mathematics

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On Best Approximations for Set-Valued Mappings in $\mathcal{G}$-convex Spaces

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Received: 8 January 2020; Accepted: 2 March 2020; Published: 4 March 2020

Abstract: In this paper we obtain a best approximations theorem for multi-valued mappings in $\mathcal{G}$-convex spaces. As applications, we derive results on the best approximations in hyperconvex and normed spaces. The obtain results generalize many existing results in the literature.

Keywords: $\mathcal{G}$-convex space; hyperconvex space; KKM map; best approximations

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1. Introduction and Preliminaries

S. Park and H. Kim [1] introduced the notion of generalized convex space or $\mathcal{G}$-convex space. In $\mathcal{G}$-convex space, many results were obtained in nonlinear analysis, see [2–25]. The aim of this paper is to obtain the best approximation theorem in $\mathcal{G}$-convex space. Our result generalize theorems of A. Amini-Harandi and A. P. Farajzadeh [3] (Theorem 2.1), W. A. Kirk, B. Sims and G. X. Z. Yuan [15] (Theorem 3.5), M. A. Khamsi [16], (Theorem 6), S. Park [18] (Theorem 5). Also we obtain that almost quasi-convex and almost affine conditions is unnecessary in results of J. B. Prolla [21] and A. Carbone [7].

A multifunction $\Phi : X \to Y$ is a map such that $\Phi(x) \subseteq Y$ for all $x \in X$. Let $S \subset X$, then $\Phi(S) = \cup\{\Phi(s) : s \in S\}$. Let $T \subset Y$, denote

$$\Phi^{-}(T) = \{s \in X : \Phi(s) \cap T \neq \emptyset\} \text{ and } \Phi^{+}(T) = \{s \in X : \Phi(s) \subset T\}$$

as lower and upper inverses of T with respect to Φ respectively. A multifunction $\Phi : X \to Y$ is upper (lower) semi-continuous on X if for every open $U \subset Y$, the set $\Phi^{+}(U)$ ($\Phi^{-}(U)$) is open. A multifunction Φ is continuous if it is upper and lower semi-continuous. A multifunction Φ with compact values is continuous if Φ is a continuous multifunction in the Hausdorff distance.

Denote $Int(S), Bd(S)$ and $\langle S\rangle$, the interior, boundary and the set of all nonempty finite subsets of S respectively.

Let $r \in \mathbb{R}^{+} \cup \{0\}$ and $\emptyset \neq S \subset X$, we denote the $r-$parallel set of S by

$$S + r = \bigcup\{B(s,r) : s \in S\},$$

where $B(s,r) = \{t \in X : d(s,t) \leq r\}$.

For nonempty subsets S and T of X, we define

$$d(S,T) = \inf\{d(s,t) : s \in S, t \in T\}.$$

We call a set K is metrically convex if for any $x, y \in K$ and positive numbers p_i and p_j such that $d(x,y) \leq p_i + p_j$, there exists $z \in K$ such that $z \in B(x,p_i) \cap B(y,p_j)$.

Denote Δ_n, the standard $n-$simplex having vertices $e_1, e_2, \ldots, e_{n+1}$, where e_i is the ith unit vector in $\mathbb{R}^{n+1}$. A $\mathcal{G}$-convex space $(X, D; \Omega)$ consists of a topological space X, a nonempty set D and a multifunction $\Omega : \langle D \rangle \to X$ such that for each $S \in \langle D \rangle$ with the cardinality $|S| = n+1$, there exists a continuous function $\varphi_S : \Delta_n \to \Omega(S)$, such that each $J \in \langle S \rangle$ implies $\varphi_S(\Delta_J) \subset \Omega(J)$, where Δ_J denotes the faces of Δ_n corresponding to $J \in \langle S \rangle$. We write $\Omega(S) = \Omega_S$ for each $S \in \langle D \rangle$. Note that S may or may not be a subset of Ω_S. For $(X, D; \Omega)$ a subset K of X is called $\Omega-$convex if for each $S \in \langle D \rangle, S \subset K$ implies $\Omega_S \subset K$. If $D = X$, then $(X, D; \Omega)$ announced as (X, Ω). For any $K \subset X$, the $\Omega-$convex hull of K is denoted and defined by

$$co_\Omega K := \bigcup\{\Omega_S : S \in \langle K \rangle\}.$$

A multifunction $\Phi : K \to X$ is a KKM map if $\Omega_S \subset \Phi(S)$ for each $S \in \langle K \rangle$, where K is a $\Omega-$convex subset of X, see for example [25]. A multifunction $\Phi : K \to X$ is called generalized KKM map if for each $S \in \langle K \rangle$, there exists a function $\varrho : S \to X$ such that $\Omega_{\varrho(T)} \subset \Phi(T)$ for each $T \in \langle S \rangle$.

H. Kim and S. Park in [14] (Theorem 3), obtained an extension of KKM theorem of Ky Fan, see [12] (Lemma 1) and [13] (Theorem 4).

Theorem 1. *Let $(X, D; \Omega)$ be a $\mathcal{G}$-convex space, S a nonempty set and $\Phi : S \to X$ a multifunction with closed (resp. open) values. If Φ is a generalized KKM map, then the class of its values has the finite intersection property (More precisely, for each $T \in \langle S \rangle$, there exists and $T' \in \langle D \rangle$ such that $\Omega_{T'} \cap \bigcap_{t \in T} \Phi(t) \neq \emptyset$.*

In this paper we use the following Corollary of Theorem 1.

Theorem 2. *Let (X, Ω) be a $\mathcal{G}$-convex space, S a nonempty set and $\Phi : S \to X$ a generalized KKM map with closed values. If there exists a nonempty compact subset L of X such that $\bigcap_{t \in T} \Phi(t) \subset L$ for some $T \in \langle S \rangle$ then $\bigcap_{s \in S} \Phi(s) \neq \emptyset$.*

2. Main Results

In this section, by using Theorem 2, we prove a new best approximation theorem in $\mathcal{G}$-convex spaces.

Theorem 3. *Let $\Phi : S \to X$ be a continuous multi map with compact values such that*

$$\Phi(x) + r \text{ is } \Omega-\text{convex for all } x \in S, r \geq 0 \tag{1}$$

and $g : S \to S$ is a continuous onto map, where (X, Ω) a $\mathcal{G}$-convex space with metric d and S a nonempty Ω-convex subset of X. If there exists a nonempty compact subset K of X such that

$$\bigcap_{y \in M} \{x \in S : d(g(x), \Phi(x)) \leq d(g(y), \Phi(x))\} \subset K \text{ for some } M \in \langle S \rangle, \tag{2}$$

then there exists $v_0 \in S$ such that

$$d(g(v_0), \Phi(v_0)) = \inf_{x \in S} d(x, \Phi(v_0)).$$

If S is metrically convex and $g(v_0) \notin \Phi(v_0)$, then $v_0 \in Bd(S)$.

Proof. Define the multimaps $H, T : S \to S$ by

$$H(x) = \{y \in S : d(g(y), \Phi(y)) \leq d(g(x), \Phi(y))\},$$

$$T(x) = (g \circ H)(x).$$

We have that $T(x)$ is nonempty for each $x \in S$, because $g(x) \in T(x)$ for each $x \in S$. We prove that T is generalized KKM map. Suppose that there exists $\{x_1, \ldots, x_n\} \in \langle K \rangle$ such that $g^{-1}(\Omega_{\{g(x_1),\ldots,g(x_n)\}})$ is not a subset of $H(\{x_1, \ldots, x_n\})$. Then there exists $z \in g^{-1}(\Omega_{\{g(x_1),\ldots,g(x_n)\}})$ such that $z \notin H(x_k)$ for every $k \in \{1, \ldots, n\}$. So, we have

$$d(g(z), \Phi(z)) > d(g(x_k), \Phi(z)) \text{ for every } k \in \{1, \ldots, n\}.$$

Let

$$r = \max_{1 \leq k \leq n} \{d(g(x_k), \Phi(z))\},$$

we have

$$g(x_k) \in \Phi(z) + r \text{ for all } k \in \{1, \ldots, n\}.$$

This implies that

$$\Omega_{\{g(x_1),\ldots,g(x_n)\}} \subset co_{\Omega}(\Phi(z) + r).$$

Since

$$z \in g^{-1}(\Omega_{\{g(x_1),\ldots,g(x_n)\}}),$$

we have

$$g(z) \in \Omega_{\{g(x_1),\ldots,g(x_n)\}},$$

so,

$$g(z) \in co_{\Omega}(\Phi(z) + r).$$

From condition (1) we obtain

$$g(z) \in \Phi(z) + r.$$

So, exists $u \in \Phi(z)$ such that

$$d(g(z), u) \leq r,$$

that is why

$$d(g(z), \Phi(z)) \leq d(g(z), u) \leq r < d(g(z), \Phi(z)).$$

This is a contradiction. Therefore, for each $D \in \langle S \rangle$ we have

$$g^{-1}(\Omega_{g(D)}) \subseteq H(D).$$

Since g is onto map we have that

$$\Omega_{g(D)} \subseteq T(D) \text{ for each } D \in \langle S \rangle.$$

This implies that T is a generalized KKM map. Since maps Φ and g are continuous we get that $T(x)$ is closed for each $x \in S$. Hence, by condition (2) and Theorem 2, there exists $v_0 \in S$ such that

$$d(g(v_0), \Phi(v_0)) = \inf_{x \in S} d(x, \Phi(v_0)).$$

If S is metrically convex and $g(v_0) \notin \Phi(v_0)$ then $v_0 \in Bd(S)$. Namely, if $v_0 \in Int(S)$, then there exists $\gamma > 0$ such that

$$B_S(v_0, \gamma) = \{x \in S : d(v_0, x) < \gamma\} \subset S$$

and

$$\gamma < d(g(v_0), \Phi(v_0)) \leq d(x, \Phi(v_0)) \text{ for all } x \in B_S(g(v_0), \gamma).$$

Let $u_0 \in \Phi(v_0)$ such that $d(g(v_0), \Phi(v_0)) = d(g(v_0), u_0)$. Then, if S is metrically convex, we obtain

$$B_S(g(v_0), \gamma) \cap B_S(u_0, d(g(v_0), \Phi(v_0)) - \gamma) \neq \emptyset.$$

Since

$$B_S(u_0, d(g(v_0), \Phi(v_0) - \gamma) \subset \Phi(v_0) + d(\Phi(v_0), g(v_0)) - \gamma,$$

we have

$$B_S(g(v_0), \gamma) \cap (\Phi(v_0) + d(\Phi(v_0), g(v_0)) - \gamma) \neq \emptyset.$$

Let $z \in S$ such that

$$z \in B_S(g(v_0), \gamma) \cap (\Phi(v_0) + d(g(v_0), \Phi(v_0) - \gamma),$$

we obtain

$$d(g(v_0), \Phi(v_0)) \leq d(z, \Phi(v_0)) \leq d(g(v_0), \Phi(v_0)) - \gamma < d(g(v_0), \Phi(v_0)),$$

a contradiction. Therefore, $v_0 \in Bd(S)$. □

Next results follows from Theorem 3.

Corollary 1. *Let $\Phi : S \to X$ be a continuous multi map with compact values such that condition (1) is satisfied and $g : S \to S$ is a continuous onto map, where (X, Ω) a $\mathcal{G}$-convex space with metric d and S a nonempty Ω-convex set contained in compact subset of X. Then there exists $v_0 \in K$ such that*

$$d(g(v_0), \Phi(v_0)) = \inf_{x \in S} d(x, \Phi(v_0)).$$

If K is metrically convex and $g(v_0) \notin \Phi(v_0)$, then $v_0 \in Bd(S)$.

Corollary 2. *Let the metric space (X, Ω) be a $\mathcal{G}$-convex space with metric d, S a nonempty $\Omega-$convex set contained in compact subset of X, $\Phi : S \to X$ is a continuous multimap with compact values such that condition (1) is satisfied. Then there exists $v_0 \in K$ such that*

$$d(v_0, \Phi(v_0)) = \inf_{x \in S} d(x, \Phi(v_0)).$$

If K is metrically convex and $v_0 \notin \Phi(v_0)$ then $v_0 \in Bd(S)$.

3. Some Applications

As some applications of our results, we give the versions of Fan' best approximation theorem in hyperconvex and normed spaces.

Recall that a metric space (X, d) is called a hyperconvex metric space if for any class of elements x_i of X and any class $p_i \in \mathbb{R}^+ \cup \{0\}$ with $d(x_i, x_j) \leq p_i + p_j$, we have

$$\bigcap_i \mathcal{B}(x_i, p_i) \neq \emptyset.$$

Let $\mathcal{U}$ be a nonempty bounded subset of a hyperconvex metric space X, denote

$$co\mathcal{U} = \bigcap\{\mathcal{V} : \mathcal{V} \text{ is closed ball in } X \text{ containing } \mathcal{U}\}.$$

Denote $\mathscr{U}(X) = \{\mathcal{U} \subset X : \mathcal{U} = co\mathcal{U}\}$, the elements of this set are known as admissible subset of X. Moreover, any hyperconvex metric space (X, d) is an $\mathcal{G}$-convex space (X, Ω), with $\Omega_{\mathcal{U}} = co\mathcal{U}$ for each $\mathcal{U} \in \langle X \rangle$. The r-parallel set of an admissible subset of a hyperconvex metric space is also an admissible set, see R. Espínola and M. A. Khamsi [11] (Lemma 4. 10). In this case the condition (1) is satisfied.

Following Corollary 1, we obtained best approximation result for hyperconvex metric spaces due to A. Amini-Harandi and A. P. Farajzadeh [3] (Theorem 2.1).

Corollary 3. *Let (X, d) be hyperconvex metric space and S be a compact admissible subset of X. Suppose that $\Phi : S \to \mathscr{U}(X)$ continuous multimmap with compact values and $g : S \to S$ is a continuous onto map. Then there exists $v_0 \in K$ such that*

$$d(g(v_0), \Phi(v_0)) = \inf_{x \in S} d(g(x), \Phi(v_0)).$$

Moreover, if $g(v_0) \notin \Phi(v_0)$ then $v_0 \in Bd(S)$.

In view of Corollary 2, the result of G. X. Z. Yuan, [25] (Theorem 2. 11. 16) and for single-valued maps, the result of M. A Khamsi, [16] (Lemma) are obtain as follows:

Corollary 4. *Let $\Phi : K \to X$ be a continuous multimap on a nonempty admissible compact set K to hyperconvex metric space X. Then there exists an element v_0 in K such that*

$$d(v_0, \Phi(v_0)) = \inf_{x \in K} d(x, \Phi(v_0)).$$

Corollary 5. *Let $\phi : K \to X$ be a continuous map on a nonempty admissible compact set K to hyperconvex metric space X. Then there exists an element v_0 in K such that*

$$d(v_0, \phi(v_0)) = \inf_{x \in K} d(x, \phi(v_0)).$$

If X is a normed linear space, then condition (1) in Theorem 3 is satisfied. So, from Theorem 3 we obtain the next result for normed linear spaces.

Theorem 4. *Let X be a normed linear space, S a nonempty convex set contained in compact subset of X, $\Phi : S \to X$ is a continuous multimap with convex compact values and $g : S \to S$ is a continuous onto map. Then there exists $v_0 \in S$ such that*

$$||g(v_0) - \Phi(v_0)|| = \inf_{x \in S} d(x, \Phi(v_0)).$$

J. B. Prolla [21] and A. Carbone [7] obtained a form of Theorem 4 using almost affine and almost quasi-convex maps in normed vector spaces.

Definition 1. *Let S a nonempty convex subset of a normed space X. A map $g : S \to X$ is*

(i) almost affine if for all $x, y \in S$ and $u \in S$

$$||g(\lambda x + (1-\lambda)y) - u|| \leq \lambda||g(x) - u|| + (1-\lambda)||g(y) - u||,$$

for each λ with $0 < \lambda < 1$.

(ii) almost quasi-convex if for all $u \in S$ and $r > 0$, the set

$$\{x \in K : ||g(x) - u|| < r\} \text{ is convex.}$$

Note that the mapping to be an almost quasi-convex is unnecessary in Theorem 4.

Corollary 6. *Let X be a normed linear space, S a nonempty convex compact subset of X, $\phi : S \to X$ is a continuous map and $g : S \to S$ is a continuous, almost affine, onto map. Then there exists $v_0 \in S$ such that*

$$||g(v_0) - \phi(v_0)|| = \inf_{x \in S} d(x, \phi(v_0)).$$

Corollary 7. *Let X be a normed linear space, S a nonempty convex compact subset of X, $\phi : S \to X$ is a continuous map and $g : S \to S$ is a continuous, almost quasi-convex, onto map. Then there exists $v_0 \in S$ such that*

$$||g(v_0) - \phi(v_0)|| = \inf_{x \in S} d(x, \phi(v_0)).$$

Example 1. *Let $X = \mathbb{R}, S = [0,1]$ and define maps $\phi : S \to S$ and $g : S \to S$ by*

$$\phi(x) = x, \; g(x) = 4x - 4x^2.$$

Then map g is not almost quasi-convex and results of J. B. Prolla [21] and A. Carbone [7] are not applicable. Note that the maps ϕ and g satisfy all hypotheses of Theorem 4 and $v_0 \in \{0, \frac{3}{4}\}$.

Author Contributions: Conceptualization, Z.D.M.; Investigation, Z.D.M., A.H., M.d.l.S. and S.R.; Methodology, S.R.; Software, Z.D.M.; Supervision, M.d.l.S. and S.R. All authors contributed equally in the preparation of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding

Acknowledgments: The third author would like to thanks Basque Government for its support of this work through Grant IT1207-19.

Conflicts of Interest: The authors declare no conflict of interest.

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