HILBERT AND THE EMERGENCE OF MODERN MATHEMATICAL LOGIC

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ABSTRACT: Hilbert's unpublished 1917 lectures on logic, analyzed here, are the beginning of modern metalogic. In them he proved the consistency and Post-completeness (maximal consistency) of propositional logic -results traditionally credited to Bernays (1918) and Post (1921). These lectures contain the first formal treatment of first-order logic and form the core of Hilbert's famous 1928 book with Ackermann. What Bernays, influenced by those lectures, did in 1918 was to chenge the emphasis from the consistency and Post-completeness of a logic to its soundness and completeness: a sentence is provable if and only if valid. By 1917, strongly influenced by *PM*, Hilbert accepted the theory of types and logicism -a surprising shift. But by 1922 he abandoned the axiom of reducibility and then drew back from logicism, returning to his 1905 approach of trying to prove the consistency of number theory syntactically.

Keywords: Hilbert, Bernays, first-order logic, metalogic, completeness, higher-order logic, axiom of reducibility, Hilbert's unpublished lectures.

1. Introduction

During the past hundred years, the most significant transformation of mathematical logic was the shift from a logical to a metalogical perspective. Whereas Frege and Russell had each been concerned to formulate a system of logic that was adequate for constructing the real numbers, Gödel was concerned with metalogical questions. The most important figure in the transition from Russell to Gödel was Hilbert. The metalogical questions with which Hilbert was occupied -consistency, independence, completeness, and decidability- were excluded from Russell's view of logic but were central to Gödel's.

The present paper is an attempt to understand the evolution of Hilbert's metalogical[1] ideas from his early treatment of logic (1905) to his mature views in his 1928 book *Grundzüge der theoretischen Logik* with his student Ackermann. We particularly emphasize unpublished materials, especially Hilbert's lecture courses from 1917 to 1923, because of the light they shed on the evolution of his views. It turns out that the most important of those lecture courses is one that he gave in 1917-18 and that survives in an authorized version at Göttingen. This course documents an important shift in his treatment of logic from his earlier courses and provides the conceptual framework for his *Grundzüge*.

Before discussing this 1917-18 course in detail, we begin by exploring how Hilbert's metalogical concerns influenced Gödel.

2. Hilbert's Influence on Gödel

As is well known, the two most important results in the emergence of modern mathematical logic were metalogical results due to Gödel: the completeness theorem for first-order logic and the incompleteness theorems for number theory and for higher-order logic (including the theory of types). The problem of showing the completeness of first-order logic first appeared in print in Hilbert and Ackermann's *Grundzüge* in 1928. This was a book which much influenced Gödel soon after its publication.

In the *Grundzüge*, Hilbert considered completeness ("Vollständigkeit") in two senses. Both senses were relative to an axiom system. The first meaning of completeness involved semantics, namely that the axiom system allows the proof of "all correct (valid) formulas of a certain domain that is characterized by its content" (1928, 33). The second meaning was what he called "completeness in the sharper sense", and was a purely syntactic notion. An axiom system was said to be complete in the sharper sense if it was consistent but became inconsistent as soon as any further axiom was adjoined to it. (Later, completeness in the sharper sense was sometimes called Post completeness; see (van Heijenoort 1967, 264).)

The *Grundzüge* showed, by using work of Bernays, that propositional logic not only was complete but also was complete in the sharper sense (1928, 33).[2] Then the book established that first-order logic is not complete in the sharper sense, and posed the problem of proving that first-order logic ("the restricted functional calculus") is complete (1928, 68). Gödel solved this problem in his doctoral dissertation, whose culmination was his completeness theorem for first-order logic.

Yet when one reads the published version (1930) of his doctoral dissertation, which was considerably revised from the dissertation itself, one has the impression that Gödel was primarily influenced by *Principia Mathematica* rather than by Hilbert and Ackermann's *Grundzüge*. In the published version, for example, Gödel claims that the axiom system of first-order logic for which he proves completeness is the same as that given in sections *1 and *10 of PM (1930, 103). Unfortunately, this claim is not correct. Rather, Gödel's axiom system (including its rules of inference) is, with one exception, symbol for symbol the same as the axiom system for first-order logic in the *Grundzüge*.[3] Yet in his (1930) Gödel did not mention the close relation between the two systems. By contrast, in the dissertation he stated clearly that his axiom system was basically that of the *Grundzüge* (1929, 65).

In fact, first-order logic is not considered at all in PM, not even as a subsystem of the theory of types. In particular, in *10 of PM, which deals with propositional functions of a single variable, Russell intends his axioms to apply to variables of any type. By contrast, in first-order logic the variables can only range over the lowest type, i.e. the type of individuals.

In his doctoral dissertation (1929) Gödel does point out the essential equivalence of the version of first-order logic found in the *Grundzüge* and that found in *1 and *10 of PM. Once again, however, he does not mention that he has modified *10 by omitting those axioms that explicitly refer to types and by restricting the other axioms to apply only to the lowest type.

Overall, what Gödel appears to have done is the following. He took the idea of first-order logic from Hilbert and Ackermann's *Grundzüge* and used the name that they gave it, namely "the restricted functional calculus". He adopted their symbolism and terminology, as he remarked in his (1930, 103). He recognized that, by suitable modifications to *1 and *10 of PM, he could formulate an axiom system for first-order logic that was essentially the same as the one in the *Grundzüge*. Finally, he took from the *Grundzüge* the metalogical problem of proving the completeness of first-order logic, and proceeded to solve it.

Surprisingly, Gödel nowhere mentions that this metalogical problem was first posed by Hilbert and Ackermann (1928) and, in more detail, in (Hilbert 1929, 8). Rather, Gödel acts as if this problem would occur to anyone, and states that, once a system of axioms has been given in a purely formal way, "the question at once arises whether the originally postulated system of axioms and principles of inference is complete" (1930, 103). But, as Dreben and van Heijenoort have argued persuasively (1986, 44), this question did not arise in the tradition leading from Frege to Russell, nor in the tradition of algebra of logic leading from Boole to Schröder. The question could only arise when one abandoned the view of Frege and Russell that logic was universal (i.e. one cannot step outside of it to study it), and studied formal axiomatic systems as mathematical objects.

To what degree was Gödel influenced by Löwenheim or Skolem to consider first-order logic? In his 1929 dissertation, Gödel certainly refers in footnote 15 to Skolem's proof of the Löwenheim-Skolem theorem, and in the published version he makes it clear that what he has in mind is Skolem's 1920 paper, but does not cite any of Löwenheim's papers (1930, 108). The 1920 paper, which proved the Löwenheim-Skolem theorem for first-order logic and then extended it to an infinitary logic, influenced Gödel by supplying a technique in his completeness proof. That paper, however, does not provide any kind of formal treatment of first-order logic. All of Skolem's theorems are semantic, and he blurs the distinction between syntax and semantics by using "contradictory" to mean "not satisfiable". In his 1923 paper Skolem does go part of the way toward a formal treatment, giving a definition of first-order formula though not any axioms or rules of inference for first-order logic. Unfortunately, Gödel did not know the 1923 paper when writing his dissertation, as his later letters make clear (1986, 51). Although Skolem's 1920 paper exerted some influence on Gödel's use of first-order logic, the decisive influence in providing a conceptual framework was Hilbert and Ackermann's book, which gave a formal treatment of first-order logic and posed the completeness problem for it.[4]

3. Background to Hilbert's 1917-18 Lectures

Hermann Weyl wrote a very insightful obituary (1944) on Hilbert. Unfortunately, this obituary gives the misleading impression that Hilbert's work was very neatly divided into periods in which he worked solely on one major area: 1885-1893 on the theory of invariants, 1893-1898 on algebraic number fields, 1898-1902 on the foundations of geometry, 1902-1912 on integral equations, 1910-1922 on physics, and 1922-1930 on the foundations of mathematics in general.

Weyl's dates give a general picture of the major themes in Hilbert's work, but are very inaccurate when it comes to Hilbert's research on logic and the foundations of mathematics. Already in his 1891 lecture course on projective geometry, Hilbert was thinking seriously about the foundations of geometry, and did so again in 1894 in a course specifically devoted to the foundations of geometry (see (Toepell 1986)). The next year Hilbert published an early version of his axiomatization of Euclidean geometry (1895). His ideas on this subject continued to evolve up to and beyond his famous monograph, *Grundlagen der Geometrie* (1899), which was devoted to metalogical questions about geometry: the consistency, independence, and completeness of his axioms. It was, after all, in geometry that nineteenth-century mathematicians first proved results about consistency and independence, mainly by using models. And in 1902, in the French translation of his book, Hilbert included a metalogical axiom (not found in the original German edition) whose purpose was to insure that a line contained points corresponding to all real numbers. This metalogical axiom, which he called the axiom of completeness, stated that a model of his geometrical axioms had to be maximal, i.e. not capable of being enlarged by new elements while still satisfying the other axioms.[5] Hilbert's earliest version of his axiom of completeness had already appeared in 1900 as part of his axiomatization of the real numbers (1900). The transformation that was underway in Hilbert by 1900, but was not complete until years later, was from thinking of axiomatization, consistency, and independence as of real concern only to geometry and the real numbers to thinking of those matters as of fundamental concern to all areas of mathematics, including mathematical logic.

Responding in 1905 to the paradoxes, Hilbert published his first paper devoted to the larger issues of the foundations of logic and mathematics. For the first time, he used a formal symbolic language. The paper showed the influence of Schröder when, within it, the quantifier "for every x" was represented by an infinitely long formula (Moore 1988, 107).

This paper of 1905 was Hilbert's last publication on logic and the foundation of mathematics until 1918. One might be forgiven for thinking, as Weyl supposes, that Hilbert did indeed devote all his intellectual energy during the intervening years to integral equations and physics.

But in fact Hilbert devoted considerable energy to foundational questions during those years. This is most apparent if one considers, not Hilbert's publications, but his lecture courses, for which an authorized version exists in many cases. These show that his interest in foundational matters continued throughout the years from 1905 to 1917. His interest in such matters did not arise out of nothing in 1917, but was intensified from an interest for which there is evidence in almost every intervening year.

Let us consider the surviving list of his lecture courses on foundational questions from 1905 to 1917, as it is found in his *Nachlass*:[6]

1905, Logische Prinzipien des mathematischen Denkens

1908, Prinzipien der Mathematik

1910, Elemente und Prinzipienfragen der Mathematik

1911-12, Logische Grundlagen der Mathematik (as well as lectures with this title by Hilbert, Toeplitz, and Haar in the Seminar in Mathematics and Physics)[7]

1913, Einige Abschnitte aus der Vorlesung über die Grundlagen der Mathematik und Physik

1914-15, Probleme und Prinzipien der Mathematik (Einlage in Elemente und Prinzipienfragen der Mathematik, 1910)

1916, Die Grundlagen der Physik I

1916-17, Die Grundlagen der Physik

1917, Mengenlehre

1917-18, Prinzipien der Mathematik und Logik

Of all these lecture courses, the most important for mathematical logic were the first one in 1905 and the last in 1917-18. The 1905 course was Hilbert's first serious attempt to provide foundations for logic and arithmetic, and is discussed in detail in (Peckhaus 1990, 50-75). But the 1905 course was abortive, since Hilbert realized, by the time he finished it, that it could not provide an adequate foundation.

By contrast, the 1917-18 course did provide such a foundation. We now turn to that course.

4. Hilbert's 1917-18 Lectures on Logic: Metamathematics

One of the most striking features of this lecture course was how much it had in common with his earlier interests. Logic was seen as a subject to study within mathematics, rather than the other way around. What he had called "Axiomenlehre" in his 1905 lecture course (1905a, 6) had in 1917 become the "axiomatische Methode". But it was a *mathematical* method that was, in his view, as appropriate for logic as it was for geometry or mechanics. It took questions that had originally been part of geometry -questions of consistency and independence- and raised them in the context of logic.

How natural it was for Hilbert to do so is clear from later developments, since almost all of mathematical logic is now concerned with metalogical questions. How unnatural it was as well, in one sense, is clear from statements in *Principia Mathematica*. In PM, it was objected that one cannot prove that if p is equivalent to q, then q may be substituted for p in any formula:

> This can be proved in each separate case, but not generally, because we have no means of specifying (with our apparatus of primitive ideas) that a function is one which can be built up out of these ideas alone.(Whitehead and Russell 1910, 120)

It is precisely here that the lack of metalogic is felt. Russell and Whitehead had no way of specifying the notion of logical formula because they had no metalanguage. For them, it was impossible to stand outside of the theory of types, and therefore it was impossible to prove the independence of the primitive propositions of logic (1910, 95). (Already in the *Principles of Mathematics*, Russell had argued that only the dependence of a primitive proposition of logic could be established, by a proof of it from the others; its independence could not be established (1903, 16).)

For Hilbert, it was vital to be able to stand outside of a system of logic by using a metalanguage. Nevertheless, in 1917 he did not have the general notion of metalanguage, although he had isolated many of its essential features. That general notion is due to Carnap (1934) and Tarski (1935), both of whom were influenced by Hilbert.

Ironically, Russell himself later accepted the need for a kind of metalanguage. In his 1940 book *An Inquiry into Meaning and Truth*, he argued that the "conception of a hierarchy of languages is involved in the theory of types, which, in some form, is necessary for the solution of the paradoxes" (1940 = 1980, 62). Such a hierarchy of languages, he insisted, was unavoidable, although he acknowledged that his hierarchy was different from Carnap's and Tarski's. Just how different becomes clear when one realizes that if p is a proposition in Russell's object language, then the proposition not-p occurs in his metalanguage rather than in his object language. Thus, in contrast to propositional logic and first-order logic and today's versions of the theory of types, Russell's 1940 object language was not closed under logical operations such as "not".

Russell had already hinted at his notion of a hierarchy of languages in his 1922 introduction to Wittgenstein's *Tractatus*. There Russell argued that

> every language has, as Mr. Wittgenstein says, a structure concerning which, *in the language*, nothing can be said, but (...) there may be another language dealing with the structure of the first language, and (...) to this hierarchy of languages there may be no limit. (1922, 23)

Nevertheless, what Russell had in mind was far from the notion of object language and metalanguage, as we understand them today.

5. The Structure of the 1917-18 Lectures

Hilbert's 1917-18 course was divided into two parts. The first of these was devoted to the axiomatic method, as embodied in geometry and in the real numbers, while the second dealt with mathematical logic. The first part bore a close relationship to material in his *Grundlagen der Geometrie* (1899). The material in the second part, however, differed substantially from anything that he had written or published previously, and consisted of five chapters:

1. Der Aussagen-Kalkül
2. Prädikaten-Kalkül und Klassen-Kalkül
3. Ueberleitung zum Funktionen-Kalkül
4. Systematische Darstellung des Funktionen-Kalküls
5. Der erweiterte Funktionen-Kalkül

It is no accident that the titles of the four chapters in Hilbert and Ackermann's book *Grundzüge der theoretischen Logik* (1928) were the same as the chapters of the 1917-18 course. The one exception was that chapters 3 and 4 of the course were combined to give chapter 3 of the book, which was entitled "Die engere Funktionenkalkül".

We should not be surprised at the close relationship between the 1917-18 course and the book. For in his preface Hilbert remarked that the book was based on that course and on two other courses, "Logikkalkül" given in 1920-21 and "Grundlagen der Mathematik", given in 1921-22 (1928, v). Yet the three courses were by no means mere duplicates of each other. The 1920-21 course was much the shortest, consisting of merely 62 pages and containing only logic. The 1921-22 course, devoted to logic and the foundations of mathematics, was about 150 pages, whereas the 1917-18 course was the longest, at 246 pages.

Of the three courses, the one given in 1917-18 has the closest textual relation to the 1928 book. In fact, there are numerous passages in the book which were taken more or less verbatim from the 1917-18 lectures. This is particularly true for chapters 2-4 of the book, whereas chapter 1 has a close relation to the 1920-21 course. For example, from the middle of p.37 to the end of p.41 of the book (in chapter 2) corresponds almost verbatim to pp.98-105 of the 1917-18 lectures. In chapter 3, matters are more complicated. Thus pp. 48-53 of the book correspond to pp.120-133 of the course, but by p.50 there was much evidence of including only selective passages from the course. As for chapter 4, pp.93-99 of the book are taken almost verbatim from pp.208-220 of the course. There are a large number of passages in the book that correspond textually with the course, but for reasons of space we omit further examples.

A sentence from the beginning of the 1917-18 lectures on logic illustrates a fundamental concept that remained basically unchanged in the 1928 book. In those lectures he wrote: "The calculus of logic consists in the application of the formal method of algebra to the field of logic" (1917-18, 63). And in his 1920-21 lecture course he repeated the same sentence, except that he now replaced the word "algebra" by "mathematics". This same sentence, slightly elaborated, opens the 1928 book: "*Theoretical logic*, also called *mathematical* or *symbolic logic*, is an application of the formal method of mathematics to the field of logic" (1928, 1). This emphasis on the formal axiomatic method, on the manipulation of uninterpreted symbols by fixed rules, carried over from the 1917-18 lectures to the book.

6. Propositional Logic in 1917-18

In his 1917-18 lectures Hilbert gave propositional logic an axiomatization which was essentially identical with the one found in his 1905 lecture course (1905a, 225-228; cf. Peckhaus 1990, 64). Yet in the 1905 course he did not raise such metalogical questions of the consistency, independence, and completeness of an axiom system for logic;[8] now, in 1917, he did so. Before we consider those metalogical matters, we wish to consider his axiomatization in some detail.

Hilbert in 1917-18 used the following symbols for propositional logic: 0 for a true proposition, 1 for a false proposition, = for equality (actually, logical equivalence of propositions), + for "and", × (or juxtaposition) for "or", an overbar for "not", and variables X, Y, Z for propositions. The axioms that Hilbert gave were the commutative, associative, and distributive laws for + and ×, the law of identity X=X, a law stating that X = 0 or X = 1, and finally four special axioms:

9.	$X+\overline{X}=1$	10.	$X\times\overline{X}=0$
11.	$1+1=1$	12.	$X\times 1=X$

These were essentially the symbols and axioms which he had given in his 1905 course, except that in 1905 he used ≡ rather than = for the logical equivalence of two propositions. Also, in 1917-18 he included the law stating that X = 0 or X = 1, as he had not done earlier (1917-18, 64-66). In neither course did he introduce rules of inference, and instead all arguments consisted of equations -an approach that went back to Boole.

In the 1917-18 course the first metalogical question that he considered was consistency. He established the consistency of his axioms for propositional logic by giving an interpretation for those axioms in terms of the numbers 0 and 1 with "A and B" interpreted as the minimum of A and B, with "A or B" interpreted as the maximum of A and B, and with "not A" interpreted as 1 - A. Since his axioms had a model, he concluded that they were consistent (1917-18, 70). This was analogous to what he had done with the consistency and independence proofs that he had given in 1899 for his axioms of Euclidean geometry; the one difference was that then the models were infinite whereas now, in 1918, they were finite.

As for independence in propositional logic, Hilbert did not get very far. He offered only one example, in which he modified the above arithmetic interpretation by letting "A or B" be interpreted as the constant function 0. In this interpretation axioms 1-10 were true but 11 and 12 were false. Thus he established that axioms 11 and 12 were independent of the remaining ten axioms (1917-18, 69).

Hilbert regarded completeness as the most important of these metalogical questions for propositional logic. Yet at first it was not altogether clear what he meant by "completeness" in this context, since he said that it essentially reduced to showing that the axiom system sufficed for traditional syllogistic logic (1917-18, 67). Almost a hundred pages later, he finally defined completeness (Vollständigkeit) as what in his 1928 book he would call completeness in the sharper sense, i.e. if any unprovable formula is appended to the axioms they become inconsistent (Post completeness). He established completeness in this sense with an arithmetic interpretation, along the lines of his consistency proof discussed two paragraphs above, by showing that all his axioms had the value 0, together with the following argument: Suppose that an unprovable formula A can be appended to the axioms without leading to a contradiction; A is provably equivalent to a formula in conjunctive normal form, at least one of whose conjuncts has among its disjuncts no propositional variable and its negation; by suitable substitutions, that conjunct is provably equivalent to a disjunction of the variable X some number of times, and hence to X itself; by substituting the negation of X for X, there is a contradiction.

It is surprising but significant that in 1918 Hilbert did not consider completeness -either for propositional logic or for first-order logic- in the modern sense that every valid formula is provable. It remained for Bernays to begin shifting the emphasis in that direction.

7. Bernays' 1918 Habilitationsschrift

In 1918 Bernays submitted at Göttingen, under Hilbert's direction, a *Habilitationsschrift* entitled "Beiträge zur axiomatischen Behandlung des Logik-Kalküls". The stimulus for this work came from Hilbert's 1917-18 course, as Bernays explicitly acknowledged (1918, iv). Bernays, unlike Hilbert, made use of a simplified version of the axioms of propositional logic given in PM. At the same time, Bernays expressed them in terms of Hilbert's 1917-18 symbolism. As rules of inference, Bernays used Modus Ponens (which was stated in PM, although not so clearly) and a rule of substitution (which was not found in PM at all) (1918, 2).

Bernays remarked (1918, iv) that in the 1917-18 course Hilbert had proved the consistency and completeness of propositional logic. This remark may be confusing until one recalls that "completeness" here meant Post completeness. To some degree, Bernays changed the emphasis from Post completeness to completeness, i.e. to showing that every valid formula is provable (1918, 9). This is the first time that completeness in the modern sense was treated by anyone as a possible property of a logic.

Bernays felt compelled to make a sharp distinction between the notion of "correct" formula and "provable" formula, a distinction that he hinted was lacking in Hilbert's earlier work. Moreover, Bernays spoke of "valid" (allgemeingültig) formulas rather than of Hilbert's "correct" formulas, with the principal aim of showing that every provable formula is valid, and conversely (1918, 6). Here Bernays took a valid formula to be one that is true with any assignment of truth-values to the propositional variables. These distinctions avoided the occasional confusion in Hilbert's 1917-18 course as to whether the "correct" formulas were the valid ones or the provable ones.

Drawing these distinctions led Bernays to put forward what we would now call the soundness of propositional logic (i.e. every provable formula is valid). He proved the soundness of his propositional logic, and showed how consistency was then an immediate consequence. The proof of soundness was quite modern; it consisted of showing that the axioms were valid and that the two rules of inference preserved validity.

When Bernays began to prove completeness, he did so by showing that Post completeness implies that every valid formula is provable (1918, 9). To establish Post completeness, he essentially used the argument that Hilbert gave in the 1917-18 lectures. As Bernays remarked, his result gave a decision procedure for determining whether an arbitrary formula is provable by reducing it to conjunctive normal form and ascertaining whether in each of the conjuncts there occurred some variable and its negation (1918, 15). By contrast, in his 1917-18 lectures, Hilbert had not mentioned decision procedures at all, although in his published lecture "Axiomatisches Denken" (1918) he did emphasize, as he had already in his 1900 Paris lecture, the importance of the solvability of a mathematical question in a finite number of steps. This would later evolve into the *Entscheidungsproblem*.

Bernays carried Hilbert's ideas much further, in regard to independence proofs in propositional logic, than Hilbert had done in his 1917-18 lectures. In particular, after showing that one of the axioms in PM for propositional logic was redundant, he established that, in his simplified version, the other four axioms in PM were independent of each other. He did so by using finite arithmetic interpretations in the style of Hilbert's own work on consistency and independence. In some cases, when an arithmetic interpretation was not at hand, he simply introduced abstract tables (generally with three or four elements) to show the result of the operations "not" and "or" on those elements. In each case, the formulas provable with the table had some distinguished value (usually 0) while the formula to be shown independent had some different value (1918, 28-41). In effect, Bernays was investigating many-valued logic, although he did not look at the matter in this way.

The results in Bernays' *Habilitationsschrift* were not published until 1926, when he emphasized the connection of his results with PM and left unspoken the strong connection with Hilbert's 1917-18 lectures. In (1926) Bernays included his 1918 proof of completeness and his analysis showing that four of the axioms of propositional logic in PM were independent while the fifth followed from the other four. But he omitted his 1918 results on various possible systems of deduction for propositional logic that relied on rules of inference rather than on axioms, and, in the simplest case that he considered, used only one axiom (1918, 52). These "natural deduction" systems were later rediscovered, and extended to first-order logic, by Gentzen.

In their 1928 book Hilbert and Ackermann used Bernays' 1918 version of the axioms for propositional logic (simplified from those in PM). Hilbert himself first used Bernays' version while giving a course on foundations of mathematics during the winter semester of 1921-22. Ironically, Hilbert had used something close to Bernays' version in the 1917-18 lectures, but within his axioms for first-order logic, not propositional logic.

8. First-Order Logic in 1917-18

One of the most striking features of Hilbert's 1917-18 lectures was their explicit and detailed treatment of first-order logic. The significance of Hilbert's use of first-order logic is only clear when we understand that, before Hilbert, first-order logic was not treated as a separate subsystem of logic. The person who is often cited as a counterexample to our claim about first-order logic is Löwenheim. It is true that Löwenheim (1915) distinguished carefully between quantification over individuals and quantification over relations. Moreover, he did indeed prove his famous theorem, which, in the form that Skolem (1920) later stated it, was that a satisfiable first-order sentence is satisfiable in a denumerable domain. But Löwenheim's proof of this theorem made essential use of infinitely long formulas, which Skolem (1920) adopted as well, only to abandon them for the usual first-order logic two years later. Löwenheim's 1915 paper used a first-order logic which included infinitely long formulas, and so was not based on first-order logic alone.

Thus, in 1917-18, when Hilbert gave his lecture course, first-order logic (without infinitely long formulas) had been considered by almost no one. Frege's original logic of the *Begriffsschrift* (1879), for example, repeatedly quantified over properties, e.g. in formulas 76 and 77, in his treatment of sequences. Likewise, Peano's postulates for the positive integers (1889) quantified over all classes. Thus both Frege's and Peano's logics were higher-order, not first-order. And the Russell-Whitehead theory of types (1910) was also a kind of higher-order logic.[9]

Other than Hilbert the only person who, during 1917-18, seems to have considered something close to first-order logic was Hermann Weyl. In his book *Das Kontinuum*, Weyl certainly rejected quantification over higher-order objects (1918, 20-21). Yet he took the natural numbers as given within his logic, which was in that sense stronger than first-order logic. Moreover, he rejected the unrestricted use of the principle of the excluded middle, and so was certainly not proposing *classical* first-order logic. No evidence is known that shows Hilbert and Weyl discussing first-order logic at that time. But Hilbert did give his well-known lecture on axiomatic thinking in Zurich in September 1917, and since Weyl taught in Zurich at that time, he may well have attended his *Doktorvater* Hilbert's lecture.

In his 1917-18 lectures, Hilbert distinguished carefully between propositional logic, first-order logic, and higher-order logic. He carefully gave a recursive definition of first-order formula, and then stated axioms for first-order logic.

Let us first consider his recursive definition. Today such recursive definitions of well-formed formula are the basis for many metatheorems about formulas, and are an essential tool for the logician. But this was not the case in 1917, and even Hilbert did not then use his definition to prove anything about all formulas (although he did do so in 1921-22; see Section 11 below). Nevertheless, Hilbert gave the first rigorous definition of first-order formula. In effect, the definition was given in a metalanguage. The importance of this definition lies in its treatment of formulas as purely syntactic objects, i.e. as strings of symbols devoid of meaning. (Recall that Russell and Whitehead argued that there could be no definition of formula of PM and insisted that there was no way to prove statements about all formulas of PM.)

Hilbert's definition was preceded by a precise statement of the primitive symbols, three kinds of variables (propositional, individual, and functional variables), the corresponding three kinds of constants, three logical signs (not, or, for every), and parentheses. Here the functional variables had "empty places", and in this Hilbert was influenced by Frege. Hilbert's definition of formula (or "expression", as he called it) then consisted of taking the six kinds of variables and constants as expressions (provided that the empty places were filled in with individual variables or constants) and, finally, with the following recursive clauses: If A is an expression, so are "not A", "A or B" and "for every x, A" (1917-18, 129-130). With small changes, this was the definition of first-order expression found in Hilbert and Ackermann's book (1928, 51-52).

Hilbert's 1917-18 axioms for first-order logic consisted of two kinds. The first kind was a simplified version of five of the axioms for propositional logic in PM (although PM was not explicitly mentioned in this context). The second kind consisted of six axioms for quantifiers (axioms that differed considerably from those in PM). Finally, Hilbert stated a rather complex set of rules of inference, including Modus Ponens and various rules on quantifiers and substitution. (Recall that he used no rules of inference for his version of propositional logic.) In Hilbert and Ackermann's book (1928, 53) the propositional axioms in first-order logic were unchanged from the 1917-18 lectures, except in the omission of the axiom that Bernays (1918) had proved redundant; the axioms for quantifiers and the rules of inference were quite different, given in the book in a much simpler form than in the lectures and credited to Bernays.

After these preliminaries, Hilbert turned to various metalogical questions about first-order logic. The first of these was consistency. He began by separating from first-order logic the part that used only propositional variables and no quantifiers. Thus, as axioms for this part, he used his first five axioms and, as rules of inference, he took substitution and Modus Ponens. (In effect, this was a system of propositional logic, but a different one from that presented earlier in his 1917-18 lectures.) To prove the consistency of this propositional part of first-order logic, he once again gave a finite arithmetic interpretation with values 0 and 1, with "or" interpreted as the arithmetic product, and with "not X" interpreted arithmetically as 1 - X. Then the five axioms had the value 0, and the rules of inference preserved the value 0, giving that all provable formulas had value 0. From this it followed that this propositional part was consistent, since if two contradictory formulas were provable, one of them would have the value 1. He then extended this proof to show that all of first-order logic is consistent, but added that there was no guarantee that first-order logic would remain consistent if unobjectionable "contentual" assumptions were included (1917-18, 150-155).

Then Hilbert turned to the question of the "completeness" of first-order logic (i.e. Post-completeness). As a first step, Hilbert proved Post-completeness for the propositional part of first-order logic, in the way discussed in Section 6 above. The arithmetic interpretation that he used to establish the consistency of first-order logic suggested to him a way to prove that first-order logic was not Post-complete. In that interpretation, he had shown that all provable formulas had value 0; hence it sufficed to find a formula that was not provable but had value 0. His candidate was "if there is an x such that F(x), then for every x, F(x)". He was able to show that this had value 0 in his interpretation, and this formula was certainly not valid, but he was unable to prove that this formula was not provable in his system (1917-18, 152-156). By 1928, Ackermann had managed to show that this formula was not provable in first-order logic, thereby establishing that first-order logic was not Post-complete; the proof was given in his joint book with Hilbert (1928, 66).

It seems surprising that in 1917-18 Hilbert did not consider at all whether first-order logic is complete in the modern sense and did not pose this problem until his book with Ackermann in 1928. But this becomes more comprehensible when we realize that, in 1917 and over the next decade, Hilbert was trying to advance simultaneously on several different but interrelated fronts within logic and number theory (consistency, decidability, Post completeness). Hilbert did not have any idea that the completeness of first-order logic would later become central to modern work in mathematical logic.

9. First-Order Logic vs. Higher-Order Logic

One of the most striking features of Hilbert's work on logic is how he attempted to establish a result for a subsystem and then to extend the result (e.g. consistency or Post-completeness) to larger and larger subsystems, with the aim of eventually extending the result to the entire system. This is how he proceeded, for example, in 1905 when he established the consistency of a very weak system of number theory. In his 1920 lecture course "Problems of Mathematical Logic", he again proceeded to show the consistency of a weak system of number theory and to sketch how the proof might be extended to all of number theory (1920, 36-46).

But in the 1917-18 lectures Hilbert paid no particular attention at all to the consistency or Post-completeness of number theory. Surprisingly, he instead considered higher-order logic, and the theory of types, in considerable detail. In that context, he treated the whole numbers within second-order logic as predicates of predicates (1917-18, 193), in a manner reminiscent of Russell and Frege. (That treatment of whole numbers carried over to Hilbert and Ackermann's book (1928, 86).)

Shortly before, when Hilbert gave his lecture "Axiomatic Thinking" in Zurich in 1917, he praised "the ingenious mathematician and logician Russell" for his very successful work in axiomatizing logic, and regarded as the "crowning achievement of axiomatization" the completion of Russell's work in axiomatizing logic (1918, 412). This positive attitude toward PM is very apparent at the end of the 1917-18 lectures, where Hilbert accepted the ramified theory of types, stating that "the introduction of the axiom of reducibility is the appropriate means to fashion the theory of types into a system in which the foundations of higher mathematics can be developed"

(1917-18, 246). Hilbert preserved this sentence almost unchanged in his book with Ackermann (1928, 113), but in the meantime he had become critical of the axiom of reducibility (1928, 115).

With that one exception, the chapter on higher-order logic in Hilbert and Ackermann's book is taken, often verbatim, from the 1917-18 lectures. The book follows those lectures in arguing for the insufficiency of first-order logic and the need to extend it to a higher-order logic, in showing how the paradoxes require this extended logic to have types, in remarking that one could stop at second-order logic, and in seeing the need for the axiom of reducibility to overcome difficulties that would arise otherwise with the real numbers (e.g. in showing that every bounded set of real numbers has a least upper bound).

By 1920, when Hilbert next gave a lecture course on foundational matters, he had some reservations about the axiom of reducibility as a way of obtaining the union of a predicate of predicates, something that was needed for least upper bounds (1920,32). Moreover, he now thought that Russell, by using this axiom, was turning from constructive logic to the axiomatic method.

Hilbert discussed this axiom once more in his lecture course during the winter semester of 1921-22. By that time, he had come to the conclusion that the axiom of reducibility was not a satisfactory solution to the problem of providing a foundation for analysis. Treating this axiom in the form that every second-order predicate of individuals is equivalent to some first-order predicate, he then took as given a domain of individuals together with certain basic properties and basic relations. For an arbitrary choice of basic properties and relations, he observed, the axiom of reducibility was certainly not satisfied. It would be necessary, in each case, to complete the system of basic relations in such a way that the axiom of reducibility would be satisfied. But one had not seen how to obtain such a completion by purely logical concept-construction or by a logically constructive process. He then argued for a return to the axiomatic standpoint and for giving up the logicist goal of founding arithmetic and analysis through logic alone, since the reduction to logic survived in name only. The axiom of reducibility was not logically self-evident, as were the general rules of deduction, but it was plausible that it did not lead to a contradiction. This plausibility, however, depended on the usual axioms for analysis, and his goal was to get beyond mere plausibility. He concluded that transfinite logic (i.e. logic which considered infinite domains) was not capable of providing a secure foundation for arithmetic: either transfinite logic was handled purely formalistically (in which case it was imprecise and offered no protection from contradictions) or it was made so precise in content ("inhaltlich") that contradictions were avoided (in which case one did not attain a foundation for the usual arguments in analysis and set theory) (1921-22, 99-100).

10. The Two 1920 Lecture Courses

During 1920 Hilbert gave two lecture courses on mathematical logic. The first of them, called "Problems in Mathematical Logic", was delivered in the summer semester, while the second, called "The Calculus of Logic", took place in the winter semester of 1920-21. There are quite substantial differences between the two courses. The first was by far the more polemical of the two, and began with criticisms of Brouwer and Weyl, apparently his first public criticism of their foundational work.[10] It was concerned with the paradoxes of logic and set theory as well as with revisions in the foundations of arithmetic that had been proposed in recent years by Poincaré, Zermelo, Russell, Weyl, and Hilbert himself. Hilbert regarded Poincaré as continuing in the tradition of Kronecker's prohibitions against set theory. By contrast, Hilbert insisted that any prohibitions must merely exclude contradictions and must allow all worthwhile results to remain and, further, that the freedom to construct new concepts must not be restricted beyond what is necessary (1920, 20).

Hilbert praised Zermelo's axiomatization of set theory as "the most brilliant example of a complete working out of the axiomatic method" (1920, 33). In this course Hilbert gave his first, and perhaps only, detailed treatment of Zermelo's axiomatization, handling it within a formal language that appeared to be first-order logic.[11] After remarking that the consistency of Zermelo's axioms was not yet proved, Hilbert asked whether, and to what extent, these axioms could be reduced to logic (1920, 22-28). He emphasized that the goal of reducing set theory, especially the methods of analysis, to logic had not been attained and perhaps could never be attained, but that the importance of the axiomatic method was independent of the question of how far mathematical axiom systems can be reduced to pure logic. Thus at this time Hilbert was an agnostic toward logicism, whereas in 1917-18 he had accepted logicism, at least insofar as it concerned analysis (1920, 33-34). This shift is reflected in his increasing doubts about the axiom of reducibility.

The final section of the 1920 course was devoted to Hilbert's first version of his new *Beweistheorie*, which he was developing in an attempt to demonstrate the consistency of number theory (1920, 39). This appears to be the first time that he publicly used the term *Beweistheorie*. As part of that version, he gave his definition of a formalized proof within number theory. Since his only rule of inference was Modus Ponens, he presented his definition of formal proof as follows: A proof is a figure or diagram ("Figur") consisting of inferences

$$\begin{array}{l} A \\ A \rightarrow B \\ \hline B \end{array}$$

where each of the formulas A and A $\rightarrow$ B is either an axiom or is the end-formula of a previous inference (1920, 37). His axioms for number theory were essentially the Peano postulates without quantifiers and with the principle of induction omitted, and his proof of consistency was purely syntactic, depending on his demonstration that any provable formula had at most two implication signs $\rightarrow$ (1920, 41). His proof for the consistency of this weak system of number theory was carried over to his (1922) article, as was his emphasis on his numerals as meaningless signs.

The second course on logic in 1920, given during the winter semester of 1920-21, was quite different from the first. Whereas the first was largely concerned with questions in the philosophy of mathematics, the second returned to his development of mathematical logic. This second course was divided into three chapters, one on propositional logic, one on the functional calculus, and one on his new approach to the foundations of number theory. The approach that he took here to the propositional calculus later served as the basis for the first four sections of Hilbert and Ackermann's book (1928, 1-12). Except for one axiom (A implies A), this approach used rules for the transformation of a propositional formula into a logically equivalent propositional formula. These rules were chosen so that every formula could be transformed into a logically equivalent formula in conjunctive normal form. To these rules he added what he called rules for correctness, one of which stated that a correct formula was transformed into a correct formula by his transformation rules (1920-21, 8-9). Here too there was some ambiguity as to whether "correctness" meant provability in his system, or validity. Bernays had made clear in his 1918 *Habilitationsschrift* that it was important to remove this ambiguity, but Hilbert had still not done so.

Hilbert offered a proof of the completeness of this particular axiomatization for propositional logic. As in 1917-18 for a different axiomatization, the proof was based on showing that the provable formulas were precisely those whose conjunctive normal form had, for each conjunct, some propositional letter and its negation (1920-21, 10).

Hilbert's treatment of first-order logic in the second chapter was more informal, and not as precise, as his 1917-18 version.[12] This time he based his axiomatization of first-order logic on the same kind of approach that he took in the first chapter for propositional logic, i.e. transformation rules and rules of "correctness" (1920-21, 31-32). This approach to developing first-order logic, by using somewhat complicated rules, did not survive into Hilbert and Ackermann's book (1928), although it can be regarded as a forerunner of Gentzen's natural deduction.

The final chapter of the 1920-21 course was his second attempt at a new foundation for number theory. This attempt, however, got even less far than the first, although it did use the principle of induction as a rule of inference (1920-21, 51). At the conclusion of the course, he remarked that if one could establish the Post-completeness of number theory, then one would have established as well the decidability of all number-theoretic questions (1920-21, 61). From this, he added, one gains an understanding of Brouwer's recent paradoxical assertion that the principle of the excluded middle fails for infinite sets. It is surprising to see Hilbert putting a positive light on Brouwer's assertion, since Hilbert reacted negatively to Brouwer's foundational views both earlier (1920) and later (1922).

11. Logic in 1921-22

The last of the lecture courses that Hilbert mentioned as relevant to his book with Ackermann was "Foundations of Mathematics", given in the winter semester of 1921-22. Once more this course was concerned with the axiomatic method, but this time the emphasis was entirely on questions of consistency. Logic was to be studied within mathematics by the axiomatic method, and was seen in principle as not essentially different from other branches of the mathematical sciences, such as geometry or classical mechanics.

Hilbert opened this course by stating that the problem of consistency "is the most important and the most difficult in the investigation of axiom systems" (1921-22, 4). He added that the other axiomatic questions, such as completeness and independence, were very closely tied to consistency. By the completeness of an axiom system he meant once again that no unprovable sentence can be adjoined without making the system inconsistent, i.e. Post-completeness.

The first chapter dealt with two earlier methods for proving consistency, the method of exhibition ("Aufweisung") and the method of reduction ("Zurückführung"). What he called the method of exhibition involved the construction of a "system of things" (a model) for which the axioms held. The one example of this method that he mentioned gave a proof of the consistency of propositional logic. Here, he referred back to the 1920-21 lectures for a treatment of propositional logic. Yet the axiomatization whose consistency he proved here was different from that in earlier lectures, and consisted of Bernays' treatment of propositional logic in his 1918 *Habilitationsschrift*. Hilbert's model of propositional logic consisted, as it had already in 1917, of the numbers 0 and 1, with arithmetical operations on them for the connectives "or" and "not". Now, however, he was concerned to emphasize that this arithmetic interpretation did not depend on the consistency of number theory, since it used "a finite, perfectly surveyable system of definitions" (1921-22, 9). Thus, in this context, the model was required not to be infinite.

The method of reduction was illustrated by using analytic geometry to reduce the consistency of Euclidean geometry to that of the real numbers. In effect, this was what is now called a relative consistency proof. What is a little surprising is that he referred the reader to the 1917-18 course (on logic and the axiomatic method) for a discussion of the axioms of geometry, rather than referring to one of his many courses on geometry (1921-22, 14). At the end of the chapter, after giving several other relative consistency proofs for parts of mathematical physics (such as thermodynamics), he stressed that all these proofs depended completely "on the assumption of the consistency of analysis, i.e. arithmetic in the wider sense (in which set theory is also included)" (1921-22, 50).

Hilbert then turned, in the second chapter, to earlier efforts to prove the consistency of analysis. Here he began by emphasizing the attempts by Dedekind, Frege, and especially Russell, to reduce arithmetic and analysis to logic (1921-22, 70). He did not regard any of these attempts as successful -the first two because of the paradoxes, and Russell's because of problems with the axiom of reducibility (see Section 9 above).

At the same time Hilbert fully accepted the need to distinguish levels (first-order, second-order, etc.) in logic and continued to base his logic on a version of the theory of types. In fact, he remarked:

> For us, the essential thing is that by distinguishing levels ("Stufen") the extended functional calculus (higher-order logic) is made precise in content and obtains the same degree of certainty as the restricted functional calculus (first-order logic). (1921-22, 84)

This was a surprising statement, since at the time Hilbert had a finitistic proof for the consistency of first-order logic but had no such proof for the consistency of second-order logic or the theory of types.

In the final chapter Hilbert made his third attempt at a new foundation for number theory. It used a new system of ten quantifier-free axioms, including six for propositional logic, two for identity, and two for number theory itself: $a+1 \neq 0$ and the predecessor of $a+1$ is a. This was exactly the system he later published in his 1923 article (1921-22, chap. 3, 3; 1923, 153). But whereas the 1923 article immediately went on to assume his transfinite axiom as a way of defining quantifiers, apparently he had not yet formulated that axiom when he gave the 1921-22 course. At least he gave no hint of that axiom, or of any other way of handling quantifiers within a consistency proof, in this final chapter.

In the chapter he modified his 1920 definition of formal proof to allow each line of the proof to be a substitution instance of an axiom or of a previous line of the proof or to be the final line of an inference (by Modus Ponens) (1921-22, chap. 3, 5). This definition, which differed from the earlier one in its use of substitution, was published essentially unchanged in his 1922 paper and was the first precise definition of "provable formula" for first-order logic.

The final result of his course was a consistency proof for his ten quantifier-free axioms. This proof, that the formula $0 \neq 0$ could not be deduced in his system, depended on showing that any proof in his system which ended with a variable-free formula could be replaced by a proof in which no variable occurred on any line of the proof (1921-22, chap. 3, 22). Then he showed that the predecessor function could be eliminated from such a proof. Thus the only formulas left in a proof were built up by propositional logic from equalities or inequalities of numerals. The equalities were said to be correct when the numerals on both sides of = were the same, false when they were different. Inequalities were defined to be correct or false in an analogous way. So the proof reduced to showing that, by using propositional logic, every substitution instance of an axiom without variables was correct and that Modus Ponens preserved correct formulas. This was done by induction on formulas, together with the conjunctive normal form for propositional logic (1921-22, chap. 3, 19-38). Apparently, this was the first time that he used induction on formulas to prove a metatheorem, although the means for doing so were already present in his 1917-18 lectures.

12. Conclusion

The evolution of Hilbert's views on metalogical questions is complex. Some of his views remained relatively unchanged over several decades, such as his emphasis on the axiomatic method as the appropriate tool in the foundations of mathematics. Even the scope of the axiomatic method, however, underwent some change. Originally, in the early 1890s, he was concerned with this method only in the context of geometry. But by (1900a) he wished the method to be used to axiomatize parts of mathematical physics, including the kinetic theory of gases. As well, by 1920 he was interested in proofs of the consistency of such theories, and not only in the consistency of geometry or arithmetic.

Hilbert's concern with metalogical questions -consistency, independence, completeness-grew out of his work on the foundations of geometry (1899) and the reduction of its consistency to that of the real numbers. Already in (1905), in his first attempt at a consistency proof for a fragment of number theory, he used the idea of treating proofs as mathematical objects, an essential feature of his *Beweistheorie*, which was only developed in the 1920s. Yet in his (1905a) course, where he struggled unsuccessfully with the question of how to deal with quantifiers in mathematical logic and so remained within propositional logic, he did not raise any metalogical questions about that logic. In his next few courses on the foundations of mathematics, such as those of 1908 and 1910, he did not get any further.

The turning point came in his course in the winter semester of 1917-18. Already in his Zurich lecture in September 1917, Hilbert had praised Russell's axiomatization of logic in PM as the crowning achievement of the axiomatic method. In the 1917-18 course he adopted the theory of types as his basic logical framework. But he did so in a way that was quite different from PM. In particular, he separated off first-order logic as a distinct subsystem of logic, and he gave a precise definition of first-order formula. Moreover, he raised metalogical questions about first-order logic, as well as for propositional logic -the first time that such metalogical questions were raised about logic or about subsystems of logic. He used a finite arithmetic interpretation to prove the consistency of propositional logic and of first-order logic. By means of such an interpretation, he showed the independence of two of his axioms for propositional logic from the others. His deepest result about propositional logic was to prove its Post completeness.

Hilbert did not publish any of his metalogical results at the time, and so the proof of the Post completeness of propositional logic was later credited to Bernays (1918; 1926) and to Post (1921). The completeness of propositional logic was closely related to its Post completeness. But, at the time, Hilbert had not formulated the notion of completeness in the modern sense. It remained for Bernays in 1918 to isolate the notion of completeness and to underline its importance.

Bernays did so by first distinguishing carefully between "provable" formulas and "valid" formulas in propositional logic. (Hilbert had blurred this distinction by speaking of "correct" formulas, which in fact he sometimes intended to be the provable ones, sometimes the valid ones.) Then Bernays introduced the concept of the soundness of propositional logic (every provable formula is valid). After proving soundness in a thoroughly modern way (by showing that every axiom was valid and that the two rules of inference preserved validity), he observed that the soundness of his logic immediately yielded its consistency. He then introduced the modern notion of completeness (every valid formula is provable). From the soundness and Post completeness of his logic he deduced its completeness. It remained to establish the Post completeness of his logic, and here Bernays essentially gave Hilbert's argument for it as found in the 1917-18 lectures. Although in 1918 he credited Hilbert with proving Post completeness, Bernays did not do so in the (1926) paper where he published his results on propositional logic. What is more surprising is that neither in 1918 nor in 1926 did he pose the problem of showing the completeness of first-order logic. Nor did Hilbert do so until 1928.

Whereas in 1917-18 Hilbert had accepted the ramified theory of types as the proper foundation for mathematics, and with it had accepted the reduction of mathematics to logic (i.e. logicism), by 1921-22 he no longer accepted the axiom of reducibility and had become doubtful about the possibility of successfully reducing mathematics to logic. Already in 1920, when his doubts about the axiom of reducibility first surfaced, he returned to his attempts (more or less abandoned since 1905) to establish the consistency of number theory directly -without the logicist reduction of numbers to classes or predicates- and added to them efforts to show the Post completeness of number theory. These attempts were repeated in the 1920-21 and 1921-22 lectures, and in the (1922) paper. The first fruit of these attempts was his precise syntactic definition of provable formula in a formal theory, and the beginnings of his *Beweistheorie*.

Hilbert's 1928 book with Ackermann did not include work on the consistency of number theory, work that appeared instead in Hilbert's articles during the 1920s and in his 1934 book with Bernays. Yet the basic structure of the 1928 book was already present in the 1917-18 lectures. All of the chapters, except the first on propositional logic, were heavily indebted to those lectures and frequently borrowed verbatim from them. (One of the few exceptions was the emphasis in the book, but not in the lectures, on the *Entscheidungsproblem* for first-order logic.)

As Church has pointed out (1956, 288), the first explicit formulation of first-order logic as a separate logical system was published in Hilbert and Ackermann's book (1928). But that explicit formulation was already present a decade earlier in the 1917-18 lectures. The lectures gave a precise definition of first-order formula, a precise axiomatization of first-order logic, and raised the first metalogical questions about a formal system of logic. In the lectures Hilbert also answered those questions by establishing the consistency and Post completeness of propositional logic, together with the consistency and the lack of Post completeness of first-order logic.

With the 1917-18 lectures, the basic paradigm for metalogical research was established. It remained for the future to develop it and bring it to maturity.

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Notes

1 We allow ourselves the liberty of referring to these ideas as "metalogical" or part of "metalogic", even though these terms date from a later period, namely the 1930s. Hilbert never used the term "metalogic" and began to speak of "metamathematics" only around 1922. (Gödel 1930a) and (Tarski 1930) both used the term "metamathematics" rather than "metalogic".

2 Post (1921) established the same result independently of Bernays, but Post had no influence on Hilbert or Gödel.

3 The one exception is Gödel's axiom 6, which differs from the corresponding axiom in the *Grundzüge*, and is found as an axiom at *10.12 of PM.

4 In fact, their book (1928, 80) refers to Skolem's 1920 paper, which was in an obscure journal. Gödel may well have been led to that paper by their book.

5 (Hilbert 1902, 25). In the second German edition of 1903, and in all later editions, he included a version of the axiom of completeness.

6 This is part of a longer list in his *Nachlass*, SUB Göttingen Cod. Ms. D. Hilbert 520.

7 For this particular course, we thank Dr. V. Peckhaus, who pointed out that it is found on p. 15 of the official list of lectures given at Göttingen University during 1911-12. Unlike the other courses mentioned above, this one is not on the list in Hilbert's *Nachlass*.

8 Although Hilbert did not mention these metalogical questions in the 1905 course, he did prove the consistency of a very weak system of arithmetic there (1905a, 263-266), just as he had done in the 1905 paper.

9 For a discussion of the ways in which the logics of Frege, Peano, Peirce, Russell, and Schröder were higher-order rather than first-order, see (Moore 1988).

10 (Hilbert 1920, 1). He first published those criticisms in his (1922) article.

11 This is the earliest treatment of set theory within first-order logic, but Hilbert does not make it clear that Zermelo's axiom of separation must, in first-order logic, become an axiom schema.

12 In particular, he did not define what it meant to be an "expression", or well-formed formula of first-order logic, as he had done in 1917-18.

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