Published for SISSA by Springer

Received: *February 2, 2022*
Accepted: *March 3, 2022*
Published: *March 21, 2022*

3D supersymmetric nonlinear multiple D0-brane action and 4D counterpart of multiple M-wave system

Igor Bandos[a,b] **and Unai D.M. Sarraga**[a]

[a] *Department of Physics, University of the Basque Country UPV/EHU, P.O. Box 644, 48080 Bilbao, Spain*
[b] *IKERBASQUE, Basque Foundation for Science, Plaza Euskadi 5, 48009 Bilbao, Spain*

E-mail: igor.bandos@ehu.eus, unaidemiguel13@gmail.com

Abstract: Found is the complete nonlinear action of multiple D0-brane system (mD0) in three dimensional type II superspace which is invariant under rigid $D = 3$ $\mathcal{N} = 2$ spacetime supersymmetry and under local worldline supersymmetry generalizing the κ-symmetry of single D0-brane action. We show that a particular representative of this family of actions can be obtained by dimensional reduction of the action of $D = 4$ non-Abelian multiwaves (nAmW), the $D = 4$ counterpart of 11D multiple M-wave (mM0) action, that we have also constructed in this paper. This reduction results in an action with is nonlinear due to the presence of a certain function $\mathcal{M}(\mathcal{H})$ of the relative motion Hamiltonian $\mathcal{H}$, the counterpart of which enters the 4D nAmW action linearly. Curiously, the action possesses double supersymmetry also for an arbitrary function $\mathcal{M}(\mathcal{H})$. In particular for $\mathcal{M} = \text{const}$ we find a dynamical system describing the sum of single D0 action and the action of 1d dimensional reduction of the $D = 3$ $\mathcal{N} = 2$ SYM coupled to the worldline supergravity induced by the embedding of the center of energy motion into the $D = 3$ $\mathcal{N} = 2$ superspace.

Keywords: D-Branes, M-Theory, Superspaces

ArXiv ePrint: 2112.14610

https://doi.org/10.1007/JHEP03(2022)122

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1 Introduction

The search for supersymmetric action describing the system of nearly coincident Dirichlet-p-branes (Dp-branes or super-Dp-branes) of string theory can be followed for almost 25 years. It is not expected to be given just by the sum of N actions of individual Dp-branes, which are known from late 90th [1–7].[1] The search is for an effective action which would describe a system of N Dp-branes and N^2 strings which have endpoints attached to the same or different Dp-branes. In [16] Witten argued that such a system should allow for the (gauge fixed) description in terms of U(N) supersymmetric Yang-Mills (SYM) multiplet.

In weak field limit a single Dp-brane allows for a gauge fixed description by the Abelian SYM action, to be precise, by the action of maximally supersymmetric $d = p + 1$ SYM. In it the scalar (and spinor) fields of SYM multiplet describe (roughly speaking) the position of the Dp-brane worldvolume in $D = 10$ dimensional spacetime (superspace), while the gauge field provides the low energy description of fundamental superstring. In the case of N (closely located) Dp-branes one should also account for the contribution of strings having their ends on different branes. The low energy description of such a string would be also provided by vector field, but with a mass gap increasing with separation between branes. In the limit of N coincident Dp-branes all the N^2 gauge fields become massless. Witten suggested [16] that in this limit of the coincident Dp-brane also the $\mathrm{U}(1)^{N^2}$ gauge symmetry of the system should be enhanced to U(N). The very low energy (gauge fixed) description of the system of N coincident super-Dp-branes is thus believed to be provided by the action of the maximally supersymmetric $d = (p + 1)$ dimensional U(N) supersymmetric gauge theory.

As far as the effective action of the bosonic Dp-brane is the Dirac-Born-Infeld action [17], in particular, the Born-Infeld action in the case of spacetime filling D9-brane [18], Tseytlin proposed in this context to study a non-Abelian generalization of the Born-Infeld action with symmetric trace prescription [19]. The widely accepted candidate for the bosonic part of the action of coincident Dp-brane is the so-called dielectric brane action proposed by Myers in [20] motivated by consistency of T-duality rules for the background and Dp-branes fields. It possess an interesting 'dielectric brane effect' which consists in polarization of higher p-brane changes in its low p-brane version (see also the related study in earlier [21]). A similar bosonic action for the system of multiple M-theory gravitons was proposed and studied in [22–25]. However, the straightforward supersymmetric generalizations of the actions from [20] and [22, 23] were searched for decades and still remain unknown.

A very interesting supersymmetric '-1 quantization' approach to the problem was developed by Howe, Lindstrom and Wulff in [26, 27] using the so-called boundary fermions. The quantization of these allows to reproduces the nonabelian algebra generators from their bilinears, so that the quantization of the dynamical system from [26–28] should reproduce the desired multiple Dp-brane action. However, the complete consistent realization of this

[1] Actually, the subject of [1] was the derivation of the equations of motion for super-p-branes, including super-Dp-branes in the frame of superembedding approach [8–10]. This is based on (and sometimes identified with) the so-called STV (worldline superfield) approach to superparticles and superstrings pioneered in [11] (see [12] for early review) and reaching its highest achievement in [13–15] where the superfield action for heterotic superstring has been constructed.

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step seems to require the quantization of the complete interacting system of supergravity and super-Dp-brane.

The search for a supersymmetric mDp counterpart of the single Dp-brane actions was the subject of study in [29–35]. In [30] a supersymmetrized version of $D = 3$ Born-Infeld action was described on the basis of superembedding approach [8, 9, 36] and generalized action principle [37]. Besides that the supersymmetric and κ-symmetric actions are known for $d = 1$ ($p = 0$) non-Abelian multibrane systems only. These include the action for the system of ten-dimensional (10D) multiple 0-branes [29, 31] (see [35] for its review and comparative discussion), the action for 11D multiple M0 (mM0 or multiple M-waves) system [32, 34] and a candidate for multiple D0-brane (mD0) action in $D = 10$ [35] (the present study suggests the existence of its nonlinear generalization).

The reason to discuss 11D mM0 system in the context of present discussion on mDp-branes is based on the observation [6] that the dimensional reduction of single M0-brane, which is 11D massless superparticle [6], reproduces 10D D0-brane action. Then it was natural to expect that the 10D mD0 action can be obtained by dimensional reduction from some 11D action which should be a non-Abelian generalization of the M0 action.

Such 11D action was constructed in [32], essentially by completing the M0 (massless superparticle) action, describing the center of energy motion, by interacting terms involving matrix fields, the same as should describe the relative motion sector of the 10D mD0 system. Namely these latter are the fields of 10D SU(N) SYM multiplet dimensionally reduced to $d = 1$ (as, according to [16], are the fields describing mD0 system in the limit of very low energy). This action was interpreted as describing the dynamical system of multiple interacting M-waves (multiple M0-branes or mM0), which is similar to the treatment of the bosonic Myers-type 11D action in [22, 23]. An evidence of meaningfulness of this mM0 action in the String/M-theory context is its invariance under two supersymmetries, the 11D spacetime supersymmetry and the worldline supersymmetry generalizing the κ-symmetry of massless superparticle (M0-brane) action. This latter guarantees that the ground state of the system is 1/2 BPS state, i.e. that it preserves 1/2 of the spacetime supersymmetry and thus saturates the so-called BPS bound (the fact that guarantees its stability).[2]

On the other hand, an attempt to obtain an action for 10D mD0 system from this 11D mM0 action, performed in [35], was not successful. Instead a candidate mD0 action was constructed in [35] directly, by coupling of 1d reduction of 10D SU(N) SYM to the supergravity induced by the embedding of the center of mass worldline in 10D type IIA superspace. The result of present study suggests the hint to resolve this problem with dimensional reduction which is important for the treatment of mM0 as a decompactification limit of type IIA mD0 system.

[2]The mM0 action of [32] can be also discussed in the context of 11D multiple Mp-brane systems, for which the allowed values of p is expired by $p = 0, 2, 5$. The progress in understanding of the mM2 and mM5 cases is even more restricted then of mDp-branes. Till 2007 it was not even clear what should play the role of $d = p + 1$ U(N) SYM description of the very low energy limit of mDp-brane. Now it is believed that the infrared fixed points of the system of N coincident M2-branes is described by Bagger-Lambert-Gustavsson (BLG) model [38–41] for $N = 2$ and by Aharony-Bergman-Jafferis-Maldacena (ABJM) model [41, 42] for $N \geq 2$. The infrared fixed point of multiple M5-brane system should reproduce an enigmatic $D = 6$ $(2, 0)$ superconformal theory which, according to conjecture of [43, 44], can be described by $D = 5$ SYM model if the nonperturbative sector of this is completely accounted for.

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The other problem is that both the 11D mM0 action of [32, 34] and the candidate 10D mD0 action of [35] are known for the case of flat target superspace only. To study this latter problem on a toy model, the $D = 3$ counterpart of mM0 action was constructed in [33], where it was called non-Abelian multiwave system (nAmW) and was also generalized for the case of curved AdS superspace. In [45] the quantization of $D = 3$ nAmW model was performed, which resulted in a system of relativistic field equations for the bosonic and fermionic fields on the space with additional non-commutative bosonic and non-anticommutative fermionic coordinates. It was noticed their that the structure of $D = 4$ counterpart of 3D nAmW quantum theory promises to be simpler and more transparent due to the intrinsic complex structure characteristic for the $D = 4$ $\mathcal{N} = 1$ superspace (and for the $\mathcal{N} = 2$ $D = 3$ one). So the construction and quantization of 4D nAmW system would provide a better basis for studying the quantization of 11D mM0 and 10D mD0 models which, in its turn, might shed a new light on the structure and properties of String/M-theory. The $D = 4$ nAmW system can also provide a conventional toy model to approach the problem of curved space generalization and dimensional reduction of 11D mM0 system.

Motivated by the above arguments, in this paper we first construct the action for $D = 4$ nAmW system in flat $\mathcal{N} = 1$ superspace, the lower dimensional counterpart of the 11D mM0-brane, and show that it is invariant under two supersymmetries: the rigid spacetime (target superspace) $D = 4$ $\mathcal{N} = 1$ supersymmetry and the local worldsheet supersymmetry generalizing the κ-symmetry of the massless $D = 4$ $\mathcal{N} = 1$ superparticle [46].[3]

Then we study its dimensional reduction and find an essentially nonlinear action for $D = 3$ $\mathcal{N} = 2$ counterpart of the 10D mD0-brane which we, abusing a bit the terminology, call 3D mD0-brane. The general form of this 3D mD0 action includes an arbitrary function $\mathcal{M}(\mathcal{H})$ of the so-called relative motion Hamiltonian $\mathcal{H}$ which in its turn is constructed from the matrix fields describing relative motion and interaction of the mD0 constituents. The action invariance under the $D = 3$ $\mathcal{N} = 2$ supersymmetry is manifest. Curiously enough, we find that the invariance under the worldline supersymmetry, which generalizes the κ-symmetry of a single D0-brane [47, 48], holds in the case of arbitrary function $\mathcal{M}(\mathcal{H})$. In particular, in the case of constant $\mathcal{M}(\mathcal{H}) = m$ we reproduce the 3D counterpart of the candidate mD0 action constructed in [35]. This suggests that a more general candidates for 10D mD0-brane should also exist in 10D type IIA superspace, and we will turn to this problem in a forthcoming paper.

The rest of this paper is organized according to the table of content.

We use the mostly minus metric conventions and Weyl spinor indices in $D = 4$ which are denoted by dotted and undotted symbols from the beginning of Greek alphabet. The symbols from the middle of the Greek alphabet denote the 4-vector indices when they do not carry tilde and $D = 3$ vector indices when are covered by tilde. The bosonic vector and fermionic spinor coordinates of $D = 4$ $\mathcal{N} = 1$ are denoted by

$$Z^M = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \,, \qquad \mu = 0,1,2,3 \,, \qquad \alpha = 1,2 \,, \qquad \dot{\alpha} = 1,2 \,, \tag{1.1}$$

[3] A bit before [46], the κ-symmetry was discovered in massive superparticle model with extended $\mathcal{N} = 2$ supersymmetry [47, 48]. Its identity as worldline supersymmetry was revealed in [11].

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while

$$Z^{\tilde{M}} = (x^{\tilde{\mu}}, \theta^\alpha, \bar{\theta}^\alpha)\,, \qquad \tilde{\mu} = 0,1,2\,, \qquad \alpha = 1,2 \tag{1.2}$$

are coordinates of the $D = 3$ $\mathcal{N} = 2$ superspace. The Weyl spinor indices are raised and lowered by Levi-Civita type symbols

$$\epsilon^{\alpha\beta} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{1.3}$$

For instance

$$\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta\,, \qquad \theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta\,, \tag{1.4}$$

which also applies to $D = 3$ spinors, and

$$\bar{\theta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}}\,, \qquad \bar{\theta}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\,. \tag{1.5}$$

We use the following representation for the relativistic Pauli matrices (rPMs)

$$\begin{aligned} \sigma_{\mu\,\alpha\dot{\beta}} = (\mathbb{I}, \sigma_1, \sigma_2, \sigma_3) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\ &= \tilde{\sigma}^{\mu\,\dot{\alpha}\beta} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\beta\alpha}\sigma_{\nu\,\alpha\dot{\beta}}\eta^{\nu\mu} \end{aligned} \tag{1.6}$$

which obey

$$(\sigma_{(\mu}\tilde{\sigma}_{\nu)})_\alpha{}^\beta := \frac{1}{2}(\sigma_\mu\tilde{\sigma}_\nu + \sigma_\nu\tilde{\sigma}_\mu)_\alpha{}^\beta = \eta_{\mu\nu}\delta_\alpha{}^\beta\,, \qquad (\tilde{\sigma}_{(\mu}\sigma_{\nu)})^{\dot{\alpha}}{}_{\dot{\beta}} = \eta_{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\,, \tag{1.7}$$

$$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1) = \eta^{\mu\nu}\,, \tag{1.8}$$

and

$$\sigma_{\mu\alpha\dot{\alpha}}\tilde{\sigma}^{\mu\dot{\beta}\beta} = 2\delta_\alpha{}^\beta\delta^{\dot{\beta}}{}_{\dot{\alpha}}\,. \tag{1.9}$$

As only one of the $D = 4$ relativistic Pauli matrices (1.6), σ_2, is complex and antisymmetric, we can identify the $D = 3$ gamma-matrices, which are known to allow for a real symmetric representation (after rising or lowering indices with the charge conjugation matrix) with rPMs carrying indices 0, 1 and 3,

$$\sigma_{\mu\,\alpha\dot{\beta}} = \left(\sigma_{\tilde{\mu}\,\alpha\dot{\beta}}, \sigma_{2\,\alpha\dot{\beta}}\right)\,, \qquad \tilde{\sigma}_\mu^{\dot{\beta}\alpha} = \left(\tilde{\sigma}_{\tilde{\mu}}^{\dot{\beta}\alpha}, \tilde{\sigma}_2^{\dot{\beta}\alpha}\right)\,, \tag{1.10}$$

$$\gamma_{\tilde{\mu}\,\alpha\beta} = \sigma_{\tilde{\mu}\,\alpha\dot{\beta}} = (\mathbb{I}, \sigma_1, \sigma_3) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)\,, \tag{1.11}$$

$$\tilde{\gamma}_{\tilde{\mu}}^{\beta\alpha} = \tilde{\sigma}_{\tilde{\mu}}^{\dot{\beta}\alpha} = (\mathbb{I}, -\sigma_1, -\sigma_3) = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\gamma_{\tilde{\mu}\,\gamma\delta}\,. \tag{1.12}$$

2 $D = 4$ $\mathcal{N} = 1$ massless superparticle in spinor moving frame (Lorentz harmonic) formulation

The $D = 4$ nAmW action is a lower dimensional counterpart of the 11D mM0 action of [32] and in principle can be obtained by dimensional reduction of this latter. However, the complicated structure of 11D spinor frame variables used to write mM0 action makes easier to construct it from the principles of symmetry, which includes super-Poincaré supergroup containing rigid spacetime $D = 4$ $\mathcal{N} = 1$ supersymmetry, local worldline supersymmetry generalizing the κ-symmetry of the massless $D = 4$ $\mathcal{N} = 1$ superparticle and SU(N) gauge symmetry realized on the fields of $d = 1$ $\mathcal{N} = 2$ SYM multiplet.

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2.1 Spinor moving frame variables (Lorentz harmonics) in $D = 4$

As the counterpart of 11D mM0, the action of 4D nAmW can be written in its simplest form with the use of spinor moving frame variables. In $D = 4$ these variables, which were also called Lorentz harmonics [49] and can be identified with Newman-Penrose diad [50] (see [51]), are the pair of nonvanishing complex spinors $v_\alpha^{\pm} = (\bar{v}_{\dot\alpha}^{\mp})^*$ obeying

$$v^{\alpha -} v_\alpha^+ = 1\,, \qquad \bar{v}^{\dot\alpha -} \bar{v}_{\dot\alpha}^+ = 1\,. \tag{2.1}$$

This condition implies that the complex matrix composed of the columns v_α^+, v_α^- is unimodular and hence belongs to the $\mathrm{SL}(2,\mathbb{C})$ group,[4]

$$\epsilon^{\alpha\beta} v_\alpha^+ v_\beta^- = 1 \qquad \Leftrightarrow \qquad V_\alpha^{(\beta)} := (v_\alpha^+, v_\alpha^-) \in \mathrm{SL}(2,\mathbb{C})\,. \tag{2.2}$$

This matrix is called spinor (moving) frame matrix because it can be considered as a kind of square root of a vector frame in the following sense. From the bilinear of these spinors one can construct two real lightlike vectors and two mutually conjugated complex lightlike vectors

$$u_\mu^{=} = v^- \sigma_\mu \bar{v}^-\,, \quad u_\mu^{\#} = v^+ \sigma_\mu \bar{v}^+\,, \quad u_\mu^{-+} = v^- \sigma_\mu \bar{v}^+\,, \quad u_\mu^{+-} = v^+ \sigma_\mu \bar{v}^- = (u_\mu^{-+})^* \tag{2.3}$$

which obey

$$u_\mu^{=} u^{\mu \#} = 2\,, \qquad u_\mu^{+-} u^{\mu\, -+} = -2\,, \qquad \text{other contractions} = 0\,. \tag{2.4}$$

These vectors form the light-like tetrade of the Newman-Penrose formalism [50] and can be collected in the $\mathrm{SO}(1,3)$ valued moving frame matrix[5]

$$U_\mu^{(a)} \in \mathrm{SO}(1,3) \qquad \Leftrightarrow \qquad U_\mu^{(a)} U^{\mu(b)} = \eta^{ab} = \mathrm{diag}(+1,-1,-1,-1)\,, \tag{2.5}$$

$$u_\mu^{\#} = U_\mu^{(0)} - U_\mu^{(3)}\,, \qquad u_\mu^{=} = U_\mu^{(0)} + U_\mu^{(3)}\,, \qquad u_\mu^{\mp\pm} = -U_\mu^1 \pm i U_\mu^2\,. \tag{2.6}$$

It is convenient to write (2.3) in equivalent form by representing vectors by 2×2 matrices which just factorize in terms of our spinor frame variables,

$$u_{\alpha\dot\beta}^{=} = u_\mu^{=} \sigma^\mu_{\alpha\dot\beta} = 2 v_\alpha^- \bar{v}_{\dot\beta}^-\,, \qquad u_{\alpha\dot\beta}^{\#} = u_\mu^{\#} \sigma^\mu_{\alpha\dot\beta} = 2 v_\alpha^+ \bar{v}_{\dot\beta}^+\,, \tag{2.7}$$

$$u_{\alpha\dot\beta}^{-+} = u_\mu^{-+} \sigma^\mu_{\alpha\dot\beta} = 2 v_\alpha^- \bar{v}_{\dot\beta}^+\,, \qquad u_{\alpha\dot\beta}^{+-} = u_\mu^{+-} \sigma^\mu_{\alpha\dot\beta} = 2 v_\alpha^+ \bar{v}_{\dot\beta}^- = (u_{\beta\dot\alpha}^{-+})^*\,. \tag{2.8}$$

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[4]The spinor frame variables are complexification of the harmonic variables of $\mathcal{N} = 2$ harmonic superspace approach to the off-shell description of the $\mathcal{N} = 2$ SYM, matter and supergravity in terms of unconstrained superfields [52–54]. Hence the name of Lorentz harmonics used in [49].

[5]To obtain these relations it is useful to notice that, as it follows from (2.2),

$$V_{(\beta)}^{-1\alpha} = \epsilon^{\alpha\gamma} V_\gamma^{(\delta)} \epsilon_{(\delta)(\beta)} = (v^{\alpha -}, -v^{\alpha +})$$

and that the moving frame matrix is expressed in terms of spinor frame by the relation

$$U_\mu^{(a)} = \frac{1}{2} \sigma_{\mu\alpha\dot\gamma} V_{(\dot\delta)}^{-1\dot\gamma} \tilde\sigma^{(a)(\dot\delta)(\beta)} V_{(\beta)}^{-1\alpha}\,.$$

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2.2 $D = 4$ massless superparticle action and its κ-symmetry

$D = 4$ massless superparticle action can be written in the form [49]

$$S^0_{\mathrm{D=4}} = \int_{\mathcal{W}^1} \rho^{\#} v^-_\alpha \bar{v}^-_{\dot\alpha} \Pi^{\alpha\dot\alpha} = \int d\tau \rho^{\#}(\tau) v^-_\alpha(\tau) \bar{v}^-_{\dot\alpha}(\tau) \Pi^{\alpha\dot\alpha}_\tau \,, \tag{2.9}$$

where

$$\Pi^{\alpha\dot\alpha} = d\tau \Pi^{\alpha\dot\alpha}_\tau \,, \qquad \Pi^{\alpha\dot\alpha}_\tau = \partial_\tau x^{\alpha\dot\alpha}(\tau) - 2i\partial_\tau \theta^\alpha(\tau)\bar\theta^{\dot\alpha}(\tau) + 2i\theta^\alpha(\tau)\partial_\tau\bar\theta^{\dot\alpha}(\tau) \tag{2.10}$$

is the pull-back to the superparticle worldline of the Volkov-Akulov (VA) 1-form

$$\Pi^{\alpha\dot\alpha} = dx^{\alpha\dot\alpha} - 2id\theta^\alpha\bar\theta^{\dot\alpha} + 2i\theta^\alpha d\bar\theta^{\dot\alpha} =: \Pi^\mu \tilde\sigma_\mu^{\dot\alpha\alpha} \,, \tag{2.11}$$

$$\Pi^\mu = dx^\mu - id\theta\sigma^\mu\bar\theta + \theta\sigma^\mu d\bar\theta \,, \tag{2.12}$$

$x^{\alpha\dot\alpha}(\tau) = x^\mu(\tau)\tilde\sigma_\mu^{\dot\alpha\alpha}$ and $\theta^\alpha(\tau) = (\bar\theta^{\dot\alpha}(\tau))^*$ are bosonic vector and fermionic spinor coordinate functions of proper time τ which define embedding of the superparticle worldline $\mathcal{W}^1$ in $D = 4$ $\mathcal{N} = 1$ superspace $\Sigma^{(4|4)}$,

$$\mathcal{W}^1 \in \Sigma^{(4|4)} \;:\qquad x^\mu = x^\mu(\tau) \,, \qquad \theta^\alpha = \theta^\alpha(\tau) \,, \qquad \bar\theta^{\dot\alpha} = \bar\theta^{\dot\alpha}(\tau) \,. \tag{2.13}$$

To simplify the notation, we use the same symbols for coordinate functions and coordinates (e.g. $x^\mu(\tau)$ and x^μ) as well as for the pull-backs of differential forms to the worldline $\mathcal{W}^1$ and the forms on target superspace $\Sigma^{(4|4)}$ (see e.g. (2.10) and (2.11)).

The bosonic spinor fields $v^\pm_\alpha = v^\pm_\alpha(\tau)$ in (2.9) are constrained by (2.1) and hence define a spinor moving frame attached to the worldline. In the light of the above discussion, we can write (2.9) in the following equivalent forms

$$S^0_{\mathrm{D=4}} = \int_{\mathcal{W}^1} \rho^{\#} \frac{1}{2} u^{=}_{\alpha\dot\alpha} \Pi^{\alpha\dot\alpha} = \int_{\mathcal{W}^1} \rho^{\#} u^{=}_\mu \Pi^\mu = \int_{\mathcal{W}^1} \rho^{\#} E^{=} \,. \tag{2.14}$$

were, at the last stage, we have introduced one of the pull-backs of the 1-forms of supervielbein adapted to the embedding of worldline in superspace

$$E^{=} = \Pi^\mu u^{=}_\mu = \frac{1}{2} u^{=}_{\alpha\dot\alpha} \Pi^{\alpha\dot\alpha} = \Pi^{\alpha\dot\alpha} v^-_\alpha \bar v^-_{\dot\alpha} \,, \tag{2.15}$$

$$E^{\#} = \Pi^\mu u^{\#}_\mu = \frac{1}{2} u^{\#}_{\alpha\dot\alpha} \Pi^{\alpha\dot\alpha} = \Pi^{\alpha\dot\alpha} v^+_\alpha \bar v^+_{\dot\alpha} \,, \tag{2.16}$$

$$E^{-+} = \Pi^\mu u^{-+}_\mu = \frac{1}{2} u^{-+}_{\alpha\dot\alpha} \Pi^{\alpha\dot\alpha} = \Pi^{\alpha\dot\alpha} v^-_\alpha \bar v^+_{\dot\alpha} = (E^{-+})^* \,, \tag{2.17}$$

$$E^{\mp} = d\theta^\alpha v^{\mp}_\alpha \,, \qquad \bar E^{\mp} = d\bar\theta^{\dot\alpha} \bar v^{\mp}_{\dot\alpha} \,, \tag{2.18}$$

The first two real forms, $E^{=}$ and $E^{\#}$, will be used to write the 4D nAmW action.

The action (2.9) is manifestly invariant under $D = 4$ $\mathcal{N} = 1$ spacetime supersymmetry. It also possesses the local fermionic κ-symmetry in its irreducible form[6]

$$\begin{aligned}
\delta_\kappa x^{\alpha\dot\alpha} &= 2i\kappa^+ v^{\alpha -}\bar\theta^{\dot\alpha} + 2i\bar\kappa^+\theta^\alpha \bar v^{\dot\alpha -}\,, \\
\delta_\kappa \theta^\alpha &= \kappa^+ v^{\alpha-}\,, \qquad \delta_\kappa \bar\theta^{\dot\alpha} = \bar\kappa^+ \bar v^{\dot\alpha -}\,, \\
\delta_\kappa v_\alpha^\mp &= 0\,, \qquad \delta_\kappa \rho^\# = 0\,,
\end{aligned} \tag{2.19}$$

which can be identified with the worldline supersymmetry [11].

Notice that the transformation of the bosonic coordinate function in (2.19) can be written in the form $\delta_\kappa x^{\alpha\dot\alpha} = 2i\delta_\kappa\theta^\alpha\bar\theta^{\dot\alpha} - 2i\theta^\alpha\delta_\kappa\bar\theta^{\dot\alpha}$ or equivalently

$$i_\kappa \Pi^{\alpha\dot\alpha} := i_{\delta_\kappa}\Pi^{\alpha\dot\alpha} := \delta_\kappa x^{\alpha\dot\alpha} - 2i\delta_\kappa\theta^\alpha\bar\theta^{\dot\alpha} + 2i\theta^\alpha\delta_\kappa\bar\theta^{\dot\alpha} = 0\,, \tag{2.20}$$

where i_δ is defined by $i_\delta \mathrm{d} = \delta$. See appendix A for the discussion on the use of this and other differential form notation.

2.3 Cartan forms and other gauge symmetries of the spinor frame action

For our discussion below it is useful to introduce the $SO(1{,}3)$ Cartan forms and to discuss the 1d supergravity induced by embedding of the superparticle worldline into the target superspace. To this end let us notice that to obtain, starting from (2.9), the complete set of equations of motion one has to vary also the spinor frame variables constrained by (2.1). The simplest way is to define the so-called admissible variations which do not break conditions (2.1). As (2.1) is one complex conditions imposed on 4 complex variables in $v_\alpha^\mp$, it is easy to conclude that there are three independent complex admissible variations. These in their turn can be related to the Cartan forms

$$\Omega^{--} = v^{\alpha-}\mathrm{d}v_\alpha^-\,, \qquad \bar\Omega^{--} = \bar v^{\dot\alpha-}\mathrm{d}\bar v_{\dot\alpha}^-\,, \tag{2.21}$$

$$\Omega^{++} = v^{\alpha+}\mathrm{d}v_\alpha^+\,, \qquad \bar\Omega^{++} = \bar v^{\dot\alpha+}\mathrm{d}\bar v_{\dot\alpha}^+\,, \tag{2.22}$$

$$\omega^{(0)} = v^{\alpha-}\mathrm{d}v_\alpha^+\,, \qquad \bar\omega^{(0)} = \bar v^{\dot\alpha-}\mathrm{d}\bar v_{\dot\alpha}^+\,, \tag{2.23}$$

which can be used to write the 'admissible' derivatives of spinor frame variables which take into account the conditions (2.1):

$$\mathrm{d}v_\alpha^- = -\omega^{(0)}v_\alpha^- + \Omega^{--}v_\alpha^+ \quad\Leftrightarrow\quad \mathrm{D}v_\alpha^- := \mathrm{d}v_\alpha^- + \omega^{(0)}v_\alpha^- = \Omega^{--}v_\alpha^+\,, \tag{2.24}$$

$$\mathrm{d}v_\alpha^+ = \omega^{(0)}v_\alpha^+ - \Omega^{++}v_\alpha^- \quad\Leftrightarrow\quad \mathrm{D}v_\alpha^+ := \mathrm{d}v_\alpha^+ - \omega^{(0)}v_\alpha^+ = -\Omega^{++}v_\alpha^-\,, \tag{2.25}$$

and their c.c. relations.

The expressions for admissible variations

$$\delta v_\alpha^- = -i_\delta\omega^{(0)}v_\alpha^- + i_\delta\Omega^{--}v_\alpha^+ \quad\Leftrightarrow\quad i_\delta\mathrm{D}v_\alpha^- := \delta v_\alpha^- + i_\delta\omega^{(0)}v_\alpha^- = i_\delta\Omega^{--}v_\alpha^+\,, \tag{2.26}$$

$$\delta v_\alpha^+ = i_\delta\omega^{(0)}v_\alpha^+ - i_\delta\Omega^{++}v_\alpha^- \quad\Leftrightarrow\quad i_\delta\mathrm{D}v_\alpha^+ := \delta v_\alpha^+ - i_\delta\omega^{(0)}v_\alpha^+ = -i_\delta\Omega^{++}v_\alpha^-\,, \tag{2.27}$$

[6] The κ-symmetry of the Brink-Schwarz form of the massless superparticle action [46] is infinitely reducible. Its basic relation can be written in the form $\delta\theta^\alpha = \tilde\kappa_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}P_\mu$ where P_μ is light-like on the mass shell, $P_\mu P^\mu = 0$. The irreducible form of the κ-symmetry (2.19) can be obtained from this by solving the light-likeness conditions in terms of spinor frame variables, $P_\mu = \rho^\# v^-\sigma_\mu\bar v^-$ and identifying $\kappa^+ = 2\tilde\kappa_{\dot\alpha}v^{\dot\alpha-}\rho^\#$.

can be obtained from (2.24) and (2.25) by formal contraction with variation symbol i_δ as described in the appendix A (in the case of 1-forms this is essentially the substitution $d \mapsto \delta$).

Eqs. (2.24), (2.25) also define covariant derivatives D in which the real and imaginary parts of $\omega^{(0)}$ (2.23) play the role of connections (gauge fields) for SO(1,1) and U(1) = SO(2) transformations. These gauge transformations can be parametrized by $i_\delta \omega^{(0)}$ and its c.c. $i_\delta \bar{\omega}^{(0)}$ in the expressions for admissible variations (2.26) and (2.27). These are the gauge symmetries of the massless superparticle action (2.9) if supplemented by the following SO(1,1) scaling of the Lagrange multiplier $\rho^{\#}$,

$$\delta \rho^{\#} = (i_\delta \omega^{(0)} + i_\delta \bar{\omega}^{(0)}) \rho^{\#} \,. \tag{2.28}$$

One more gauge symmetry of this action is complex two parametric transformation parametrized by $i_\delta \Omega^{++}$ and by its c.c. $i_\delta \bar{\Omega}^{++}$,

$$\begin{aligned} \mathbb{K}_2 \, : \qquad & \delta_{K_2} v_\alpha^- = 0 \,, \qquad \delta_{K_2} v_\alpha^+ = -i_\delta \Omega^{++} v_\alpha^- \,, \\ & \delta_{K_2} \bar{v}_{\dot{\alpha}}^- = 0 \,, \qquad \delta_{K_2} \bar{v}_{\dot{\alpha}}^+ = -i_\delta \bar{\Omega}^{++} \bar{v}_{\dot{\alpha}}^- \,. \end{aligned}$$

In the model invariant under the above described $[\mathrm{SO}(1,1) \otimes \mathrm{U}(1)] \ltimes \mathbb{K}_2$ transformations, the variables $v_\alpha^\mp$ can be considered as a kind of homogeneous coordinate for the coset of the $\mathrm{Spin}(1,3) = \mathrm{SL}(2,\mathbb{C})$ group isomorphic to the $\mathbb{S}^2$ sphere [55, 56]

$$\{v_\alpha^\mp\} \; = \; \frac{\mathrm{SL}(2,\mathbb{C})}{[\mathrm{SO}(1,1) \otimes \mathrm{U}(1)] \ltimes \mathbb{K}_2} = \mathbb{S}^2 \,, \tag{2.29}$$

which can be recognized as the celestial sphere of an observer living in four dimensional spacetime. The reason to introduce the Lagrange multiplier $\rho^{\#}$, which has a Stückelberg-type transformation (2.28) under (i.e. can be gauged to unity by) the SO(1,1) gauge symmetry, is to have the above described clear group-theoretical meaning of the bosonic spinor variables.

2.4 Equations of motion and induced supergravity on the worldline

The equations of motion which follow from the action (2.9) include (see eqs. (A.24)–(A.31) in appendix A for the complete list of equations)

$$E^{=} = E^{+-} = E^{-+} = 0 \,, \qquad E^- = \bar{E}^- = 0 \,, \tag{2.30}$$

so that only the pull-backs of one bosonic and two fermionic projections of the target superspace supervielbein forms remains nonvanishing on the mass shell. These forms

$$E^{\#} = \Pi^{\alpha\dot{\alpha}} v_\alpha^+ \bar{v}_{\dot{\alpha}}^+ = \mathrm{d}\tau E_\tau^{\#} \,, \qquad E^+ = \mathrm{d}\theta^\alpha v_\alpha^+ = \mathrm{d}\tau E_\tau^+ \,, \qquad \bar{E}^+ = \mathrm{d}\bar{\theta}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}}^+ = \mathrm{d}\tau \bar{E}_\tau^+ \tag{2.31}$$

describe 1d extended $\mathcal{N} = 2$ supergravity induced on the worldline by its embedding into the target $D = 4$ $\mathcal{N} = 1$ superspace. This statement is aimed to reflect the fact that under the κ-symmetry (2.19), which can be identified with the worldline supersymmetry, these forms are transformed by

$$\delta_\kappa E^{\#} = -4iE^+ \bar{\kappa}^+ + 4i\kappa^+ \bar{E}^+ \,, \qquad \delta_\kappa E^+ = \mathrm{D}\kappa^+ \,, \qquad \delta_\kappa \bar{E}^+ = \mathrm{D}\bar{\kappa}^+ \,. \tag{2.32}$$

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This implies

$$\delta_\kappa E_\tau^{\#} = -4iE^+\bar{\kappa}^+ + 4i\kappa^+\bar{E}^+\,, \qquad \delta_\kappa E_\tau^+ = \mathrm{D}_\tau\kappa^+\,, \qquad \delta_\kappa \bar{E}_\tau^+ = \mathrm{D}_\tau\bar{\kappa}^+\,, \tag{2.33}$$

which is the characteristic transformation of 1d $\mathcal{N}=2$ supergravity supermultiplet.

These 1-forms are constructed from the coordinate functions of the superparticle which define the embedding of the worldline into the target superspace (2.13) and spinor moving frame fields. Hence the name of supergravity induced by embedding.

The use of induced supergravity allows to couple a one-dimensional supersymmetric matter multiplets to massless superparticle in a way that preserves its local fermionic κ-symmetry. Basically, the way consists in coupling of a given multiplet to the above described induced supergravity. The action for 4D nAmW system, the counterpart of 11D mM0 action from [32], is constructed on this way using the 1d reduction of the 3d $\mathcal{N}=2$ by coupling of $SU(\mathrm{N})$ SYM multiplet to the induced supergravity. We present this action and its worldline supersymmetry in the next section.

3 $D=4$ $\mathcal{N}=1$ non-Abelian multiwave action and its worldline supersymmetry

The action of 4D non-Abelian multiwave (nAmW) system, which serves as a counterpart of the 11D mM0 action, can in principle be obtained from this latter by dimensional reduction. However, as we have already noticed, due to complicated structure of the 11D moving frame variables, it was easier to construct it directly using the procedure described in [33] for $D=11$ and in [34] for $D=3$. On this way we found the following 4D nAmW action

$$\begin{aligned} S^{\mathrm{D}=4}_{\mathrm{nAmW}} = \int\limits_{\mathcal{W}^1} \rho^{\#}E^{=} + \frac{1}{\mu^6}\int\limits_{\mathcal{W}^1}(\rho^{\#})^3\mathrm{tr}\left(\bar{\bar{\widetilde{\mathbb{P}}}}\mathrm{D}\widetilde{\mathbb{Z}} + \widetilde{\mathbb{P}}\mathrm{D}\bar{\bar{\widetilde{\mathbb{Z}}}} - \frac{i}{8}\mathrm{D}\widetilde{\Psi}\,\bar{\bar{\widetilde{\Psi}}} + \frac{i}{8}\widetilde{\Psi}\mathrm{D}\bar{\bar{\widetilde{\Psi}}}\right) + \\ + \frac{1}{\mu^6}\int\limits_{\mathcal{W}^1}(\rho^{\#})^3\left(E^{\#}\widetilde{\mathcal{H}} + iE^+\mathrm{tr}(\bar{\bar{\widetilde{\Psi}}}\bar{\bar{\widetilde{\mathbb{P}}}} + \widetilde{\Psi}[\widetilde{\mathbb{Z}},\bar{\bar{\widetilde{\mathbb{Z}}}}]) + i\bar{E}^+\mathrm{tr}(\widetilde{\Psi}\widetilde{\mathbb{P}} + \bar{\bar{\widetilde{\Psi}}}[\widetilde{\mathbb{Z}},\bar{\bar{\widetilde{\mathbb{Z}}}}]\right)\,, \end{aligned} \tag{3.1}$$

where

$$\widetilde{\mathcal{H}} = \mathrm{tr}\left(\widetilde{\mathbb{P}}\bar{\bar{\widetilde{\mathbb{P}}}} + [\widetilde{\mathbb{Z}},\bar{\bar{\widetilde{\mathbb{Z}}}}]^2 - \frac{i}{2}\widetilde{\mathbb{Z}}\widetilde{\Psi}\widetilde{\Psi} + \frac{i}{2}\bar{\bar{\widetilde{\mathbb{Z}}}}\bar{\bar{\widetilde{\Psi}}}\bar{\bar{\widetilde{\Psi}}}\right)\,. \tag{3.2}$$

The first term of (3.1) coincides with the $D=4$ massless superparticle action (2.9), (2.14) which can be interpreted now as describing the center of energy movement of the interacting multiwave system. The remaining part of the nAmW action, proportional to the dimensional parameter $\frac{1}{\mu^6}$,[7] contains bosonic and fermionic matrix fields. These are traceless $N\times N$ (i.e. $[su(N)]^c = sl(N,\mathbb{C})$ valued) bosonic matrix fields

$$\widetilde{\mathbb{Z}} = \widetilde{\mathbb{Z}}_{0|\#} := \widetilde{\mathbb{Z}}_{0|+2}\,, \qquad \bar{\bar{\widetilde{\mathbb{Z}}}} = \bar{\bar{\widetilde{\mathbb{Z}}}}_{\#|0} := \bar{\bar{\widetilde{\mathbb{Z}}}}_{+2|0} = (\widetilde{\mathbb{Z}})^\dagger\,, \tag{3.3}$$

$$\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_{+|+3}\,, \qquad \bar{\bar{\widetilde{\mathbb{P}}}} = \bar{\bar{\widetilde{\mathbb{P}}}}_{+3|+} = (\widetilde{\mathbb{P}})^\dagger\,, \tag{3.4}$$

[7]Notice that μ is of dimension of mass in our physical system of units with $c=\hbar=1$. Then $[\mu^{-6}] = M^{-6} = L^6$. See below for comment on the dimension of matrix fields.

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[8] and traceless $N \times N$ fermionic matrix fields

$$\widetilde{\Psi} = \widetilde{\Psi}_{\#|+}\,, \qquad \bar{\widetilde{\Psi}} = \bar{\widetilde{\Psi}}_{+|\#} = (\widetilde{\Psi})^{\dagger}\,. \tag{3.5}$$

Here the sign subindices (equivalent to the opposite sign superindices, e.g. $\widetilde{\Psi}_{\#|+} = \widetilde{\Psi}^{=|-}$) indicate the transformation properties of the matrix fields under the $\mathrm{GL}(1,\mathbb{C}) = \mathrm{SO}(1,1) \times \mathrm{U}(1)$ transformations acting on the spinor frame variables $v_\alpha^{\mp} = v_\alpha^{(\mp|0)}$ and $\bar{v}_{\dot{\alpha}}{}^{\mp} = \bar{v}_{\dot{\alpha}}{}^{(0|\mp)}$. These latter enter, besides the first term, also in the remaining part of the action (3.1) through the induced supergravity 1-forms: 1d supervielbein $E^{\#}$, $E^{+}, \bar{E}^{+}$ (2.31) and Cartan forms $\omega^{(0)}$, $\bar{\omega}^{(0)}$ (2.23) which enter the covariant derivatives

$$\mathrm{D}\widetilde{\mathbb{Z}} = \mathrm{d}\widetilde{\mathbb{Z}} + 2\bar{\omega}^{(0)}\widetilde{\mathbb{Z}} + [\,\mathbb{A}\,,\,\widetilde{\mathbb{Z}}\,]\,, \qquad \mathrm{D}\bar{\widetilde{\mathbb{Z}}} = \mathrm{d}\bar{\widetilde{\mathbb{Z}}} + 2\omega^{(0)}\bar{\widetilde{\mathbb{Z}}} + [\,\mathbb{A}\,,\,\bar{\widetilde{\mathbb{Z}}}\,]\,, \tag{3.6}$$

$$\mathrm{D}\widetilde{\Psi} = \mathrm{d}\widetilde{\Psi} + (2\omega^{(0)} + \bar{\omega}^{(0)})\widetilde{\Psi} + [\,\mathbb{A}\,,\,\widetilde{\Psi}\,]\,, \quad \mathrm{D}\bar{\widetilde{\Psi}} = \mathrm{d}\bar{\widetilde{\Psi}} + (\omega^{(0)} + 2\bar{\omega}^{(0)})\bar{\widetilde{\Psi}} + [\,\mathbb{A}\,,\,\bar{\widetilde{\Psi}}\,]\,, \tag{3.7}$$

as connection for the $\mathrm{GL}(1,\mathbb{C}) = \mathrm{SO}(1,1) \times \mathrm{U}(1)$ transformations.

The $\mathrm{SU}(N)$ connection 1-form $\mathbb{A} = \mathrm{d}\tau\mathbb{A}_\tau$, with the traceless antihermitian $N \times N$ matrix field $\mathbb{A}_\tau$, enters the action (3.1) inside the covariant derivatives (3.6), (3.7) and their hermitian conjugate.

The dimensions of the matrix matter fields are

$$[\widetilde{\mathbb{Z}}] = M = [\bar{\widetilde{\mathbb{Z}}}]\,, \qquad [\bar{\widetilde{\mathbb{P}}}] = M^2 = [\widetilde{\mathbb{P}}]\,, \qquad [\widetilde{\Psi}] = M^{3/2} = [\bar{\widetilde{\Psi}}]\,. \tag{3.8}$$

The choice of such non-canonical dimension allows us to reduce the dependence on the dimensional parameter μ, with dimension of mass $[\mu] = M$, to an overall multiplier in front of the part of the action including the matrix fields.

The action (3.1) is invariant under the local worldline supersymmetry transformations which act on the center of energy fields as a bit deformed version of the κ-symmetry of massless superparticle

$$\begin{aligned} \delta_\epsilon x^{\alpha\dot{\alpha}} &= 2i\delta_\epsilon\theta^\alpha\bar{\theta}^{\dot{\alpha}} - 2i\theta^\alpha\delta_\epsilon\bar{\theta}^{\dot{\alpha}} + v^{\alpha+}\bar{v}^{\dot{\alpha}+}i_\epsilon E^{=}\,, \\ \delta_\epsilon\theta^\alpha &= \epsilon^{+}v^{\alpha-}\,, \qquad \delta_\epsilon\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}^{+}\bar{v}^{\dot{\alpha}-}\,, \\ \delta_\epsilon v_\alpha^{\mp} &= 0\,, \qquad \delta_\epsilon\bar{v}_{\dot{\alpha}}^{\mp} = 0\,, \qquad \delta_\epsilon\rho^{\#} = 0\,, \end{aligned} \tag{3.9}$$

with

$$i_\epsilon E^{=} = \frac{3i}{2\mu^6}(\rho^{\#})^2\,\mathrm{tr}\left(\epsilon^{+}\bar{\widetilde{\Psi}}\bar{\widetilde{\mathbb{P}}} + \bar{\epsilon}^{+}\widetilde{\Psi}\widetilde{\mathbb{P}} - (\epsilon^{+}\widetilde{\Psi} + \bar{\epsilon}^{+}\bar{\widetilde{\Psi}})[\widetilde{\mathbb{Z}}, \bar{\widetilde{\mathbb{Z}}}]\right)\,, \tag{3.10}$$

and on the matrix fields by

$$\delta_\epsilon\widetilde{\mathbb{Z}} = -i\epsilon^{+}\bar{\widetilde{\Psi}}\,, \qquad \delta_\epsilon\bar{\widetilde{\mathbb{Z}}} = -i\bar{\epsilon}^{+}\widetilde{\Psi}\,, \tag{3.11}$$

$$\delta_\epsilon\widetilde{\mathbb{P}} = i\epsilon^{+}[\widetilde{\Psi}, \widetilde{\mathbb{Z}}] + i\bar{\epsilon}^{+}[\bar{\widetilde{\Psi}}, \widetilde{\mathbb{Z}}]\,, \qquad \delta_\epsilon\bar{\widetilde{\mathbb{P}}} = -i\epsilon^{+}[\widetilde{\Psi}, \bar{\widetilde{\mathbb{Z}}}] - i\bar{\epsilon}^{+}[\bar{\widetilde{\Psi}}, \bar{\widetilde{\mathbb{Z}}}]\,, \tag{3.12}$$

$$\delta_\epsilon\widetilde{\Psi} = 4\epsilon^{+}\bar{\widetilde{\mathbb{P}}} + 4\bar{\epsilon}^{+}[\widetilde{\mathbb{Z}}, \bar{\widetilde{\mathbb{Z}}}]\,, \qquad \delta_\epsilon\bar{\widetilde{\Psi}} = 4\bar{\epsilon}^{+}\widetilde{\mathbb{P}} + 4\epsilon^{+}[\widetilde{\mathbb{Z}}, \bar{\widetilde{\mathbb{Z}}}]\,, \tag{3.13}$$

$$\delta_\epsilon\mathbb{A} = iE^{\#}(\epsilon^{+}\widetilde{\Psi} - \bar{\epsilon}^{+}\bar{\widetilde{\Psi}}) + 8iE^{+}\epsilon^{+}\bar{\widetilde{\mathbb{Z}}} - 8i\bar{E}^{+}\bar{\epsilon}^{+}\,\widetilde{\mathbb{Z}}\,. \tag{3.14}$$

[8]The symbols of the matrix fields of nAmW model are covered by tilde here to distinguish them form the matrix fields of 3D mD0 model which are inert under $\mathrm{SO}(1,1)$ but carry $\mathrm{U}(1)$ charges, see (4.43)–(4.45). One might wonder why we do not make this redefinition in this section, without accompanying it by dimensional reduction. The answer is that such a redefinition will result in appearance of the derivative of the Stückelberg fields in the action and we found such form of the action less convenient.

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An interesting problem is to construct the Hamiltonian formalism and to perform the quantization of the $D=4$ nAmW system as a preliminary step to approach the problem of quantization of the 11D mM0 system which in its turn might give new insights in the features of String/M-theory. Leaving this for future, below we will study the dimensional reduction of the nAmW action down to $D=3$ which produces a candidate action for the three dimensional counterpart of 10D multiple D0-brane, which we call 3D mD0-brane.

4 Nonlinear mD0-action in 3D by dimensional reduction of the 4D nAmW

4.1 From $D=4$ massless superparticle to $D=3$ (counterpart of) super-D0-brane

Now we would like to obtain the action for massive $D=3$ $\mathcal{N}=2$ superparticle by dimensional reduction of the $D=4$ action (2.9). To be precise we would like to have the $D=3$ action invariant, besides the extended $\mathcal{N}=2$ spacetime supersymmetry, under κ-symmetry and thus being the $D=3$ (counterpart of 10D) super-D0-brane.

4.1.1 Reduction of Brink-Schwarz superparticle

In the case of Brink-Schwarz superparticle action

$$S_{\rm BS}^{\rm D=4} = \int \left(p_\mu \Pi^\mu + \frac{1}{2} \mathrm{d}\tau e p_\mu p^\mu \right) , \tag{4.1}$$

the dimensional reduction can be performed by using equations of motion for x^2 coordinate, which reads $\mathrm{d}p_2 = 0$ and implies that p_2 is a constant,

$$\mathrm{d}p_2 = 0 \qquad \Rightarrow \qquad p_2 = m = \mathrm{const}\,. \tag{4.2}$$

The choice of x^2 as a direction of reduction,

$$x^\mu = (x^{\tilde\mu}, x^2)\,, \qquad p_\mu = (p_{\tilde\mu}, p_2)\,, \qquad \tilde\mu = 0,1,3 \qquad \leftrightarrow \qquad \mu = 0,1,2,3\,, \tag{4.3}$$

is convenient because it is in consonance with the representation of the $D=4$ rPMs (1.6) and its relation with $D=3$ gamma matrices (1.11). It corresponds to the split of the 4D VA forms on

$$\Pi^2 = \mathrm{d}x^2 - i\mathrm{d}\theta\sigma^2\bar\theta + i\theta\sigma^2\mathrm{d}\bar\theta = \mathrm{d}x^2 - \epsilon_{\alpha\beta}(\mathrm{d}\theta^\alpha\bar\theta^\beta - \theta^\alpha\mathrm{d}\bar\theta^\beta) \tag{4.4}$$

and 3D VA 1-forms

$$\Pi^{\tilde\mu} = \mathrm{d}x^{\tilde\mu} - i\mathrm{d}\theta\gamma^{\tilde\mu}\bar\theta + i\theta\gamma^{\tilde\mu}\mathrm{d}\bar\theta = \frac{1}{2}\Pi^{\alpha\beta}\gamma^{\tilde\mu}_{\alpha\beta}\,, \tag{4.5}$$

which can be represented by symmetric spin-tensor 1-form

$$\Pi^{\alpha\beta} = \Pi^{\tilde\mu}\tilde\gamma^{\alpha\beta}_{\tilde\mu} = \mathrm{d}x^{\alpha\beta} - 2i\mathrm{d}\theta^{(\alpha}\bar\theta^{\beta)} + 2i\theta^{(\alpha}\mathrm{d}\bar\theta^{\beta)}\,. \tag{4.6}$$

Substituting the solution (4.2) back into the action (4.1), using (4.4) and omitting the total derivative term we find

$$S^{\mathrm{D}=3}_{\mathrm{dAL}} = \int \left(p_{\tilde{\mu}}\Pi^{\tilde{\mu}} + \frac{1}{2}\mathrm{d}\tau e(p_{\tilde{\mu}}p^{\tilde{\mu}} - m^2) \right) + m \int (\mathrm{d}\theta^{\gamma}\bar{\theta}_{\gamma} - \theta^{\gamma}\mathrm{d}\bar{\theta}_{\gamma})\,. \tag{4.7}$$

This is the $D=3$ counterpart of the De Azcárraga-Lukierski $\mathcal{N}=2$ massive superparticle action [47, 48]. This possesses κ-symmetry and actually the κ-symmetry was first discovered in this example [47], a bit earlier than for the massless superparticle [46].

In the modern perspective (4.7) is $D=3$ counterpart of super-D0-brane action [6] or, simplifying terminology, 3D D0-brane. The second term in (4.7) is the Wess-Zumino term of the D0-brane, the prototype of the Wess-Zumino term of the superstring [57] and of higher super-p-branes (see [2–7, 58–60] and refs. therein).

4.1.2 Illuminating ansatz for dimensional reduction of spinor moving frame formalism

Our task now it to perform dimensional reduction of the spinor moving frame action for $D=4$ massless superparticle in such a way that it reproduce 3D D0-brane action. (See appendix B for the warm-up exercise of dimensional reduction reproducing 3D massless superparticle). To this end it is sufficient to use the ansatz (cf. (B.5))

$$\begin{cases} v^-_{\alpha} = \mathrm{v}^-_{\alpha} - i\mathrm{v}^+_{\alpha}\mu^{=}\,, \\ v^+_{\alpha} = \mathrm{v}^+_{\alpha}\,, \end{cases} \qquad \begin{cases} \bar{v}^-_{\dot{\alpha}} = \mathrm{v}^-_{\alpha} + i\mathrm{v}^+_{\alpha}\mu^{=}\,, \\ \bar{v}^+_{\dot{\alpha}} = \mathrm{v}^+_{\alpha}\,, \end{cases} \tag{4.8}$$

with real spinors $\mathrm{v}^{\pm}_{\alpha}$ obeying

$$\mathrm{v}^{-\alpha}\mathrm{v}^+_{\alpha} = 1\,, \qquad (\mathrm{v}^{\pm}_{\alpha})^* = \mathrm{v}^{\pm}_{\alpha}\,, \tag{4.9}$$

and some real 1d field $\mu^{=} = \mu^{=}(\tau) = (\mu^{=})^*$.

Let us stress that this ansatz is not suitable for dimensional reduction of the nAmW action due to reasons which we will explain below. However, we find it quite illuminative, and decided to describe it briefly with hope to create a feeling of dimensional reduction in spinor moving frame formalism.

Eqs. (4.8) imply

$$u^{=}_{\alpha\dot{\beta}} = 2\mathrm{v}^-_{\alpha}\mathrm{v}^-_{\beta} + 2\mathrm{v}^+_{\alpha}\mathrm{v}^+_{\beta}(\mu^{=})^2 + 2i\epsilon_{\alpha\beta}\mu^{=} \quad \left(= (u^{=}_{\beta\dot{\alpha}})^*\ \right), \tag{4.10}$$

and the action (2.14) naturally splits onto the parts containing only 3d VA 1-forms (4.5) and containing Π^2 (4.4),

$$S^0_{\mathrm{D}=4}|_{(4.8)} = \frac{1}{2}\int \rho^{\#}(\mathrm{v}^-_{\alpha}\mathrm{v}^-_{\beta} + (\mu^{=})^2\mathrm{v}^+_{\alpha}\mathrm{v}^+_{\beta})\Pi^{\alpha\beta} - \int \rho^{\#}\mu^{=}\left(\mathrm{d}x^2 - \epsilon_{\alpha\beta}(\mathrm{d}\theta^{\alpha}\bar{\theta}^{\beta} - \theta^{\alpha}\mathrm{d}\bar{\theta}^{\beta})\right)\,. \tag{4.11}$$

Now let us use the equation of motion for x^2 coordinate,

$$\mathrm{d}(\rho^{\#}\mu^{=}) = 0 \qquad \Rightarrow \qquad \rho^{\#}\mu^{=} = m = \mathrm{const}\,. \tag{4.12}$$

Substituting its solution into (4.11) we find

$$S^0_{\mathrm{D}=4}|_{(4.8),\ (4.12)} = \frac{1}{2}\int \rho^{\#}\left(\mathrm{v}^-_\alpha \mathrm{v}^-_\beta + \frac{m^2}{(\rho^{\#})^2}\mathrm{v}^+_\alpha \mathrm{v}^+_\beta\right)\Pi^{\alpha\beta} + m\int \epsilon_{\alpha\beta}(\mathrm{d}\theta^\alpha\bar{\theta}^\beta - \theta^\alpha \mathrm{d}\bar{\theta}^\beta)\,. \tag{4.13}$$

The term with x^2 has disappeared as it has become the total derivative, and thus we have arrived at the action for $D=3$ superparticle.

The second term in (4.13) is the Wess-Zumino term of the D0-brane, the same as in (4.7). Notice that $\rho^{\#}$ can be just removed from the first, kinetic term if we redefine the 3D spinor moving frame variables, by

$$\mathrm{v}^2_\alpha = \sqrt{\frac{\rho^{\#}}{m}}\mathrm{v}^-_\alpha\,, \qquad \mathrm{v}^1_\alpha = \sqrt{\frac{m}{\rho^{\#}}}\mathrm{v}^+_\alpha. \tag{4.14}$$

The redefined spinors still obey

$$\mathrm{v}^{\alpha 2}\mathrm{v}^1_\alpha = 1 \qquad \Longleftrightarrow \qquad \mathrm{v}^{\alpha p}\mathrm{v}^q_\alpha = -\epsilon^{pq}\,, \qquad q=1,2 \tag{4.15}$$

and hence form the $\mathrm{SL}(2,\mathbb{R})$ valued matrix

$$\mathrm{v}^q_\alpha = (\mathrm{v}^1_\alpha, \mathrm{v}^2_\alpha) \ \in \ \mathrm{SL}(2,\mathbb{R})\,. \tag{4.16}$$

Thus the dimensional reduction of the 4D massless superparticle action (2.14) results in

$$\begin{aligned} S^{\mathrm{D0}}_{\mathrm{D}=3} =: \int \mathcal{L}^0_{\mathrm{D}=3} &= \frac{1}{2}m\int \mathrm{v}^q_\alpha \mathrm{v}^q_\beta \Pi^{\alpha\beta} + m\int \epsilon_{\alpha\beta}(\mathrm{d}\theta^\alpha\bar{\theta}^\beta - \theta^\alpha\mathrm{d}\bar{\theta}^\beta) \\ &= \frac{1}{2}m\int \mathrm{u}^0_{\alpha\beta}\Pi^{\alpha\beta} + m\int \epsilon_{\alpha\beta}(\mathrm{d}\theta^\alpha\bar{\theta}^\beta - \theta^\alpha\mathrm{d}\bar{\theta}^\beta) \end{aligned} \tag{4.17}$$

which is the $D=3$ D0-brane action in spinor moving frame formulation [35, 61]. In the second line of (4.17) we have introduced the matrix $\mathrm{u}^0_{\alpha\beta} = \mathrm{v}^q_\alpha \mathrm{v}^q_\beta$ representing timelike normalized vector from the 3d moving frame attached to the worldline,

$$\mathrm{u}^0_{\alpha\beta} = \mathrm{u}^{0\tilde{\mu}}\gamma_{\tilde{\mu}\alpha\beta} = \mathrm{v}^1_\alpha \mathrm{v}^1_\beta + \mathrm{v}^2_\alpha \mathrm{v}^2_\beta\,, \tag{4.18}$$

$$\mathrm{u}^1_{\alpha\beta} = \mathrm{u}^{1\tilde{\mu}}\gamma_{\tilde{\mu}\alpha\beta} = 2\mathrm{v}^1_{(\alpha}\mathrm{v}^2_{\beta)}\,, \qquad \mathrm{u}^2_{\alpha\beta} = \mathrm{u}^{2\tilde{\mu}}\gamma_{\tilde{\mu}\alpha\beta} = \mathrm{v}^1_\alpha \mathrm{v}^1_\beta - \mathrm{v}^2_\alpha \mathrm{v}^2_\beta\,. \tag{4.19}$$

These vectors obey the orthogonality and normalization conditions

$$\mathrm{u}^0_{\tilde{\mu}}\mathrm{u}^{0\tilde{\mu}} = 1\,, \qquad \mathrm{u}^0_{\tilde{\mu}}\mathrm{u}^{I\tilde{\mu}} = 0\,, \qquad \mathrm{u}^I_{\tilde{\mu}}\mathrm{u}^{J\tilde{\mu}} = -\delta^{IJ}\,, \qquad I,J=1,2 \tag{4.20}$$

which imply that they form $\mathrm{SO}(1,2)$ valued 3D moving frame matrix

$$\mathrm{u}_{\tilde{\mu}}{}^{\tilde{a}} = \left(\mathrm{u}_{\tilde{\mu}}{}^0, \mathrm{u}_{\tilde{\mu}}{}^I\right) \ \in \ \mathrm{SO}(1,2)\,. \tag{4.21}$$

4.1.3 Irreducible κ-symmetry of 3D super-D0-brane

If we take the exterior derivative (see appendix A) of the Lagrangian form of the action (4.17) and do not concentrate on the derivatives of the spinor moving frame variables, we find

$$\mathrm{d}\mathcal{L}^0_{\mathrm{D}=3} = -2im(\mathcal{E}^1 + i\mathcal{E}^2)\wedge(\bar{\mathcal{E}}^1 - i\bar{\mathcal{E}}^2) + \frac{1}{2}m\Pi^{\alpha\beta}\wedge \mathrm{d}\mathrm{u}^0_{\alpha\beta}\,, \tag{4.22}$$

where we have used the fermionic part of the pull-back of 3D supervielbein adapted to the embedding

$$\mathcal{E}^q = (\mathcal{E}^1, \mathcal{E}^2) = \mathrm{d}\theta^\alpha \mathrm{v}_\alpha^q \,, \qquad \bar{\mathcal{E}}^q = (\bar{\mathcal{E}}^1, \bar{\mathcal{E}}^2) = \mathrm{d}\bar{\theta}^\alpha \mathrm{v}_\alpha^q \,. \tag{4.23}$$

Below we will also need one of three bosonic 1-forms of this supervielbein

$$\mathrm{E}^0 = \Pi^{\tilde{\mu}} \mathrm{u}_{\tilde{\mu}}^0 = \frac{1}{2}\Pi^{\alpha\beta}\mathrm{u}_{\alpha\beta}^0 = \frac{1}{2}\Pi^{\alpha\beta}(\mathrm{v}_\alpha^1 \mathrm{v}_\beta^1 + \mathrm{v}_\alpha^2 \mathrm{v}_\beta^2)\,, \qquad \mathrm{E}^I = \Pi^{\tilde{\mu}}\mathrm{u}_{\tilde{\mu}}^I = (\mathrm{E}^1, \mathrm{E}^2)\,. \tag{4.24}$$

Using the formalism described in appendix A we can conclude from (4.22) that the action (4.17) is invariant under κ-symmetry defined by

$$\begin{aligned} &i_\kappa \mathcal{E}^1 = \kappa\,, && i_\kappa \mathcal{E}^2 = i\kappa\,, && i_\kappa \bar{\mathcal{E}}^1 = \bar{\kappa}\,, && i_\kappa \bar{\mathcal{E}}^2 = -i\bar{\kappa}\,, \\ &i_\kappa \Pi^{\alpha\beta} = 0\,, && \delta_\kappa \mathrm{v}_\alpha^q = 0 && (\Rightarrow \delta_\kappa \mathrm{u}_{\alpha\beta}^0 = 0)\,, && \end{aligned} \tag{4.25}$$

or, equivalently,

$$\begin{aligned} &\delta_\kappa \theta^\alpha = \kappa(\mathrm{v}^{\alpha 2} - i\mathrm{v}^{\alpha 1})\,, && \delta_\kappa \bar{\theta}^\alpha = \bar{\kappa}(\mathrm{v}^{\alpha 2} + i\mathrm{v}^{\alpha 1})\,, \\ &\delta_\kappa x^{\alpha\beta} = 2i(\delta_\kappa \theta^{(\alpha}\,\bar{\theta}^{\beta)} - \theta^{(\alpha}\delta_\kappa \bar{\theta}^{\beta)})\,, && \delta_\kappa \mathrm{v}_\alpha^q = 0\,. \end{aligned} \tag{4.26}$$

Let us also observe that the action (4.17) is invariant under the local SO(2) rotation of the spinor frame variables (4.16), which also produce SO(2) rotations mixing the spacelike vectors ${\mathrm{u}_{\tilde{\mu}}}^I = ({\mathrm{u}_{\tilde{\mu}}}^1, {\mathrm{u}_{\tilde{\mu}}}^2)$. Using this SO(2) as an identification relation on the set of spinor moving frame fields v_α^q, we can consider these as homogeneous coordinates of the coset

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \simeq \frac{\mathrm{SO}(1,2)}{\mathrm{SO}(2)}\,. \tag{4.27}$$

This is suitable for the description of the massive particle as SO(2) is the small group of 3D timelike momentum.

Another observation is that the above SO(2) symmetry is not seen in the original reduction ansatz (4.8) for 4D spinor frame variables. It appears in the final action as a kind of emergent symmetry. However, if we try to use this ansatz for dimensional reduction of a more complicated nAmW system, such a symmetry will not emerge in the final answer. This would imply that the spinor frame variables in this 3D action parametrize the $\mathrm{SL}(2,\mathbb{R})$ group rather then coset (4.27) and hence carry one extra degree of freedom which actually is not unwanted. This is why for dimensional reduction of nAmW system we will use a bit more complicated ansatz which we will first describe on a simpler example of the reduction of massless superparticle to $D = 3$ super-D0-brane.

4.2 Revising the 4D M0 reduction to 3D D0. Properties of 3D spinor frame

The above described dimensional reduction was based on the ansatz (4.8) for spinor moving frame variables which breaks explicitly the U(1) subgroup of the $\mathrm{GL}(1,\mathbb{C})$ gauge symmetry of the 4D spinor moving frame formalism. Furthermore with it, an important SO(2) gauge symmetry of the $D = 3$ Lorentz harmonic approach was restored only at final stage as an emergent symmetry of the single super-D0-brane action (4.17). The origin of this emergent gauge symmetry can be followed to the fact that the original $D = 4$ massless superparticle action involves only one of four moving frame vectors, $u_\mu^=$; and this would not happen if we apply that ansatz to the reduction of nAmW action (3.1) which also involves $u_\mu^\#$.

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4.2.1 $\mathrm{U}(1) = \mathrm{SO}(2)$ invariant reduction of $D = 4$ spinor frame formalism

In this section we describe a more complicated $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ invariant ansatz for the reduction of the $D = 4$ spinor moving frame formalism down to $D = 3$ which is characterized by the following expressions for reduced light-like vectors of the 4D moving frame

$$\rho^{\#} u^{=}_{\alpha\dot{\beta}} = \mathcal{M} \mathrm{u}^0_{\alpha\beta} + i\mathcal{M}\epsilon_{\alpha\beta} \, , \tag{4.28}$$

$$\frac{1}{\rho^{\#}} u^{\#}_{\alpha\dot{\beta}} = \mathcal{M} \mathrm{u}^0_{\alpha\beta} - i\mathcal{M}\epsilon_{\alpha\beta} \, , \tag{4.29}$$

where $\mathcal{M} = \mathcal{M}(\tau)$ is a new real field on the worldline. As only these moving frame vectors are present explicitly in the nAmW action, and their reduction does not contain the vectors $\mathrm{u}^I_{\alpha\beta} = (\mathrm{u}^1_{\alpha\beta}, \mathrm{u}^2_{\alpha\beta})$ we can expect that the reduced action will be invariant under the $\mathrm{SO}(2)$ rotation mixing these vectors.

The explicit form of the $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ invariant ansatz expressing the $D = 4$ spinor moving frame variables (2.2) in terms of 3D Lorentz harmonics $\mathrm{v}^q_\alpha = (\mathrm{v}^1_\alpha, \mathrm{v}^2_\alpha)$ (4.16), which produces (4.28) and (4.29) reads

$$\begin{aligned} \sqrt{\rho^{\#}} v^-_\alpha &= \frac{\sqrt{\mathcal{M}}}{\sqrt{2}} \left(\mathrm{v}^2_\alpha - i \mathrm{v}^1_\alpha \right) , & \frac{1}{\sqrt{\rho^{\#}}} v^+_\alpha &= \frac{1}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\mathrm{v}^1_\alpha - i \mathrm{v}^2_\alpha \right) , \\ \sqrt{\rho^{\#}} \bar{v}^-_{\dot{\alpha}} &= \frac{\sqrt{\mathcal{M}}}{\sqrt{2}} \left(\mathrm{v}^2_\alpha + i \mathrm{v}^1_\alpha \right) , & \frac{1}{\sqrt{\rho^{\#}}} \bar{v}^+_{\dot{\alpha}} &= \frac{1}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\mathrm{v}^1_\alpha + i \mathrm{v}^2_\alpha \right) . \end{aligned} \tag{4.30}$$

Notice that this implies

$$\sqrt{\rho^{\#}} \bar{v}^-_{\dot{\alpha}} = i \frac{\mathcal{M}}{\sqrt{\rho^{\#}}} v^+_\alpha \, , \qquad \frac{1}{\sqrt{\rho^{\#}}} \bar{v}^+_{\dot{\alpha}} = i \frac{\sqrt{\rho^{\#}}}{\mathcal{M}} v^-_\alpha \tag{4.31}$$

which reflects the $\mathrm{SO}(2) = \mathrm{U}(1)$ invariance of the ansatz (4.30).

Actually, (4.31) shows that the complete $GL(2, \mathbb{C})$ gauge symmetry of the 4D Lorentz harmonic formalism is preserved by the ansatz (4.30). Its $\mathrm{SO}(1,1)$ subgroup leaves invariant the l.h.s.-s and does not act on the r.h.s.-s, while $\mathrm{U}(1) \subset GL(2, \mathbb{C})$ transformations of the l.h.s. produce the $\mathrm{U}(1) = \mathrm{SO}(2)$ transformations of 3D spinor frame variables in the r.h.s.

Using (4.30), the (pull-backs of) relevant 4D supervielbein forms (2.15)–(2.18) can be expressed in terms of (pull-backs of) 3D supervielbein forms (4.24), (4.23):

$$\begin{aligned} \rho^{\#} E^{=} &= \mathcal{M} \mathrm{E}^0 + \frac{i}{2} \mathcal{M} \epsilon_{\alpha\beta} \Pi^{\alpha\dot{\beta}} = \mathcal{M} \mathrm{E}^0 - \mathcal{M} \Pi^2 \\ &= \mathcal{M} \mathrm{E}^0 - \mathcal{M} (\mathrm{d}x^2 - \mathrm{d}\theta^\gamma \bar{\theta}_\gamma + \theta^\gamma \mathrm{d}\bar{\theta}_\gamma) \, , \end{aligned} \tag{4.32}$$

$$\begin{aligned} \frac{1}{\rho^{\#}} E^{\#} &= \frac{1}{\mathcal{M}} \mathrm{E}^0 - \frac{i}{2} \frac{1}{\mathcal{M}} \epsilon_{\alpha\beta} \Pi^{\alpha\dot{\beta}} = \frac{1}{\mathcal{M}} \mathrm{E}^0 + \frac{1}{\mathcal{M}} \Pi^2 \\ &= \frac{1}{\mathcal{M}} \mathrm{E}^0 + \frac{1}{\mathcal{M}} (\mathrm{d}x^2 - \mathrm{d}\theta^\gamma \bar{\theta}_\gamma + \theta^\gamma \mathrm{d}\bar{\theta}_\gamma) \, , \end{aligned} \tag{4.33}$$

$$\frac{1}{\sqrt{\rho^{\#}}} E^{+} = \frac{1}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\mathcal{E}^1 - i \mathcal{E}^2 \right), \qquad \frac{1}{\sqrt{\rho^{\#}}} \bar{E}^{+} = \frac{1}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\bar{\mathcal{E}}^1 + i \bar{\mathcal{E}}^2 \right) , \tag{4.34}$$

The dimensional reduction of the massless superparticle action requires to use only the first of these expressions, (4.32). Substituting it into the action and using the equations of motion for x^2,

$$\mathrm{d}\mathcal{M} = 0 \qquad \Rightarrow \qquad \mathcal{M} = m = \mathrm{const} \tag{4.35}$$

we arrive at the 3D super-D0-brane action (4.17).

4.2.2 Cartan forms and covariant derivatives in 3D

To describe the dimensional reduction of the nAmW action we have to introduce the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ Cartan forms

$$f^{pq} := \mathrm{v}^{\alpha p}\mathrm{dv}_\alpha^q = +f^{qp}\,, \qquad \mathrm{v}^{\alpha p} = \epsilon^{pq}\mathrm{v}_q^\alpha = \epsilon^{\alpha\beta}\mathrm{v}_\beta^p\,. \tag{4.36}$$

Due to (4.15) their matrix is symmetric, $f^{pq} = f^{qp}$ and the derivative of the 3D spinor frame variables are expressed in terms of these by

$$\mathrm{dv}_\alpha^q = \mathrm{v}_{\alpha p} f^{pq}\,, \tag{4.37}$$

where $\mathrm{v}_{\alpha p} = \epsilon_{pq}\mathrm{v}_\alpha^q$, or in more details

$$\mathrm{dv}_\alpha^1 = \mathrm{v}_\alpha^1 f^{21} - \mathrm{v}_\alpha^2 f^{11}\,, \qquad \mathrm{dv}_\alpha^2 = \mathrm{v}_\alpha^1 f^{22} - \mathrm{v}_\alpha^2 f^{12}\,. \tag{4.38}$$

It is important that f^{qq} transforms as $\mathrm{SO}(2)$ connection, while $f^{12} = f^{21}$ and $f^{11} - f^{22}$ forms are covariant under $\mathrm{SO}(2)$. This can be easily seen using the 3d vector frame (4.18), (4.19) which allows to identify the $\mathrm{SO}(2)$ connection

$$\mathrm{u}^1\mathrm{du}^2 = -\mathrm{u}^2\mathrm{du}^1 = \frac{1}{2}\mathrm{u}^{1\alpha\beta}\mathrm{du}_{\alpha\beta}^2 = f^{qq} \qquad \Leftrightarrow \qquad \mathrm{u}^I\mathrm{du}^J = \epsilon^{IJ} f^{qq}\,, \tag{4.39}$$

and the covariant forms $f^I = (f^1, f^2)$ forming the vielbein of the $\mathrm{SO}(1,2)/\mathrm{SO}(2)$ coset

$$f^1 := \mathrm{u}^0\mathrm{du}^1 = -f^{11} + f^{22}\,, \qquad f^2 := \mathrm{u}^0\mathrm{du}^2 = 2f^{12}\,. \tag{4.40}$$

For our discussion in the next section it is important that (4.30) and (2.23) imply the following expressions for 4D connections

$$\omega^{(0)} - \bar{\omega}^{(0)} = i\mathrm{v}^{\alpha q}\mathrm{dv}_\alpha^q =: i f^{qq}\,, \qquad \omega^{(0)} + \bar{\omega}^{(0)} = \frac{\mathrm{d}\rho^{\#}}{\rho^{\#}} - \frac{\mathrm{d}\mathcal{M}}{\mathcal{M}}\,. \tag{4.41}$$

The second of these equations implies that the covariant derivative of the Stükelberg field of the 4D nAmW system is expressed in terms of derivative of the field $\mathcal{M}$:

$$\mathrm{D}\rho^{\#} = \mathrm{d}\rho^{\#} - (\omega^{(0)} + \bar{\omega}^{(0)})\rho^{\#} = \rho^{\#}\frac{\mathrm{d}\mathcal{M}}{\mathcal{M}}\,. \tag{4.42}$$

4.3 $\mathcal{N}=2$ mD0-brane action from dimensional reduction of the 4D nAmW

Let us turn to the problem of the dimensional reduction of the $D=4$ nAmW action (3.1) This procedure simplifies if, before substituting the above discussed ansatz (4.30) for 4D spinor frame variables, we redefine the matrix fields passing to the SO(1,1) invariant ones by multiplication on suitable powers of the $D=4$ Stückelberg field $\rho^{\#}$:

$$\tilde{\mathbb{Z}} = \frac{1}{\rho^{\#}}\mathbb{Z}\,, \qquad \tilde{\bar{\mathbb{Z}}} = \frac{1}{\rho^{\#}}\bar{\mathbb{Z}}\,, \tag{4.43}$$

$$\tilde{\mathbb{P}} = \frac{1}{(\rho^{\#})^2}\mathbb{P}\,, \qquad \tilde{\bar{\mathbb{P}}} = \frac{1}{(\rho^{\#})^2}\bar{\mathbb{P}}\,, \tag{4.44}$$

$$\tilde{\Psi} = \frac{1}{(\rho^{\#})^{\frac{3}{2}}}\Psi\,, \qquad \tilde{\bar{\Psi}} = \frac{1}{(\rho^{\#})^{\frac{3}{2}}}\bar{\Psi}\,, \tag{4.45}$$

Using (4.43)–(4.45) one can check that the left and right 'charges' of the new matrix fields are opposite,

$$\mathbb{Z} = \mathbb{Z}_{-|+}\,, \qquad \bar{\mathbb{Z}} = \bar{\mathbb{Z}}_{+|-}\,, \tag{4.46}$$

$$\mathbb{P} = \mathbb{P}_{-|+}\,, \qquad \bar{\mathbb{P}} = \bar{\mathbb{P}}_{+|-}\,, \tag{4.47}$$

$$\Psi = \Psi_{\frac{1}{2}|-\frac{1}{2}}\,, \qquad \bar{\Psi} = \bar{\Psi}_{-\frac{1}{2}|\frac{1}{2}}\,, \tag{4.48}$$

which is tantamount to the statement of their SO(1,1) invariance.

The above redefinition clearly produces the terms proportional to $\mathrm{d}\rho^{\#}$ in the action. However, after it is accompanied by the reduction of the $D=4$ spinor moving frame sector with the ansatz (4.30), such derivatives becomes replaced by the derivatives of the new field $\mathcal{M}$ inert under both SO(1,1) and SO(2) transformations. Indeed, one can easily check, using (4.42), that the covariant derivatives of old and new matrix fields are related by

$$\mathrm{D}\tilde{\mathbb{Z}} = \frac{1}{\rho^{\#}}\left(\mathrm{D}\mathbb{Z} - \frac{\mathrm{d}\mathcal{M}}{\mathcal{M}}\mathbb{Z}\right)\,, \tag{4.49}$$

$$\mathrm{D}\tilde{\bar{\mathbb{Z}}} = \frac{1}{\rho^{\#}}\left(\mathrm{D}\bar{\mathbb{Z}} - \frac{\mathrm{d}\mathcal{M}}{\mathcal{M}}\bar{\mathbb{Z}}\right)\,, \tag{4.50}$$

$$\mathrm{D}\tilde{\Psi} = \frac{1}{(\rho^{\#})^{\frac{3}{2}}}\left(\mathrm{D}\Psi - \frac{3}{2}\frac{\mathrm{d}\mathcal{M}}{\mathcal{M}}\Psi\right)\,, \tag{4.51}$$

$$\mathrm{D}\tilde{\bar{\Psi}} = \frac{1}{(\rho^{\#})^{\frac{3}{2}}}\left(\mathrm{D}\bar{\Psi} - \frac{3}{2}\frac{\mathrm{d}\mathcal{M}}{\mathcal{M}}\bar{\Psi}\right)\,, \tag{4.52}$$

where (see (4.41))

$$\mathrm{D}\mathbb{Z} = \mathrm{d}\mathbb{Z} - if^{qq}\mathbb{Z} + [\mathbb{A},\mathbb{Z}]\,, \tag{4.53}$$

$$\mathrm{D}\Psi = \mathrm{d}\Psi + \frac{i}{2}f^{qq}\Psi + [\mathbb{A},\Psi]\,. \tag{4.54}$$

Now, writing the 4D nAmW action in terms of new matrix variables and reduced spinor frame variables (4.30), and using the new covariant derivatives (4.53), (4.54) and supervielbein forms (4.32)–(4.34), we arrive at

$$
\begin{aligned}
S_{\mathrm{nAmW}}|_{(4.30)} = & \int\limits_{\mathcal{W}^1} \mathrm{E}^0 \left(\mathcal{M} + \frac{1}{\mathcal{M}}\frac{\mathcal{H}}{\mu^6}\right) + \\
& + \int\limits_{\mathcal{W}^1} \left(-\mathcal{M} + \frac{1}{\mathcal{M}}\frac{\mathcal{H}}{\mu^6}\right) (\mathrm{d}x^2 - \mathrm{d}\theta^\gamma \bar{\theta}_\gamma + \theta^\gamma \mathrm{d}\bar{\theta}_\gamma) + \\
& + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \mathrm{tr}\left(\bar{\mathbb{P}}\mathrm{D}\mathbb{Z} + \mathbb{P}\mathrm{D}\bar{\mathbb{Z}} - \frac{i}{8}\mathrm{D}\Psi\,\bar{\Psi} + \frac{i}{8}\Psi\mathrm{D}\bar{\Psi}\right) - \\
& - \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{\mathrm{d}\mathcal{M}}{\mathcal{M}} \mathrm{tr}\left(\bar{\mathbb{P}}\mathbb{Z} + \mathbb{P}\bar{\mathbb{Z}}\right) + \\
& + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{i}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\mathcal{E}^1 - i\mathcal{E}^2\right) \mathrm{tr}(\bar{\Psi}\bar{\mathbb{P}} + \Psi[\mathbb{Z},\bar{\mathbb{Z}}]) + \\
& + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{i}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2\right) \mathrm{tr}(\Psi\mathbb{P} + \bar{\Psi}[\mathbb{Z},\bar{\mathbb{Z}}]).
\end{aligned}
\tag{4.55}
$$

Here $\mathcal{H}$ is of the same form as $\widetilde{\mathcal{H}}$ (3.2), but written in terms of new $\mathrm{SO}(1,1)$ invariant matrix variables,

$$
\mathcal{H} = \mathrm{tr}\left(\mathbb{P}\bar{\mathbb{P}} + [\mathbb{Z},\bar{\mathbb{Z}}]^2 - \frac{i}{2}\mathbb{Z}\Psi\Psi + \frac{i}{2}\bar{\mathbb{Z}}\bar{\Psi}\bar{\Psi}\right), \tag{4.56}
$$

and the real bosonic E^0 and complex fermionic $\mathcal{E}^q = (\bar{\mathcal{E}}^q)^*$ 1-forms are defined in (4.24) and (4.23), respectively.

As we have already discussed on simpler examples, the dimensional reduction procedure implies the use of the equations of motion for one of the bosonic coordinate functions, $x^2(\tau)$ in our case. This coordinate function enters only once, in the second term of (4.55), and its equation of motion

$$
\mathrm{d}\left(\mathcal{M} - \frac{1}{\mathcal{M}}\frac{\mathcal{H}}{\mu^6}\right) = 0 \tag{4.57}
$$

implies

$$
\mathcal{M} - \frac{1}{\mathcal{M}}\frac{\mathcal{H}}{\mu^6} = m = \mathrm{const} \tag{4.58}
$$

with constant m of dimension of mass. Eq. (4.58) is solved by

$$
\mathcal{M}_\pm = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} + \frac{\mathcal{H}}{\mu^6}}\,. \tag{4.59}
$$

In our context only the solution with plus sign makes sense, as it has nonvanishing value when $\mathcal{H} = 0$ (in particular when all matrix field vanish),

$$
\mathcal{M} = \mathcal{M}(\mathcal{H}/\mu^6) := \frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{\mathcal{H}}{\mu^6}} = m\left[1 + \frac{\mathcal{H}}{m^2\mu^6} + \mathcal{O}\left(\left(\frac{\mathcal{H}}{m^2\mu^6}\right)^2\right)\right]. \tag{4.60}
$$

The second equality gives the decomposition of the solution in power series in $\frac{1}{\mu^6}$. Notice that this power series can be also treated as weak field decomposition in the matrix fields,

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as decomposition of low energy of relative motion or as decomposition in $\frac{1}{m^2}$. Below we will refer on it as on decomposition in coupling constant $\frac{1}{\mu^6}$ just for convenience.

Substituting the solution (4.60) of the x^2 equation of motion back to the action (4.55) we find the following candidate action for the description of 3D mD0 system:

$$\begin{aligned} S^{3\mathrm{D}}_{\mathrm{mD0}} = & \int\limits_{\mathcal{W}^1} \left(m\mathrm{E}^0 + m(\mathrm{d}\theta^\gamma \bar{\theta}_\gamma - \theta^\gamma \mathrm{d}\bar{\theta}_\gamma) \right) + \\ & + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \mathrm{tr}\left(\bar{\mathbb{P}}\mathrm{D}\mathbb{Z} + \mathbb{P}\mathrm{D}\bar{\mathbb{Z}} - \frac{i}{8}\mathrm{D}\Psi\,\bar{\Psi} + \frac{i}{8}\Psi\mathrm{D}\bar{\Psi} \right) - \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{\mathrm{d}\mathcal{M}}{\mathcal{M}} \mathrm{tr}\left(\bar{\mathbb{P}}\mathbb{Z} + \mathbb{P}\bar{\mathbb{Z}} \right) + \\ & + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \mathrm{E}^0 \frac{2}{\mathcal{M}} \mathcal{H} + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{i}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\mathcal{E}^1 - i\mathcal{E}^2 \right) \mathrm{tr}(\bar{\Psi}\bar{\mathbb{P}} + \Psi[\mathbb{Z}, \bar{\mathbb{Z}}]) + \\ & + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{i}{\sqrt{2}\sqrt{\mathcal{M}}} \left(\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2 \right) \mathrm{tr}(\Psi\mathbb{P} + \bar{\Psi}[\mathbb{Z}, \bar{\mathbb{Z}}]) \end{aligned} \tag{4.61}$$

with $\mathcal{H}$ defined in (4.56), bosonic and fermionic 1-forms defined in (4.24) and (4.23), covariant derivatives defined in (4.53), (4.54) and $\mathcal{M} = \mathcal{M}(\mathcal{H}/\mu^6)$ given in (4.60).

Actually, as we will discuss below, the action (4.61) with arbitrary (nonvanishing) function $\mathcal{M}(\mathcal{H}/\mu^6)$ also makes sense.

Notice that, as the derivative of $\mathcal{M} = \mathcal{M}(\mathcal{H}/\mu^6)$ (independently of whether it is given by (4.60) or considered to be arbitrary) is proportional to $\frac{1}{\mu^6}$ (actually to $\frac{1}{m^2\mu^6}$),

$$\mathrm{d}\mathcal{M} = \frac{1}{\mu^6} \frac{\mathrm{d}\mathcal{H}}{2\sqrt{\frac{m^2}{4} + \frac{\mathcal{H}}{\mu^6}}}\,, \tag{4.62}$$

so that at the first order in $\frac{1}{\mu^6}$ the action does not contain $\mathrm{d}\mathcal{H}$ and reads

$$\begin{aligned} S^{3\mathrm{D}}_{\mathrm{mD0}}|_{\mathcal{M}\mapsto m} = & \int\limits_{\mathcal{W}^1} \left(m\mathrm{E}^0 + m(\mathrm{d}\theta^\gamma \bar{\theta}_\gamma - \theta^\gamma \mathrm{d}\bar{\theta}_\gamma) \right) + \\ & + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \mathrm{tr}\left(\bar{\mathbb{P}}\mathrm{D}\mathbb{Z} + \mathbb{P}\mathrm{D}\bar{\mathbb{Z}} - \frac{i}{8}\mathrm{D}\Psi\,\bar{\Psi} + \frac{i}{8}\Psi\mathrm{D}\bar{\Psi} \right) + \\ & + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \mathrm{E}^0 \frac{2}{m} \mathcal{H} + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{i}{\sqrt{2}\sqrt{m}} \left(\mathcal{E}^1 - i\mathcal{E}^2 \right) \mathrm{tr}(\bar{\Psi}\bar{\mathbb{P}} + \Psi[\mathbb{Z}, \bar{\mathbb{Z}}]) + \\ & + \frac{1}{\mu^6} \int\limits_{\mathcal{W}^1} \frac{i}{\sqrt{2}\sqrt{m}} \left(\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2 \right) \mathrm{tr}(\Psi\mathbb{P} + \bar{\Psi}[\mathbb{Z}, \bar{\mathbb{Z}}])\,. \end{aligned} \tag{4.63}$$

5 Worldline supersymmetry of the $D = 3$ $\mathcal{N} = 2$ mD0-brane action

In this section we will show that the properties expected from mD0 system are possessed by a more generic system described by the functional (4.61) with an arbitrary but nonvanishing function $\mathcal{M} = \mathcal{M}(\mathcal{H}/\mu^6)$. Namely we will show that such action possesses, besides the manifest $D = 3$ $\mathcal{N} = 2$ supersymmetry, also the worldline supersymmetry generalizing the κ-symmetry (4.25) of single D0-brane action (4.17).

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5.1 Worldline supersymmetry transformations of the center of energy variables

The previous experience with 4D nAmW system and with 10D action of [35] suggests, when searching for worldline supersymmetry of mD0-brane, to assume that it acts on the center of energy variables of the mD0 system, i.e. on the coordinate functions and spinor frame variables, in the same manner as the κ-symmetry of single D0 brane,

$$\begin{aligned} \delta_\epsilon \theta^\alpha &= \frac{1}{\sqrt{2}}(\mathrm{v}^{\alpha 2} - i\mathrm{v}^{\alpha 1})\,\epsilon\,, \qquad \delta_\epsilon \bar{\theta}^\alpha = \frac{1}{\sqrt{2}}(\mathrm{v}^{\alpha 2} + i\mathrm{v}^{\alpha 1})\,\bar{\epsilon}\,, \\ \delta_\epsilon x^{\alpha\beta} &= 2i\delta_\epsilon \theta^{(\alpha}\bar{\theta}^{\beta)} - 2i\theta^{(\alpha}\delta_\epsilon\bar{\theta}^{\beta)}\,, \\ \delta_\epsilon \mathrm{v}_\alpha^q &= 0\,. \end{aligned} \tag{5.1}$$

In the formalism of appendix A this is tantamount to stating that

$$\begin{aligned} &i_\epsilon(\mathcal{E}^1 + i\mathcal{E}^2) = 0\,, \qquad i_\epsilon(\bar{\mathcal{E}}^1 - i\bar{\mathcal{E}}^2) = 0\,, \qquad i_\epsilon \mathrm{E}^0 = 0\,, \qquad i_\epsilon \mathrm{E}^I = 0\,, \\ &i_\epsilon f^I = 0\,, \qquad i_\epsilon f^{qq} = 0 \;\Rightarrow\; \delta_\epsilon \mathrm{v}_\alpha^\mp = 0\,, \end{aligned} \tag{5.2}$$

and, consequently,

$$\begin{aligned} i_\epsilon \mathcal{E}^1 = \epsilon/\sqrt{2} \quad &\Rightarrow \quad i_\epsilon\mathcal{E}^2 = i\epsilon/\sqrt{2}\,, \qquad i_\epsilon(\mathcal{E}^1 - i\mathcal{E}^2) = \sqrt{2}\epsilon\,, \\ i_\epsilon \bar{\mathcal{E}}^1 = \bar{\epsilon}/\sqrt{2} \quad &\Rightarrow \quad i_\epsilon\bar{\mathcal{E}}^2 = -i\bar{\epsilon}/\sqrt{2}\,, \qquad i_\epsilon(\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2) = \sqrt{2}\bar{\epsilon}\,. \end{aligned} \tag{5.3}$$

In this form it is easier to find the following transformation properties of the bosonic and fermionic 1-forms entering the part of the action containing the matrix fields

$$\begin{aligned} \delta_\epsilon \mathrm{E}^0 &= -2i\frac{1}{\sqrt{2}}(\mathcal{E}^1 - i\mathcal{E}^2)\bar{\epsilon} - 2i\frac{1}{\sqrt{2}}(\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2)\epsilon\,, \\ \delta_\epsilon(\mathcal{E}^1 - i\mathcal{E}^2) &= \sqrt{2}\mathrm{D}\epsilon\,, \qquad \delta_\epsilon(\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2) = \sqrt{2}\mathrm{D}\bar{\epsilon}\,, \end{aligned} \tag{5.4}$$

which are the typical transformations of the $d = 1$ $\mathcal{N} = 2$ supergravity multiplet. This supergravity is induced by embedding of the worldline into $D = 3$ $\mathcal{N} = 2$ superspace (cf. (2.32) and discussion around it).

Eqs. (5.4) are useful to search for the worldline supersymmetry invariance of the part of the action containing the matrix fields.

5.2 Worldline supersymmetry transformations of the matrix matter fields

Now, writing the variation of the action (4.61) with (5.1)–(5.4) and extracting from this variation the terms proportional to $\mathrm{D}\mathbb{P}$, $\mathrm{D}\mathbb{Z}$, $\mathrm{D}\Psi$, and their hermitian conjugates, we find the following *equations* for the basic κ-symmetry variations of matrix 'matter' fields

$$\delta_\epsilon \mathbb{Z} = -\frac{i}{\sqrt{\mathcal{M}}}\epsilon\bar{\Psi} + \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\left(\mathbb{Z}\delta_\epsilon\tilde{\mathcal{H}} - \mathbb{P}\Delta_\epsilon\mathcal{K}\right), \tag{5.5}$$

$$\delta_\epsilon \bar{\mathbb{Z}} = -\frac{i}{\sqrt{\mathcal{M}}}\bar{\epsilon}\Psi + \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\left(\bar{\mathbb{Z}}\delta_\epsilon\tilde{\mathcal{H}} - \bar{\mathbb{P}}\Delta_\epsilon\mathcal{K}\right), \tag{5.6}$$

$$\begin{aligned}\delta_\epsilon \mathbb{P} = &+\frac{i}{\sqrt{\mathcal{M}}}\left(\epsilon[\Psi,\mathbb{Z}] + \bar{\epsilon}[\bar{\Psi},\mathbb{Z}]\right) - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\,\mathbb{P}\delta_\epsilon\tilde{\mathcal{H}} + \\ &+\frac{1}{\mu^6}\frac{2\mathcal{M}'}{\mathcal{M}}\left([[\mathbb{Z},\bar{\mathbb{Z}}],\mathbb{Z}] + \frac{i}{4}\bar{\Psi}\bar{\Psi}\right)\Delta_\epsilon\mathcal{K},\end{aligned} \tag{5.7}$$

$$\begin{aligned}\delta_\epsilon \bar{\mathbb{P}} = &+\frac{i}{\sqrt{\mathcal{M}}}\left(\epsilon[\bar{\mathbb{Z}},\Psi] + \bar{\epsilon}[\bar{\mathbb{Z}},\bar{\Psi}]\right) - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\,\bar{\mathbb{P}}\delta_\epsilon\tilde{\mathcal{H}} - \\ &-\frac{1}{\mu^6}\frac{2\mathcal{M}'}{\mathcal{M}}\left([[\mathbb{Z},\bar{\mathbb{Z}}],\bar{\mathbb{Z}}] + \frac{i}{4}\Psi\Psi\right)\Delta_\epsilon\mathcal{K},\end{aligned} \tag{5.8}$$

$$\delta_\epsilon \Psi = +\frac{4}{\sqrt{\mathcal{M}}}\left(\epsilon\bar{\mathbb{P}} + \bar{\epsilon}[\mathbb{Z},\bar{\mathbb{Z}}]]\right) - \frac{1}{\mu^6}\frac{2\mathcal{M}'}{\mathcal{M}}[\bar{\Psi},\bar{\mathbb{Z}}]\Delta_\epsilon\mathcal{K}, \tag{5.9}$$

$$\delta_\epsilon \bar{\Psi} = +\frac{4}{\sqrt{\mathcal{M}}}\left(\epsilon[\mathbb{Z},\bar{\mathbb{Z}}] + \bar{\epsilon}\mathbb{P}]\right) + \frac{1}{\mu^6}\frac{2\mathcal{M}'}{\mathcal{M}}[\Psi,\mathbb{Z}]\Delta_\epsilon\mathcal{K}. \tag{5.10}$$

These are equations because their right hand sides contain variations $\delta_\epsilon\mathcal{H}$ of $\mathcal{H}$ from (4.56) and

$$\Delta_\epsilon\mathcal{K} = \delta_\epsilon\mathcal{K} - \frac{i}{2\sqrt{\mathcal{M}}}\left(\epsilon\,\nu + \bar{\epsilon}\bar{\nu}\right) \tag{5.11}$$

where

$$\nu = \mathrm{tr}\left(\bar{\Psi}\bar{\mathbb{P}} + \Psi[\mathbb{Z},\bar{\mathbb{Z}}]\right), \qquad \bar{\nu} = \mathrm{tr}\left(\Psi\mathbb{P} + \bar{\Psi}[\mathbb{Z},\bar{\mathbb{Z}}]\right) \tag{5.12}$$

and $\delta_\epsilon\mathcal{K}$ is the variation of

$$\mathcal{K} = \mathrm{tr}(\bar{\mathbb{P}}\mathbb{Z} + \mathbb{P}\bar{\mathbb{Z}}). \tag{5.13}$$

As both these composite variations enters with coefficients $\propto \frac{1}{\mu^6}$, the equations certainly have the solution, at least as power series in $\frac{1}{\mu^6}$. But moreover, there exists a simple way to solve these equations which consists in firstly, using them to obtain closed algebraic equations for $\delta_\epsilon\mathcal{H}$ and $\delta_\epsilon\mathcal{K}$, secondly, solve these and, thirdly, substituting these solutions for the last terms in (5.5)–(5.10).

Indeed, using (5.5)–(5.10) we find

$$\delta_\epsilon\mathcal{H} = \frac{i}{\sqrt{\mathcal{M}}}\,\mathrm{tr}\left[\left(\bar{\epsilon}\bar{\Psi} - \epsilon\Psi\right)\left([\mathbb{Z},\bar{\mathbb{P}}] + [\bar{\mathbb{Z}},\mathbb{P}] - \frac{i}{4}\{\Psi,\bar{\Psi}\}\right)\right] + \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\,\mathfrak{H}\,\delta_\epsilon\mathcal{H} \tag{5.14}$$

and

$$\delta_\epsilon\mathcal{K} = -\frac{i}{\sqrt{\mathcal{M}}}\,\epsilon\,\mathrm{tr}\left(\bar{\Psi}\bar{\mathbb{P}} - 2\Psi[\mathbb{Z},\bar{\mathbb{Z}}]\right) - \frac{i}{\sqrt{\mathcal{M}}}\,\bar{\epsilon}\,\mathrm{tr}\left(\Psi\mathbb{P} - 2\bar{\Psi}[\mathbb{Z},\bar{\mathbb{Z}}]\right) + \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\,\mathfrak{H}\,\Delta_\epsilon\mathcal{K}, \tag{5.15}$$

where

$$\mathfrak{H} = \mathrm{tr}\left(-2\mathbb{P}\bar{\mathbb{P}} + 4[\mathbb{Z},\bar{\mathbb{Z}}]^2 - \frac{i}{2}\bar{\mathbb{Z}}\Psi\Psi + \frac{i}{2}\mathbb{Z}\bar{\Psi}\bar{\Psi}\right) \tag{5.16}$$

(cf. (4.56)). Eqs. (5.14) and (5.15) are closed algebraic equations for $\delta_\epsilon \mathcal{H}$ and $\delta_\epsilon \mathcal{K}$, respectively, which are solved by

$$\delta_\epsilon \mathcal{H} = \frac{i}{\sqrt{\mathcal{M}}} \frac{1}{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \operatorname{tr}\left[\left(\bar{\epsilon}\bar{\Psi} - \epsilon\Psi\right) \left([\mathbb{Z}, \bar{\mathbb{P}}] + [\bar{\mathbb{Z}}, \mathbb{P}] - \frac{i}{4}\{\Psi, \bar{\Psi}\}\right)\right] \tag{5.17}$$

and

$$\Delta_\epsilon \mathcal{K} = -\frac{3i}{\sqrt{\mathcal{M}}} \frac{1}{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \left[\epsilon \operatorname{tr}\left(\bar{\Psi}\bar{\mathbb{P}} - \Psi[\mathbb{Z}, \bar{\mathbb{Z}}]\right) + \bar{\epsilon} \operatorname{tr}\left(\Psi\mathbb{P} - \bar{\Psi}[\mathbb{Z}, \bar{\mathbb{Z}}]\right)\right] . \tag{5.18}$$

Thus the worldline supersymmetry transformations of the matrix matter fields are given by (5.5)–(5.10) with (5.17) and (5.18).

5.3 Worldline supersymmetry transformations of the non-Abelian gauge field

Taking into account the above relations, we find that the remaining expression of the action variation contains the terms proportional to the pull-backs of bosonic and fermionic supervielbein forms to the worldline, namely to

$$\mathrm{E}^0 , \qquad (\mathcal{E}^1 - i\mathcal{E}^2) , \qquad (\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2) \tag{5.19}$$

and to the variation of $\mathrm{SU}(N)$ gauge field 1-form $\delta\mathbb{A}$. This implies that, if the action is invariant under worldline supersymmetry, then

$$\delta_\epsilon \mathbb{A} = \mathrm{E}^0 \delta_\epsilon \mathbb{A}_0 + (\mathcal{E}^1 - i\mathcal{E}^2)\delta_\epsilon \mathbb{A}_\eta + (\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2)\delta_\epsilon \mathbb{A}_{\bar{\eta}} . \tag{5.20}$$

It is important to notice that $\delta_\epsilon \mathbb{A}$ enters the action variation in the trace of its product with $\left([\mathbb{Z}, \bar{\mathbb{P}}] + [\bar{\mathbb{Z}}, \mathbb{P}] - \frac{i}{4}\{\Psi\bar{\Psi}\}\right)$,

$$\operatorname{tr}\left[\delta_\epsilon \mathbb{A} \left([\mathbb{Z}, \bar{\mathbb{P}}] + [\bar{\mathbb{Z}}, \mathbb{P}] - \frac{i}{4}\{\Psi, \bar{\Psi}\}\right)\right] , \tag{5.21}$$

so that the possibility to compensate all remaining terms in the action variation by choosing appropriate $\delta_\epsilon \mathbb{A}_0$, $\delta_\epsilon \mathbb{A}_\eta$ and $\delta_\epsilon \mathbb{A}_{\bar{\eta}}$ is a nontrivial check of consistency of our calculations.

For instance, this implies that the condition of vanishing the contribution $\propto \mathrm{E}^0$ in the κ-symmetry variation of the action,

$$\operatorname{tr}\left[\delta_\epsilon \mathbb{A}_0 \left([\mathbb{Z}, \bar{\mathbb{P}}] + [\bar{\mathbb{Z}}, \mathbb{P}] - \frac{i}{4}\{\Psi, \bar{\Psi}\}\right)\right] = -\frac{2}{\mathcal{M}} \left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}}\right) \delta_\epsilon \mathcal{H} , \tag{5.22}$$

can be solved because $\delta_\epsilon \mathcal{H}$ (5.17) is also given by the trace of certain expression with $\left([\mathbb{Z}, \bar{\mathbb{P}}] + [\bar{\mathbb{Z}}, \mathbb{P}] - \frac{i}{4}\{\Psi, \bar{\Psi}\}\right)$. This allows to obtain

$$\delta_\epsilon \mathbb{A}_0 = -\frac{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathcal{H}\right)}{\left(1 - \frac{1}{\mu^6} \frac{\mathcal{M}'}{\mathcal{M}} \mathfrak{H}\right)} \frac{2i}{\mathcal{M}\sqrt{\mathcal{M}}} \left(\bar{\epsilon}\bar{\Psi} - \epsilon\Psi\right) . \tag{5.23}$$

Similarly studying the terms $\propto (\mathcal{E}^1 - i\mathcal{E}^2)$ and their c.c.-s we obtain

$$\begin{aligned}
\delta_\epsilon \mathbb{A}_\eta &= \frac{8i}{\sqrt{2\mathcal{M}}}\, \epsilon\bar{\mathbb{Z}} - \frac{1}{\mu^6}\, \frac{\mathcal{M}'}{2\sqrt{2}\mathcal{M}^2} \left(2i\Psi\sqrt{\mathcal{M}}\Delta_\epsilon\mathcal{K} - \frac{3\left(\epsilon\Psi - \bar{\epsilon}\bar{\Psi}\right) \mathrm{tr}\left(\bar{\mathbb{P}}\bar{\Psi} - \Psi[\mathbb{Z},\bar{\mathbb{Z}}]\right)}{\left(1 - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathfrak{H}\right)} \right) \\
&= \frac{8i}{\sqrt{2\mathcal{M}}}\, \epsilon\bar{\mathbb{Z}} - \frac{1}{\mu^6}\, \frac{3\mathcal{M}'}{2\sqrt{2}\mathcal{M}^2} \frac{1}{\left(1 - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathfrak{H}\right)} \times \\
&\qquad \times \left[(\bar{\epsilon}\bar{\Psi} - 2\epsilon\Psi)\mathrm{tr}\left(\bar{\mathbb{P}}\bar{\Psi} - \Psi[\mathbb{Z},\bar{\mathbb{Z}}]\right) + \bar{\epsilon}\Psi\mathrm{tr}\left(\bar{\mathbb{P}}\bar{\Psi} - \Psi[\mathbb{Z},\bar{\mathbb{Z}}]\right) \right]
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
\delta_\epsilon \mathbb{A}_{\bar\eta} &= -\frac{8i}{\sqrt{2\mathcal{M}}}\, \bar{\epsilon}\mathbb{Z} + \frac{1}{\mu^6}\, \frac{3\mathcal{M}'}{2\sqrt{2}\mathcal{M}^2} \frac{\epsilon\Psi - \bar{\epsilon}\bar{\Psi}}{\left(1 - \frac{1}{\mu^6}\frac{\mathcal{M}'}{\mathcal{M}}\mathfrak{H}\right)}\, \mathrm{tr}\left(\mathbb{P}\Psi - \bar{\Psi}[\mathbb{Z},\bar{\mathbb{Z}}]\right) + \\
&\quad + \frac{1}{\mu^6}\, \frac{i\mathcal{M}'}{\sqrt{2}\sqrt{\mathcal{M}}\mathcal{M}}\, \bar{\Psi}\, \Delta_\epsilon\mathcal{K}\,.
\end{aligned} \tag{5.25}$$

The second of these equations can be further specified with the use of (5.18).

5.4 Resume on the candidate mD0 action(s)

Thus we have shown that the actions (4.61) with arbitrary nonvanishing function $\mathcal{M}(\mathcal{H}/\mu^6)$ of the matrix field Hamiltonian (4.56) possesses, besides the target superspace $D = 3$ $\mathcal{N} = 2$ supersymmetry, also local worldline supersymmetry which generalizes the κ-symmetry of the single-D0-brane action. Hence any of these can be considered as a candidate for the role of $D = 3$ counterpart of the multiple D0-brane (mD0) action. A special representative of this family is the action with $\mathcal{M}(\mathcal{H}/\mu^6)$ given in (4.60), as this is obtained by dimensional reduction from the $D = 4$ counterpart of the mM0-brane system (4D nAmW action). Generically the actions (4.61) are essentially nonlinear but the simplest representation of the family with $\mathcal{M}(\mathcal{H}/\mu^6) = m =$ const. This simplest case provide us with the counterpart of the 10D action considered as a candidate on the role of mD0 action in [35]. Our study suggests to search for more generic essentially nonlinear candidates on the role of 10D mD0 action, and this problem is presently under investigation.

6 Conclusion

The main result of this paper is doubly supersymmetric (i.e. possessing both rigid target superspace supersymmetry and worldline supersymmetry) candidate action(s) for the description of 3D counterpart of 10D multiple-D0-brane (mD0), (4.61). It includes all bosonic and fermionic fields which are expected to be present in the 3D mD0 system, which can be restored from the very low energy gauge fixed description by the U(N) SYM action [16] and the known actions for single super-Dp-branes [2–7]. Furthermore it is invariant under both the target superspace $D = 3$ $\mathcal{N} = 2$ rigid supersymmetry and local worldline supersymmetry generalizing the κ-symmetry of the single super-D0-brane (in its irreducible form characteristic for the spinor moving frame formulation [61]). This latter is

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important because it guarantees the preservation of a part of target space supersymmetry by the ground state of the system, and this is what is expected from the multiple D0-brane.

Curiously enough the action includes an arbitrary nonvanishing function $\mathcal{M}(\mathcal{H})$ of the 'relative motion Hamiltonian' $\mathcal{H}$ which in its turn is constructed from the bosonic and fermionic matrix fields of mD0 system. Generically, the action is essentially nonlinear. An exception is the case of $\mathcal{M}(\mathcal{H}) = m =$ const (4.63) which is given by the sum of single D0-brane action (the variables of which is now describing the center of mass motion of the system) and the action of $d = 1$ $\mathcal{N} = 2$ SYM the supersymmetry of which is made local by coupling to the supergravity induced by the center of energy motion (see (5.4)). The $D = 10$ counterpart of such an action was considered in [35] where, in particular, its comparison with the multiple 0-brane action from [31] was carried out. Our present study suggests the existence of other candidates on the role of 10D mD0 actions and we will turn to the problem of their construction and studying in a forthcoming publication.

One might wonder whether it is possible to write also the generalization of 4D nAmW action and of its 11D mM0 prototype with arbitrary function $\tilde{\mathcal{M}}(\tilde{\mathcal{H}})$. At least presently we do not know such actions possessing, besides target space supersymmetry, also the worldline supersymmetry. In the case of 3D mD0-brane the possibility to write the doubly supersymmetric action with arbitrary function $\mathcal{M}(\mathcal{H})$ was found in a way occasionally: we first obtained the action with definite function (4.60) by dimensional reduction of the $D = 4$ nAmW (4D mM0) action, and found that it is more convenient to check for the presence of worldline supersymmetry first without specification of the form of $\mathcal{M}(\mathcal{H})$. Then we have found that the deserved worldline supersymmetry is actually present in the case of arbitrary invertible $\mathcal{M}(\mathcal{H})$.

The problem of choice of the function $\mathcal{M}(\mathcal{H})$ which leads to the true mD0-brane action should be addressed in a more general perspective of String/M-theory. It is tempting to use the T-duality (which was the main argument for construction of bosonic actions in [20]) to make this choice. However, to this end we need also to have a complete, doubly supersymmetric action for multiple super-D1-brane (mD1) system. For the moment, in our $D = 3$ case a special role is played by the action with $\mathcal{M}(\mathcal{H})$ given in (4.60) because it is obtained from the $D = 4$ action for a nAmW system (3.1), the 4D counterpart of the multiple M0-brane system [32] which was also obtained for the first time in the present paper.

Let us also notice that, if we froze all the center of energy degrees of freedom in our mD0 action with an arbitrary invertible $\mathcal{M}(\mathcal{H})$, we find a family of nonlinear generalizations of the (first order) action for 1d SYM which for our knowledge are new. In higher dimensions the known nonlinear generalizations of the non-Abelian gauge field action are quite a few, to the best of our knowledge, all related to the symmetric trace BI action by Tseytlin [19]. The study of the properties of new 1d nonlinear supersymmetric gauge field models and their applications looks interesting on its own.

The other interesting directions for the development of the line of present study is, first of all, the already mentioned search for 10D generalization of the action (4.61), proving its worldline supersymmetry, studying its properties and its possible relation with 11D mM0-action of [32], as well as addressing the problem of choice of the best candidate for the role of 10D mD0-brane action. This latter issue is related to the problem of the search for higher p multiple Dp-brane actions, beginning from $p = 1$.

As far as the study of $D = 3$ and $D = 4$ supersymmetric systems is concerned, an interesting problem is the development of Hamiltonian approach and quantization of $D = 4$ $\mathcal{N} = 1$ nAmW system as well as of $D = 3$ $\mathcal{N} = 2$ mD0-brane. These should result in new interesting supersymmetric systems of relativistic wave equations in the (super)spaces with noncommutative coordinates which promise to be simpler than the result of quantization of $D = 3$ $\mathcal{N} = 1$ nAmW system in [45] due to the complex structure characteristic for the both $D = 4$ $\mathcal{N} = 1$ and $D = 3$ $\mathcal{N} = 2$ superspaces.

As far as quantization of $D = 4$ nAmW system is concerned, it can be considered as a preliminary step to approach the quantum description of 11D mM0 system which in its turn might shed a new light on the properties of String/M-theory. In $D = 3$ $\mathcal{N} = 2$ case especially intriguing looks the question on the influence of the choice of the function $\mathcal{M}(\mathcal{H})$ on the results of the quantization.

Acknowledgments

Work of IB has also been partially supported by the Spanish MICINN/FEDER (ERDF EU) grant PGC2018-095205-B-I00, by the Basque Government Grant IT-979-16, and by the Basque Country University program UFI 11/55.

A Notice on differential forms and variations

In this appendix we present some basic equations of the differential form formalism.

Let Ξ_q and Ξ'_p be differential q-form and p-form in a superspace with coordinates $Y^{\mathfrak{M}}$. This is to say

$$\Xi_q = \frac{1}{q!} \mathrm{d}Y^{\mathfrak{M}_q} \wedge \ldots \wedge \mathrm{d}Y^{\mathfrak{M}_1} \Xi_{\mathfrak{M}_1 \ldots \mathfrak{M}_q}(Y) \,, \quad \Xi'_p = \frac{1}{p!} \mathrm{d}Y^{\mathfrak{M}_p} \wedge \ldots \wedge \mathrm{d}Y^{\mathfrak{M}_1} \Xi'_{\mathfrak{M}_1 \ldots \mathfrak{M}_p}(Y) \,, \tag{A.1}$$

where $\wedge$ is the exterior product of the forms,

$$\mathrm{d}Y^{\mathfrak{M}} \wedge \mathrm{d}Y^{\mathfrak{N}} = -(-1)^{\varepsilon(\mathfrak{M})\,\varepsilon(\mathfrak{N})} \mathrm{d}Y^{\mathfrak{N}} \wedge \mathrm{d}Y^{\mathfrak{M}} \,, \quad Y^{\mathfrak{M}} Y^{\mathfrak{N}} = (-1)^{\varepsilon(\mathfrak{M})\,\varepsilon(\mathfrak{N})} Y^{\mathfrak{N}} Y^{\mathfrak{M}} \tag{A.2}$$

and $\varepsilon(\mathfrak{M})$ is the Grassmann parity of $Y^{\mathfrak{M}}$. For instance, in the case of $D = 4$ $\mathcal{N} = 1$ superspace, $Y^{\mathfrak{M}} \mapsto Z^M = (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ (1.1) and

$$\varepsilon(x^\mu) = 0, \qquad \varepsilon(\theta^\alpha) = 1, \qquad \varepsilon(\bar{\theta}^{\dot{\alpha}}) = 1 \,. \tag{A.3}$$

For bosonic p- and q-forms,

$$\Xi_q \wedge \Xi_p = (-1)^{pq} \Xi_p \wedge \Xi_q \qquad \text{when} \qquad \varepsilon(\Xi_q) = 0 \,. \tag{A.4}$$

In the wedge product of two fermionic forms an additional minus sign appears, so that, e.g.

$$\mathrm{d}\theta^\alpha \wedge \mathrm{d}\theta^\beta = \mathrm{d}\theta^\beta \wedge \mathrm{d}\theta^\alpha \,, \quad \mathrm{d}x^\mu \wedge \mathrm{d}\theta^\alpha = -\mathrm{d}\theta^\alpha \wedge \mathrm{d}x^\mu \,, \quad \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \,. \tag{A.5}$$

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The exterior derivative of the differential forms is defined by

$$\begin{aligned} \mathrm{d}\Xi_q &= \frac{1}{q!}\mathrm{d}Y^{\mathfrak{M}_q}\wedge\ldots\wedge\mathrm{d}Y^{\mathfrak{M}_1}\wedge\mathrm{d}Y^{\mathfrak{M}_0}\partial_{\mathfrak{M}_0}\Xi_{\mathfrak{M}_1\ldots\mathfrak{M}_q}(Y) \\ &\equiv \frac{1}{(q+1)!}\mathrm{d}Y^{\mathfrak{M}_{q+1}}\wedge\ldots\wedge\mathrm{d}Y^{\mathfrak{M}_2}\wedge\mathrm{d}Y^{\mathfrak{M}_1}(q+1)\partial_{[\mathfrak{M}_2}\Xi_{\mathfrak{M}_2\ldots\mathfrak{M}_{q+1}\}}(Y)\,. \end{aligned} \tag{A.6}$$

Here the mixed brackets $[\ldots\}$ denote graded antisymmetrization over the enclosed indices, which implies symmetrization for exchanging the pair of fermionic indices and antisymmetrization for the pair including at least one bosonic index. The exterior derivative is nilpotent

$$\mathrm{dd}=0 \tag{A.7}$$

and acts on the product of differential forms according to the following Leibniz rules

$$\mathrm{d}(\Xi_p\wedge\Xi'_q)=\Xi_p\wedge\mathrm{d}\Xi'_q+(-)^q\mathrm{d}\Xi_p\wedge\Xi'_q\,. \tag{A.8}$$

The variations of differential forms can be calculated with the use of Lie derivative formula,

$$\delta\Xi_q=i_\delta\mathrm{d}\Xi_q+\mathrm{d}i_\delta\Xi_q\,, \tag{A.9}$$

where i_δ is the contraction with variation symbol defined by

$$i_\delta\Xi_q=\frac{1}{(q-1)!}\mathrm{d}Y^{\mathfrak{M}_q}\wedge\ldots\wedge\mathrm{d}Y^{\mathfrak{M}_2}\delta Y^{\mathfrak{M}_1}\Xi_{\mathfrak{M}_1\ldots\mathfrak{M}_q}(Y)\,, \tag{A.10}$$

and hence obeying

$$i_\delta(\Xi_p\wedge\Xi'_q)=\Xi_p\wedge i_\delta\Xi'_q+(-)^q i_\delta\Xi_p\wedge\Xi'_q\,. \tag{A.11}$$

To use the Lie derivative expression for the variation of the Lagrangian D-form $\mathcal{L}$ in D-dimensional field theory,

$$\delta\mathcal{L}=i_\delta(\mathrm{d}\mathcal{L})+\mathrm{d}(i_\delta\mathcal{L})\,, \tag{A.12}$$

the exterior derivative should be considered as formally taken in the space of more dimensions: better in the space where all the fields are treated on the equal footing with coordinates (coordinate-field democracy).

In particular in the case of superparticle Lagrangian, to obtain its variation from the Lie derivative formula (A.12), the exterior derivative should be calculated with considering all the 1d fields, $x^\mu(\tau)$, $v^-_\alpha(\tau)$ etc., to be replaced by independent variables x^μ, v^-_α etc. rather than being the functions of proper time τ only. Also the second term is the total derivative and hence is not essential when we derive the equations of motion.

Often it is convenient to use the covariant Lie derivative formula, e.g.

$$\delta\mathcal{L}=i_\delta(\mathrm{D}\mathcal{L})+\mathrm{D}(i_\delta\mathcal{L})\,, \tag{A.13}$$

which is equivalent to (A.13) in the case when connection in the covariant derivative corresponds to gauge transformations which leave invariant the Lagrangian form.

Examples of the use of simple and covariant Lie derivative formula are

$$\mathrm{d}\Pi^{\alpha\dot\alpha}=-4i\mathrm{d}\theta^\alpha\wedge d\bar\theta^{\dot\alpha}\quad\Rightarrow\quad\delta\Pi^{\alpha\dot\alpha}=+4i\delta\theta^\alpha\mathrm{d}\bar\theta^{\dot\alpha}-4i\mathrm{d}\theta^\alpha\delta\bar\theta^{\dot\alpha}+\mathrm{d}(i_\delta\Pi^{\alpha\dot\alpha})\,, \tag{A.14}$$

as well as

$$
\begin{aligned}
&\mathrm{D}E^{\#} = -4iE^{+}\wedge \bar{E}^{+} - E^{-+}\wedge \Omega^{++} - E^{+-}\wedge \bar{\Omega}^{++} \quad \Rightarrow \\
\Rightarrow \quad &\delta E^{\#} = -4iE^{+}i_\delta \bar{E}^{+} + 4ii_\delta E^{+}\bar{E}^{+} + \left(i_\delta E^{-+}\Omega^{++} - E^{-+}i_\delta\Omega^{++} + \text{c.c.}\right) , \qquad (A.15)\\
&\mathrm{D}E^{+} = -E^{-}\wedge \Omega^{++} \quad \Rightarrow \quad \delta E^{+} = i_\delta E^{-}\Omega^{++} - E^{-}i_\delta \Omega^{++} + \mathrm{D}i_\delta E^{+} , \qquad (A.16)\\
&\mathrm{D}\bar{E}^{+} = -\bar{E}^{-}\wedge \bar{\Omega}^{++} \quad \Rightarrow \quad \delta \bar{E}^{+} = i_\delta \bar{E}^{-}\bar{\Omega}^{++} - \bar{E}^{-}i_\delta \bar{\Omega}^{++} + \mathrm{D}i_\delta \bar{E}^{+} , \qquad (A.17)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{D}E^{=} = -4iE^{-}\wedge \bar{E}^{-} + E^{+-}\wedge \Omega^{--} + E^{-+}\wedge \bar{\Omega}^{--} \quad \Rightarrow \\
\Rightarrow \quad &\delta E^{=} = -4iE^{-}i_\delta \bar{E}^{-} + 4ii_\delta E^{-}\bar{E}^{-} + \left(-i_\delta E^{+-}\Omega^{--} + E^{+-}i_\delta\Omega^{--} + \text{c.c.}\right) , \qquad (A.18)\\
&\mathrm{D}E^{-} = +E^{+}\wedge \Omega^{--} \quad \Rightarrow \quad \delta E^{-} = -i_\delta E^{+}\Omega^{--} + E^{+}i_\delta \Omega^{--} + \mathrm{D}i_\delta E^{-} , \qquad (A.19)\\
&\mathrm{D}\bar{E}^{-} = +\bar{E}^{+}\wedge \bar{\Omega}^{--} \quad \Rightarrow \quad \delta \bar{E}^{-} = -i_\delta \bar{E}^{+}\bar{\Omega}^{--} + \bar{E}^{+}i_\delta \bar{\Omega}^{--} + \mathrm{D}i_\delta \bar{E}^{-} . \qquad (A.20)
\end{aligned}
$$

To illustrate the above formalism let us use it to obtain equations of motion for the massless superparticle in $D = 4$ $\mathcal{N} = 1$ superspace. The formal exterior derivative of its Lagrangian form

$$
\mathcal{L}^{0\ \mathrm{D=4}} = \rho^{\#}E^{=} = \rho^{\#}\Pi^{\alpha\dot{\beta}}v_\alpha^{-}\bar{v}_{\dot{\beta}}^{-} \qquad (A.21)
$$

is

$$
\mathrm{d}\mathcal{L}^{0\ \mathrm{D=4}} = -\mathrm{D}\rho^{\#}\wedge E^{=} - 4i\rho^{\#}E^{-}\wedge \bar{E}^{-} + \rho^{\#}E^{+-}\wedge \Omega^{--} + \rho^{\#}E^{-+}\wedge \bar{\Omega}^{--} . \qquad (A.22)
$$

From this, using (A.12), we find

$$
\begin{aligned}
\delta\mathcal{L}^{0\,\mathrm{D=4}} = &\, i_\delta \mathrm{D}\rho^{\#}E^{=} - \mathrm{D}\rho^{\#}i_\delta E^{=} - 4i\rho^{\#}E^{-}i_\delta\bar{E}^{-} + 4i\rho^{\#}i_\delta E^{-}\bar{E}^{-} + \\
&+ \rho^{\#}E^{+-}i_\delta\Omega^{--} - \rho^{\#}i_\delta E^{+-}\Omega^{--} + \rho^{\#}E^{-+}i_\delta\bar{\Omega}^{--} - \rho^{\#}i_\delta E^{-+}\bar{\Omega}^{--} \qquad (A.23)
\end{aligned}
$$

which allows to easily find that the nontrivial equations of motion of the superparticle in the spinor moving frame formulations:

$$
\begin{aligned}
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta \mathrm{D}\rho^{\#}} = \frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta\rho^{\#}} = 0 \quad \Rightarrow \quad E^{=} = 0 , \qquad (A.24)\\
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta \Omega^{--}} = v_\alpha^{+}\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta v_\alpha^{-}} = 0 \quad \Rightarrow \quad \rho^{\#}E^{+-} = 0 , \qquad (A.25)\\
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta \bar{\Omega}^{--}} = \bar{v}_{\dot{\alpha}}^{+}\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta \bar{v}_{\dot{\alpha}}^{-}} = 0 \quad \Rightarrow \quad \rho^{\#}E^{-+} = 0 , \qquad (A.26)\\
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta E^{=}} = \frac{1}{2}u_\mu^{\#}\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta x^\mu} = 0 \quad \Rightarrow \quad \mathrm{D}\rho^{\#} = 0 , \qquad (A.27)\\
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta E^{+-}} = \frac{1}{2}u_\mu^{-+}\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta x^\mu} = 0 \quad \Rightarrow \quad \rho^{\#}\Omega^{--} = 0 , \qquad (A.28)\\
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta E^{-+}} = \frac{1}{2}u_\mu^{+-}\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta x^\mu} = 0 \quad \Rightarrow \quad \rho^{\#}\bar{\Omega}^{--} = 0 , \qquad (A.29)\\
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta E^{-}} = -v^{\alpha +}\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta \theta^\alpha} = 0 \quad \Rightarrow \quad \rho^{\#}\bar{E}^{-} = 0 , \qquad (A.30)\\
&\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{i_\delta \bar{E}^{-}} = -\bar{v}^{\dot{\alpha} +}\frac{\delta\mathcal{L}^{0\ \mathrm{D=4}}}{\delta \bar{\theta}^{\dot{\alpha}}} = 0 \quad \Rightarrow \quad \rho^{\#}E^{-} = 0 . \qquad (A.31)
\end{aligned}
$$

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B $D = 3$ massless superparticle by dimensional reduction of spinor moving frame sector of the $D = 4$ massless superparticle

In this appendix we discuss the derivation of the $D = 3$ massless superparticle action from its $D = 4$ counterpart in spinor frame formulation. This can be considered as an illuminating warm up exercise suggesting the way of dimensional reductions of more complex systems discussed in the main text.

B.1 Brink-Schwarz formulation

In the Brink-Schwarz formulation of the massless superparticle action (4.1) the dimensional reduction can be performed by, firstly, separating, say p_2 component (we make this choice for future convenience),

$$p_\mu = (p_{\tilde{\mu}}, p_2)\,, \qquad \tilde{\mu} = 0,1,3 \qquad \leftrightarrow \qquad \mu = 0,1,2,3\,, \tag{B.1}$$

and setting this component equal to zero,

$$p_2 = 0\,. \tag{B.2}$$

Substituting (B.2) as an ansatz into (4.1), we arrive at the action for $D = 3$ massless superparticle,

$$S^{\mathrm{D=3}}_{\mathrm{BS}} = \int \left(p_{\tilde{\mu}} \Pi^{\tilde{\mu}} + \frac{1}{2} \mathrm{d}\tau e p_{\tilde{\mu}} p^{\tilde{\mu}} \right)\,. \tag{B.3}$$

B.2 Spinor moving frame formulation

As in the spinor moving frame formulation the role of additional momentum variable are taken by bilinear of bosonic spinors, v^-_α and $\bar{v}^-_{\dot{\alpha}}$, it is natural to expect that similar dimensional reduction can be performed just by choosing a suitable ansatz for the $D = 4$ harmonics in terms of $D = 3$ ones.

Actually the ansatz consists in imposing reality conditions

$$v^-_\alpha = \bar{v}^-_{\dot{\alpha}}\,, \qquad v^+_\alpha = \bar{v}^+_{\dot{\alpha}}\,. \tag{B.4}$$

This choice is invariant under $\mathrm{SL}(2,\mathbb{R}) = \mathrm{Spin}(1,2)$ subgroup of $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Spin}(1,3)$. It is convenient to complete (B.4) to

$$v^-_\alpha = \bar{v}^-_{\dot{\alpha}} = \mathrm{v}^-_\alpha\,, \qquad v^+_\alpha = \bar{v}^+_{\dot{\alpha}} = \mathrm{v}^+_\alpha\,, \tag{B.5}$$

were we have introduced real 3D spinor moving frame variables $\mathrm{v}^\pm_\alpha$ which form the $\mathrm{SL}(2,\mathbb{R})$ valued matrix, i.e. obey

$$\mathrm{v}^{-\alpha}\mathrm{v}^+_\alpha = 1\,, \qquad (\mathrm{v}^\pm_\alpha)^* = \mathrm{v}^\pm_\alpha\,. \tag{B.6}$$

Using the split of the $D = 4$ relativistic Pauli matrices (1.6) as in (1.10) and identifying the $D = 3$ gamma matrices as in (1.11), (1.12) we find that the ansatz (B.5) implies

$$u^{=}_2 = 0\,, \qquad u^{\#}_2 = 0\,, \qquad u^{-+}_2 = -u^{+-}_2 = (u^{+-}_2)^*\,, \tag{B.7}$$

while the real three vectors formed from $0, 1, 3$ components of the 4D moving frame vectors are identified with the components of the 3D moving frame vectors

$$\mathrm{u}^{=}_{\tilde{\mu}} = \mathrm{v}^{-}\gamma_{\tilde{\mu}}\mathrm{v}^{-} \equiv \mathrm{v}^{-\alpha}\gamma_{\tilde{\mu}\,\alpha\beta}\mathrm{v}^{-\beta}\,, \qquad \mathrm{u}^{\#}_{\tilde{\mu}} = \mathrm{v}^{+}\gamma_{\tilde{\mu}}\mathrm{v}^{+}\,, \qquad \mathrm{u}^{\perp}_{\tilde{\mu}} = \mathrm{v}^{+}\gamma_{\tilde{\mu}}\mathrm{v}^{-} \tag{B.8}$$

as follows

$$u^{=}_{\tilde{\mu}} = \mathrm{u}^{=}_{\tilde{\mu}}\,, \qquad u^{\#}_{\tilde{\mu}} = \mathrm{u}^{\#}_{\tilde{\mu}}\,, \qquad u^{-+}_{\tilde{\mu}} = u^{+-}_{\tilde{\mu}} = u^{\perp}_{\tilde{\mu}}\,. \tag{B.9}$$

Now, let us turn to the 4D massless superparticle action (2.14). With the ansatz (B.5), which results in (B.7), one of four VA 1-forms,

$$\Pi^{2} = \mathrm{d}x^{2} - i\mathrm{d}\theta\sigma^{2}\bar{\theta} + i\theta\sigma^{2}\mathrm{d}\bar{\theta} = \mathrm{d}x^{2} - \epsilon_{\alpha\beta}(\mathrm{d}\theta^{\alpha}\bar{\theta}^{\beta} - \theta^{\alpha}\mathrm{d}\bar{\theta}^{\beta})\,, \tag{B.10}$$

just disappears from the action which becomes 3D massless superparticle action in its spinor moving frame formulation

$$S^{0}_{\mathrm{D}=3} = \int \rho^{\#}\mathrm{v}^{-}_{\alpha}\bar{\mathrm{v}}^{-}_{\beta}\Pi^{\alpha\beta} = \int \rho^{\#}\mathrm{u}_{\tilde{\mu}}\Pi^{\tilde{\mu}} = \int \rho^{\#}\mathrm{E}^{=}\,. \tag{B.11}$$

Here we have introduced 3D Volkov-Akulov (VA) 1-forms

$$\Pi^{\alpha\beta} = \Pi^{\tilde{\mu}}\tilde{\gamma}^{\alpha\beta}_{\tilde{\mu}} = \mathrm{d}x^{\alpha\beta} - 2i\mathrm{d}\theta^{(\alpha}\bar{\theta}^{\beta)} + 2i\theta^{(\alpha}\mathrm{d}\bar{\theta}^{\beta)}\,, \quad \Pi^{\tilde{\mu}} = \mathrm{d}x^{\tilde{\mu}} - i\mathrm{d}\theta\gamma^{\tilde{\mu}}\bar{\theta} + i\theta\gamma^{\tilde{\mu}}\mathrm{d}\bar{\theta}\,, \tag{B.12}$$

as well as one of 3D counterparts of the pull-backs of the 4D supervielbein forms (2.15)–(2.17) adapted to the embedding,

$$\mathrm{E}^{=} = \Pi^{\tilde{\mu}}\mathrm{u}^{=}_{\tilde{\mu}} = \frac{1}{2}\mathrm{u}^{=}_{\alpha\beta}\Pi^{\alpha\beta} = \Pi^{\alpha\beta}\mathrm{v}^{-}_{\alpha}\mathrm{v}^{-}_{\beta}\,, \tag{B.13}$$

$$\mathrm{E}^{\#} = \Pi^{\tilde{\mu}}\mathrm{u}^{\#}_{\tilde{\mu}} = \frac{1}{2}\mathrm{u}^{\#}_{\alpha\beta}\Pi^{\alpha\beta} = \Pi^{\alpha\beta}\mathrm{v}^{+}_{\alpha}\mathrm{v}^{+}_{\beta}\,, \tag{B.14}$$

$$\mathrm{E}^{\perp} = \Pi^{\tilde{\mu}}\mathrm{u}^{\perp}_{\tilde{\mu}} = \frac{1}{2}\mathrm{u}^{\perp}_{\alpha\beta}\Pi^{\alpha\beta} = \Pi^{\alpha\beta}\mathrm{v}^{-}_{(\alpha}\mathrm{v}^{+}_{\beta)}\,. \tag{B.15}$$

C Some properties and applications of SL(2, ℝ)/SO(2) Cartan forms

Inverting the relations (4.39) and (4.40) we find

$$\left.\begin{array}{l}\mathrm{du}^{0}_{\alpha\beta} = \mathrm{u}^{1}_{\alpha\beta}f^{1} + \mathrm{u}^{2}_{\alpha\beta}f^{2}\,,\\ \mathrm{du}^{1}_{\alpha\beta} = \mathrm{u}^{0}_{\alpha\beta}f^{1} + \mathrm{u}^{2}_{\alpha\beta}f^{qq},\\ \mathrm{du}^{2}_{\alpha\beta} = \mathrm{u}^{0}_{\alpha\beta}f^{2} - \mathrm{u}^{1}_{\alpha\beta}f^{qq}.\end{array}\right\} \quad \Longleftrightarrow \quad \left\{\begin{array}{l}\mathrm{du}^{0}_{\alpha\beta} = \mathrm{u}^{1}_{\alpha\beta}f^{1} + \mathrm{u}^{2}_{\alpha\beta}f^{2}\,,\\ \mathrm{Du}^{1}_{\alpha\beta} = \mathrm{du}^{1}_{\alpha\beta} - \mathrm{u}^{2}_{\alpha\beta}f^{qq} = \mathrm{u}^{0}_{\alpha\beta}f^{1},\\ \mathrm{Du}^{2}_{\alpha\beta} = \mathrm{du}^{2}_{\alpha\beta} + \mathrm{u}^{1}_{\alpha\beta}f^{qq} = \mathrm{u}^{0}_{\alpha\beta}f^{2}.\end{array}\right. \tag{C.1}$$

and

$$f^{11} = \frac{1}{2}f^{qq} - \frac{1}{2}f^{1}\,, \qquad f^{22} = \frac{1}{2}f^{qq} + \frac{1}{2}f^{1}\,, \qquad f^{12} = \frac{1}{2}f^{2}\,. \tag{C.2}$$

Notice also that

$$\mathrm{d}f^{qp} = -\epsilon_{q'p'}f^{qq'} \wedge f^{pp'} = f^{q}{}_{p'} \wedge f^{pp'}\,. \tag{C.3}$$

In particular this implies

$$\begin{aligned} \mathrm{d}f^{qq} &= -\epsilon_{q'p'} f^{qq'} \wedge f^{qp'} = 2(f^{11} - f^{22}) \wedge f^{12} \,, & \text{(C.4)} \\ \mathrm{d}(f^{11} - f^{22}) &= 2f^{qq} \wedge f^{12} \,, & \text{(C.5)} \\ \mathrm{d}f^{12} &= f^{11} \wedge f^{22} = -\frac{1}{2} f^{qq} \wedge (f^{11} - f^{22}) \,. & \text{(C.6)} \end{aligned}$$

Equivalently, we can write this relations as

$$\mathrm{D}f^1 = \mathrm{d}f^1 + f^{qq} \wedge f^2 = 0 \,, \quad \mathrm{D}f^2 = \mathrm{d}f^3 - f^{qq} \wedge f^1 = 0 \,, \quad \mathrm{d}f^{qq} = -f^1 \wedge f^2 \,. \qquad \text{(C.7)}$$

In terms of covariant derivatives and SO(2) Cartan forms the derivatives of 3d spinor harmonics are

$$\mathrm{Dv}^1_\alpha = \mathrm{dv}^1_\alpha + \frac{1}{2}\mathrm{v}^2_\alpha f^{qq} = \frac{1}{2}\mathrm{v}^1_\alpha f^2 + \frac{1}{2}\mathrm{v}^2_\alpha f^1 \,, \qquad \mathrm{Dv}^2_\alpha = \mathrm{dv}^2_\alpha - \frac{1}{2}\mathrm{v}^1_\alpha f^{qq} = +\frac{1}{2}\mathrm{v}^1_\alpha f^1 - \frac{1}{2}\mathrm{v}^2_\alpha f^2 \,. \qquad \text{(C.8)}$$

Using the above equations we can easily find

$$\mathrm{D}\mathcal{E}^1 = \frac{1}{2}\mathcal{E}^1 \wedge f^2 + \frac{1}{2}\mathcal{E}^2 \wedge f^1 \,, \qquad \mathrm{D}\mathcal{E}^2 = \frac{1}{2}\mathcal{E}^1 \wedge f^1 - \frac{1}{2}\mathcal{E}^2 \wedge f^2 \,, \qquad \text{(C.9)}$$

which imply

$$\begin{aligned} \mathrm{D}(\mathcal{E}^1 - i\mathcal{E}^2) &= \frac{1}{2}(\mathcal{E}^1 + i\mathcal{E}^2) \wedge (f^2 - if^1) \,, & \text{(C.10)} \\ \mathrm{D}(\bar{\mathcal{E}}^1 + i\bar{\mathcal{E}}^2) &= \frac{1}{2}(\bar{\mathcal{E}}^1 - i\bar{\mathcal{E}}^2) \wedge (f^2 + if^1) \,. & \text{(C.11)} \end{aligned}$$

These equations and the Ricci identities

$$\begin{aligned} \mathrm{DD}\mathbb{Z} &= +if^1 \wedge f^2 \mathbb{Z} + [\mathbb{F}, \mathbb{Z}] \,, & \text{(C.12)} \\ \mathrm{DD}\bar{\mathbb{Z}} &= -if^1 \wedge f^2 \bar{\mathbb{Z}} + [\mathbb{F}, \bar{\mathbb{Z}}] \,, & \text{(C.13)} \\ \mathrm{DD}\Psi &= -\frac{i}{2} f^1 \wedge f^2 \Psi + [\mathbb{F}, \Psi] \,, & \text{(C.14)} \\ \mathrm{DD}\bar{\Psi} &= +\frac{i}{2} f^1 \wedge f^2 \bar{\Psi} + [\mathbb{F}, \bar{\Psi}] \,, & \text{(C.15)} \end{aligned}$$

are useful in search for κ-symmetry of the 3D mD0 action.

The equations

$$\mathrm{dE}^0 = \mathrm{E}^1 \wedge f^1 + \mathrm{E}^2 \wedge f^2 - 2i(\mathcal{E}^1 \wedge \bar{\mathcal{E}}^1 + \mathcal{E}^2 \wedge \bar{\mathcal{E}}^2) \,, \qquad \text{(C.16)}$$

and

$$\mathrm{d}\left(\mathrm{d}\theta^\alpha \bar{\theta}_\alpha - \theta^\alpha \mathrm{d}\bar{\theta}_\alpha\right) = -2\mathcal{E}^1 \wedge \bar{\mathcal{E}}^2 + 2\mathcal{E}^2 \wedge \bar{\mathcal{E}}^1 \qquad \text{(C.17)}$$

are also useful to search for κ-symmetry of the single D0-brane in 3D. The formal exterior derivative of the Lagrangian 1-form of this system reads

$$\begin{aligned} \mathrm{d}\mathcal{L}^{3\mathrm{d}\ \mathrm{D0}} &= m\mathrm{d}\left(\mathrm{E}^0 + \mathrm{d}\theta^\alpha \bar{\theta}_\alpha - \theta^\alpha \mathrm{d}\bar{\theta}_\alpha\right) \\ &= -2im(\mathcal{E}^1 + i\mathcal{E}^2) \wedge (\bar{\mathcal{E}}^1 - i\bar{\mathcal{E}}^2) + m\mathrm{E}^1 \wedge f^1 + m\mathrm{E}^2 \wedge f^2 \\ &= -2im(\mathcal{E}^2 - i\mathcal{E}^1) \wedge (\bar{\mathcal{E}}^2 + i\bar{\mathcal{E}}^1) + m\mathrm{E}^1 \wedge f^1 + m\mathrm{E}^2 \wedge f^2 \,. \qquad \text{(C.18)} \end{aligned}$$

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