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Fixed-Point Study of Generalized Rational Type Multivalued Contractive Mappings on Metric Spaces with a Graph

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Abstract: The main result of this paper is a fixed-point theorem for multivalued contractions obtained through an inequality with rational terms. The contraction is an F-type contraction. The results are obtained in a metric space endowed with a graph. The main theorem is supported by illustrative examples. Several results as special cases are obtained by specific choices of the control functions involved in the inequality. The study is broadly in the domain of setvalued analysis. The methodology of the paper is a blending of both graph theoretic and analytic methods.

Keywords: fixed-point; multivalued maps; F-contraction; directed graph; metric space

MSC: 47H10; 54H10; 54H25

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1. Introduction and Mathematical Preliminaries

Let (X, d) be a metric space. The following standard notations and definitions will be used. $N(X)$ is the family of all nonempty subsets of X, $B(X)$ is the family of all nonempty bounded subsets of X, $CB(X)$ is the family of all nonempty closed and bounded subsets of X, $K(X)$ is the family of all nonempty compact subsets of X and

$$D(x, B) = \inf\{d(x, y) : y \in B\}, \text{ where } x \in X \text{ and } B \in B(X),$$

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}, \text{ where } A, B \in CB(X).$$

H is known as the Hausdorff metric induced by the metric d on $CB(X)$ [1]. Furthermore, if (X, d) is complete then $(CB(X), H)$ is also complete.

Let X be a nonempty set and $\mho = \{(x, x) : x \in X\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\mho \subseteq E(G)$. Assume that G has no parallel edges. By G^{-1} we denote the graph obtained from G by reversing the directions of the edges. Thus, $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. By $\widetilde{G}$ we denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the $V(\widetilde{G}) = V(G)$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$. A nonempty set X is said to be endowed with a directed graph $G(V, E)$ if $V(G) = X$ and $\mho \subseteq E(G)$.

Let $F : (0, \infty) \to \mathbb{R}$ be a function with the following properties:
$(F1)$ F is strictly increasing, i.e., $x < y \Longrightarrow F(x) < F(y)$;
$(F2)$ For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $(0, \infty)$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0} \alpha^k F(\alpha) = 0$;

$(F4)$ $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.
We denote the set of all functions F satisfying $(F1 - F3)$ by $\Im$ and the set of all functions F satisfying $(F1 - F4)$ by $\Im_*$.

Wardowski [2] introduced the notion of F-contraction and established a new type of generalization of the Banach's contraction mapping principle.

Definition 1 ([2]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exist $F \in \Im$ and $\tau > 0$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

holds for any $x, y \in X$ with $d(Tx, Ty) > 0$.

Theorem 1 ([2]). *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point ξ in X.*

Definition 2 ([3]). *Let (X, d) be a metric space endowed with a directed graph $G(V, E)$. A mapping $T : X \to X$ is graph-preserving if*

$$(x, y) \in E(G), \text{ for } x, y \in X \Longrightarrow (Tx, Ty) \in E(G).$$

Definition 3 ([4]). *Let (X, d) be a metric space endowed with a directed graph $G(V, E)$. A mapping $T : X \to X$ is said to be an GF-contraction if T is graph-preserving and there exist $F \in \Im$ and $\tau > 0$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

holds for any $x, y \in X$ with $(x, y) \in E(G)$ and $d(Tx, Ty) > 0$.

Definition 4 ([5]). *Let (X, d) be a metric space endowed with a directed graph $G(V, E)$. A multivalued mapping $T : X \to CB(X)$ is graph-preserving if*

$$(x, y) \in E(G), \text{ for } x, y \in X \Longrightarrow (u, v) \in E(G), \text{ whenever } u \in Tx \text{ and } v \in Ty.$$

Lemma 1 ([5]). *Let (X, d) be a metric space and $T : X \to N(X)$ be an upper semi-continuous mapping such that Tx is closed for all $x \in X$. If $x_n \to x_0$, $y_n \to y_0$ and $y_n \in Tx_n$, then $y_0 \in Tx_0$.*

Definition 5 ([5]). *Let (X, d) be a metric space endowed with a directed graph $G(V, E)$. A multivalued mapping $T : X \to CB(X)$ is weakly graph-preserving if $(x, y) \in E(G)$ where $x \in X$ and $y \in Tx$, implies that $(y, z) \in E(G)$ for all $z \in Ty$.*

Let X be a nonempty set and $T : X \to N(X)$ be a multivalued mapping. We define

$$P_T = \{x \in X : x \in Tx\}, \quad T_G = \{(x, y) \in E(G) : H(Tx, Ty) > 0\} \text{ and}$$

$$X_T = \{x \in X : (x, y) \in E(G) \text{ for some } y \in Tx\}.$$

The following class of functions will be used in our results in the next section.

Let ψ: $[0, \infty)^5 \to [0, \infty)$ be such that (i) ψ is continuous and monotone nondecreasing in each coordinate, (ii) $\psi(t, t, t, t, t) \leq t$ for all $t \geq 0$. We denote the collection of such functions ψ by the symbol Ψ.

Let ϕ: $[0, \infty)^4 \to [0, \infty)$ be such that (i) ϕ is continuous and monotone nondecreasing in each coordinate, (ii) $\phi(x_1, x_2, x_3, x_4) = 0$ if $x_1 x_2 x_3 x_4 = 0$. We denote the collection of such functions ϕ by the symbol Φ.

Using the above mathematical notions in this paper we establish an F-contraction type multivalued fixed-point result in a metric space with a graph. Fixed-point theory on metric spaces with the additional structure of a graph is a recent development. Some works from this line of research can be found in works such as [3,5–10]. We make specific choices of a particular function used in the metric inequality to discuss special cases of the main theorem. This demonstrates the generality of our result. It may be further mentioned that F-contractions are new concepts in metric fixed-point theory which have been extended in various ways in works such as [2,4–6,11,12]. Essentially our results are in the domain of setvalued analysis to which the Banach contraction mapping principle was extended by Nadler [1]. In his result Nadler used the Hausdorff distance. The work was followed by several other works such as [5,6,13–15]. The contractive inequality which we use in our problem involves some rational terms. Dass and Gupta [16] generalized the Banach's contraction mapping principle by using a contractive condition of rational type. Fixed-point theorems for contractive type conditions satisfying rational inequalities in metric spaces have been developed in several works [17–20]. Finally, we support our main theorem with illustrative examples.

2. Main Result

Theorem 2. *Let $(X,\ d)$ be a complete metric space endowed with a directed graph G and $T : X \to K(X)$ be a multivalued map. Suppose that (i) T is upper semi-continuous and weakly graph-preserving, (ii) X_T is nonempty, (iii) there exist $\tau > 0$, $F \in \Im$, $\psi \in \Psi$ and $\phi \in \Phi$ such that for $x, y \in X$ with $(x, y) \in T_G$,*

$$\tau + F(H(Tx, Ty)) \leq F(M(x, y) + N(x, y)),$$

where $N(x,y) = \phi\Big(D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\Big)$ *and*

$$M(x,y) = \psi\Big(d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Tx)D(y,Ty) + D(x,Ty)D(y,Tx)}{1+d(x,y)}, \frac{D(x,Tx)D(y,Ty) + D(x,Ty)D(y,Tx)}{1+H(Tx,Ty)}\Big).$$

Then P_T is nonempty.

Proof. Let us assume T has no fixed point. Then $D(x, Tx) > 0$ for all $x \in X$. Let $x_0 \in X_T$. Then there exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. Now $0 < D(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, which implies that $(x_0, x_1) \in T_G$. Using the assumption (iii) and a property of F, we have

$$F(D(x_1,Tx_1)) \leq F(H(Tx_0,Tx_1)) \leq F(M(x_0,x_1) + N(x_0,x_1)) - \tau, \tag{1}$$

where

$$\begin{aligned} M(x_0,x_1) &= \psi\Big(d(x_0,x_1), D(x_0,Tx_0), D(x_1,Tx_1), \\ &\qquad \frac{D(x_0,Tx_0)D(x_1,Tx_1) + D(x_0,Tx_1)D(x_1,Tx_0)}{1+d(x_0,x_1)}, \\ &\qquad \frac{D(x_0,Tx_0)D(x_1,Tx_1) + D(x_0,Tx_1)D(x_1,Tx_0)}{1+H(Tx_0,Tx_1)}\Big) \\ &\leq \psi\Big(d(x_0,x_1), d(x_0,x_1), D(x_1,Tx_1), \frac{D(x_0,Tx_0)D(x_1,Tx_1)}{1+d(x_0,x_1)}, \\ &\qquad \frac{D(x_0,Tx_0)D(x_1,Tx_1)}{1+H(Tx_0,Tx_1)}\Big) \\ &\leq \psi\Big(d(x_0,x_1), d(x_0,x_1), D(x_1,Tx_1), \frac{D(x_0,Tx_0)D(x_1,Tx_1)}{1+d(x_0,x_1)}, \\ &\qquad \frac{d(x_0,x_1)D(x_1,Tx_1)}{1+D(x_1,Tx_1)}\Big) \\ &\leq \psi\Big(d(x_0,x_1), d(x_0,x_1), D(x_1,Tx_1), D(x_1,Tx_1), d(x_0,x_1)\Big) \end{aligned} \tag{2}$$

and

$$0 \le N(x_0, x_1) = \phi(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0))$$
$$\le \phi\Big(d(x_0, x_1), D(x_1, Tx_1), D(x_0, Tx_1), d(x_1, x_1)\Big) = 0,$$

that is, $N(x_0, x_1) = 0$. □

If possible, suppose that $d(x_0, x_1) \le D(x_1, Tx_1)$. Then from (2), using the properties of ψ, we have

$$M(x_0, x_1) \le \psi\Big(D(x_1, Tx_1), D(x_1, Tx_1), D(x_1, Tx_1), D(x_1, Tx_1), D(x_1, Tx_1)\Big)$$
$$\le D(x_1, Tx_1).$$

Using (1) and a property of F, we have

$$F(D(x_1, Tx_1)) \le F(D(x_1, Tx_1)) - \tau,$$

which is a contradiction. Thus, $D(x_1, Tx_1) < d(x_0, x_1)$. Using (2) and the properties of ψ, we have

$$M(x_0, x_1) \le \psi\Big(d(x_0, x_1), d(x_0, x_1), d(x_0, x_1), d(x_0, x_1), d(x_0, x_1)\Big)$$
$$\le d(x_0, x_1).$$

By (1) and a property of F, we have

$$F(D(x_1, Tx_1)) \le F(d(x_0, x_1)) - \tau. \tag{3}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = D(x_1, Tx_1)$. Hence from (3), we have

$$F(d(x_1, x_2)) \le F(d(x_0, x_1)) - \tau. \tag{4}$$

As T is weakly graph-preserving, $(x_0, x_1) \in E(G)$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, we have $(x_1, x_2) \in E(G)$. Now, $0 < D(x_2, Tx_2) \le H(Tx_1, Tx_2)$, which implies that $(x_1, x_2) \in T_G$. By the assumption (iii) and a property of F, we have

$$F(D(x_2, Tx_2)) \le F(H(Tx_1, Tx_2)) \le F(M(x_1, x_2) + N(x_1, x_2)) - \tau. \tag{5}$$

Arguing similarly as before, we have

$$F(D(x_2, Tx_2)) \le F(d(x_1, x_2)) - \tau. \tag{6}$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) = D(x_2, Tx_2)$. From (6), we have

$$F(d(x_2, x_3)) \le F(d(x_1, x_2)) - \tau. \tag{7}$$

Continuing this process, we construct a sequence $\{x_n\}$ such that for all $n \ge 0$,

$$x_{n+1} \in Tx_n, \quad (x_n, x_{n+1}) \in T_G \tag{8}$$

and

$$F(d(x_n, x_{n+1})) \le F(d(x_{n-1}, x_n)) - \tau. \tag{9}$$

Let $\gamma_n = d(x_n, x_{n+1})$ for all $n \ge 0$.

From (9), we have

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq ... \leq F(\gamma_0) - n\tau. \tag{10}$$

Taking limit as $n \to \infty$ in the above inequality, we get $\lim\limits_{n\to\infty} F(\gamma_n) = -\infty$, which by property (F_2) of F, implies that $\lim\limits_{n\to\infty} \gamma_n = 0$. Then by property (F_3) of F, there exists $k \in (0,1)$ such that $\lim\limits_{n\to\infty} \gamma_n^k F(\gamma_n) = 0$. Now, using (10), we have

$$\gamma_n^k F(\gamma_n) - \gamma_n^k F(\gamma_0) \leq -\gamma_n^k n\tau \leq 0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\gamma_n^k = 0.$$

Then there exists $n_1 \in N$ such that $n\gamma_n^k \leq 1$ for all n $\geq n_1$, which implies that $\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_1$. Then we have

$$\sum_{n=n_1}^{\infty} d(x_n, x_{n+1}) = \sum_{n=n_1}^{\infty} \gamma_n \leq \sum_{n=n_1}^{\infty} \frac{1}{n^{\frac{1}{k}}}.$$

As $0 < k < 1$, $\sum_{n=n_1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent. Then it follows that $\sum d(x_n, x_{n+1})$ is convergent. This implies that $\{x_n\}$ is a Cauchy sequence. As X is complete, there exists $z \in X$ such that $\lim\limits_{n\to\infty} x_n = z$. Since T is upper semi-continuous, by Lemma 1, we have $z \in Tz$, which contradicts the assumption that T has no fixed point. Hence T has a fixed point, i.e., P_T is nonempty.

Remark 1. *Varying the functions ψ and ϕ in the assumption (iii) of Theorem 2, we have different form of F-contractions for which Theorems 2 hold. For some examples, choosing*

(a) $\psi(t_1, t_2, t_3, t_4, t_5) = t_1$ *and* $\phi(x_1, x_2, x_3, x_4) = 0$,
(b) $\psi(t_1, t_2, t_3, t_4, t_5) = \max\{t_2, t_3\}$ *and* $\phi(x_1, x_2, x_3, x_4) = 0$,
(c) $\psi(t_1, t_2, t_3, t_4, t_5) = \max\{t_4, t_5\}$ *and* $\phi(x_1, x_2, x_3, x_4) = 0$,
(d) $\psi(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\}$ *and* $\phi(x_1, x_2, x_3, x_4) = 0$,
respectively, we have the following form of F-contractions respectively

$$1_{(a)} : \tau + F(H(Tx, Ty)) \leq F(d(x, y)),$$

$$2_{(b)} : \tau + F(H(Tx, Ty)) \leq F\Big(max\ \{D(x, Tx), D(y, Ty)\}\Big),$$

$$3_{(c)} : \tau + F(H(Tx, Ty)) \leq F\Big(max\Big\{\frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + d(x, y)}, \frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + H(Tx, Ty)}\Big\}\Big),$$

$$4_{(d)} : \tau + F(H(Tx, Ty)) \leq F(M(x, y)),$$

where $M(x, y) = \max\Big\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + d(x, y)}, \frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + H(Tx, Ty)}\Big\}.$

Remark 2. *Theorem 2 is a generalization of Theorem 2 in [6].*

Remark 3. *Theorem 2 is true for the class of functions $T : X \to CB(X)$ under the consideration* *of the class of function $\Im_*$ instead of $\Im$. Arguing similarly as in the proof of Theorem 2 and taking into account the condition $(F4)$ of F, we get*

$$\begin{aligned} F(D(x_1, Tx_1)) &= F(inf\,\{d(x_1, z) : z \in Tx_1\}) \\ &= inf(F(\{d(x_1, z) : z \in Tx_1\})). \end{aligned}$$

From (3), we have

$$inf(F(\{d(x_1, z) : z \in Tx_1\})) \le F(d(x_0, x_1)) - \tau < F(d(x_0, x_1)) - \frac{\tau}{2}.$$

Then there exists $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) \le F(d(x_0, x_1)) - \frac{\tau}{2}.$$

Arguing similarly as in the proof of Theorem 2, it can be proved that P_T is nonempty.

Example 1. *Take the metric space $X = [0, \infty)$ with usual metric d. Assume that G is a directed graph with $V(G) = X$ and $E(G) = \{(x, y) :\ if\ x, y \in [0, 1]\} \cup \{(x, x) : x > 1\}$. Define a multivalued mapping $T : X \to K(X)$ as $Tx = \begin{cases} [0, \frac{e^{-\tau}}{5}x] & if \quad x \in [0, 1], \\ \{\frac{e^{-\tau}}{5}\} & if \quad x > 1. \end{cases}$*
Let $F(x) = \ln(x)$, $\psi(x_1, x_2, x_3, x_4, x_5) = x_1$, $\phi(x_1, x_2, x_3, x_4) = 0$ and $\tau > 0$. Then T is upper semi-continuous and weakly graph-preserving. Let $x, y \in X$ with $(x, y) \in E(G)$ and $H(Tx, Ty) > 0$. Then $x, y \in [0, 1]$ with $x \ne y$. Without loss of generality, assume that $y < x$. Then

$$H(Tx, Ty) = e^{-\tau}\frac{1}{5}|x - y| \le e^{-\tau}|x - y| = e^{-\tau} d(x, y).$$

Taking 'ln' on both sides of the above equation, we get

$$\begin{aligned} F(H(Tx, Ty)) &\le -\tau + F(d(x, y)), \\ \tau + F(H(Tx, Ty)) &\le F(d(x, y)), \\ \tau + F(H(Tx, Ty)) &\le F(M(x, y) + N(x, y)), \end{aligned}$$

where $N(x, y) = \phi\Big(D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\Big)$ and

$$\begin{aligned} M(x, y) = \psi\Big(d(x, y), D(x, Tx), D(y, Ty), &\frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + d(x, y)}, \\ &\frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + H(Tx, Ty)}\Big). \end{aligned}$$

Thus, all the conditions of Theorem 2 are satisfied and here $P_T = \{0\}$ is the fixed-point set of T.

Example 2. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and G be a directed graph with $V(G) = X$ and $E(G) = \{(0, 0), (0, 1), (0, 4), (0, 5), (1, 1), (1, 0), (1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6), (6, 7), (7, 1), (7, 7), (8, 7), (8, 8)\}$. Let d be a metric defined on X as $d(x, y) = \begin{cases} 0 & if \quad x = y, \\ x + y & if \quad x \ne y. \end{cases}$*
Let $T : X \to K(X)$ be defined as

$$T(x) = \begin{cases} \{4, 5\}, & if \quad x \in \{0, 4, 5\}, \\ \{2, 3\}, & if \quad x \in \{1, 2, 3\}, \\ \{7\}, & if \quad x \in \{6, 8\}, \\ \{1\}, & if \quad x = 7. \end{cases}$$

Let $F(x) = \ln(x)$, $\psi(x_1, x_2, x_3, x_4, x_5) = max(x_1, x_2, x_3, x_4, x_5)$, $\phi(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4$ *and* $\tau = 0.2$. *Then all the conditions of Theorem 2 are satisfied and here* $P_T = \{2, 3, 4, 5\}$ *is the fixed-point set of* T.

Remark 4. *Take* $x = 0$ *and* $y = 1$. *Then* $H(T0, T1)$= 7, $d(0, 1) = 1$, $D(0, T0) = 4$, $D(1, T1) = 3$, $D(0, T1) = 2$, $D(1, T0) = 5$. *It is easy to verify that the inequality (3.1) of Theorem 2 in [6] is not satisfied when* $x = 0$ *and* $y = 1$. *Therefore, the above example is not applicable in case of Theorem 2 in [6]. Hence Theorem 2 is an actual extension of Theorem 2 in [6].*

3. Conclusions

In this paper, we combine several concepts which have featured prominently in the recent literature of fixed-point theory. Fixed-point theory has many applications as, for instances, those in [10,21]. It is our perception that the structure of graph on the metric space allows us to obtain fixed-point results with more flexibility and for making some new applications. These problems are supposed to be taken up in our future works.

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