carnap's usage -called *protosyntax*. Quine concluded his discussion by saying that

> Logical truth, as syntactical truth, is syntactically undefinable. Logical truth can be said (...) to be informal. (Quine 1940, §60, 318)

We might have expected this. There was no reason to *require* that logic be a formal system, so, for a logicist, a natural response to Gödel's incompleteness was to accept that not only mathematical, but also logical truth is informal (on the complex reception of Gödel's theorems, see (Dawson 1985)). This calls into question the common assumption that Gödel's theorem was a fatal blow to logicism (see, e.g., Grattan-Guinness 1984); at the very least, its effect was not immediate.

2.3. Mathematical Logic

Quine's interest was not in obtaining new mathematical results, but in bringing to maturity the program of *Principia Mathematica*. This is clear from his two most important contributions, 'New Foundations' (see above) and the 1940 book *Mathematical Logic*. The importance of that "highly technical undertaking" is stressed in (Quine 1936), and the structure of the 1940 book, going from the logic of statements and quantifiers through class theory to relations and numbers, is expressive enough. Quine also included comments on how analysis and algebra fit into the logicist reduction of mathematics (Quine 1940, vi, 125-126, 278-279). He was interested in clarifying what the requirements are of the classical mathematics used by the sciences, particularly physics. Quine was not doing metatheoretical work, as were his friend Tarski and Gödel; he was more interested in advancing and refining Carnap's program for the epistemological foundations of science. Quine was a philosophical logician.

When in September 1938 he sailed to the Azores, with thoughts of writing a general book on mathematical logic, he "meant at first merely to follow" the course outlined in NF for the treatment of classes (Quine 1986, 20). This is certainly notable, for he decided to do so in spite of the limitations discovered during 1937 (see above). Obviously, he must have hoped to be able to show that V is indeed infinite, and to find a suitable definition of the class of natural numbers. But then a paper by Rosser was published, showing that the problems with the axiom of infinity are related to trouble with mathematical induction (Rosser 1939b). Within NF, induction works only for stratified formulas, so there is no general theorem of mathematical induction; the only possibility is to reestablish it by means of special axioms (see Rosser 1953). That has important side-effects: one cannot prove the existence of a class with n members, for every natural number n, so the existence of the class of natural numbers, under the Frege-Russell definition, cannot be proven. In other words, we have no proof that there is no last natural number, i.e., one cannot prove that $n+1$ is different from n for all natural numbers n. Such a system is clearly ill-suited as a foundation for mathematics.

Thus Quine was led to look for some modification of NF that might be stronger and more satisfactory. He found inspiration in von Neumann's version of set theory, which allows for classes that are not sets, i.e., that cannot be elements of a class. Some classes are sets, but some are 'proper classes', including the problematic instances that led to paradoxes (the classes of all ordinals and all cardinals, and the universal class). Quine decided to complement NF analogously, so in *Mathematical Logic* he distinguished between classes and "elements" (which we will call sets). He introduced them by means of two different axioms (Quine 1940, *200-*202), which we will formulate in a notation similar to the one we have used for NF, and using a two-sorted language with capital letters for non-sets (proper classes). The *axiom of class comprehension* would be:

$\exists Y \forall x (x \in Y \leftrightarrow \phi)$ where Y is not free in ϕ,

which forms the class of all elements that satisfy any condition ϕ, stratified or not. The *axiom of set comprehension* would be:

$\exists y \forall x (x \in y \leftrightarrow \phi)$ where ϕ is stratified and contains no free class variables, and y is not free in ϕ.[21]

The resulting system is stronger than NF, since now we have the class of natural numbers (which, however, is not proved to be a set), and we can prove that two different natural numbers have different successors (Quine 1940, §46, 250-253). We will call this system ML*, to distinguish it from the now customary system ML, adopted in the 2nd edition of the book (1951) following a modification suggested by Hao Wang. Quine was confident that the paradoxes would not be derivable in ML*, since both Russell's paradox and the paradox of cardinals depend on unstratified formulas. But in the fall of 1941 Rosser phoned him saying that he had been able to derive the paradox known under the name of Burali-Forti (see Rosser 1942). The system was inconsistent! The only way out that he found was to drop the axiom of set comprehension and replace it by new axioms: seven of its proved consequences, that were sufficent for the developments in the book (Quine 1986, 21-22). It was emergency repair, and a correction slip was added to all unsold copies of the book.

The deeper reason for the trouble was that, contrary to Quine's intentions (see 1940, 165), the sets of ML* are not the classes of NF: ML* was not a conservative extension of NF. With the expanded language, we allow for formulas ϕ that contain bound class variables, so the above axiom of set comprehension is stronger than NF's. It was Wang who became aware of the situation and proposed an alternative formulation: in the axiom of set comprehension, simply require that ϕ contain no class variables at all, whether free or bound (Wang 1950, 27). This ensures that the sets of ML (the modified system) are exactly the classes of NF, and Rosser's derivation of the Burali-Forti paradox is blocked. Wang went on to suggest that his modified system "is the system that Quine originally intended to present but that he made a mistake in his presentation" (ibid.). Wang was also able to prove that ML is consistent relative to NF (Wang 1950, 28-31). Quine adopted the modified system as the basis for the 1951 edition of *Mathematical Logic*, now free from flaws and restored to its original elegance.

One further comment is in order. A system such as ML, in which the class of natural numbers is not a set, can hardly be regarded as a convenient foundation for mathematics. This is obvious in case we are interested in developing set-theoretical results, but also standard mathematical theories, such as the theory of Lebesgue integration, require that $\mathbb{N}$ be a set (Rosser 1952, 240). Surprisingly, realization of this problem seems to have been slow. In his review of Quine's book, Church (1940, 164) remarked on the "striking and important" fact that arithmetic and analysis seemed to be derivable from the logic of ML* without "the introduction of special assumptions such as the axiom of infinity".[22] But, as we have seen, we would need to complement ML with an axiom stating that the class of natural numbers is a set. This, however, would have disastrous effects, since then (as shown by Wang and Rosser) it would become possible to prove the consistency of ML within the system itself -that is, by Gödel's second incompleteness theorem, ML would be inconsistent (Rosser 1952, 240-241). This is probably the reason why Rosser preferred NF, rather than ML, for his book (1953). If ML is to be consistent, the class of natural numbers *cannot* be a set, which mars the elegance of the system and its interest as a basis for mathematics.

2.4. Further developments and metatheory

Quine's class-theoretic systems have been somewhat out of the main stream of set theory, but over the years there has been quite a lot of work on them. The extremely simple NF system has attracted most of the attention. In the 1940s and 50s, J.B. Rosser (1907-) and H. Wang (1921-) were the main figures in the development. Particularly important were the results concerning ML* and ML that we have already discussed.

In 1953 Rosser published a textbook, *Logic for Mathematicians*, that took NF as the base system, and showed how to develop considerable parts of mathematics in suitable extensions of it. This made explicit the limitations of the system: mathematical induction for all formulas had to be added as an additional axiom schema, and an equivalent of the axiom of infinity was assumed. In order to derive the theory of ordinals and cardinals, following a scheme similar to that of *Principia Mathematica*, it was necessary to introduce a kind of limitation of size, since big classes like V are not well-behaved, and to assume further axioms concerning the "relatively small" classes, e.g., that they can be well-ordered (Rosser 1953). Although the main advantage of NF is its axiomatic or syntactical simplicity and elegance, it is lost when we use NF as the base system for mathematics.

In 1950 Rosser and Wang showed that NF has no standard model, in the sense that no model of NF preserving the standard identity relation '=' can have both the ordinals and the finite cardinals well-ordered by the less-than relation '<' (Rosser & Wang 1950). In later developments, E. Specker and R.B. Jensen established quite astonishing results using more powerful techniques. Most notably, Specker was able to prove that the axiom of choice is refutable in NF (!); the behavior of "relatively large" sets in NF is complex, and they do not admit well-orderings. Since, at the same time, the axiom of choice is provable for finite sets in NF, by combining both results Specker was finally able to prove the axiom of infinity that Rosser had postulated (Specker 1953). Jensen showed that a slight modification in NF's axiom of extensionality yields a system so weak that it can be proved consistent in ordinary arithmetic.[23]

Quine himself discussed most of these findings in his 1963 book, *Set Theory and Its Logic*, where he somewhat favored the iterative conception of set underlying ZFC, showing how to see it as a natural extension of type theory. He has later said that, after 1963, he "plumped for the primacy of the iterative concept"; in his view, "plumping" is the most one can do, which is another way of emphasizing the bankruptcy theory (Hahn & Schilpp 1986, 646).

3. Farewell to logicism

As we have seen, many of Quine's pronouncements throughout the period we are discussing, 1930 to perhaps 1947 (see below), seemed to be those of a convinced logicist. His devotion to the program of *Principia Mathematica*, whose importance he stressed in (1936), was already present in his dissertation. And we have found what look like clear logicist statements in (1937) and (1940) (see above); in particular, the confident assertion "mathematics reduces to logic" was preserved in the introduction to the 1951 edition of *Mathematical Logic*. But this is not to say that the logicist thesis was felt by Quine as a particularly enlightening one. On the contrary, everything suggests that by the late 1930s the logicist viewpoint was losing its significance, beginning to look somewhat void to him.

Indeed, a gradual shift away from logicism seems to have been typical of his generation. Since Quine was such a prominent logician and seeming logicist, the peculiarities of the path that took him away from it are of particular interest. At stake was the question of the scope of logic, but in this connection it is crucial to understand that logicism ought not to be a matter of mere convention. The idea that mathematics reduces to logic should help clarify our conception of mathematics itself; the thesis is intrinsically epistemological, or else void. Thus, in order to be interesting, the logic system that we use for a reduction of mathematics ought to form a perspicuous unity, so to say -a unity not only for technical reasons (simplicity or elegance, formal properties, etc.) but also when viewed epistemologically. Quine had been able to present elegant systems based on three primitives corresponding to the three parts of logic. This is fine from a technical point of view, but the philosophical question was: do these three theories form a perspicuous unity? Is it clarifying to put them together?

Quine was led to answer in the negative as a result of considerations involving some famous points in his philosophy of logic: the issue of analyticity, the question of existential assumptions ("ontology"), and, underlying his reactions, Quine's empiricistic and nominalistic preferences. The crucial metatheoretical results obtained during the 1930s by Gödel and others played also an important part in his thoughts, as did the availability of a handful of different set theories, and the gradual realization of the shortcomings of his own systems.

Indeed, we have to take into account that by the time Quine decided to narrow the scope of 'logic', after the war, his systems -particularly ML*- were facing difficulties. We have seen that Quine's diagnosis of the situation in set theory was determined by his "bankruptcy theory". He believed that the only natural approach here would be to define classes as the counterparts of open sentences (Quine 1963, 5). Since this procedure was blocked by the "antinomies", set theory confronted a situation of bankruptcy, and the only way out was to place arbitrary restrictions in order to reestablish consistency while losing as little as possible (Quine 1953, 121-122, 127). "These contradictions had to be obviated by unintuitive, *ad hoc* devices; our mathematical myth-making became deliberate and evident to all" (Quine 1948, 18).

This implies that the reduction of mathematics to the logic of NF, or for that matter to ZFC, could hardly be seen by him as a "reduction of the recondite to the obvious".[24] In view of the problems encountered by the attempt to accomplish Russell's program on the basis of NF and ML*, and given the peculiar metatheoretical properties of these systems, one may be tempted to think that bankruptcy was a prophecy that fulfilled itself. The availability of many different axiomatic set theories, largely incompatible with one another (Quine 1963, viii), entailed in Quine's eyes a lack of definiteness in set theory. This contrasted with the solidity of logic, when narrowly conceived: all characterizations of logical truths attempted during the 1930s, be they semantic or proof-theoretic, whatever the system employed, turned out to be essentially equivalent (compare Quine 1970, chap. 4). Here was one reason to trace a demarcation between mathematics and logic, cutting along quantification theory.

But one should resist the easy conclusion that the problems with NF and ML* were the main reason why Quine took the step of narrowing his definition of logic. Notably, several other reasons for taking it had been formulated before 1942, in the period of Quine's absolute devotion to

Russell's program, before ML* had turned out to be contradictory. Let us historically analyze these factors.

As regards the analytic, this is of course the central issue in Quine's confrontation with his philosophical master, Rudolf Carnap. After meeting Carnap at Prague in 1933, Quine became a follower of his views, particularly those expressed in *Logische Syntax der Sprache*. In his 'Lectures on Carnap', delivered at Harvard late in 1934 (see Creath 1990, 47-103), Quine supported Carnap's view that the propositions of logic, and therefore of mathematics, are true by convention (or, otherwise stated, strictly analytical). This, as we have mentioned, was a revised version of Wittgenstein's thought that logic was a collection of tautologies. The doctrine of the analytic character of logic certainly turned logicism into an interesting philosophical or epistemological doctrine: *if* logic is strictly analytical, then a reduction of mathematics to pure logic is certainly of utmost importance. However, Quine started to question that assumption quite early; the first printed expression of his misgivings can be found in 'Truth by Convention' (Quine 1936).

Here, Quine regards Russell's logicist program as extremely important and difficult (1936, 337, 354). Russell's logicism is taken to be the thesis that mathematical truths are conventional in the sense of following logically from logical truths and definitions (ibid. 331). But one should be careful here: in order to establish the analytic character of logic itself we need the stronger thesis that logical truths, in their turn, rest merely upon convention. And this stronger thesis seems to Quine plagued with difficulties -either it is untenable or applicable to almost any science. Actually, given the wide conception of logic that was customary in the 1930s, it seems that a sensitive logician ought to have had the impression that his discipline afforded an example of a slow, continuous transition from the analytic (truth functions) to the synthetic (set theory). It seems plausible that this may have been behind Quine's views on the analytic/synthetic distinction in his famous 'Two Dogmas of Empiricism' (1951, in (Quine 1953)).

The issue of analyticity turned up again in the second half of 1940, when Carnap, Tarski and Quine met for discussion from time to time at Harvard:

> Halcyon days. (...) By way of providing structure for our discussions, Carnap proposed reading the manuscript of his *Introduction to Semantics* for criticism. Midway in the first page, Tarski and I took issue with Carnap on analyticity. The controversy continued through subsequent sessions, without resolution and without progress in the reading of Carnap's manuscript. (Quine 1986, 19)

The question came up in letters of 1943 (Creath 1990, 294 ff) which contain some information on the 1940 discussion. According to Quine, it turned out that the "distinguishing feature of analyticity" for Carnap "was its epistemological immediacy in some sense". But Tarski and Quine "urged that the only logic to which we could attach any seeming epistemological immediacy would be some sort of finitistic logic" (ibid. 295). In the terminology that we have used above, the unity of the logic system was in question. To put it briefly and simplistically, truth-function theory and quantification theory (perhaps also a wider theory involving finite sets) might still be regarded as analytic, but not so for full set theory. Epistemologically considered, a system of logic that is strong enough to include set theory no longer forms a perspicuous unity, and this mars the clarification that logicism was supposed to bring.

Quine also argued, "supported by Tarski",

> that there remains a kernel of technical meaning in the old controversy about reality or irreality of universals, and that in this respect we find ourselves on the side of the Platonists insofar as we hold to the full non-finitistic logic. (ibid. 295)

Here we find a second theme that is also crucial. Quine is assimilating classes or sets, like numbers and functions, to the attributes of traditional logicians, so that the problem of their mode of existence becomes the old medieval problem of universals (see Quine 1948, 9). He will defend the thesis that (objectually) quantified variables presuppose the existence of their values.[25] Take a proposition of number theory, for instance that there are prime numbers larger than a million, or take the proposition that any bounded set of real numbers has a least upper bound. These examples show that classical mathematics is full of commitments to an ontology of abstract entities. And here we may see another reason to question the unity of the logic system, since truth-function theory and quantification theory may be understood to impose no ontological commitment, except for the extremely weak assumption that may be involved in the truth of quantification theory with identity, $\exists x\ (x = x)$.

One should realize that Quine is not talking about "ontology" in the usual sense of the word: his so-called ontological criterion (look for the values of bound variables) does not concern what there *really* is, if Quine may permit us so to speak, but what a given doctrine or theory *says* there is (Quine 1948, 15-16). The search for different formulations of a given theory that may involve less ontology has always been very important to Quine, because he has always been a nominalist at heart. As early as 1933 he "felt a nominalist's discontent with classes" (1986, 14) and began a series of attempts that were to lead him to the device of "virtual classes" employed in *Set Theory and Its Logic*. Although Quine tries to do as much as possible with the most austere "ontology", he has always been very clear about the need to accept important ontological commitments for some purposes -the basic example being classical mathematics, i.e., analysis.

Quine did not immediately move to a new definition of logic, but the two issues that we have mentioned gradually affected his conception of the subject, over a period of perhaps a decade. Already in *Mathematical Logic* there is a remark on the possibility of narrowing the scope of logic, and regarding set theory as mathematics proper; but he stuck to the customary, wider conception of logic (Quine 1940, §23, 127-128). In this connection Quine made some remarks that relate to the question of ontology: on the narrower conception, logic would deal with formulas syntactically considered, and with metatheorems, while set theory would use formulas as theorems dealing with a mathematical subject matter (ibid.). The issue of analyticity is briefly touched upon in the book, casting doubt on it, and no claim is made that logical truths are analytic (ibid., end of §10)

We have not yet exhausted the range of issues that would lead to Quine's restriction of logic to first-order predicate logic. A very important element consisted in the metatheoretical results achieved during the 1930s. Truth-function theory was shown to be complete and decidable; quantification theory was shown by Gödel to be complete, but Church established its undecidability; finally, Gödel's theorem proved the incompleteness of super-systems of primitive recursive arithmetic, including all systems of set theory. On the one hand, this was taken to show the convenience of distinguishing those three parts of logic (Quine 1953, 96), but in the end Quine was led to decree logic to be complete, while mathematics (including set theory) is incomplete (see Quine 1970). In the terminology of *Mathematical Logic* that we have seen above (section 3.2), Quine found it convenient to decree logic to be formal, while mathematics was left informal.

Now, the historical question is, when did all of these factors finally prompt Quine to reject his earlier conceptions? It seems to have been somewhere around 1950, when he made several attempts at formulating a satisfactory nominalism, which can be seen as the program of founding mathematics and science upon a finitistic logic (see above). One of the first was a joint paper with Nelson Goodman, 'Steps Toward a Constructive Nominalism', where they tried to formulate mathematics within an ontology of finitely many physical objects. In the same year, Quine read a paper 'On the Problem of Universals' to the Association for Symbolic Logic, where he explores the possibilities of nominalism as compared with platonistic set theory and predicative set theory, which he calls "conceptualism".[26] In the joint paper with Goodman, they settled for a formalist account of mathematics, trying to accomodate it into a finitistic ontology. This seems to mark Quine's definitive farewell to logicism, which is given explicit form in 'On What There Is'.

Rather simplistically, that paper presents logicism, intuitionism, and formalism as modern counterparts to medieval realism, conceptualism, and nominalism regarding the universals (Quine 1948, 14-15). Quine mentions Frege, Russell, Whitehead, Church and Carnap as the representatives of logicism, but he himself is conspicuously absent from the list, in spite of having probably been the most significant active representative as of 1940, thanks to his systems NF and ML*. Everything suggests that he now preferred to be associated with formalism, linked with two of his objectives: to preserve as much as possible from classical mathematics, especially in view of its utility to physics and technology; and also, on a different line, to try to dispense with abstract entities altogether (ibid. 15).

Although the 1951 edition of *Mathematical Logic* still reads logicist, Quine's research was moving along formalistic and nominalistic lines. In the end, he settled for the narrower conception of logic, which is the most consistent with his basic philosophical positions, and he abandoned strict nominalism, as becomes clear in *Set Theory and Its Logic* (Quine 1963). As he says, nominalism "would be my actual position if I could make a go of it. But when I quantify irreducibly over classes, as I usually do, I am not playing the nominalist. Quite the contrary." (Quine 1986, 26)

4. Summary

The issue of the border between logic and set theory has always been, and still is, a bone of contention. Nowadays philosophers tend to be convinced that mathematics begins with the infinite and set theory; in this they follow the views espoused by Quine after 1950 (e.g., Quine 1970, 64-70). Mathematicians, on the other hand, classify set theory as one of the central areas of mathematical logic (see, e.g., Barwise 1977), which of course is normally taken to be a branch of mathematics. In the present paper we have seen some of the reasons that historically led logicians to concentrate upon first-order logic (hereafter FOL), to prefer set theory rather than type theory, and in some cases to establish a borderline between logic and set theory.

Some of the main motives for abandoning higher-order logic (TT) in favor of first-order are the following. The general feeling of insecurity regarding foundational matters, together with the Hilbertian concentration upon proof theory, led to the preference for strictly formal systems with good proof-theoretical properties. Metatheoretical results, especially the perfect fit between semantics and proof theory established by Gödel's completeness theorem (1930), together with the incompleteness of stronger systems (1931), gave powerful reasons for choosing FOL. This was complemented by the conviction that FOL sufficed for codifying practically all classical mathematical reasoning, and in particular for codifying set-theoretical proofs (see Tarski 1956, 271; Gödel 1940, 34). Moreover, there was still a distrust of the (supposedly antinomical) notion of set, that collaborated in the abandonment of second- and higher-order logic, although this conclusion was resisted by such influential logicians as Hilbert and Church. Skeptical and relativistic motives were visible above all in Skolem, but even the platonistic Gödel felt that FOL was the best underlying logic for a metatheoretical analysis of 'higher' logical notions such as those of set and class (or concept) (see Gödel 1940, 1944).[27]

Furthermore, the advent of model theory around 1950 afforded new motives for special concentration upon FOL. Here, it was precisely its weakness -reflected in the compactness and Löwenheim-Skolem theorems- that was turned into a powerful tool for algebraic applications. But this was no matter of principle: it involved no rejection of higher-order logic, but simply suggested a concentration upon first-order for the purposes of an interesting research program (Shapiro 1991, 193). Thus, around 1950, notwithstanding (Church 1956), the panorama of logic was basically the present one: first-order logic was being regarded as the main logic system, the paradigm for all of logic, and the only system considered in introductory logic courses and expositions of subjects such as set theory, model theory, etc. One may feel that such was, in the long run, the most powerful argument for FOL -after all, what we regard as natural is what we are familiar with.

As we have indicated, the situation in the foundations of mathematics called for decisions regarding the scope of logic and its relations to mathematics. The confusion that still existed over set-, class- and type-theoretic systems around 1940 (with the alternatives of Zermelo-Fraenkel, Bernays-Gödel still in its beginnings, ramified type theory, simple type theory, Quine's NF and ML*, etc.) suggested that one could not appeal to any kind of absoluteness here, as with the theory of propositional connectives and quantifiers. Metatheoretical results, in particular the incompleteness of any form of set theory sufficient for mathematical purposes, and the platonistic ontological implications of set theory, as exposed by Quine -all colaborated in suggesting that the border ought to be traced at least at the level of the axiom of infinity, if not with the emergence of the notion of set. In any event, set theory, in the technical sense of the expression, clearly belonged to mathematics and was just one more mathematical theory that could be based on first-order logic.

Given that logicism required set theory to be an integral part of logic, those developments undermined its appeal. The 'unity' of the grand logic of *Principia* was in question. Logicians such as Carnap and the young Quine had still regarded membership as a logical notion, while the skeptic Skolem argued its non-logical character as early as 1922. In 1936, Tarski expressed his opinion that there was a problem of essential vagueness here (1956, 419-20; see 1986, vol.4, 723). Thus the intrinsic arbitrariness of a definition of logic was acknowledged. Above all, once Carnap's tenet of analyticity was rejected, the thesis that mathematics is logic afforded no epistemological clarification. Even from a strictly mathematical standpoint, classical mathematics was clearer than the very abstract and strong theory of sets. There was no point, then, in insisting that mathematics can be reduced to a logic that embraces set theory.

However, all that was not enough to eradicate the classical, entrenched talk of set theory as a part of logic (alongside nascent proof theory and embryonic model theory). There is evidence that Gödel, Tarski and many other logicians resisted Quine's neat differentiation between logic and set theory, which explains the customary usage of regarding set theory as one the main branches of mathematical logic.

Finally, we can review in a few words the arguments that favored set theory over type theory. We have seen that the main motive for prefering TT was an appearance of greater naturalness and above all greater security, while ZFC was technically more convenient. Zermelo's cumulative hierarchy, later defended by Gödel (1947), amounted to turning ZFC into a transfinite extension of simple TT, with the peculiarity of being formulated in a first-order language. Besides, arguments for security were undermined by Gödel's result that the axiom of choice and the generalized continuum hypothesis are consistent relative to ZF without choice. Since the Bernays-Gödel theory of sets and classes later turned out to be a conservative extension of ZFC, after 1940 it was a only a matter of time before logicians would join Zermelo, Tarski and Gödel. Meanwhile, Quine's systems, which seemed promising at first, became increasingly dissatisfactory as more was known about their metatheoretical properties and their use in reconstructing classical mathematics.

Notes

1 Agradezco a Gregory H. Moore e Ignacio Jané sus valiosos comentarios de una versión anterior de este trabajo. El profesor Quine tuvo la amabilidad de contestar a mis cartas sobre el tema, lo que no quiere decir que esté de acuerdo en todos los puntos con la reconstrucción que sigue; además, no ha tenido ocasión de leer el artículo.

2 Actually Löwenheim allowed infinitely long sentences, see (Moore 1995).

3 See (Moore 1980; 1982 chap. 4.8; 1988), and his contribution in this volume.

4 For a more detailed discussion of topics in this section, see (Ferreirós 1993, 1996), or the brief account in (Ferreirós 1994).

5 Realization that set theory formed the core of logicism is not new. Quine, for instance, has emphasized that the logicists only attained a reduction of pure mathematics to logic *and* set theory. One should also recall that not all partisans of set theory were logicists; most notably, Cantor was an opponent (see his correspondence with Hilbert and Dedekind during the 1890s in (Meschkowski & Nilson 1992)).

6 A brief but penetrating account of Russell's logicism can be found in (Grattan-Guinness 1984).

7 To Zermelo it was actually more than just a "picture" or a preferred model, since he formulated set theory within higher-order logic (see Moore 1980). Further details on the evolution of the axiomatization of set theory, including the gradual emergence of the axiom of foundation (already considered by von Neumann), can be found in (Moore 1982, 260-272).

8 Here we can only give a hint at Russell's original theory of types, since later we only have to consider the simplified version due to Ramsey. For further details, see (Russell 1908, Whitehead & Russell 1910-13, Introduction, chap. 2).

9 A discussion of logicism in the 1920s, particularly of Ramsey and related work by Leon Chwistek, can be found in (Grattan-Guinness 1984).

10 This view was, strictly speaking, abandoned in (Carnap 1937).

11 Recall that the propositions of the *Tractatus* are finally declared meaningless, 'unsinnig' (Wittgenstein 1921, 6.54). This meant the impossibility of a genuine metatheory of a logic system.

12 We cannot enter here into a detailed exposition of *Logical Syntax*. For details on its logical aspects, see the excellent discussion in (Sarkar 1992), and also Hintikka's contribution to the same volume.

13 *AMS Bulletin* 45, 1939, 822. Also see the last chapter of (Church 1956), and (ibid. 48, 332-333).

14 For examples from the years 1930 to 1935, see (Tarski 1956, 61, 113-115, 241-243, 297, 384). In general, see entries 'set theory' and 'type, logical' in the analytical index (entries which, by the way, might have been put together in the context of this sample of papers -though not, for instance, in the wider context of vols. 1-2 of (Tarski 1986)).

15 The reader will find a slight difficulty here since type theory is formulated in the metatheoretic symbolism, and Tarski's choice of variables and connectives makes his version more complex than Gödel's (above).

16 'On the Nature of Mathematical Truth', reprinted in (Benacerraf & Putnam 1983).

17 For analysis of the dissertation in relation to Quine's later work, and an interesting discussion of Harvard around 1930, see (Dreben 1990).

18 We might also mention points in Quine's later philosophy of language that are intimately related to this. E.g., in 1934 he had already defended sentences vs. propositions as referents of propositional variables; see 'Ontological remarks on the propositional calculus', *Mind* 43, 1934, reprinted in *The Ways of Paradox,* Harvard University Press, 1966, 265-271.

19 Quine's doctrines on reference were treated in 'A Logistical Approach to the Ontological Problem', *Journal of Unified Science* 9, 1940, and 'Designation and Existence', *Journal of Philosophy* 36, 1939. But it was only with 'On What There Is' (Quine 1948) that they became famous and widely-known.

20 'On Cantor's Theorem', *Journal of Symbolic Logic* 2, 1937, 120-124.

21 The condition on free class variables is needed because otherwise all classes would be elements.

22 He was also careful to remark that "detailed development of the theory of real numbers would require the axiom of choice (not mentioned by Quine)" (Church 1940, 163). Wang formulated this axiom for ML in his (1950, 27).

23 More details about these and other results can be found in (Fraenkel, Bar-Hillel & Levy 1973, 161-71) (Ullian 1986, 576-82) and (Wang 1986, 640-42). For a detailed survey, see M. Boffa in *Journal of Symbolic Logic* 42, 1977.

24 Quine has emphasized this point in a letter to the author, May 29, 1996.

25 Quine has said that he already discussed related issues with Leśniewski in 1933 (1986, 13), and he prepared related arguments for publication in 1939 -'A Logistical Approach to the Ontological Problem', which was to appear in the *Journal of Unified Science*, stopped by the war, and 'Designation and Existence', *Journal of Philosophy* 36, 1939. However, the most famous one came out as 'On What There Is' (Quine 1948).

26 'Steps Toward a Constructive Nominalism', *Journal of Symbolic Logic* 12, 1947, 105-122. 'On the Problem of Universals' was merged with other papers, written in 1939 and 1951, to form 'Logic and the Reification of Universals', in (Quine 1953, chap. IV). These efforts culminated in the finitistic theory of sets that he used as a basis for exploring the alternative full systems in *Set Theory and Its Logic*, where he wrote: "in the present book I defend the finite classes against the ultimate classes" (Quine 1963, ix).

27 Some of the relevant reasons for preferring first-order logic are analyzed in Jané's contribution to this volume. It has recently been stated (Shapiro 1991, 178, 191-193) that Gödel was, with Skolem, the main early proponent of first-order logic, but I see no reason to liken their viewpoints. Certainly Gödel showed a clear preference for *formal systems*, which in his view had to be of a combinatorial character and in a certain sense finitary; thus he objected to infinitary sentences and rules (see Grattan-Guinness 1979, Dawson 1985, Moore 1980 & 1996). But he was no strict partisan of first-order logic like Skolem and later Quine; his conception of logic was actually very wide and seems difficult to analyze (see Gödel 1944).

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