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A study of localic subspaces, separation, and variants of normality and their duals

Doctoral thesis in cotutelle between the University of Coimbra and the University of the Basque Country UPV/EHU, supervised by Professor Jorge Picado and Professor Javier Gutiérrez García.

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Abstract

As in classical topology, in localic topology one often needs to restrict to locales satisfying a certain degree of separation. In fact, the study of separation in the category of locales constitutes a non-trivial and important piece of the theory. For instance, it is sometimes impossible to give an exact counterpart of a classical axiom, while other times a single property for spaces yields multiple non-equivalent localic versions.

The main goal of this thesis is to investigate several classes of separated locales and their connections with different classes of sublocales, that is, the regular subobjects in the category of locales.

In particular, we introduce a new diagonal separation and show that it is, in a certain sense, dual to Isbell's (strong) Hausdorff property. The duality between suplattices and preframes, and that between normality and extremal disconnectedness, turn out to be of special interest in this context.

Regarding higher separation, we introduce cardinal generalizations of normality and their duals (e.g., properties concerning extensions of disjoint families of cozero elements), and give characterizations via suitable insertion or extension results.

The lower separation property known as the T_D-axiom, also plays an important role in the thesis. Namely, we investigate the T_D-duality between the category of T_D-spaces and a certain (non-full) subcategory of the category of locales, identifying the regular subobjects in the localic side, and provide several applications in point-free topology.

Resumo

Tal como na topologia clássica, também na topologia dos locales (reticulados locais) é frequente termos que nos restringir a locales que satisfaçam um certo grau de separação. De facto, o estudo de axiomas de separação na categoria dos locales constitui um aspecto não trivial e relevante da teoria. Por exemplo, em alguns casos é impossível termos a contrapartida exacta de um axioma clássico, enquanto noutros casos uma única propriedade para espaços topológicos produz, na categoria dos locales, diversas versões não equivalentes entre si.

O objectivo principal desta tese é investigar várias classes de locales separados e suas conexões com diferentes classes de sublocales (os subobjetos regulares na categoria dos locales).

Em particular, introduzimos uma nova propriedade de separação diagonal e mostramos que se trata, em certo sentido, de uma propriedade dual do axioma (forte) de Hausdorff introduzido por Isbell. As dualidades entre semi-reticulados e reticulados pré-locais, e entre normalidade e desconexão extrema, acabam por ter um papel relevante neste contexto.

Relativamente a axiomas de separação fortes, introduzimos generalizações de normalidade, em função de um cardinal arbitrário, e suas duais (por exemplo, propriedades envolvendo extensões de famílias disjuntas de elementos co-zero), e apresentamos caracterizações em termos de propriedades de inserção ou extensão de funções.

O axioma T_D, uma propriedade de separação muito fraca, também desempenha um papel importante nesta tese. Especificamente, investigamos a dualidade T_D entre a categoria dos espaços topológicos T_D e uma determinada subcategoria (não plena) da categoria dos locales, identificando os subobjetos regulares na subcategoria de locales, e apresentamos várias aplicações à topologia sem pontos.

Resumen

Tal y como ocurre en topología clásica, en topología locálica frecuentemente uno tiene que restringir su atención a locales que cumplen cierto grado de separación. De hecho, el estudio de la separación en la categoría de locales es un aspecto no trivial y relevante de la teoría. En algunos casos, es imposible dar una contrapartida exacta a un axioma clásico, mientras que en otros casos, una sola propiedad produce multitud de versiones locálicas no equivalentes entre sí.

El principal objetivo de esta tesis es investigar varias clases de locales separados y sus relaciones con diferentes clases de sublocales, esto es, los subobjetos regulares en la categoría de locales.

En particular, introducimos una nueva separación diagonal, y probamos que es, en cierto sentido, dual al axioma Hausdorff (fuerte) de Isbell. En este contexto, la dualidad entre retículos completos y premarcos, y aquella entre la normalidad y la desconexión extrema resultan ser de especial interés.

En cuanto a la separación más fuerte, introducimos generalizaciones cardinales de la normalidad y sus duales (por ejemplo, propiedades que consisten en la extensión de familias disjuntas de elementos cozero), y damos caracterizaciones de las mismas en términos de teoremas de extensión o inserción.

Ciertas propiedades de separación más débiles, especialmente el axioma T_D, también desempeñan un papel importante en esta tesis. Específicamente, investigamos la dualidad T_D entre la categoría de espacios topológicos T_D y cierta subcategoría (no plena) de la categoría de locales, identificando los subobjetos regulares en la categoría de locales, y proporcionamos algunas aplicaciones en la topología sin puntos.

Laburpena

Topologia klasikoan gertatzen den antzera, topologia lokalikoan ohikoa da banantze-maila jakin bat duten lokaleetara murriztu behar izatea. Izan ere, banantzearen azterketa lokaleen kategorian aspektu ez-tribiala eta garrantzitsua da. Honela, batzuetan ezinezkoa da axioma klasiko baten analogo zehatza ematea eta beste batzuetan propietate bakar batek hainbat bertsio lokaliko ez-baliokide izan ditzake.

Tesi honen helburu nagusia zenbait lokale bananduren klase ikertzea da, eta horien erlazioak aztertzea azpilokaleen klase desberdinekin, azken hauek lokaleen kategoriako azpiobjektu erregularak direlarik.

Besteak beste, diagonal erako banantze mota berri bat aurkeztuko dugu, eta frogatuko dugu, neurri batean, Isbell-en Hausdorff axioma gogorraren duala dela. Testuinguru honetan, interes berezikoak dira erretikulu osoen eta aurremarkoen arteko dualtasuna, baita normaltasunaren eta muturreko ez-konexutasunaren artekoa ere.

Banantze-propietate gogorrenei dagokienez, normaltasunaren orokortze kardinalak eta beraien dualak aztertuko ditugu (esate baterako, kozero familia disjuntuen hedatze-propietateak), eta hedatze- edo txertatze-teoremen bidez hauen karakterizazioak emango ditugu.

Banantze-propietate ahulei, eta bereziki T_D-axiomari, ere arreta jarriko diegu. Esaterako, T_D-espazio topologikoen kategoriaren eta lokaleen kategoriaren azpikategoria (ez-oso) jakin baten arteko T_D-dualtasuna ikertuko dugu, bide batez lokaleen azpikategoria horretako azpiobjektu erregularrak identifikatuz, eta hainbat aplikazio aztertuko ditugu punturik gabeko topologian.

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Introduction

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> Mi estética actual es transformar con matemática de espejo cóncavo las normas clásicas.
>
> Ramón María del Valle-Inclán, *Luces de bohemia*

A *frame* is a complete lattice L satisfying the infinite distributivity law

$$(\bigvee_{i\in I} a_i) \wedge b = \bigvee_{i\in I} a_i \wedge b, \qquad \text{for any } \{a_i\}_{i\in I} \subseteq L \text{ and } b \in L.$$

Frames and their homomorphisms —maps preserving arbitrary joins and finite meets— form the category of frames, denoted by Frm. A *locale* is the same thing as a frame, but their morphisms go in the opposite direction — i.e., locales and their morphisms form a category Loc which is dual to Frm. It turns out that Loc contains a very substantial part of the category Top of topological spaces as a full subcategory, and hence one may regard locales as generalized spaces. Accordingly, *point-free topology* (also known as *localic topology* or *locale theory*) is the study of the category of locales (and its dual, the category of frames).

There are good reasons for studying locales as generalized spaces. Isbell pointed out in his pioneering article [71] that the category of locales is not only a generalization but an improvement of the category of classical topological spaces, and it has since become clear that localic topology yields a better theory in many respects. In these pages, it is not our purpose to discuss in depth the advantages of localic topology, and for a detailed account we refer to any of the excellent works [78, 79, 95]. However, let us mention for example the fact that locale theory is inherently constructive — in fact, the localic Tychonoff's theorem ensuring that products of compact locales are compact is completely choice-free, a situation that contrasts strikingly with its classical counterpart. Another pleasant feature is the behaviour of products of locales. Unlike in classical topology, products of paracompact (resp. Lindelöf) locales are paracompact (resp. Lindelöf). Finally, we also emphasize a fact which plays an important role in this dissertation: the category of frames is an algebraic category (i.e., the forgetful functor of Frm into Set is monadic). In particular, free frames exist and one may present frames by generators and relations as in other familiar algebraic structures. This is a very useful tool in point-free topology, not available in the realm of classical topology. Notably, it allows the construction of the frame of (extended) reals [22, 26] (for more examples of this technique, we refer to [87] and the references therein).

Generalized subspaces: sublocales

Regular subobjects (i.e., isomorphism classes of regular monomorphisms with a fixed codomain) in the category of topological spaces correspond to subspace inclusions. Analogously, generalized or point-free subspaces (known as *sublocales*) are defined to be the regular subobjects in the category of locales. A major difference between the category of topological spaces and the category of locales is the nature of their lattices of regular subobjects. Whereas the lattice of subspaces of a topological space is of a very special nature — a complete and atomic Boolean algebra — the lattice of sublocales of a locale is merely a coframe (i.e., the order-theoretic dual of a frame), and in particular, not every sublocale has a complement. Hence, lattices of regular subobjects in the category of locales are more complicated objects and they are one of the fundamental areas of study in point-free topology.

Throughout this work, we shall adopt the concrete approach to the category of locales, as in [91] — i.e., we shall regard morphisms in Loc not simply as frame homomorphisms formally turned around, but as actual mappings. In particular, sublocales will be seen as special subsets of the locale in question, thus providing a pleasant framework to deal with the geometry of sublocales.

As with the classical topological spaces and their separation axioms, in point-free topology one often needs to restrict to smaller classes of locales satisfying certain degree of separation. Actually, the study of separation is one of the intricate topics in localic topology. Sometimes, a single axiom from classical topology has multiple non-equivalent localic counterparts; other times, the classical axiom may be too point-dependent to admit a direct lattice-theoretic formulation. The recent monograph [97] contains an extensive and detailed account of separation properties studied in the literature.

In this context, the main goal of this thesis is to study and introduce several classes of sublocales, with a special emphasis on their relations with different classes of separated locales. Our contributions may be summarized as follows:

Lower separation: the T_D axiom.

The classical T_D axiom of Aull and Thron [11] plays a fundamental role in localic topology (for instance, it ensures that subspaces of a topological space have a proper representation as sublocales of the associated locale). In this thesis, we shall study several families of sublocales connected to the T_D property: the family of smooth sublocales and that of D-sublocales. Among other results, we will give new criteria of T_D-spatiality and total T_D-spatiality of a locale. The study of T_D-spatiality will rely heavily on a certain subspace of the prime spectrum consisting of covered primes [31, 32]. We shall also emphasize the differences of the theory when one replaces covered primes by maximal elements.

Furthermore, by using our results we will be able to provide some new insights on the duality of T_D-spaces from [31] (in particular, we identify and study the appropriate notion of generalized subspace in this duality).

Higher separation: normality and its variants.

Among higher separation axioms, we shall pay special attention to variants of normality and their duals. It has long been known that normality and extremal disconnectedness mirror each other. Indeed, not only are these concepts lattice-theoretically dual, but there are also several pairs of parallel insertion and extension results in classical topology characterizing normality and extremal disconnectedness. The source of this duality was investigated point-freely in [68], where the authors used the technique of fixing a class of complemented sublocales in order to produce relative notions of upper and lower semicontinuity which behave dually under complementation. In this work, we will use this technique in order to study certain variants of normality and their duals.

We will introduce point-freely a new cardinal generalization of normality, referred to as *total collectionwise normality*, and subsequently, its associated insertion and extension theorems will be proved. For that purpose, we will take advantage of the algebraic nature of the category of frames in order to present (by generators and relations) a point-free cardinal generalization of the ordered structure (in the sense of the Lawson topology) on the extended real line. We will refer to it as the *frame of the compact hedgehog*. Furthermore, the theory of semicontinuity of compact hedgehog-valued functions will be developed, thereby establishing connections with disjoint families of cozero elements. The dual notions and some other cardinal generalizations (e.g., collectionwise normality, or the infinite variants of De Morgan laws for frames) and the relations between them will also be discussed.

Diagonal separation.

In the category of topological spaces (resp. the category of locales), a number of important separation axioms can be expressed by requiring that the diagonal of the space (resp. locale) in question belong to a given class of subspaces (resp. sublocales). Locales whose diagonal is closed are known as *strongly Hausdorff* (introduced by Isbell). In this thesis, we shall study a new class of separated locales corresponding to those whose diagonal is an intersection of open sublocales — we speak about $\mathcal{F}$-separated locales. We will show that the strong Hausdorff property and $\mathcal{F}$-separatedness are, in a certain sense, dual to each other. As a consequence of this study, the parallel between suplattices and preframes, and that between (the hereditary variants of) normality and extremal disconnectedness will emerge naturally.

Outline of the thesis

Chapter 1 contains the necessary preliminaries and notation that will be needed throughout this dissertation. More specific background will also be provided in some of the chapters.

Chapter 2 discusses two diagonal separation properties, the so-called strong Hausdorff property — corresponding to locales with closed diagonal — and a new one — corresponding to locales whose diagonal is an intersection of open sublocales — thus revealing a strong structural parallel between both notions. Certain characterizations in terms of relaxed morphisms are proved. As a result of this study, the duality between (the hereditary variants of) normality and extremal disconnectedness arises.

Chapter 3 begins with an exposition of the relative approach to normality as a tool for formalizing the duality between normality and extremal disconnectedness. This approach is then used for studying two point-free cardinal generalizations of normality, namely collectionwise normality and total collectionwise normality. The dual notions are then explored and it is proved that there is no real cardinal generalization: they collapse to the base cases. On the way, different cardinal generalizations of z-embeddings are discussed.

Chapter 4 deals with a different cardinal generalizations of extremal disconnectedness. We introduce two new classes of frames called infinitely extremally disconnected frames and infinitely De Morgan frames. It is shown that the latter is the conjunction of the former and a weak scatteredness condition. For frames which are additionally coframes, the finite second De Morgan law implies infinite extremal disconnectedness.

Chapters 5 and 6 are devoted to a localic theory of the compact hedgehog, as well as its connections with some variants of normality studied in previous chapters.

More precisely, in Chapter 5 the compact localic hedgehog is introduced as a frame presented by suitable generators and relations. We motivate the compact hedgehog as the natural cardinal generalization of the ordered structure on the (extended) real line and its associated Lawson topology. It is shown that it is a compact regular frame, and other basic properties are established. Subsequently, we develop the theory of compact hedgehog-valued functions and it is proved that continuous compact hedgehog-valued functions correspond bijectively to disjoint families of cozero elements. We also prove new point-free insertion and extension theorems characterizing normality and total collectionwise normality.

Chapter 6 develops the theory of compact hedgehog-valued functions in the more general and unifying setting of sublocale selections. This way, generalized insertion and extension theorems and their dual results are obtained.

The theory of real-valued (resp. compact hedgehog-valued) functions makes an essential use of the idea of discretization of a given locale. Whereas in Chapters 5 and 6 the discretization is taken to be the whole coframe of sublocales, in Chapter 7 an alternative construction is explored. We systematically study the family of smooth sublocales proving that it is a subcolocale of the coframe of sublocales which forms a complete Boolean algebra. We investigate its relations with the T_D axiom (providing new characterizations of T_D-spatiality), its functoriality properties and related topics. In particular, the theory of compact hedgehog-valued functions is revisited in this context, thus showing that the notion of semicontinuity from Chapter 5 does not change after replacing the former discretization by the new one.

Another family of sublocales is studied in Chapter 8. We introduce the concept of D-sublocale and show that it is the appropriate notion of generalized subspace in the duality of T_D-spaces. We also prove that the family of D-sublocales of a locale forms a zero-dimensional subcolocale of the coframe of all sublocales. Following a parallel structure with the previous chapter, we then move on to proving a strong connection with the T_D axiom (in particular providing a Niefield-Rosenthal type theorem for characterizing total T_D-spatiality) and discussing its functoriality properties.

Chapter 9 presents a short discussion relating the two families of sublocales studied in Chapters 7 and 8. We end up with the observation that inclusion relations between these (and other) subcolocales of the coframe of sublocales of a given locale characterize well-known locale-theoretic properties of the locale itself.

Finally, this dissertation contains the Appendix A. This appendix is based on [68] and it includes some technical results on relative continuity of extended real-valued functions. The reason to include the appendix is to make this thesis as self-contained as possible and to present the results from [68] with a slightly modified terminology more similar to that used in Chapters 3 and 6.

List of publications

This thesis consists of the material contained in the following six papers; all of them are already accepted for publication in different journals.

[4] I. Arrieta, On infinite variants of De Morgan law in locale theory, *Journal of Pure and Applied Algebra*, vol. 225 (1), art. no. 106460, 2021.

[5] I. Arrieta, On joins of complemented sublocales, *Algebra Universalis*, vol. 83, art. no. 1, 2022.

[6] I. Arrieta and J. Gutiérrez García, On the categorical behaviour of locales and D-localic maps, *Quaestiones Mathematicae*, accepted for publication.

[7] I. Arrieta, J. Gutiérrez García, and J. Picado, Frame presentations of compact hedgehogs and their properties, *Quaestiones Mathematicae*, accepted for publication.

[8] I. Arrieta, J. Picado, and A. Pultr, A new diagonal separation and its relations with the Hausdorff property, *Applied Categorical Structures*, vol. 30, pp. 247–263, 2022.

[9] I. Arrieta and A. L. Suarez, The coframe of D-sublocales of a locale and the T_D-duality, *Topology and its Applications*, vol. 291, art. no. 107614, 2021.

Besides the mentioned research papers, an introductory survey paper on locale theory has also been published:

[10] I. Arrieta and A. Zozaya, ¿Qué es un espacio? *La Gaceta de la Real Sociedad Matemática Española*, vol. 24 (2), pp. 249–271, 2021, in spanish.

Chapter 1

Preliminaries

In this chapter we present some general background concerning the categories of locales and frames. Our goal is to provide the necessary preliminaries and to fix the notation that will be used throughout the dissertation. The main references on the topic are Johnstone [77] and the more recent Picado and Pultr [91] (see also the shorter [98, 95]). Another important reference is the pioneering paper by Isbell [71]. The article [78] describes the main advantages of localic topology (i.e., topology done within the category of locales rather than in the category of topological spaces). For general category theory, we refer the reader to [86, 1].

1.1 The categories of locales and frames

A *locale* (or *frame*) is a complete lattice L with the property that $(\bigvee_{i\in I} a_i) \wedge b = \bigvee_{i\in I} a_i \wedge b$ for all $\{a_i\}_{\in I} \subseteq L$ and $b \in L$. If L and M are frames, a map $h\colon L \to M$ is a *frame homomorphism* if it preserves arbitrary joins (including the bottom element 0) and finite meets (including the top element 1). Frames and frame homomorphisms form a category denoted by Frm. The category of locales, denoted by Loc, is by definition the opposite of the category of frames — i.e., $\mathsf{Loc} = \mathsf{Frm}^{op}$. Since every frame homomorphism $h\colon L \to M$ preserves arbitrary joins, it has a unique right adjoint, denoted by $h_*\colon M \to L$, given by the equivalence

$$h(a) \leq b \iff a \leq h_*(b)$$

for any $a \in L$ and $b \in M$. As a right adjoint, h_* preserves arbitrary meets and is given by

$$h_*(b) = \bigvee\{a \in L \mid h(a) \leq b\}$$

for any $b \in M$. A *localic map* is a map of the form h_* (i.e., a meet preserving map whose left adjoint is a frame homomorphism). Given a localic map $f\colon L \to M$, we shall denote its left adjoint frame homomorphism by $f^*\colon M \to L$. This left adjoint is given by

$$f^*(b) = \bigwedge\{a \in L \mid b \leq f(a)\}$$

for any $b \in M$. Accordingly, we may regard Loc as a concrete category whose objects are locales and whose morphisms are localic maps. Note that a frame homomorphism h is surjective (resp. injective) if and only if h_* is injective (resp. surjective) if and only if $h \circ h_*$ (resp. $h_* \circ h$) is the identity.

For a topological space X, its lattice $\Omega(X)$ of open sets is a frame. Given a continuous map $f\colon X \to Y$ between topological spaces, the preimage operator $f^{-1}[-]\colon \Omega(Y) \to \Omega(X)$ is a frame homomorphism; hence we obtain a functor $\Omega\colon \mathsf{Top} \to \mathsf{Frm}^{op} = \mathsf{Loc}$.

An element $p \neq 1$ in a frame L is *prime* if for all $a, b \in L$, $a \wedge b = p$ implies $a = p$ or $b = p$ (equivalently, if $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$). We denote the set of prime elements in a frame L by $\mathsf{pt}(L)$. For every $a \in L$, we set $\Sigma_a = \{p \in \mathsf{pt}(L) \mid a \nleq p\}$. It turns out that the family $\{\Sigma_a \mid a \in L\}$ is a topology on $\mathsf{pt}(L)$. This topology is denoted by $\Sigma(L)$ and it is referred to as the *spectrum* of L. The notions of prime and spectrum will be discussed further in Subsection 1.2.5. It is easy to show that localic maps send primes into primes. Therefore, given a localic map $f\colon L \to M$, we may restrict and co-restrict it to obtain a map $\mathsf{pt}(L) \to \mathsf{pt}(M)$ which is easily seen to be continuous with respect to the topologies of the spectra; this continuous map will be denoted by $\Sigma(f)\colon \Sigma(L) \to \Sigma(M)$. Accordingly, we have the *spectrum functor* $\Sigma\colon \mathsf{Loc} \to \mathsf{Top}$. Furthermore, there is an important adjunction

$$\begin{tikzcd} \mathsf{Top} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc} \arrow[l, bend left, "\Sigma"] \end{tikzcd} \tag{1.1.1}$$

If X is a topological space, for each $x \in X$ the element $X - \overline{\{x\}}$ is prime in $\Omega(X)$. Moreover, the unit η of the adjunction has components $\eta_X\colon X \to \Sigma(\Omega(X))$ (called the *soberification* of X) given by $\eta_X(x) = X - \overline{\{x\}}$. A space X is then *sober* if η_X is a bijection (equivalently, if η_X is a homeomorphism). For example, every Hausdorff space is sober. Moreover, every space of the form $\Sigma(L)$ is sober.

The axiom of sobriety has a remarkable role in point-free topology, as it allows one to reconstruct a topological space X from the lattice-theoretic structure of $\Omega(X)$ (as it was just seen, $X \cong \Sigma(\Omega(X))$ for a sober space X).

The counit ϵ of the adjunction has components ϵ_L whose left adjoints are surjections $\epsilon_L^*\colon L \twoheadrightarrow \Omega(\Sigma(L))$ (called the *spatialization* of L) given by $\epsilon_L^*(a) = \Sigma_a$. Then, L is *spatial* if ϵ_L is an isomorphism (equivalently, if there exists an isomorphism $L \cong \Omega(X)$).

The adjunction (1.1.1) is idempotent and therefore it restricts to an equivalence between the full subcategories of Top and Loc consisting of sober topological spaces and spatial locales, respectively. The functor Ω is full and faithful when restricted to sober spaces, and so we may regard locales as generalized sober spaces. Since most of the spaces in practice are sober, we simply regard locales as generalized topological spaces. Following this viewpoint, a property $\mathcal{P}$ of locales will be said to be a *conservative* extension of a topological property $\mathcal{Q}$ if X satisfies $\mathcal{Q}$ if and only if $\Omega(X)$ satisfies $\mathcal{P}$.

A *cover* of L is a subset $C \subseteq L$ with the property that $\bigvee C = 1$. A subset $B \subseteq L$ is a $\bigvee$*-base*. of L if every $a \in L$ can be expressed as $a = \bigvee B'$ with $B' \subseteq B$.

1.1.1 Frm is an algebraic category

The category Frm is *algebraic* (i.e., the forgetful functor $\mathsf{Frm} \to \mathsf{Set}$ is monadic). This has a number of pleasant consequences (see for instance [77, Proposition 3.8] or [95, 4.3]). Among others, Frm is complete and cocomplete, monomorphisms in Frm are precisely the injective frame homomorphisms and regular epimorphisms are precisely the surjective frame homomorphisms. Moreover, free frames exist and so one can present frames by generators and relations. We shall not go into the details of the constructions as for our purposes we just need the fact that given a set S of *symbols* and a set R of *relations* (i.e., formal equalities $\bigvee_{i\in I} \bigwedge F_i = \bigvee_{j\in J} \bigwedge G_j$ with $F_i, G_j \subseteq S$ finite subsets for all $i \in I$, $j \in J$), there is a frame $\langle S \mid R\rangle$ and a map $\varphi\colon S \to \langle S \mid R\rangle$ with the property that for every frame L and map $f\colon S \to L$ which sends the relations to identities in L, there exists a unique frame homomorphism $h\colon \langle S \mid R\rangle \to L$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "f"'] & \langle S \mid R\rangle \arrow[d, dashed, "h"] \\ & L \end{tikzcd}$$

is commutative.

1.1.2 The Heyting operator

For each a in a frame L, the map $a \wedge (-)\colon L \to L$ preserves arbitrary joins. Hence it has a right adjoint $a \to (-)\colon L \to L$ called the *Heyting operator*, thus making L into a *complete Heyting algebra*. Conversely, every complete Heyting algebra is a frame. The Heyting operator is characterized by the condition

$$a \wedge b \leq c \iff a \leq b \to c$$

for all $a, b, c \in L$.

This operator satisfies the following properties (see [91, III 3.1.1]):

(H1) $1 \to a = a$;

(H2) $a \leq b$ if and only if $a \to b = 1$;

(H3) $a \leq b \to a$;

(H4) $a \to b = a \to (a \wedge b)$;

(H5) $a \wedge (a \to b) = a \wedge b$;

(H6) $a \wedge b = a \wedge c$ if and only if $a \to b = a \to c$;

(H7) $(a \wedge b) \to c = a \to (b \to c) = b \to (a \to c)$;

(H8) $a = (a \vee b) \wedge (b \to a)$;

(H9) $a \leq (a \to b) \to b$;

(H10) $((a \to b) \to b) \to b = a \to b$.

In particular, for each $a \in L$, the element $a^* := a \to 0$ is called the *pseudocomplement* of a. Pseudocomplements satisfy standard properties such as

$$a \leq a^{**}, \quad a^{***} = a^* \quad \text{or} \quad b^* \leq a^* \text{ whenever } a \leq b.$$

An infinite *first De Morgan law* is also satisfied — i.e,

$$\Big(\bigvee_{i\in I} a_i\Big)^* = \bigwedge_{i\in I} a_i^* \tag{FDM}$$

for every $\{a_i\}_{i\in I} \subseteq L$.

An element $a \in L$ is *regular* if $a = b^*$ for some $b \in L$ (or, equivalently, if $a^{**} = a$).

The Heyting operator can be used for providing a more concrete description of localic maps:

Proposition 1.1.1. *Let $f\colon L \to M$ be a meet-preserving map between frames and let f^* denote its left adjoint. Then f is a localic map if and only if the following two properties are satisfied*:

(1) $f(a) = 1$ *implies* $a = 1$ *for all* $a \in L$;

(2) $f(f^*(b) \to a) = b \to f(a)$ *for all* $a \in L$ *and* $b \in M$.

1.1.3 Some important properties

The following notions are all topologically inspired. A frame L is said to be

- *subfit* if for every $a, b \in L$ with $a \nleq b$, there is a $c \in L$ such that $a \vee c = 1 \neq b \vee c$.
- T_1 if every prime element $p \in L$ is maximal (that is, $p \leq a$ implies $p = a$ or $a = 1$).
- *fit* if for every $a, b \in L$ with $a \nleq b$, there is a $c \in L$ such that $a \vee c = 1$ and $c \to b \nleq b$.
- *regular* if for every $a \in L$, one has $a = \bigvee\{b \in L \mid b \prec a\}$ where $b \prec a$ means $b^* \vee a = 1$.
- *normal* if for every $a, b \in L$ with $a \vee b = 1$, there exist $c, d \in L$ such that $c \wedge d = 0$, $a \vee c = 1$ and $b \vee d = 1$.
- *extremally disconnected* if for every $a, b \in L$ with $a \wedge b = 0$, there exist $c, d \in L$ such that $c \vee d = 1$, $a \wedge c = 0$ and $b \wedge d = 0$.
- *compact* if for every $\{a_i\}_{i\in I}$ with $\bigvee_{i\in I} a_i = 1$, there is a finite $F \subseteq I$ such that $\bigvee_{i\in F} a_i = 1$.

- *zero-dimensional* if every $a \in L$ is a join of complemented elements of L.

Regularity, normality, extremal disconnectedness, compactness and zero-dimensionality are conservative extensions of the homonymous topological properties. For a topological intuition of subfitness and fitness, we refer to Subsection 1.2.1 below. A sober space X is T_1 if and only if $\Omega(X)$ is a T_1-frame. Moreover, the implications

$$\text{zero-dimensional} \implies \text{regular} \implies \text{fit} \implies \text{subfit}$$

hold; and normality together with subfitness implies regularity. For a comprehensive account of separation properties in point-free topology, we refer to [97].

1.2 Sublocales

A generalized subspace (or point-free subspace) of a locale L is defined to be a regular subobject of L in Loc. Regular subobjects in Loc are isomorphism classes of injective localic maps with codomain L (equivalently, isomorphism classes of frame surjections with domain L). There are several equivalent representations of generalized subspaces. We shall mostly use that of sublocales, which allows us to represent generalized subspaces as actual subsets of L closed under certain operations. Specifically, a subset $S \subseteq L$ is a *sublocale* if it is closed under arbitrary meets and $a \to s \in S$ for all $a \in L$ and $s \in S$. A subset $S \subseteq L$ is a sublocale if and only if it is a frame with the order inherited from L and the embedding $j_S \colon S \hookrightarrow L$ is a localic map. The associated frame surjection $\nu_S \colon L \twoheadrightarrow S$ is given by $\nu_S(a) = \bigwedge\{s \in S \mid a \leq s\}$ for any $a \in L$, and the identity

$$\nu_S(a) \to s = a \to s \tag{1.2.1}$$

is satisfied for any $a \in L$ and $s \in S$.

A *nucleus* on L is a monotone, increasing and idempotent map $\nu \colon L \to L$ which preserves finite meets. Nuclei on L are in bijection with sublocales of L. Given a nucleus ν, its set of fixpoints $\{a \in L \mid \nu(a) = a\}$ is a sublocale of L. Conversely, given a sublocale $S \subseteq L$, the associated nucleus is $j_S \circ \nu_S$. Sublocales should not be confused with subframes; the latter are subobjects in Frm. Up to isomorphism, a *subframe* of L is a subset of L closed under arbitrary joins and finite meets.

The system $\mathsf{S}(L)$ of all sublocales of L, partially ordered by inclusion, is a *coframe*, that is, its dual lattice is a frame.[1] Infima and suprema are given by

$$\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i, \quad \bigvee_{i\in I} S_i = \Big\{\bigwedge M \mid M \subseteq \bigcup_{i\in I} S_i\Big\}. \tag{1.2.2}$$

[1]When considering coframes, we shall use lattice-theoretic duals of notions from frame theory, with the terminology modified just by adding the prefix “co-”, e.g., we will speak about the co-Heyting operator, coframe homomorphisms, subcolocales, and so on.

The least element is the sublocale $\mathsf{O} = \{1\}$ and the greatest element is the entire locale L. Occasionally, it will be convenient to work with the frame $\mathsf{S}(L)^{op} := (\mathsf{S}(L), \leq \equiv \subseteq^{op})$. We shall denote meets and joins in $\mathsf{S}(L)^{op}$ respectively by $\bigsqcap_{i\in I} S_i := \bigvee_{i\in I} S_i$ and $\bigsqcup_{i\in I} S_i := \bigcap_{i\in I} S_i$.

Since $\mathsf{S}(L)$ is a coframe, there is a *co-Heyting operator* giving the *difference* $S \smallsetminus T$ of two sublocales $S, T \in \mathsf{S}(L)$; this operator is characterized by the condition

$$S \smallsetminus T \subseteq R \iff S \subseteq T \vee R$$

for any $S, T, R \in \mathsf{S}(L)$. In particular, the *supplement* of $S \in \mathsf{S}(L)$ is $S^{\#} := L \smallsetminus S$ — i.e., the smallest sublocale of L whose join with S is L. A sublocale $S \subseteq L$ is *complemented* if it is complemented as an element of the lattice $\mathsf{S}(L)$ — i.e., if $S \cap S^{\#} = \mathsf{O}$. Importantly, complemented sublocales are *linear*, that is, if $C \subseteq L$ is complemented, then

$$C \cap \bigvee_{i\in i} S_i = \bigvee_{i\in I} C \cap S_i \tag{1.2.3}$$

for every $\{S_i\}_{i\in I} \subseteq \mathsf{S}(L)$. We shall freely use some of the following properties (a comprehensive list of properties may be found in [57]).

Properties 1.2.1. Let $S, T, R, C \in \mathsf{S}(L)$, with C being complemented and $\{S_i\}_{i\in I} \subseteq \mathsf{S}(L)$. Then:

(1) $S \smallsetminus T \subseteq S$;

(2) $S \smallsetminus T = 0$ if and only if $S \subseteq T$;

(3) $S \smallsetminus C = S \cap C^{\#}$;

(4) $S \smallsetminus \bigcap_{i\in I} S_i = \bigvee_{i\in I}(S \smallsetminus S_i)$;

(5) $(S \smallsetminus T) \smallsetminus R = (S \smallsetminus R) \smallsetminus T$;

(6) If $T \subseteq S$, then the supplement of T in $\mathsf{S}(S)$ is $S \smallsetminus T$.

If X is a topological space, every subspace $A \subseteq X$ induces a sublocale of $\Omega(X)$ given by $\widetilde{A} := \Omega(\iota)_*[\Omega(A)]$ where $\iota\colon A \hookrightarrow X$ is the inclusion; we shall speak of $\widetilde{A}$ as the *induced sublocale* by the subspace A (cf. [91, VI 1.1]). The concept of induced sublocale should not be confused with that of *spatial sublocale*. In this text, the latter simply means a sublocale that is spatial as a frame in its own right. If $L = \Omega(X)$, every sublocale that is induced by a subspace of X is a spatial sublocale, but the converse is not true in general.

1.2.1 Closed and open sublocales

For any $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = {\uparrow} a = \{\, b \in L \mid b \geq a \,\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{\, a \to b \mid b \in L \,\}$$

are the *closed* and *open* sublocales of L, respectively (that we shall denote simply by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ when there is no danger of confusion). We note that $\mathfrak{o}(a)$ is isomorphic to the frame ${\downarrow}a = \{ b \in L \mid b \le a \}$ via the isomorphism $\varphi\colon \mathfrak{o}(a) \to {\downarrow}a$ given by $\varphi(b) = b \wedge a$.

For each $a \in L$, $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathsf{S}(L)$ and satisfy the identities

$$\mathfrak{c}(1) = \mathsf{O}, \quad \mathfrak{c}(0) = L, \quad \bigwedge_{i\in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in I} a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \tag{1.2.4}$$

$$\mathfrak{o}(1) = L, \quad \mathfrak{o}(0) = \mathsf{O}, \quad \bigvee_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in I} a_i) \quad \text{and} \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b). \tag{1.2.5}$$

For any sublocale S of L and any $a \in S$, the closed (resp. open) sublocales $\mathfrak{c}_S(a)$ (resp. $\mathfrak{o}_S(a)$) of S are precisely the intersections $\mathfrak{c}(a) \cap S$ (resp. $\mathfrak{o}(a) \cap S$) and for any $a \in L$ we have $\mathfrak{c}(a) \cap S = \mathfrak{c}_S(\nu_S(a))$ and $\mathfrak{o}(a) \cap S = \mathfrak{o}_S(\nu_S(a))$.

A sublocale is *locally closed* if it is the intersection of an open sublocale and a closed sublocale.

If S is a sublocale of L, the *closure* of S (in L), denoted by $\overline{S}$, is the least closed sublocale containing S, and it can be computed as $\overline{S} = \mathfrak{c}(\bigwedge S)$. Further, S is *dense* (in L) if $\overline{S} = L$ — i.e., if $\bigwedge S = 0$ (or, equivalently, if $0 \in S$). Moreover, the *interior* of S (in L) is the largest open sublocale contained in S, we shall denote it by $\mathrm{int}(S)$.

A sublocale S of L is said to be *fitted* if it is an intersection of open sublocales. Now, given a sublocale S of L, set

$$\mathring{S} := \bigcap_{S \subseteq \mathfrak{o}(a)} \mathfrak{o}(a).$$

It is easy to show that it defines the smallest fitted sublocale containing S. This operator was studied in [43], where it was referred to as the "other" or "dual" closure (cf. also Subsection 2.2.3 below). We warn the reader that, unlike in other texts, the notation $\mathring{S}$ does *not* refer to the interior $\mathrm{int}(S)$ of S.

Recall the subfitness and fitness from Subsection 1.1.3. A locale L is subfit if and only if every open sublocale is a join of closed sublocales; and it is fit if and only if every closed sublocale is fitted (equivalently, if and only if every sublocale is fitted).

The frame $\mathsf{S}(L)^{op}$ is zero-dimensional. More precisely, for each sublocale S of L, one has

$$S = \bigcap_{i\in I} \mathfrak{c}(a_i) \vee \mathfrak{o}(b_i)$$

for suitable $\{a_i\}_{i\in I}, \{b_i\}_{i\in I} \subseteq L$.

For any frame L, the canonical map $\mathfrak{c}_L\colon L \to \mathsf{S}(L)^{op}$ sending $a \in L$ to $\mathfrak{c}_L(a)$ is an injective frame homomorphism by (1.2.4). Therefore the restriction of the map $\mathfrak{c}_L$ to its image $\mathfrak{c}_L[L] = (\{ \mathfrak{c}(a) \mid a \in L \}, \le \equiv \subseteq^{op})$ yields an isomorphism $L \cong \mathfrak{c}_L[L]$.

Moreover, $\mathfrak{c}_L$ satisfies the following universal property. If $h\colon L \to M$ is a frame homomorphism which sends every element of L to a complemented element of M, there exists a unique $\overline{h}\colon \mathsf{S}(L)^{op} \to M$ making the diagram

$$\begin{tikzcd} L \arrow[r, "\mathfrak{c}_L"] \arrow[dr, "h"'] & \mathsf{S}(L)^{op} \arrow[d, dashed, "\overline{h}"] \\ & M \end{tikzcd}$$

commutative.

1.2.2 Preimages and images

Given a localic map $f\colon L \to M$ and a sublocale S of M, the set-theoretic preimage $f^{-1}[S]$ is generally not a sublocale of L. However, $f^{-1}[S]$ is closed under meets in L and it is then easy to show that there exists the largest sublocale $f_{-1}[S]$ contained in $f^{-1}[S]$. It will be referred to as the *localic preimage* of S under f. In this situation, there is a pullback diagram

$$\begin{tikzcd} f_{-1}[S] \arrow[r, hook] \arrow[d] & L \arrow[d, "f"] \\ S \arrow[r, hook] & M \end{tikzcd}$$

in Loc. Moreover, the operation $S \mapsto f_{-1}[S]$ defines a map $f_{-1}[-]\colon \mathsf{S}(M) \to \mathsf{S}(L)$ which turns out to be a coframe homomorphism. Given a sublocale S of L, the set-theoretic image $f[S]$ is a sublocale of M, thus obtaining a map $f[-]\colon \mathsf{S}(L) \to \mathsf{S}(M)$ which is the left adjoint of $f_{-1}[-]$ — i.e., we have an adjunction

$$f[-] \dashv f_{-1}[-].$$

Furthermore, the identities

$$f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a)) \quad \text{and} \quad f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$$

are satisfied for each $a \in L$. Therefore, for a frame homomorphism $h\colon L \to M$, there is a commutative square

$$\begin{tikzcd} \mathsf{S}(L)^{op} \arrow[r, "(h_*)_{-1}[-]^{op}"] & \mathsf{S}(M)^{op} \\ L \arrow[u, tail, "\mathfrak{c}_L"] \arrow[r, "h"] & M \arrow[u, tail, "\mathfrak{c}_M"'] \end{tikzcd}$$

in Frm. This way, we have an endofunctor $\mathsf{S}(-)^{op}\colon \mathsf{Frm} \to \mathsf{Frm}$ together with a natural transformation $\mathfrak{c}\colon 1_{\mathsf{Frm}} \to \mathsf{S}(-)^{op}$ whose components are the frame homomorphisms $\mathfrak{c}_L$.

We shall make repeated use of the following lemma.

Lemma 1.2.2. *A localic map $f\colon L \to M$ factors through $\mathfrak{o}(a) \subseteq M$ (resp. $\mathfrak{c}(a) \subseteq M$) if and only if $f^*(a) = 1$ (resp. $f^*(a) = 0$).*

Proof. One has $f[L] \subseteq \mathfrak{o}(a)$ if and only if $L \subseteq f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$, that is, if and only if $f^*(a) = 1$. The closed case is similar. □

1.2.3 Open and closed localic maps

A localic map $f\colon L \to M$ is said to be *open* if $f[\mathfrak{o}(a)]$ is an open sublocale of M for every $a \in L$. A result due to Joyal and Tierney [83] states that a localic map f is open if and only if its left adjoint f^* is a *complete Heyting homomorphism* (i.e., a map preserving arbitrary joins, arbitrary meets and the Heyting operator).

We shall also need to consider a couple of weaker variants of openness. Following [28], a frame homomorphism $h\colon L \to M$ is called *weakly open* if $h(a^{**}) \le h(a)^{**}$ for every $a \in L$, and it is called *nearly open* if $h(a^*) = h(a)^*$ for every $a \in L$. If f is an open localic map, then its left adjoint f^* is nearly open; and near openness implies weak openness.

Finally, a localic map is *closed* if $f[\mathfrak{c}(a)]$ is a closed sublocale of M for every $a \in L$.

1.2.4 Boolean sublocales

For each $a \in L$, we denote

$$\mathfrak{b}_L(a) = \{ b \in L \mid (b \to a) \to a = b \} = \{ b \to a \mid b \in L \}$$

(or simply $\mathfrak{b}(a)$ when there is no danger of confusion). It turns out that $\mathfrak{b}_L(a)$ is the least sublocale of L containing a, and it is always a Boolean algebra. Conversely, every Boolean sublocale S of L is equal to $\mathfrak{b}_L(\bigwedge S)$. In particular, we have the *Booleanization* of L, namely

$$B_L = \mathfrak{b}_L(0) = \{ a \in L \mid a^{**} = a \} = \{ a^* \mid a \in L \}.$$

It can be characterized as the least dense sublocale of L, or the unique Boolean dense sublocale of L. The frame surjection associated to B_L is given by the map $(-)^{**}\colon L \twoheadrightarrow B_L$ which sends an $a \in L$ to a^{**}. Similarly, the associated nucleus is the double negation nucleus $(-)^{**}\colon L \to L$. Consequently, the join of a family $\{a_i\}_{i\in I} \subseteq B_L$ in B_L is given by $\bigvee^{B_L}_{i\in I} a_i = (\bigvee_{i\in I} a_i)^{**} = (\bigwedge_{i\in I} a_i^*)^*$ (the second equality follows from an application of the first De Morgan law (FDM)).

1.2.5 More on primes, spatiality and sublocales

If $p \in \mathsf{pt}(L)$, it is readily verified that $\mathfrak{b}(p) = \{1, p\}$, and consequently sublocales $\mathfrak{b}(p)$ are often referred to as *one-point* sublocales. Conversely, given a sublocale of the form $\{1, a\}$ with $a \neq 1$, then we must have $a \in \mathsf{pt}(L)$. Therefore there is a bijection between prime elements of L and one-point sublocales of L. The following properties are all well known:

Properties 1.2.3. Let L be a frame and S and T sublocales.

(1) A frame L is spatial if and only if each element of L is a meet of primes;

(2) The primes in S are precisely the primes in L which belong to S, that is, $\mathsf{pt}(S) = \mathsf{pt}(L) \cap S$;

(3) The equality $\mathsf{pt}(S \vee T) = \mathsf{pt}(S) \cup \mathsf{pt}(T)$ holds;

(4) Fit frames are T_1;

(5) The map $\mathsf{pt}(L) \to \mathsf{pt}(\mathsf{S}(L)^{op})$ given by $p \mapsto \mathfrak{b}(p)$ is a bijection.

For a frame L, we consider the sublocale

$$\mathsf{sp}(L) = \bigvee_{p \in \mathsf{pt}(L)} \mathfrak{b}(p) = \Big\{ \bigwedge_{i \in I} p_i \mid \{p_i\}_{i \in I} \subseteq \mathsf{pt}(L) \Big\}, \tag{1.2.6}$$

where the second equality holds by the formula for joins in (1.2.2). By Properties 1.2.3 (1) and (2), it follows clearly that $\mathsf{sp}(L)$ is the largest spatial sublocale of L. Recall now the spatialization $\epsilon_L^* \colon L \twoheadrightarrow \Omega(\Sigma(L))$ of a frame L (namely, the counit of the adjunction (1.1.1)). Since it is a surjection, it corresponds to a sublocale of L. Now, by the universality of the counit it is readily verified that ϵ_L^* corresponds to the largest spatial sublocale of L. Hence the sublocale to which it corresponds is precisely $\mathsf{sp}(L)$ (see also [95, 5.8]). It is therefore justified to speak of $\mathsf{sp}(L)$ as the *spatialization sublocale* of L.

In particular, since $\mathsf{S}(L)^{op}$ is a frame, we may compute its spatialization as a concrete sublocale of $\mathsf{S}(L)^{op}$. By (1.2.6) and Property 1.2.3 (5) the following is immediate:

Proposition 1.2.4 ([110, Proposition 3.14])**.** *The equality* $\mathsf{sp}(\mathsf{S}(L)^{op}) = \{ S \in \mathsf{S}(L) \mid S = \mathsf{sp}(S) \}$ *holds — i.e., the spatialization of* $\mathsf{S}(L)^{op}$ *is the subset of* $\mathsf{S}(L)$ *consisting of those sublocales which are spatial.*

In practice, we shall mostly consider the lattice $\mathsf{sp}(\mathsf{S}(L)^{op})$ with the dual ordering (that is, inclusion between sublocales) — i.e., we shall view it as a subcolocale of $\mathsf{S}(L)$ rather than as a sublocale of $\mathsf{S}(L)^{op}$. In this case, we denote it by $\mathsf{sp}[\mathsf{S}(L)]$.

1.3 The axiom T_D

A space X is said to be T_D if every point $x \in X$ has a neighborhood U with $U - \{x\}$ open. This axiom is stronger than T_0 and weaker than T_1 and it was introduced by Aull and Thron in [11]. The axiom T_D has a fundamental role in point-free topology, for example:

- Similarly as for sobriety (cf. Section 1.1), a T_D-space can be completely recovered from its frame of opens (this will be explained in detail in Subsection 1.3.2, see also [103] for related topics).

- If X is a T_D-space, then its induced sublocales constitute a one-to-one representation of subspaces — i.e., if A and B are subspaces of X with $\widetilde{A} = \widetilde{B}$, then one has $A = B$ (for sobriety one has a similar fact: if X is a sober space, then every spatial sublocale $S \subseteq \Omega(X)$ is of the form $\widetilde{A}$).

The similar behaviour between sobriety and the T_D-axiom is actually suggested by the fact that both properties are somehow symmetric (see [31]):

- A space X is sober if and only if there is no non-trivial subspace inclusion $\iota \colon X \hookrightarrow Y$ such that $\Omega(\iota)$ is an isomorphism.

- A space X is T_D if and only if there is no non-trivial subspace inclusion $\iota\colon Y \hookrightarrow X$ such that $\Omega(\iota)$ is an isomorphism.

Hence, it is perhaps not surprising that constructions concerning the sober-spatial duality of Section 1.1 have parallel counterparts for the T_D-case. Indeed, we now give more examples by exhibiting the T_D-analogues of the notions of spatial locale, prime element, frame homomorphism, and, finally prove a suitable duality for T_D-spaces. In Chapters 7 and 8 we shall provide more examples.

1.3.1 T_D-spatiality and covered primes

Recall that a spatial locale L is isomorphic to $\Omega(\Sigma(L))$ and $\Sigma(L)$ is always sober. However, not every spatial locale is isomorphic to $\Omega(X)$ for a T_D-space (see [31]). In view of this, a locale is said to be *T_D-spatial* if it is isomorphic to $\Omega(X)$ for some T_D-space X.

An element $p \in L$ with $p \neq 1$ is said to be a *covered prime* if for every $\{a_i\}_{i\in I} \subseteq L$ with $p = \bigwedge_{i\in I} a_i$, there is an $i \in I$ with $p = a_i$. The subset $\mathsf{pt}_D(L) \subseteq \mathsf{pt}(L)$ will denote the set of covered primes of L.

Remark 1.3.1. Elements $p \in L$ such that $p = \bigwedge_{i\in I} a_i$ implies the existence of an $i \in I$ with $p = a_i$ were also referred to as *completely prime* elements in [31]. However, this terminology was corrected because that term usually means that $\bigwedge_{i\in I} a_i \leq p$ implies the existence of an $i \in I$ with $a_i \leq p$. In a general frame, the notions are not equivalent (see also [32, Remark 1]).

An alternative characterization of covered primes was given in [31, Proposition 2.1.1]; it is the equivalence between (i) and (ii) of the following proposition. For our purposes it will be convenient to present a modification of this characterization for $\bigvee$-bases:

Proposition 1.3.2. *Let L be a frame and $B \subseteq L$ a $\bigvee$-base of L. If $p \in \mathsf{pt}(L)$, then the following are equivalent*:

(i) *p is covered*;

(ii) *there is an $a \in L$ with $p < a$ such that $p \leq b \leq a$ implies $b = p$ or $b = a$*;

(iii) *there is an $a \in L$ with $p < a$ such that for all $b \in B$ with $b \leq a$, either $b \leq p$ or $a \leq b \vee p$.*

Proof. (i) $\Longrightarrow$ (ii). Let $a = \bigwedge\{b \in L \mid p < b\}$. Since p is a covered prime, we have $p < a$. If $p < b \leq a$, we have $a \leq b$ and hence $b = a$.
(ii) $\Longrightarrow$ (iii). Let $b \in B$ and set $b' = (b \vee p) \wedge a$. Then $p \leq b' \leq a$, so either $b' \leq p$ or $a \leq b'$. In the former case, since p is prime we have $b \leq b \vee p \leq p$, and the latter case is equivalent to $a \leq b \vee p$.
(iii) $\Longrightarrow$ (i). Let $a \in L$ with $p < a$ such that for all $b \in B$ with $b \leq a$, either $b \leq p$ or $a \leq b \vee p$ and suppose that $p = \bigwedge_{i\in I} a_i$. Then there is an $i_0 \in I$ such that $a \nleq a_{i_0}$. Now write $a \wedge a_{i_0} = \bigvee B_{i_0}$ with $B_{i_0} \subseteq B$ and let $b \in B_{i_0}$. Then $b \leq a$ and $a \nleq b \vee p$ (if $a \leq b \vee p$ then $a \leq (a \wedge a_{i_0}) \vee p \leq a_{i_0}$) and so $b \leq p$. Consequently $a \wedge a_{i_0} = \bigvee B_{i_0} \leq p$. By primality of p, we then necessarily have $a_{i_0} \leq p$ and so $p = a_{i_0}$. $\square$

If p is a covered prime, it is not difficult to show that the element $a > p$ in (ii) and (iii) of Proposition 1.3.2 must coincide with $\bigwedge\{b \in L \mid p < b\}$ and hence it is uniquely determined. We shall therefore refer to it as the *cover* of p and we denote it by p^+.

Covered primes also have the following very useful characterization in terms of one-point sublocales:

Lemma 1.3.3 ([57, Proposition 10.2]). *A prime p is covered in a frame L if and only if $\mathfrak{b}(p)$ is a complemented sublocale of L.*

Moreover, coveredness of primes captures the T_D-property:

Lemma 1.3.4 ([31, Proposition 2.3.2]). *A T_0-space X is T_D if and only if $X - \overline{\{x\}}$ is a covered prime in $\Omega(X)$ for every $x \in X$.*

Recall from Section 1.1 that a localic map always sends prime elements into prime elements. However the analogous assertion for covered primes is not generally true (cf. [31, 32]). We shall therefore say that a localic map $f \colon L \to M$ is *D-localic* if $f(p) \in \mathsf{pt}_D(L)$ for each $p \in \mathsf{pt}_D(M)$ — i.e., if it sends covered primes into covered primes. Following [31] we shall also say that a frame homomorphism is a *D-homomorphism* if its right adjoint is a D-localic map.

Lemma 1.3.5 ([31, 3.2]). *If X and Y are T_D-spaces and $f \colon X \to Y$ is a continuous map, then $\Omega(f) \colon \Omega(X) \to \Omega(Y)$ is a D-localic map.*

1.3.2 The T_D-duality of Banaschewski and Pultr

The material in this subsection is due to Banaschewski and Pultr [31]. Let L be a frame, and for every $a \in L$, set $\Sigma'_a = \{p \in \mathsf{pt}_D(L) \mid a \nleq p\}$. It turns out that the family $\{\Sigma'_a \mid a \in L\}$ is a topology on $\mathsf{pt}_D(L)$. This topology is denoted by $\Sigma'(L)$ and referred to as the T_D*-spectrum* of L. It is not difficult to show that $\Sigma'(L)$ is always a T_D-space (see [31, Proposition 3.3.2]). We now define the following categories:

- Frm_D is the category consisting of frames and D-homomorphisms between them. Loc_D is by definition the dual of Frm_D — i.e., $\mathsf{Loc}_D = \mathsf{Frm}_D^{op}$. We regard Loc_D as a concrete category whose morphisms are D-localic maps;
- Top_D is the full subcategory of Top consisting of T_D-spaces.

Because of Lemma 1.3.5, the functor Ω from Section 1.1 can be restricted to a functor $\Omega \colon \mathsf{Top}_D \to \mathsf{Loc}_D$. If $f \colon L \to M$ is a D-localic map, it may be restricted and co-restricted to a map $\mathsf{pt}_D(L) \to \mathsf{pt}_D(M)$ which is easily seen to be continuous with respect to the topologies of the T_D-spectra, and so one obtains a morphism $\Sigma'(f) \colon \Sigma'(L) \to \Sigma'(M)$ in Top_D and a functor $\Sigma' \colon \mathsf{Loc}_D \to \mathsf{Top}_D$. Moreover, there is an adjunction

$$\begin{tikzcd} \mathsf{Top}_D \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc}_D \arrow[l, bend left, "\Sigma'"] \end{tikzcd} \tag{1.3.1}$$

Furthermore, the unit η' of the adjunction is a natural isomorphism (and therefore Ω is full and faithful, so one can regard Loc_D as a category of generalized T_D-spaces). Specifically, η' has components $\eta'_X \colon X \to \Sigma'(\Omega(X))$ which are homeomorphisms and send $x \in X$ to $X - \overline{\{x\}}$.

Hence, we may reconstruct T_D-spaces from their lattices of open sets by considering the homeomorphism $X \cong \Sigma'(\Omega(X))$.

The counit of the adjunction has components ϵ'_L whose left adjoints are surjective D-homomorphisms $(\epsilon'_L)^* \colon L \twoheadrightarrow \Omega(\Sigma'(L))$ sending $a \in L$ to Σ'_a. The map ϵ'_L is called the T_D*-spatialization* of L. Moreover, L is T_D-spatial if and only if ϵ'_L is an isomorphism, and so the adjunction restricts to an equivalence between Top_D and the full subcategory of Loc_D consisting of T_D-spatial locales. We shall need the following easy consequence (cf. Property 1.2.3 (1)):

Lemma 1.3.6. *A frame L is T_D-spatial if and only if every element of L is a meet of covered primes.*

Proof. If X is a T_D-space, $U = \bigwedge_{x \notin U} X - \overline{\{x\}}$ with each $X - \overline{\{x\}}$ covered by Lemma 1.3.4. For the converse, assume that every element in a locale L is a meet of covered primes. Then obviously $\Sigma'_a = \Sigma'_b$ implies $a = b$, hence the map $(\epsilon'_L)^*$ defined above is also injective and thus an isomorphism. Hence, $L \cong \Omega(\Sigma'(L))$ with $\Sigma'(L)$ a T_D-space. □

Corollary 1.3.7. *Let $f \colon L \to M$ be a surjective D-localic map. If L is T_D-spatial then so is M.*

Remark 1.3.8. One should not confuse T_D-spatiality with the following stronger notion. We will say that a frame L is *strongly* T_D*-spatial* if it is of the form $\Omega(X)$ for a sober T_D-space X. The following is straightforward from the discussion above.

Lemma 1.3.9. *The following are equivalent for a frame L:*

(i) *L is strongly T_D-spatial;*

(ii) *L is (T_D-)spatial and* $\mathsf{pt}(L) = \mathsf{pt}_D(L)$ (*i.e., every prime is covered*);

(iii) *Every element of L is a meet of* (*covered*) *primes and* $\mathsf{pt}(L) = \mathsf{pt}_D(L)$;

(iv) *The frame L is spatial and* $\mathsf{pt}(L)$ *is a T_D-space.*

1.4 Relations and congruences

In the category Frm there is a fairly simple description of the sublocale associated to (the congruence generated by) a binary relation. We summarize this construction in what follows (see [91, III 11] for details).

Let R be a binary relation on L (here this simply means a subset $R \subseteq L \times L$). An element $s \in L$ is *R-saturated* if

$$\forall a, b, c \in L, \quad aRb \implies a \wedge c \le s \text{ if and only if } b \wedge c \le s.$$

We set

$$L/R = \{s \in L \mid s \text{ is } R\text{-saturated}\}.$$

Then it turns out that L/R is a sublocale of L with associated surjection $\nu_R \colon L \twoheadrightarrow L/R$ given by $\nu_R(a) = \bigwedge\{s \in L \mid s \text{ is } R\text{-saturated and } a \leq s\}$. Moreover, the surjection ν_R satisfies

$$aRb \implies \nu_R(a) = \nu_R(b),$$

for any $a, b \in L$, and it is universal among frame homomorphisms with that property:

Theorem 1.4.1. *Let R be a relation on a frame L and let $h\colon L \to M$ be a frame homomorphism such that*

$$aRb \implies h(a) = h(b)$$

for all $a, b \in L$. Then there is a unique frame homomorphism $\overline{h}\colon L/R \to M$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\nu_R"] \arrow[dr, "h"'] & L/R \arrow[d, dashed, "\overline{h}"] \\ & M \end{tikzcd}$$

commutes.

1.5 Binary coproducts of frames

We shortly describe a construction of (binary) coproducts of frames. For more information and the infinite case, we refer to [91, IV 4].

1.5.1 Semilattices

In the category SLat_{01} of bounded meet semilattices (that is, posets having binary meets, a top and a bottom) the cartesian product with the injections

$$\iota_1'\colon L_1 \longrightarrow L_1 \times L_2 \quad \text{and} \quad \iota_2'\colon L_2 \longrightarrow L_1 \times L_2$$

given by $\iota_1'(a) = (a, 1)$ and $\iota_2'(b) = (1, b)$ constitutes a coproduct of L_1 and L_2.

1.5.2 Downset frames

For a bounded meet semilattice L, we consider the family

$$\mathcal{D}(L) := \{U \subseteq L \mid \varnothing \neq U = \downarrow U\}$$

of non-empty downsets and the map

$$\lambda_L\colon L \to \mathcal{D}(L)$$

given by $\lambda_L(a) = {\downarrow}a$. Clearly, $\mathcal{D}(L)$ is a frame (ordered under set theoretic inclusion) and λ_L is a bounded meet semilattice homomorphism. The construction $L \mapsto \mathcal{D}(L)$ can be extended to a functor which turns out to be the left adjoint of the forgetful functor $\mathsf{Frm} \to \mathsf{SLat}_{01}$ — i.e., $\mathcal{D}(L)$ is the free frame on the bounded meet semilattice L.

1.5.3 Coproducts of frames

Let now L_1 and L_2 be frames. Then, their coproduct $L_1 \oplus L_2$ can be obtained as $\mathcal{D}(L_1 \times L_2)/R$, where R is the relation given by

$$R = \Big\{\Big(\bigcup_{i\in I}{\downarrow}(a_i,b), {\downarrow}(\bigvee_{i\in i} a_i, b)\Big) \mid \{a_i\}_{i\in I} \subseteq L_1,\ b \in L_2\Big\} \cup \Big\{\Big(\bigcup_{i\in I}{\downarrow}(a,b_i), {\downarrow}(a, \bigvee_{i\in I} b_i)\Big) \mid a \in L_1,\ \{b_i\}_{i\in I} \subseteq L_2\Big\},$$

with coproduct injections ι_i given by the composites

$$L_i \xrightarrow{\iota_i'} L_1 \times L_2 \xrightarrow{\lambda_{L_1\times L_2}} \mathcal{D}(L_1 \times L_2) \xrightarrow{\nu_R} \mathcal{D}(L_1 \times L_2)/R = L_1 \oplus L_2$$

It is readily verified that a downset $U \in \mathcal{D}(L_1 \times L_2)$ is R-saturated if and only if for all $\{a_i\}_{i\in I} \subseteq L_1$, $a \in L_1$, $\{b_i\}_{i\in I} \subseteq L_2$ and $b \in L_2$,

$$\begin{aligned} (a_i, b) \in U \quad &\text{for all } i \in I \implies \Big(\bigvee_{i\in I} a_i, b\Big) \in U, \quad \text{and} \\ (a, b_i) \in U \quad &\text{for all } i \in I \implies \Big(a, \bigvee_{i\in I} b_i\Big) \in U. \end{aligned}$$

R-saturated downsets (i.e., members of $L_1 \oplus L_2$) are called *cp-ideals*. In particular, we have cp-ideals

$$a \oplus b := {\downarrow}(a,b) \cup \{(x,y) \in L_1 \times L_2 \mid x = 0 \text{ or } y = 0\}$$

which are clearly the smallest R-saturated downsets containing the ${\downarrow}(a,b)$. The following basic properties will be used:

Properties 1.5.1. Let $U \in L_1 \oplus L_2$, $a, a' \in L_1$, $\{a_i\}_{i\in I} \subseteq L_1$, $\{b_i\}_{i\in I} \subseteq L_2$ and $b, b' \in L_2$. Then:

(1) $a \oplus b \leq U$ if and only if $(a,b) \in U$;

(2) $U = \bigvee\{u \oplus v \mid (u,v) \in U\}$;

(3) $\iota_1(a) = a \oplus 1$ and $\iota_2(b) = 1 \oplus b$;

(4) $\bigwedge_{i\in I} a_i \oplus b_i = (\bigwedge_{i\in I} a_i) \oplus (\bigwedge_{i\in I} b_i)$;

(5) $(\bigvee_{i\in I} a_i) \oplus b = \bigvee_{i\in I}(a_i \oplus b)$ and $a \oplus (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} a \oplus b_i$;

(6) If $a, b \neq 0$, then $a \oplus b \leq a' \oplus b'$ if and only if $a \leq a'$ and $b \leq b'$.

Observe that given frame homomorphisms $h_i\colon L_i \to M$, $(i = 1,2)$, the induced map $\langle h_1,h_2\rangle\colon L_1 \oplus L_2 \to N$ is given by

$$\langle h_1,h_2\rangle(U) = \bigvee_{(a,b)\in U} h_1(a) \wedge h_2(b)$$

for each $U \in L_1 \oplus L_2$.

1.5.4 Diagonals in Loc

Localic diagonals will be of importance in this thesis, hence we set some notation and properties in what follows. Let L be a frame. By the general formula in Subsection 1.5.3, the codiagonal frame homomorphism $\delta_L := \langle 1_L, 1_L\rangle\colon L\oplus L \to L$ is given by

$$\delta_L(U) = \bigvee_{(a,b)\in U} a \wedge b$$

for each $U \in L\oplus L$. Now, since δ_L is a surjection, it defines a sublocale of $L\oplus L$, namely

$$D_L := (\delta_L)_*[L] \subseteq L\oplus L,$$

where $(\delta_L)_*(a) = \bigvee\{\, u\oplus v \in L\oplus L \mid u\wedge v \le a\,\} = \{\,(u,v)\in L\times L \mid u \wedge v \le a\,\}$ (this follows easily from the adjunction $\delta_L \dashv (\delta_L)_*$ and the fact that every *cp*-ideal can be written as a join of basic generators — cf. Property 1.5.1 (2)).

With this description, it is clear that *cp*-ideals contained in the diagonal have the following symmetry property.

Lemma 1.5.2. *For a cp-ideal $U \in D_L$, one has $(a,b) \in U$ if and only if $(b,a) \in U$.*

The least element of D_L is

$$d_L := (\delta_L)_*(0) = \bigvee_{u\wedge v=0} u\oplus v$$

and therefore the closure of the diagonal may be expressed as $\overline{D_L} = \mathfrak{c}(d_L)$.

1.6 The frame of (extended) reals

We first recall the frame of reals $\mathfrak{L}(\mathbb{R})$ from [22]. Here we define it, equivalently, as the frame presented by generators $(r,\text{—})$ and $(\text{—},r)$ for all rationals $r \in \mathbb{Q}$, and relations

(r0) $(r,\text{—}) \wedge (\text{—},s) = 0$ if $s \le r$;

(r1) $(r,\text{—}) \vee (\text{—},s) = 1$ if $r < s$;

(r2) $(r,\text{—}) = \bigvee_{s>r}(s,\text{—})$;

(r3) $(—,r) = \bigvee_{s<r}(—,s)$;

(r4) $\bigvee_{r\in\mathbb{Q}}(r,—) = 1$;

(r5) $\bigvee_{r\in\mathbb{Q}}(—,r) = 1$.

By dropping relations (r4) and (r5) one has the *frame of extended reals* $\mathfrak{L}(\overline{\mathbb{R}})$ (see [26] for a detailed study of this frame). For all rationals $r < s$, $\mathfrak{L}(\overline{\mathbb{R}})$ is isomorphic to the closed sublocale $\uparrow(s,—)\vee(—,r)$ of $\mathfrak{L}(\mathbb{R})$. Hence, $\mathfrak{L}(\overline{\mathbb{R}})$ is a compact and regular frame (recall Subsection 1.1.3).

1.6.1 (Extended) real-valued functions

A *real-valued* (resp. *extended real-valued*) *continuous function* on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ (resp. $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$). We shall denote the set of all real-valued (resp. extended real-valued) continuous functions by $\mathsf{C}(L)$ (resp. $\overline{\mathsf{C}}(L)$).

Furthermore, following [62, 26], a *real-valued* (resp. *extended real-valued*)

- *function* on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ (resp. $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathsf{S}(L)^{op}$);
- *lower semicontinuous function* on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ (resp. $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathsf{S}(L)^{op}$) such that $f(r,—)$ is closed for every $r \in \mathbb{Q}$;
- *upper semicontinuous function* on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ (resp. $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathsf{S}(L)^{op}$) such that $f(—,r)$ is closed for every $r \in \mathbb{Q}$.

The corresponding classes of real-valued (resp. extended real-valued) functions will be denoted by

$$\mathsf{F}(L),\ \mathsf{LSC}(L),\ \text{and}\ \ \mathsf{USC}(L) \quad (\text{resp. } \overline{\mathsf{F}}(L),\ \overline{\mathsf{LSC}}(L),\ \text{and}\ \ \overline{\mathsf{USC}}(L)).$$

By the isomorphism $L \cong \mathfrak{c}_L[L]$ of Subsection 1.2.1, we may regard real-valued (resp. extended real-valued) continuous functions on L as frame homomorphisms $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ (resp. $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathsf{S}(L)^{op}$) such that $f(r,—)$ and $f(—,r)$ are closed for every $r \in \mathbb{Q}$. Under this identification, we note that $\mathsf{C}(L) = \mathsf{LSC}(L) \cap \mathsf{USC}(L)$ and $\overline{\mathsf{C}}(L) = \overline{\mathsf{LSC}}(L) \cap \overline{\mathsf{USC}}(L)$.

The family $\mathsf{C}(L)$ (resp. $\overline{\mathsf{C}}(L)$) is partially ordered by

$$f \le g \iff f(r,—) \le g(r,—) \ \text{ for all } \ r \in \mathbb{Q} \iff g(—,r) \le f(—,r) \ \text{ for all } \ r \in \mathbb{Q}. \qquad (1.6.1)$$

Since $\mathsf{F}(L) = \mathsf{C}(\mathsf{S}(L)^{op})$ (resp. $\overline{\mathsf{F}}(L) = \overline{\mathsf{C}}(\mathsf{S}(L)^{op})$), we also have a partial order in $\mathsf{F}(L)$ (resp. $\overline{\mathsf{F}}(L)$). Specifically, it is given by

$$f \le g \iff f(—,r) \subseteq g(—,r) \ \text{ for all } \ r \in \mathbb{Q} \iff g(r,—) \subseteq f(r,—) \ \text{ for all } \ r \in \mathbb{Q}. \qquad (1.6.2)$$

There is a useful way of specifying continuous (extended) real-valued functions with the help of scales ([62, 4]). An *extended scale* in L is a map $\sigma\colon \mathbb{Q} \to L$ such that $\sigma(r) \vee \sigma(s)^* = 1$

whenever $r < s$. An extended scale is a *scale* if $\bigvee_{r\in\mathbb{Q}} \sigma(r) = 1 = \bigvee_{r\in\mathbb{Q}} \sigma(r)^*$. For each extended scale σ in L, the formulas

$$f(r,\text{—}) = \bigvee_{s>r} \sigma(s) \quad \text{and} \quad f(\text{—},r) = \bigvee_{s<r} \sigma(s)^* \tag{1.6.3}$$

(for any $r,s \in \mathbb{Q}$) determine an $f \in \overline{\mathsf{C}}(L)$ ([26, Lemma 1]); f is in $\mathsf{C}(L)$ if and only if σ is a scale. If $f, g \in \overline{\mathsf{C}}(L)$ are generated by extended scales σ_f and σ_g respectively, then one has

$$f \leq g \;\text{ if and only if }\; \sigma_f(r) \leq \sigma_g(s), \quad \text{for all } r > s. \tag{1.6.4}$$

Let $S \subseteq L$ be a sublocale and $f \in \mathsf{C}(S)$. An $\overline{f} \in \mathsf{C}(L)$ is said to be a *continuous extension* of f to L if $\nu_S \circ \overline{f} = f$ — i.e., if the following diagram commutes:

$$\begin{tikzcd} \mathfrak{L}(\mathbb{R}) \arrow[d, dashed, "\overline{f}"'] \arrow[dr, "f"] & \\ L \arrow[r, two heads, "\nu_S"] & S \end{tikzcd}$$

The extended real-valued case is defined analogously.

Before defining cozero elements, we need to recall the notion of σ-frame. A *σ-frame* is a bounded lattice L with countable joins such that $(\bigvee_{i\in I} a_i) \wedge b = \bigvee_{i\in I} a_i \wedge b$ for each countable family $\{a_i\}_{\in I} \subseteq L$ and $b \in L$. Clearly every frame is a σ-frame but not conversely. A *σ-frame homomorphism* is a map which preserves countable joins (including the bottom) and finite meets (including the top). If $a, b \in L$, we write $b \prec a$ if there is a $c \in L$ with $b \wedge c = 0$ and $a \vee c = 1$. If L happens to be a frame, then this relation coincides with the one introduced in Subsection 1.1.3. A σ-frame L is then *regular* if every $a \in L$ can be written as $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$. For a frame, being regular as a σ-frame is generally stronger than being regular as a frame (cf. Subsection 1.1.3).

1.6.2 Some special classes of sublocales

An $a \in L$ is said to be a *cozero element* if $a = f((\text{—},0) \vee (0,\text{—}))$ for some $f \in \mathsf{C}(L)$. Equivalently, a is a cozero element if $a = f(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))$ for some $f \in \overline{\mathsf{C}}(L)$. It is well known that cozero elements are closed under countable joins and finite meets; thus they form a σ-frame $\operatorname{Coz} L \subseteq L$. Moreover, a *zero sublocale* (resp. *cozero sublocale*) is one of the form $\mathfrak{c}(a)$ (resp. $\mathfrak{o}(a)$) with $a \in \operatorname{Coz} L$. We warn the reader that this terminology differs from that used by other authors (e.g., [69, 14]).

Finally, an element $a \in L$ is said to be *δ-regular* [68] if $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$ (where $\prec$ is the relation defined in Subsection 1.1.3, that is, $b \prec a$ if and only if $b^* \vee a = 1$). δ-regular elements are also closed under countable joins and finite meets. A sublocale $\mathfrak{c}(a)$ (resp. $\mathfrak{o}(a)$) with a δ-regular will be called a *δ-regular closed* (resp. *δ-regular open*) sublocale. Since $b \prec a$ implies $b^{**} \prec a$, it follows that every δ-regular element is a countable join of regular elements (see Subsection 1.1.2). A sublocale of the form $\mathfrak{c}(a)$ (resp. $\mathfrak{o}(a)$) with a regular is usually called *regular closed* (resp. *regular open*).

1.6.3 Cozero elements in σ-frames

Clarke and Gilmour studied cozero elements in σ-frames [40]. Let L be a σ-frame. An $a \in L$ is said to be a *cozero element* if $a = f((—,0) \vee (0,—))$ for some σ-frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$. Equivalently, a is a cozero element if $a = f(\bigvee_{r\in\mathbb{Q}}(r,—))$ for some σ-frame homomorphism $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$. One of the main results of [40] is the following:

Theorem 1.6.1 ([40, Corollary 1])**.** *Let L be a σ-frame. Then the collection* $\operatorname{Coz} L$ *of all cozero elements of L is a regular σ-frame.*

1.6.4 C-, C^*- and z-embeddings

A sublocale S is *C-embedded* (resp. *C^*-embedded*) if every $f \in \mathsf{C}(S)$ (resp. $f \in \overline{\mathsf{C}}(S)$) has a continuous extension to L. A sublocale S is *z-embedded* if for every cozero element $a \in S$ there is a cozero element $b \in L$ with the property that $\nu_S(b) = a$. Generally, the implications

$$S \text{ is } C\text{-embedded} \implies S \text{ is } C^*\text{-embedded} \implies S \text{ is } z\text{-embedded}$$

hold. The following is well known (see [20, 14]):

Theorem 1.6.2. *The following are equivalent for a locale L:*

(i) *L is normal;*

(ii) *Every closed sublocale is C-embedded;*

(iii) *Every closed sublocale is C^*-embedded;*

(iv) *Every closed sublocale is z-embedded.*

Chapter 2

Diagonal separation in the category of locales: fitted diagonals and closed diagonals

2.1 Introduction

Let $\mathcal{P}$ be a property of subobjects relevant in a category C with finite products. An object $X \in \mathsf{ob}(C)$ is said to be *$\mathcal{P}$-separated* if the diagonal $\Delta_X = (1_X, 1_X)\colon X \rightarrowtail X \times X$ has property $\mathcal{P}$.

In the category $C = \mathsf{Top}$ of topological spaces, typical examples are described in Table 2.1. For instance, we have Hausdorff spaces (those in which the diagonal is closed), T_1-spaces (those in which the diagonal is an intersection of open subspaces) or discrete spaces (those in which the diagonal is open).

Table 2.1 Examples of $\mathcal{P}$-separation in Top

Property $\mathcal{P}$ of subobjects	$\mathcal{P}$-separation property
Closed subspace	Hausdorff
Locally closed subspace	Locally Hausdorff
Intersection of open subspaces	T_1
Intersection of locally closed subspaces	T_1
Open subspace	Discrete

In the category $C = \mathsf{Loc}$, locales whose diagonal is closed are known as *strongly Hausdorff* (for brevity (sH)), originally introduced by Isbell [71]. Moreover, locales with open diagonal were characterized by Joyal and Tierney [83]: they are precisely the complete and atomic Boolean algebras. However, the theory of locales whose diagonal is fitted (i.e., an intersection of open sublocales — we will speak about $\mathcal{F}$-separated or locales satisfying ($\mathcal{F}$-sep)) has not been developed so far.

The aim of this chapter is to study $\mathcal{F}$-separatedness *vis-á-vis* with the strong Hausdorff property. It will turn out that there is a strong structural parallel between both properties. This parallel holds in a wide range of situations, for example:

- Frames L which satisfy (sH) (resp. ($\mathcal{F}$-sep)) can be characterized by means of a Dowker-Strauss-type condition involving the combinatorial structure of the frame homomorphisms $L \to M$;
- Both properties (sH) and ($\mathcal{F}$-sep) can be decomposed as the conjunction of the axiom (T_U) and the property that certain weakened frame homomorphisms are frame homomorphisms. In the former case, the weakened homomorphisms are a subclass of suplattice morphisms whereas in the latter case they are a subclass of preframe homomorphisms.
- Perhaps somewhat unexpectedly, this parallel between properties (sH) and ($\mathcal{F}$-sep) is also related to another phenomenon which will be widely discussed in this thesis: the parallel between normality and extremal disconnectedness. For example, we shall prove that every hereditarily extremally disconnected (T_U) locale is $\mathcal{F}$-separated, whereas every hereditarily normal (T_U) locale is (sH).

The parallel between ($\mathcal{F}$-sep) and (sH) will also be analyzed in the context of singly generated frame extensions, (co)density, first-order separation properties, etc.

Table 2.2 Examples of $\mathcal{P}$-separation in Loc

PROPERTY $\mathcal{P}$ OF SUBOBJECTS	$\mathcal{P}$-SEPARATION PROPERTY	REFERENCE
Closed sublocale	Strongly Hausdorff	[71]
Locally closed sublocale	Locally strongly Hausdorff	[88]
Fitted sublocale	$\mathcal{F}$-separated	[8], Section 2.2
Semifitted sublocale	$\mathcal{S}$-separated	Section 2.10
Open sublocale	Complete and atomic Boolean algebra	[83]

The material in this chapter in based on a joint work with Jorge Picado and Aleš Pultr and has been published in the following article:

[8] I. Arrieta, J. Picado, and A. Pultr, A new diagonal separation and its relations with the Hausdorff property, *Applied Categorical Structures*, vol. 30, pp. 247–263, 2022.

Some minor changes have been included (for example, Lemma 2.4.6 has been constructivized by avoiding the use of ordinals). Moreover, the material in Sections 2.5, 2.7, 2.8, 2.9 and 2.10 is unpublished. Section 2.11 contains a table summarizing the parallel situation.

2.2 Specific preliminaries

2.2.1 Closure operators in a category

We start by recalling the definition of a closure operator in a category (in the sense of Dikranjan and Giuli [44], we also refer to [41, 42] and the references there for more information on closure operators).

Definition 2.2.1. Let C be a category equipped with a proper factorization system $(\mathcal{E}, \mathcal{M})$. A *closure operator c* with respect to $\mathcal{M}$ is a family of maps

$$\{c_X : \mathsf{Sub}_{\mathcal{M}}(X) \longrightarrow \mathsf{Sub}_{\mathcal{M}}(X)\}_{X \in \mathsf{ob}(C)}$$

that satisfies the following properties:

(i) it is monotone, that is, $m \leq n$ implies $c_X(m) \leq c_X(n)$ for all $m, n \in \mathsf{Sub}_{\mathcal{M}}(X)$;

(ii) it is inflationary, that is, $m \leq c_X(m)$ for all $m \in \mathsf{Sub}_{\mathcal{M}}(X)$;

(iii) and such that for every $f : X \to Y$ in C and for all $m \in \mathsf{Sub}_{\mathcal{M}}(X)$, the identity $f(c_X(m)) \leq c_Y(f(m))$ holds (where $f(m)$ refers to the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$).

2.2.2 $\mathcal{P}$-separation

Recall Section 2.1: given a property $\mathcal{P}$ of subobjects in a category with finite products, an object $X \in \mathsf{ob}(C)$ is *$\mathcal{P}$-separated* if the diagonal has property $\mathcal{P}$. Moreover, we shall denote by $\mathsf{Sep}_{\mathcal{P}}(C)$ the full subcategory of C consisting of $\mathcal{P}$-separated objects.

Every closure operator has an associated property $\mathcal{P}_c$ of subobjects, namely $m \in \mathsf{Sub}(X)$ has property $\mathcal{P}_c$ if it is an $\mathcal{M}$-subobject which is *c-closed* — i.e., if $c_X(m) = m$. If $\mathcal{P} = \mathcal{P}_c$ for some closure operator c, we shall simply speak of *c-separated objects* and the corresponding full subcategory will be denoted by $\mathsf{Sep}_c(C)$.

In the latter case, separated objects enjoy useful categorical properties. We recall here that a family $\{p_i : X \to X_i\}_{i \in I}$ of morphisms in C is a *mono-source* or *jointly monic* if for each two parallel morphisms $f, g : Y \to X$ with $p_i \circ f = p_i \circ g$ for all $i \in I$, one has $f = g$. For example, limit cones are always jointly monic.

Theorem 2.2.2 ([42, Proposition 4.2] and [41, Propositions 10.1 and 10.7])**.** *Let C be a category equipped with a proper factorization system $(\mathcal{E},\mathcal{M})$ and a closure operator c with respect to $\mathcal{M}$. Then the following properties hold*:

(1) *An object $X \in \mathsf{ob}(C)$ is c-separated if and only if for every pair of morphisms $f,g\colon Y \to X$ the equalizer $\mathsf{equ}(f,g) \rightarrowtail Y$ is c-closed*;

(2) *The category $\mathsf{Sep}_c(C)$ is closed under mono-sources in C. In particular, it is closed under limits and closed under monomorphisms — i.e., if $Y \rightarrowtail X$ is monic and X is c-separated, then so is Y*;

(3) *If C is $\mathcal{E}$-well-copowered, $\mathsf{Sep}_c(C)$ is extremally epireflective in C.*

Remark 2.2.3. We emphasize that the closure under monomorphisms in Theorem 2.2.2 (2) does *not* refer only to $\mathcal{M}$-subobjects, but to arbitrary monomorphisms. Hence it is quite a strong property and in categories with a complex structure of monomorphisms, it can be an interesting one. For example, it is well known that monomorphisms in $C = \mathsf{Loc}$ have a rather wild behaviour (recall the fact that Loc is not well-powered), so this property can be non-trivial.

2.2.3 Back to the category of locales

After this preparation we can now introduce the main example to be studied in this chapter. Recall from Subsection 1.2.1 that the closure and the "other" closure of a sublocale S are given by

$$\overline{S} = \bigcap_{S\subseteq \mathfrak{c}(a)} \mathfrak{c}(a) \qquad \text{and} \qquad \mathring{S} = \bigcap_{S\subseteq \mathfrak{o}(a)} \mathfrak{o}(a)$$

and they define the smallest closed sublocale and the smallest fitted sublocale containing S.

As is well known, (Regular monomorphisms, Surjections) is a proper factorization system in Loc, and the assignment $S \mapsto \overline{S}$ defines a closure operator whose separated objects are the strongly Hausdorff ones.

On the other hand, it was shown in [43, 2.4], among other properties of the "other" closure, that the assignment $S \mapsto \mathring{S}$ defines another closure operator in Loc. As usual, $\nu_{\mathring{S}}$ will denote the nucleus associated to the sublocale $\mathring{S} \subseteq L$. Moreover, we shall simply denote by

$$\nu_{\circ L} \quad \text{(or by } \nu_{\circ} \text{ if there is no danger of confusion)}$$

the nucleus $\nu_{\mathring{D}_L}$ on $L\oplus L$ (recall that $D_L \subseteq L\oplus L$ denotes the diagonal sublocale, cf. Subsection 1.5.4). Clearly, separation with respect to this "other" closure amounts to the diagonal of the locale in question being fitted. We formalize this in the following definition:

Definition 2.2.4. A locale L is $\mathcal{F}$*-separated* (or it satisfies property ($\mathcal{F}$-sep)) if its diagonal is fitted in $L\oplus L$.

As a consequence of Theorem 2.2.2, we immediately have the following:

Theorem 2.2.5 (Categorical properties of $\mathcal{F}$-separatedness)**.** *The following assertions hold*:

(1) *If* $f,g\colon M \to L$ *are localic maps and* L *is* $\mathcal{F}$*-separated, then their equalizer* $\mathsf{equ}(f,g)$ *is a fitted sublocale of* M;

(2) $\mathcal{F}$*-separated locales are closed under mono-sources in* Loc*. In particular, if* $f\colon M \to L$ *is a monomorphism in* Loc *and* L *is* $\mathcal{F}$*-separated, then so is* M;

(3) $\mathcal{F}$*-separated locales are closed under limits in* Loc;

(4) $\mathcal{F}$*-separated locales are extremally epireflective in* Loc.

Moreover, recall from Subsection 1.2.1 that a locale is fit if each of its sublocales is fitted. Since fitness is closed under products (this is proved in [92, Theorem 3]), every fit locale has a fitted diagonal, hence:

Proposition 2.2.6. *Every fit locale is* $\mathcal{F}$*-separated.*

The question naturally arises whether the converse also holds. It does not, as we shall show in Subsection 2.6.2.

2.2.4 Equalizers and pullbacks

In any finitely complete category, the equalizer of two parallel morphisms $f,g\colon X \to Y$ can be computed as the pullback of the induced morphism $\langle f,g\rangle\colon X \to Y\times Y$ along the diagonal.

Since pullback along a sublocale in Loc is given by localic preimage, for any localic maps $f,g\colon M \to L$ there is a pullback diagram

$$\begin{tikzcd} \langle f,g\rangle_{-1}[D_L] \arrow[r, hook] \arrow[d] & M \arrow[d, "\langle f,g\rangle"] \\ D_L \arrow[r, hook] & L\oplus L \end{tikzcd}$$

and therefore the equalizer can be expressed as a concrete sublocale, namely

$$\mathsf{equ}(f,g) = \langle f,g\rangle_{-1}[D_L] \subseteq M.$$

For more details and applications of this categorical setting, we refer to [96].

2.2.5 Preframes and suplattices

It is well known (cf. [83], see also [51, 97]) that the category Sup of suplattices — i.e., the category of posets with joins of arbitrary subsets, and join-preserving maps — has a tensor product in the conventional sense: for all complete lattices L and M, there is a complete lattice $L\otimes M$ and a map $t_{L,M}\colon L\times M \to L\otimes M$ preserving joins in each variable which is universal among maps $L\times M \to N$ preserving joins in each variable.

The elements $t_{L,M}(a,b)$ are denoted by $a \otimes b$ and every element of $L \otimes M$ can be expressed as a join of elements of the form $a \otimes b$.

Importantly, it turns out that when L and M are frames, the tensor product $L \otimes M$ is also a frame which is isomorphic to the frame coproduct $L \oplus M$ (where the isomorphism identifies the generators $a \otimes b$ and $a \oplus b$).

There is a "dual" approach to the frame coproduct obtained by building another tensor product in a different category. A *preframe* is a poset having finite meets (including the top) and directed joins, in which directed joins distribute over binary meets. A *preframe homomorphism* is a function preserving finite meets (including the top) and directed joins. Preframes and their homomorphisms form a category PreFrm that contains Frm as a non-full subcategory.

It was proved by Johnstone and Vickers (see [82]) that the category PreFrm also has a tensor product — i.e., for all preframes L and M, there is a preframe $L \parr M$ and a map $t'_{L,M} \colon L \times M \to L \parr M$ preserving directed joins and finite meets in each variable which is universal among maps $L \times M \to N$ preserving directed joins and finite meets in each variable.

The elements $t'(a,b)$ are denoted by $a \parr b$ and every element of $L \parr M$ can be expressed as a directed join of finite meets of elements of the form $a \parr b$.

For our purpose, it is important to note that if L_i and M_i are preframes and $h_i \colon L_i \to M_i$ are preframe homomorphisms ($i = 1,2$), then there is a preframe homomorphism

$$h_1 \parr h_2 \colon L_1 \parr L_2 \to M_1 \parr M_2$$

which sends $a \parr b$ to $h_1(a) \parr h_2(b)$.

Moreover, when L and M are frames, once again the tensor product $L \parr M$ is a frame isomorphic to the coproduct $L \oplus M$. The isomorphism maps $a \parr b$ to $(a \oplus 1) \vee (1 \oplus b)$ and $a \oplus b$ to $(a \parr 0) \wedge (0 \parr b)$.

Accordingly, we shall frequently identify $L \oplus M = L \parr M$ and we will write

$$a \parr b = (a \oplus 1) \vee (1 \oplus b)$$

which can obviously be reversed to

$$a \oplus b = (a \parr 0) \wedge (0 \parr b).$$

The parallel between the suplattice and preframe approaches to locale theory has been explored by several authors (e.g. for obtaining dual results for open maps and proper maps), see [111] for more information.

2.2.6 Prenuclei

In this chapter we shall make use of the notion of prenucleus (in the sense of Simmons [109] or Escardó [54][1]).

Definition 2.2.7. Let L be a frame. A mapping $\tau\colon L \to L$ is called a *prenucleus* on L if

(i) it is monotone,

(ii) it is increasing — i.e., $a \leq \tau(a)$ for all $a \in L$,

(iii) $\tau(a) \wedge \tau(b) = \tau(a \wedge b)$ for all $a, b \in L$.

Note that an idempotent prenucleus is precisely a nucleus. Prenuclei are ordered pointwisely and the set $\mathsf{pN}(L)$ of all prenuclei on L is a complete lattice. Moreover, a directed join of a family of prenuclei in $\mathsf{pN}(L)$ is given by their pointwise (directed) join (this is of course not true when one replaces prenuclei by nuclei).

Given a prenucleus τ on L, its set of fixpoints $\{a \in L \mid \tau(a) = a\}$ is a sublocale of L. Moreover, every prenucleus τ has a least nucleus $\overline{\tau}$ above it (we speak of the *nuclear reflection* of τ, or the *nucleus generated* by τ), and this nuclear reflection has the same set of fixpoints. Usually the nuclear reflection is constructed by transfinite iteration, and then one uses transfinite induction for proving facts about it. However, this approach makes use of ordinals and hence it is not constructively valid.

Here we briefly describe a similar, but constructive, induction principle due to Escardó [54] (cf. also an alternative approach based on an explicit formula due to Banaschewski [23]). A set Q of prenuclei on L is called *inductive* if $1_L \in Q$ and it is closed under directed joins.

Then one has

Theorem 2.2.8 ([54, Corollary 3.1])**.** *Let τ be a prenucleus on L and $Q \subseteq \mathsf{pN}(L)$ be an inductive subset. Suppose that $\lambda \in Q$ implies $\tau \circ \lambda \in Q$. Then $\overline{\tau} \in Q$.*

2.3 Dowker-Strauss-type characterizations

Dowker and Strauss proved in [45] that the strong Hausdorff property on a frame L can be characterized by a suitable property of the family of frame homomorphisms with domain L. We start by recalling this characterization (with slightly adapted terminology).

Definition 2.3.1. Let $h, k\colon L \to M$ be frame homomorphisms. The pair (h, k) is said to *respect disjoint pairs* if $h(a) \wedge k(b) = 0$ whenever $a \wedge b = 0$ (i.e., if $\bigvee_{a \wedge b = 0} h(a) \wedge k(b) = 0$).

Theorem 2.3.2 ([45, Proposition 4])**.** *A frame L is strongly Hausdorff if and only if no pair of distinct frame homomorphisms with domain L respects disjoint pairs.*

[1] Banaschewski uses a slightly more general notion with the same terminology, see [23] for details.

Recall now that a frame L is T_U (T_U for *totally unordered* [77]; also known as *unordered* in [72]) if for every pair of frame homomorphisms $h,k\colon L \to M$ one has

$$h \le k \implies h = k$$

(the homsets in Frm are ordered pointwisely).

We also introduce the following terminology:

Definition 2.3.3. Let $h,k\colon L \to M$ be frame homomorphisms. The pair (h,k) is said to be *bounded above* (resp. *bounded below*) if there is another frame homomorphism $h'\colon L \to M$ with $h \le h' \ge k$ (resp. $h \ge h' \le k$).

The following two lemmas are obvious:

Lemma 2.3.4. *A frame L is T_U if and only if no pair of distinct frame homomorphisms with domain L is bounded above (equivalently, bounded below).*

Lemma 2.3.5. *If a pair of frame homomorphisms is bounded above, then it respects disjoint pairs.*

From Theorem 2.3.2 and Lemmas 2.3.4 and 2.3.5 we immediately obtain the following

Corollary 2.3.6 ([77, Corollary III 1.5])**.** *A strongly Hausdorff frame is T_U.*

In what follows, we show that there is a "dual" to the setting just described by replacing the strong Hausdorff property by $\mathcal{F}$-separatedness.

We begin by stating some straightforward properties concerning *cp*-ideals that we shall need later.

Lemma 2.3.7. *Let $f,g\colon M \to L$ be localic maps. The following assertions hold*:

(1) *For each $U \in L\oplus L$ set $\widehat{U} := \{a \wedge b \mid (a,b) \in U\}$. Then*

$$\langle f,g\rangle^*(U) \ge \bigvee\{f^*(c) \wedge g^*(c) \mid c \in \widehat{U}\}.$$

Moreover, $D_L \subseteq \mathfrak{o}(U)$ if and only if $\widehat{U}$ is a cover of L.

(2) *For each $C \subseteq L$ set $\widetilde{C} := \bigvee\{a \oplus a \mid a \in C\}$. Then*

$$\langle f,g\rangle^*(\widetilde{C}) = \bigvee\{f^*(a) \wedge g^*(a) \mid a \in C\}.$$

Moreover, $D_L \subseteq \mathfrak{o}(\widetilde{C})$ if and only if C is a cover of L.

We also have the following technical characterization of $\mathcal{F}$-separatedness:

Theorem 2.3.8. *A frame L is $\mathcal{F}$-separated if and only if for any localic maps $f,g\colon M \to L$, the equality $\langle f,g\rangle^*(U) = 1$ holds for all $U \in L\oplus L$ with $D_L \subseteq \mathfrak{o}(U)$ only if $f = g$.*

Proof. $\Rightarrow$: Take $f, g\colon M \to L$ as in the statement and let $\langle f, g\rangle$ be the induced localic map. Then, using the formula in Subsection 2.2.4 and the fact that preimage commutes with arbitrary intersections,

$$\mathsf{equ}(f,g) = \langle f,g\rangle_{-1}[D_L] = \langle f,g\rangle_{-1}[\mathring{D}_L] = \bigcap\{\langle f,g\rangle_{-1}[\mathfrak{o}(U)] \mid D_L \subseteq \mathfrak{o}(U)\}.$$

Since $\langle f,g\rangle_{-1}[\mathfrak{o}(U)] = \mathfrak{o}(\langle f,g\rangle^*(U)) = M$ whenever $D_L \subseteq \mathfrak{o}(U)$ it follows that $\mathsf{equ}(f,g) = M$ and therefore $f = g$.

$\Leftarrow$: Recall that

$$\mathring{D}_L = \bigcap\{\mathfrak{o}(U) \mid D_L \subseteq \mathfrak{o}(U)\}$$

and let $\iota\colon \mathring{D}_L \hookrightarrow L \oplus L$ be the embedding. If $U \in L\oplus L$ is such that $D_L \subseteq \mathfrak{o}(U)$ then $\langle p_1 \circ \iota, p_2 \circ \iota\rangle^*(U) = \iota^*(\langle p_1, p_2\rangle^*(U)) = \iota^*(U) = 1$ by Lemma 1.2.2 and the obvious fact that $\langle p_1, p_2\rangle = 1_{L\oplus L}$. By the assumption it follows that $p_1 \circ \iota = p_2 \circ \iota$. Since D_L is the equalizer of the product projections p_1 and p_2 it follows that $\mathring{D}_L \subseteq D_L$, but the reverse inclusion is obvious. □

The following concept is parallel to that of Definition 2.3.1.

Definition 2.3.9. Let $h, k\colon L \to M$ be frame homomorphisms. The pair (h, k) is said to *respect covers* if for every cover C of L one has $\bigvee_{a\in C} h(a) \wedge k(a) = 1$.

Lemma 2.3.10. *Let $f, g\colon L \to M$ be localic maps. Then, the pair (f^*, g^*) respects covers if and only if $\langle f, g\rangle^*(U) = 1$ for all $U \in L\oplus L$ with $D_L \subseteq \mathfrak{o}(U)$.*

Proof. $\Rightarrow$: Assume the pair (f^*, g^*) respects covers and let $U \in L\oplus L$ be such that $D_L \subseteq \mathfrak{o}(U)$. By Lemma 2.3.7 (1) it follows that $\widehat{U}$ is a cover of L and $\langle f,g\rangle^*(U) \geq \bigvee\{f^*(c) \wedge g^*(c) \mid c \in \widehat{U}\}$. Since (f^*, g^*) respects covers, one has $\langle f,g\rangle^*(U) = 1$.

$\Leftarrow$: Suppose that $\langle f,g\rangle^*(U) = 1$ for all $U \in L\oplus L$ with $D_L \subseteq \mathfrak{o}(U)$ and let C be a cover of L. By Lemma 2.3.7 (2) it follows that $D_L \subseteq \mathfrak{o}(\widetilde{C})$ and $\langle f,g\rangle^*(\widetilde{C}) = \bigvee\{f^*(a) \wedge g^*(a) \mid a \in C\}$. By assumption with $U = \widetilde{C}$, one has $\bigvee\{f^*(a) \wedge g^*(a) \mid a \in C\} = 1$. Thus the pair (f^*, g^*) respects covers. □

Lemma 2.3.11. *If a pair of frame homomorphisms is bounded below, then it respects covers.*

Proof. Let C be a cover of L and let $h, k\colon L \to M$ be frame homomorphisms such that there is another frame homomorphism $h'\colon L \to M$ with $h \geq h' \leq k$. Then

$$\bigvee_{a\in C} h(a) \wedge k(a) \geq \bigvee_{a\in C} h'(a) = h'(1) = 1.$$ □

Now we are able to prove the main results in this section. On the one hand, combining Theorem 2.3.8 and Lemma 2.3.10 we obtain our Dowker-Strauss-type characterization for $\mathcal{F}$-separatedness:

Theorem 2.3.12. *A frame L is $\mathcal{F}$-separated if and only if no pair of distinct frame homomorphisms with domain L respects covers.*

On the other hand, Lemmas 2.3.4 and 2.3.11 immediately yield

Corollary 2.3.13. *An $\mathcal{F}$-separated frame is T_U.*

Remarks 2.3.14. (1) Observe that Theorem 2.3.12 (parallel to Theorem 2.3.2) does not require any knowledge of localic products.

(2) We emphasize that the concepts are not (and are not expected to be) order-theoretically dual (in Definition 2.3.1 we only consider disjoint *pairs*, whereas in Definition 2.3.9 the joins are possibly infinite). When one works with frames and frame homomorphisms, the relevant operations are finite meets and arbitrary joins and, accordingly, the results are "dual to the extent that they can be".

(3) Regarding Corollary 2.3.13, it is now apparent that $\mathcal{F}$-separatedness implies T_U for exactly the same reason as (sH) implies T_U (one just replaces bounded above by bounded below). Since fitness implies $\mathcal{F}$-separatedness (see Proposition 2.2.6), in particular we obtain a new proof of the fact that fitness implies T_U due to Isbell [72, 4.4].

2.4 Relaxed morphisms

In this section, we shall show that there is another parallel between the strong Hausdorff property and $\mathcal{F}$-separatedness which involves certain relaxed frame homomorphisms.

For that purpose, we first recall the notion of *weak homomorphism* introduced in [30]. Precisely, a mapping $h\colon L \to M$ between frames is a *weak homomorphism* if

(i) it is a morphism in Sup — i.e., a join preserving map,

(ii) $h(1) = 1$, and

(iii) it preserves disjoint pairs — i.e., if $a \wedge b = 0$ in L then $h(a) \wedge h(b) = 0$.

Furthermore, a frame L satisfies *property* (W) if

$$\text{each weak homomorphism } h\colon L \to M \text{ is a frame homomorphism.} \tag{W}$$

Among other results, Banaschewski and Pultr proved in [30] that a frame is strongly Hausdorff if and only if it is T_U and satisfies property (W). Our goal in the following subsections is to obtain a parallel result by exploiting the preframe approach to frame coproducts (cf. Subsection 2.2.5).

2.4.1 Almost homomorphisms and the property (A)

We will say that a mapping $h\colon L \to M$ between frames is an *almost homomorphism* if

(1) it is a morphism in PreFrm — i.e., it preserves finite meets (including the top) and directed joins,

(2) $h(0) = 0$, and

(3) it preserves covers — i.e., if C is a cover of L then $h[C]$ is a cover of M.

We say that a frame L satisfies *property* (A) if

$$\text{each almost homomorphism } h\colon L \to M \text{ is a frame homomorphism.} \tag{A}$$

Remark 2.4.1. Since a join of a family $B \subseteq L$ can be expressed as the directed join of the joins of finite subsets of B, an almost homomorphism $h\colon L \to M$ will be a frame homomorphism if and only if it preserves binary joins.

We begin by noting that under property (A), the converse of Lemma 2.3.11 holds as well:

Proposition 2.4.2. *Let L be a frame satisfying property* (A)*. If a pair of frame homomorphisms with domain L respects covers, then it is bounded below.*

Proof. Let $h,k\colon L \to M$ be frame homomorphisms and suppose that the pair (h,k) respects covers. Consider the "pointwise meet" mapping

$$h \wedge k\colon L \to M$$

given by $a \mapsto h(a) \wedge k(a)$. It obviously preserves finite meets. Next, if $D \subseteq L$ is directed then $(h \wedge k)(\bigvee D) = \bigvee\{h(a) \wedge k(b) \mid a,b \in D\} = \bigvee\{(h \wedge k)(c) \mid c \in D\}$ and so $h \wedge k$ is a preframe homomorphism. Since $(h \wedge k)(0) = 0$ and the pair respects covers, it is an almost homomorphism, and by property (A) a frame homomorphism. As $h \geq h \wedge k \leq k$, the pair is bounded below. □

The following corollary is a consequence of Lemma 2.3.4, Theorem 2.3.12 and Proposition 2.4.2.

Corollary 2.4.3. *A frame satisfying T_U and property* (A) *is $\mathcal{F}$-separated.*

The implication of the last corollary is actually an equivalence. Note that we have already shown that $\mathcal{F}$-separatedness implies T_U (cf. Corollary 2.3.13). Showing that $\mathcal{F}$-separatedness implies property (A) is more involved, and it makes use of the "preframe tensor" viewpoint of the frame coproducts, as we shall see next.

2.4.2 $\mathcal{F}$-separatedness implies property (A)

We need a series of lemmas in order to show the main result.

Lemma 2.4.4. *Let L be a frame and define $\tau_L \colon L\oplus L \to L\oplus L$ by setting*

$$\tau_L(U) = \bigvee\{\, V \to U \mid D_L \subseteq \mathfrak{o}(V)\,\}$$

for each $U \in L\oplus L$. Then

(1) *τ_L is a prenucleus with $\tau_L(U) = U$ if and only if $U \in \mathring{D}_L$ — i.e., $\overline{\tau_L} = \nu_{\circ L}$ (the nuclear reflection of τ_L is the nucleus associated to the sublocale $\mathring{D}_L$);*

(2) *If $\phi \colon L\oplus L \to M\oplus M$ is a preframe homomorphism such that for each $U \in L\oplus L$ with $D_L \subseteq \mathfrak{o}(U)$ one has $D_M \subseteq \mathfrak{o}(\phi(U))$, then $\phi \circ \tau_L \leq \tau_M \circ \phi$.*

Proof. (1) Obviously τ_L is monotone and by the Heyting fact (H3) one has that $U \leq \tau_L(U)$, that is, τ_L is increasing. Now, for each $U, U' \in L\oplus L$ we have

$$\tau_L(U) \wedge \tau_L(U') = \bigvee_{D_L \subseteq \mathfrak{o}(V\wedge V')} (V \to U) \wedge (V' \to U') \leq \bigvee_{D_L \subseteq \mathfrak{o}(V\wedge V')} (V \wedge V') \to (U \wedge U') \leq \tau_L(U \wedge U')$$

and since the other inequality follows by monotonicity, τ_L is a prenucleus. Moreover, $U = \tau_L(U)$ if and only if for every $V \in L\oplus L$ with $D_L \subseteq \mathfrak{o}(V)$ one has $U = V \to U$, that is, $U \in \mathfrak{o}(V)$.

(2) Let $U \in L\oplus L$. The join $\bigvee\{\, V \to U \mid D_L \subseteq \mathfrak{o}(V)\,\}$ is obviously directed and since ϕ preserves finite meets one has

$$\phi(\tau_L(U)) = \bigvee_{D_L \subseteq \mathfrak{o}(V)} \phi(V \to U) \leq \bigvee_{D_L \subseteq \mathfrak{o}(V)} \phi(V) \to \phi(U).$$

by an application of (H5). Now let $V \in L\oplus L$ with $D_L \subseteq \mathfrak{o}(V)$. By hypothesis $D_M \subseteq \mathfrak{o}(\phi(V))$ and so $\phi(V) \to \phi(U) \leq \tau_M(\phi(U))$. Hence $\phi(\tau_L(U)) \leq \tau_M(\phi(U))$. □

Lemma 2.4.5. *Let $h \colon L \to M$ be an almost homomorphism and $U \in L\oplus L$. If $D_L \subseteq \mathfrak{o}(U)$ then $D_M \subseteq \mathfrak{o}((h \parr h)(U))$.*

Proof. By Lemma 2.3.7 (1) we have to prove that if $\{\, a \wedge b \mid (a,b) \in U\,\}$ is a cover of L, then $\{\, u \wedge v \mid (u,v) \in (h \parr h)(U)\,\}$ is a cover of M.

Let $(a,b) \in U$ (that is, $a \oplus b \subseteq U$). By the formulas in Subsection 2.2.5 and since h preserves finite meets and the bottom element,

$$h(a) \oplus h(b) = (h(a) \parr 0) \wedge (0 \parr h(b)) = (h \parr h)((a \parr 0) \wedge (0 \parr b)) = (h \parr h)(a \oplus b) \subseteq (h \parr h)(U),$$

that is, $(h(a), h(b)) \in (h \parr h)(U)$. Finally, since h preserves covers we have

$$\bigvee\{\, u \wedge v \mid (u,v) \in (h \parr h)(U)\,\} \geq \bigvee\{\, h(a) \wedge h(b) \mid (a,b) \in U\,\} = \bigvee\{\, h(a \wedge b) \mid (a,b) \in U\,\} = 1. \qquad \square$$

Lemma 2.4.6. *If $h \colon L \to M$ is an almost homomorphism then $\nu_{\circ M} \circ (h \parr h) \circ \nu_{\circ L} = \nu_{\circ M} \circ (h \parr h)$.*

Proof. Set
$$Q := \{\, \lambda \in \mathsf{pN}(L \oplus L) \mid (h \parr h) \circ \lambda \le \nu_{\circ M} \circ (h \parr h) \,\}$$
and observe that Q is an inductive subset as $1_L \in Q$ (because $\nu_{\circ M}$ is increasing) and it is closed under directed joins (because directed joins of prenuclei are computed pointwise and $h \parr h$ preserves directed joins).

Let $\lambda \in Q$. By Lemmas 2.4.5 and 2.4.4 (2) we have $(h \parr h) \circ \tau_L \le \tau_M \circ (h \parr h)$. Moreover, by Lemma 2.4.4 (1) we have $\tau_M \circ \nu_{\circ M} \le \nu_{\circ M} \circ \nu_{\circ M} = \nu_{\circ M}$. Hence
$$(h \parr h) \circ \tau_L \circ \lambda \le \tau_M \circ (h \parr h) \circ \lambda \le \tau_M \circ \nu_{\circ M} \circ (h \parr h) \le \nu_{\circ M} \circ (h \parr h)$$
— i.e., $\tau_L \circ \lambda \in Q$. By the induction principle in Theorem 2.2.8 and Lemma 2.4.4 (1) again, one concludes that $\overline{\tau_L} = \nu_{\circ L} \in Q$, that is, $(h \parr h) \circ \nu_{\circ L} \le \nu_{\circ M} \circ (h \parr h)$. Finally
$$\nu_{\circ M} \circ (h \parr h) \circ \nu_{\circ L} \le \nu_{\circ M} \circ \nu_{\circ M} \circ (h \parr h) = \nu_{\circ M} \circ (h \parr h) \le \nu_{\circ M} \circ (h \parr h) \circ \nu_{\circ L}. \quad \square$$

Lemma 2.4.7. *Let L be an $\mathcal{F}$-separated frame. Then $\nu_\circ(a \parr b) = \nu_\circ((a \vee b) \parr 0)$ for any $a, b \in L$.*

Proof. If L is $\mathcal{F}$-separated then $\mathring{D}_L = D_L$, hence there is an isomorphism α making the diagram
$$\begin{tikzcd} L \oplus L \arrow[r, two heads, "\delta"] \arrow[d, two heads, "\nu_\circ"'] & L \arrow[dl, "\alpha"] \\ D_L & \end{tikzcd}$$
commutative. Let $a, b \in L$. By the formulas in Subsection 2.2.5 we have
$$\delta(a \parr b) = \delta((a \oplus 1) \vee (1 \oplus b)) = a \vee b = \delta(((a \vee b) \oplus 1) \vee (1 \oplus 0)) = \delta((a \vee b) \parr 0).$$
Hence $\nu_\circ(a \parr b) = \alpha(\delta(a \parr b)) = \alpha(\delta(((a \vee b) \parr 0))) = \nu_\circ((a \vee b) \parr 0)$. $\square$

We are now in position to prove the main result of this subsection.

Theorem 2.4.8. *Every $\mathcal{F}$-separated frame satisfies property* (A).

Proof. Let L be $\mathcal{F}$-separated and let $h \colon L \to M$ be an almost homomorphism. First, D_M is a sublocale of $\mathring{D}_M$ so there is a frame homomorphism β such that the diagram
$$\begin{tikzcd} M \oplus M \arrow[r, two heads, "\delta_M"] \arrow[d, two heads, "\nu_{\circ M}"'] & M \\ \mathring{D}_M \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes. Now, let $a,b \in M$. From the formulas in Subsection 2.2.5, Lemma 2.4.6 and Lemma 2.4.7 we obtain

$$\begin{aligned}\nu_{\circ M}((h(a)\oplus 1)\vee(1\oplus h(b))) &= \nu_{\circ M}(h(a)\parr h(b)) = (\nu_{\circ M}\circ(h\parr h))(a\parr b) = (\nu_{\circ M}\circ(h\parr h)\circ\nu_{\circ L})(a\parr b)\\ &= (\nu_{\circ M}\circ(h\parr h)\circ\nu_{\circ L})((a\vee b)\parr 0) = (\nu_{\circ M}\circ(h\parr h))((a\vee b)\parr 0)\\ &= \nu_{\circ M}(h(a\vee b)\parr 0) = \nu_{\circ M}((h(a\vee b)\oplus 1)\vee(1\oplus 0)).\end{aligned}$$

Hence $h(a)\vee h(b) = \delta_M((h(a)\oplus 1)\vee(1\oplus h(b))) = \delta_M((h(a\vee b)\oplus 1)\vee(1\oplus 0)) = h(a\vee b)$. By Remark 2.4.1, it follows that h is a frame homomorphism. □

By Proposition 2.2.6 and Theorem 2.4.8 we also have the following:

Corollary 2.4.9. *Every fit frame satisfies property* (A).

At this point, we have all the necessary results for a characterization of $\mathcal{F}$-separatedness. One half of the following corollary is a consequence of Corollary 2.4.3 and the other one follows by Corollary 2.3.13 and Theorem 2.4.8.

Corollary 2.4.10. *A frame is $\mathcal{F}$-separated if and only if it is T_U and satisfies property* (A).

Remarks 2.4.11. (1) Closer scrutiny of the proofs of the pair of results

$$(\mathcal{F}\,\mathrm{sep}) \equiv (\mathrm{A}) + (T_U) \quad \text{and} \quad (\mathrm{sH}) \equiv (\mathrm{W}) + (T_U)$$

reveals that the parallel is in fact deeper.

In [30] the crucial result was that $(a\oplus b)\vee d_L = ((a\wedge b)\oplus 1)\vee d_L$ for all $a,b\in L$, where $d_L = \bigwedge D_L$, and it was instrumental for proving that h preserves *binary meets*. Now realize that $(-)\vee d_L$ is the nucleus ν_- on $L\oplus L$ associated with the closure $\overline{D_L}$, hence it amounted to

$$\nu_-(a\oplus b) = \nu_-((a\wedge b)\oplus 1) \qquad \text{for all } a,b\in L.$$

Now the crucial step in Theorem 2.4.8 is that

$$\nu_\circ(a\parr b) = \nu_\circ((a\vee b)\parr 0) \qquad \text{for all } a,b\in L,$$

and it is instrumental for proving that h preserves *binary joins*. Of course, the proof for the "other closure" is technically more involved, as the corresponding nucleus does not have an explicit description (whereas for the usual closure it has a very simple one).

(2) Since regularity implies fitness, it also implies $\mathcal{F}$-separatedness, and therefore regular frames satisfy property (A). It might be worth showing that a direct proof of the latter fact is much easier. Let $h\colon L\to M$ be an almost homomorphism with L regular. Let $a,b\in L$. We first note that, by regularity, $a\vee b = \bigvee\{x\vee y \mid x\prec a,\ y\prec b\}$, and since $\{x\vee y\mid x\prec a,\ y\prec b\}$ is a

directed set, we have

$$h(a \vee b) = h(\bigvee\{x \vee y \mid x \prec a,\ y \prec b\}) = \bigvee\{h(x \vee y) \mid x \prec a,\ y \prec b\}.$$

Now let $x, y \in L$ such that $x \prec a$ and $y \prec b$. Then $(x \vee y)^* \vee (a \vee b) \geq (x^* \vee a) \wedge (y^* \vee b) = 1$ and thus $h((x \vee y)^*) \vee h(a) \vee h(b) = 1$ (since h preserves covers). Hence

$$h(x \vee y) = h(x \vee y) \wedge \Big(h((x \vee y)^*) \vee h(a) \vee h(b)\Big) \leq h(a) \vee h(b).$$

It follows that $h(a \vee b) \leq h(a) \vee h(b)$ and we conclude that h preserves binary joins. Finally, by Remark 2.4.1, h is a frame homomorphism.

2.4.3 Sufficient conditions: normality versus extremal disconnectedness

In this subsection we shall use earlier work by Banaschewski and Pultr in order to establish dual sufficient conditions for properties (W) and (A). Recall that elements a, b of a distributive lattice L are *normally separated* ([30]) if

$$\exists u, v \in L \text{ such that } u \wedge v = 0,\ a \leq u \vee b \text{ and } b \leq a \vee v. \tag{NS}$$

Dually, we will also say that elements a, b are *extremally separated* if

$$\exists u, v \in L \text{ such that } u \vee v = 1,\ a \wedge v \leq b \text{ and } u \wedge b \leq a. \tag{ES}$$

Observe that normal separation and extremal separation are dual to each other in the sense that $a, b \in L$ are normally separated in L if and only if they are extremally separated in L^{op}.

Remark 2.4.12. Elements $a, b \in L$ are extremally separated if and only if $(a \to b) \vee (b \to a) = 1$.

The notions of normal separation (resp. extremal separation) are related with normality (resp. extremal disconnectedness), see for example [64, Corollaries 5.3 and 5.5]):

Proposition 2.4.13. *The following conditions are equivalent for a frame L:*

(i) *L is hereditarily normal;*

(ii) *Every open sublocale of L is normal;*

(iii) *Every $a, b \in L$ are normally separated.*

Proposition 2.4.14. *The following conditions are equivalent for a frame L:*

(i) *L is hereditarily extremally disconnected;*

(ii) *Every closed sublocale of L is extremally disconnected;*

(iii) *Every $a, b \in L$ are extremally separated.*

The following result is due to Banaschewski and Pultr.

Lemma 2.4.15 ([30, Proposition 3.2]). *Let L and M be distributive lattices. Then an $h\colon L \to M$ preserving finite joins and meets of disjoint pairs, also preserves meets of normally separated pairs.*

Since the result holds at the level of distributive lattices, it can obviously be dualized as follows; we include a proof for the sake of completeness:

Lemma 2.4.16. *Let L and M be distributive lattices. Then an $h\colon L \to M$ preserving finite meets and joins of covering pairs, also preserves joins of extremally separated pairs.*

Proof. Let $a, b \in L$ be extremally separated and pick $u, v \in L$ as in (ES). Since h preserves meets of joins of covering pairs, one has $h(u) \vee h(v) = 1$, and we therefore have

$$h(a \vee b) = h(a \vee b) \wedge (h(u) \vee h(v)) \leq h((a \vee b) \wedge u) \vee h((a \vee b) \wedge v) \leq h(a) \vee h(b).$$

The other inequality is trivial. □

Combining Proposition 2.4.13 with Lemma 2.4.15 one obtains the following:

Corollary 2.4.17 ([30, Proposition 4.1]). *Every hereditarily normal frame has property* (W).

Similarly, Lemma 2.4.16, Proposition 2.4.14 and Remark 2.4.1 yield the following:

Corollary 2.4.18. *Every hereditarily extremally disconnected frame has property* (A).

As we have just seen, the parallel or "duality" between properties (W) and (A) studied in this chapter is also connected to the parallel between (the herditary variants of) normality and extremal disconnectedness.

2.5 Downset frames, hereditary normality and extremal disconnectedness

For the class of downset frames, the implications in Corollary 2.4.17 and Corollary 2.4.18 are actually equivalences. Once again, one half was proved in [30] and we shall prove the dual result. If X is a poset and $a \in X$, we denote ${\downarrow}a = \{b \in X \mid b \leq a\}$ and ${\uparrow}a = \{b \in X \mid b \geq a\}$. Moreover, we write $\mathsf{Dwn}(X)$ for the frame of all the downsets of X.

Now, we recall (cf. [18]) that X is a *forest* if whenever $a, b \in X$ are incomparable, one has ${\downarrow}a \cap {\downarrow}b = \varnothing$.

Proposition 2.5.1 ([30, Proposition 6.1]). *The following are equivalent for a poset X:*

(i) *X is a forest;*

(ii) $\mathsf{Dwn}(X)$ *is hereditarily normal;*

(iii) $\mathsf{Dwn}(X)$ *has property* (W).

We now prove a dual result. Let X be a poset and observe that $\mathsf{Dwn}(X)$ is hereditarily extremally disconnected if and only if $\mathsf{Dwn}(X)^{op} \cong \mathsf{Dwn}(X^{op})$ is hereditarily normal — i.e., if X^{op} is a forest. We shall say that X is a *coforest* if X^{op} is a forest — i.e., $\uparrow a \cap \uparrow b = \varnothing$ whenever $a, b \in X$ are incomparable.

Proposition 2.5.2. *The following are equivalent for a poset X:*

(i) *X is a coforest;*

(ii) $\mathsf{Dwn}(X)$ *is hereditarily extremally disconnected;*

(iii) $\mathsf{Dwn}(X)$ *has property* (A).

Proof. (i) $\iff$ (ii) follows by the comments before the statement, and (ii) $\Longrightarrow$ (iii) by Corollary 2.4.18. Hence we are only left with the task of showing that (iii) implies (i). Let $a, b \in X$ be incomparable. Define a map $h\colon \mathsf{Dwn}(X) \longrightarrow \mathbf{2} = \{0,1\}$ by

$$h(U) = \begin{cases} 1 & \text{if } a, b \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h(\downarrow a \cup \downarrow b) = 1$ while, by incomparability, $h(\downarrow a) \vee h(\downarrow b) = 0$, hence h is not a frame homomorphism.

On the other hand, h obviously preserves finite meets and the bottom element. Let us check that it preserves also directed joins. Let $\{U_i\}_{i\in I}$ be a directed family of downsets and suppose that $a, b \in \bigcup_i U_i$. Then there are $i_1, i_2 \in I$ such that $a \in U_{i_1}$ and $b \in U_{i_2}$. By directedness there is an $i \in I$ such that $a, b \in U_i$. This shows that $h(\bigcup_i U_i) = \bigvee_i h(U_i)$. Thus h is a preframe homomorphism.

Since $\mathsf{Dwn}(X)$ has property (A) it follows that h does not preserve covers — i.e., there exists a family $\{U_i\}_{i\in I}$ of downsets of $\mathsf{Dwn}(X)$ such that $\bigcup_i U_i = X$ and $\bigvee_i h(U_i) = 0$. Hence for each $i \in I$ either $a \notin U_i$ or $b \notin U_i$ and so $\uparrow a \cap \uparrow b = \bigcup_i \uparrow a \cap \uparrow b \cap U_i = \varnothing$. □

Some further equivalent conditions will be given in Chapter 4, see Corollary 4.4.8.

2.6 Relation with other separation properties

Recall that fitness implies $\mathcal{F}$-separatedness (Proposition 2.2.6). In light of the formal similarity of both notions (fitness means that every sublocale is fitted whereas $\mathcal{F}$-separatedness means that just the diagonal is fitted), it is natural to ask whether both properties are equivalent. It was left as an open problem by Picado and Pultr in [96, 2.7] and in this section we shall answer it negatively. We first briefly recall a few facts about simple extensions [24, Appendix] (for a detailed account see also [97, IV 4]).

2.6.1 Simple extensions

In this section, Y will be a topological space and $X \subseteq Y$ a fixed subspace. One defines

$$E_{X,Y}$$

to be the set Y endowed with the coarsest topology finer that $\Omega(Y)$ and the *X-included topology* $\{A \subseteq Y \mid X \subseteq A\} \cup \{\varnothing\}$, namely

$$\{U \cap A \mid U \in \Omega(Y),\ X \subseteq A,\ A \smallsetminus X \subseteq U\}.$$

We speak of $E_{X,Y}$ as a *simple extension* of X. Banaschewski used this construction for providing examples of non-fit strongly Hausdorff frames:

Theorem 2.6.1 ([24, Corollary A.8]). *If Y is a regular space and both X and $Y \smallsetminus X$ are dense in Y then $E_{X,Y}$ is strongly Hausdorff but not fit.*

2.6.2 A counterexample

We shall now use the theory of simple extensions in order to provide an example of an $\mathcal{F}$-separated frame which is not fit. We first need several lemmas:

Lemma 2.6.2. *Let $A, B \subseteq Y$ such that $A \cap B = X$. Then A and B are extremally separated in $\Omega(E_{X,Y})$.*

Proof. We check the identity in Remark 2.4.12. Since $X \subseteq B \cup (Y \smallsetminus A)$, the subset $B \cup (Y \smallsetminus A)$ is open in $E_{X,Y}$ and so $A \to B = \mathrm{int}_{E_{X,Y}}(B \cup (Y \smallsetminus A)) = B \cup (Y \smallsetminus A)$. Symetrically $B \to A = A \cup (Y \smallsetminus B)$ and hence $(A \to B) \cup (B \to A) = A \cup B \cup (Y \smallsetminus B) \cup (Y \smallsetminus A) = Y$. □

Lemma 2.6.3. *If $h\colon \Omega(E_{X,Y}) \to L$ is an almost homomorphism and $A, B \supseteq X$, then $h(A \cup B) = h(A) \vee h(B)$.*

Proof. By Lemma 2.6.2 and Lemma 2.4.16 one has the non-trivial inequality — i.e., $h(A \cup B) = h(A \cup (X \cup (B \smallsetminus A))) = h(A) \vee h(X \cup (B \smallsetminus A)) \leq h(A) \vee h(B)$. □

Lemma 2.6.4. *Let $\Omega(Y)$ satisfy property* (A) *and $h\colon \Omega(E_{X,Y}) \to M$ be an almost homomorphism. Further, let $U, V \in \Omega(Y)$ and $A \supseteq X$. Then, $h((U \cap A) \cup (V \cap A)) = h(U \cap A) \vee h(V \cap A)$.*

Proof. Consider the subframe embedding $\iota\colon \Omega(Y) \subseteq \Omega(E_{X,Y})$. The composite $h \circ \iota\colon \Omega(Y) \to M$ is obviously an almost homomorphism and Y satisfies property (A), hence it is a frame homomorphism. Thus $h(U \cup V) = h(U) \vee h(V)$ and so $h((U \cap A) \cup (V \cap A)) = h((U \cup V) \cap A) = h(U \cup V) \wedge h(A) = (h(U) \vee h(V)) \wedge h(A) = h(U \cap A) \cap h(V \cap A)$, as desired. □

Lemma 2.6.5. *Let $\Omega(Y)$ satisfy property* (A)*, Y be T_1 and $h\colon \Omega(E_{X,Y}) \to M$ be an almost homomorphism. Further, let $U, V \in \Omega(Y)$ and $A, B \supseteq X$ such that $A \smallsetminus X \subseteq U$ and $B \smallsetminus X \subseteq V$ and both $A \smallsetminus X$ and $B \smallsetminus X$ are finite. Then $h((U \cap A) \cup (V \cap B)) = h(U \cap A) \vee h(V \cap B)$.*

Proof. Let $U_1 := U \cap A \cap B$, $V_1 := V \cap A \cap B$, $U_2 := (U \smallsetminus (B \smallsetminus X)) \cap (X \cup (A \smallsetminus B) \cup (B \smallsetminus A))$ and $V_2 := (V \smallsetminus (A \smallsetminus X)) \cap (X \cup (A \smallsetminus B) \cup (B \smallsetminus A))$. Obviously $U_1, V_1 \in \Omega(E_{X,Y})$ and since $A \smallsetminus X$ and $B \smallsetminus X$ are finite and Y is T_1, also $U_2, V_2 \in \Omega(E_{X,Y})$. Now since $A \smallsetminus X \subseteq U$ and $B \smallsetminus X \subseteq V$, it follows that $(U \cap A) \cup (V \cap B) = (U \cup V) \cap (A \cup B)$ and that $(U \cup V) \cap (X \cup (A \smallsetminus B) \cup (B \smallsetminus A)) = U_2 \cup V_2$. Therefore,

$$\begin{aligned} h((U \cap A) \cup (V \cap B)) &= h((U \cup V) \cap ((A \cap B) \cup X \cup (A \smallsetminus B) \cup (B \smallsetminus A))) \\ &= h(U \cup V) \wedge (h(A \cap B) \vee h(X \cup (A \smallsetminus B) \cup (B \smallsetminus A))) = h(U_1 \cup V_1) \vee h(U_2 \cup V_2), \end{aligned}$$

where we have used Lemma 2.6.3 and the fact that h preserves finite meets.

Now, since both U_1 and V_1 and U_2 and V_2 clearly satisfy the additional condition in the statement of Lemma 2.6.4 and $U_1 \subseteq U \cap A$, $V_1 \subseteq V \cap B$, $U_2 = (U \smallsetminus (B \smallsetminus X)) \cap (X \cup (A \smallsetminus B)) \subseteq U \cap A$ and $V_2 = (V \smallsetminus (A \smallsetminus X)) \cap (X \cup (B \smallsetminus A)) \subseteq V \cap B$, we obtain the non-trivial inequality — i.e., $h((U \cap A) \cup (V \cap B)) = h(U_1) \vee h(V_1) \vee h(U_2) \vee h(V_2) \leq h(U \cap A) \vee h(V \cap B)$. □

After all these preliminary results, we can prove the main theorems.

Theorem 2.6.6. *Let Y be a T_1-space such that $\Omega(Y)$ satisfies property* (A). *Then so does $\Omega(E_{X,Y})$.*

Proof. Let $h \colon \Omega(E_{X,Y}) \to M$ be an almost homomorphism. We will show that it preserves binary joins. Let $U, V \in \Omega(Y)$ and $A, B \supseteq X$ such that $A \smallsetminus X \subseteq U$ and $B \smallsetminus X \subseteq V$. Then, for any finite subsets $F \subseteq A \smallsetminus X$ and $G \subseteq B \smallsetminus X$, by Lemma 2.6.5 we have

$$h((U \cap (X \cup F)) \cup (V \cap (X \cup G)) = h(U \cap (X \cup F)) \vee h(V \cap (X \cup G)) \leq h(U \cap A) \vee h(V \cap B).$$

Since $(U \cap A) \cup (V \cap B) = \bigcup \{(U \cap (X \cup F)) \cup (V \cap (X \cup G)) \mid F \subseteq A \smallsetminus X \text{ and } G \subseteq B \smallsetminus X \text{ are finite}\}$, and this union is directed, it follows that $h((U \cap A) \cup (V \cap B)) \leq h(U \cap A) \vee h(V \cap B)$. □

Theorem 2.6.7. *There exists a strongly Hausdorff $\mathcal{F}$-separated spatial frame which is not fit.*

Proof. Pick a regular T_0-space Y with a subspace X such that both X and $Y \smallsetminus X$ dense in Y (e.g. the real line with the subspace of rationals). By Theorem 2.6.1 the simple extension $E_{X,Y}$ is strongly Hausdorff but not fit. Moreover, in view of Theorem 2.6.6, $\Omega(E_{X,Y})$ also satisfies property (A). Now, a strongly Hausdorff frame is T_U (Corollary 2.3.6). Therefore, $\Omega(E_{X,Y})$ satisfies property (A) and is T_U, so it is $\mathcal{F}$-separated by Corollary 2.4.3. □

We now know that $\mathcal{F}$-separatedness is strictly weaker than fitness, and therefore it makes sense to compare ($\mathcal{F}$-sep) with other known weaker variants of fitness. In particular there is the *subfitness* (arguably even more important than fitness itself, see [97, Chapter II])

$$a \nleq b \implies \exists c \in L \text{ such that } a \vee c = 1 \neq b \vee c \tag{sfit}$$

or the weaker *weak subfitness*

$$a \neq 0 \implies \exists c \in L \text{ such that } a \vee c = 1 \neq c \tag{wsfit}$$

or, finally, Picado and Pultr's *prefitness*

$$a \neq 0 \quad \Longrightarrow \quad \exists c \in L \text{ such that } a \vee c = 1 \neq c = c^{**} \tag{pfit}$$

(see [92] or [97] for a comprehensive treatment of these separation properties; in Section 2.9 we shall meet a new one also given by a first-order formula). Now we have the following:

Proposition 2.6.8. *None of the properties* (sfit), (wsfit) *or* (pfit) *coincides with* ($\mathcal{F}$sep).

Proof. None of them is hereditary (see [92]) while each c-separation is even closed under monomorphisms (recall Theorem 2.2.2). ◻

2.7 Density and codensity

It is well known that dense frame homomorphisms are monomorphisms in the category of strongly Haudorff frames (see for example [91, Proposition V 2.5.3]). There is also a "dual" result for $\mathcal{F}$-separated frames. Recall that a frame homomorphism h is *codense* if for any $a \in L$,

$$h(a) = 1 \quad \Longrightarrow \quad a = 1.$$

Proposition 2.7.1. *Let $h\colon L \to M$ be a codense frame homomorphism, and $g, k\colon N \to L$ with N an $\mathcal{F}$-separated frame. If $h \circ g = h \circ k$ then $g = k$.*

Proof. Let $q\colon L \twoheadrightarrow S$ be the coequalizer of g, k in Frm. Then there is a unique u making the diagram

$$\begin{tikzcd} N \arrow[r, shift left, "k"] \arrow[r, shift right, "g"'] & L \arrow[r, "q"] \arrow[dr, "h"'] & S \arrow[d, dashed, "u"] \\ & & M \end{tikzcd}$$

commutative. Because of Theorem 2.2.5 (1), since N is $\mathcal{F}$-separated, S corresponds to a fitted sublocale of L, say $S = \bigcap_{i \in I} \mathfrak{o}(a_i)$. Now, by Lemma 1.2.2, for each $i \in I$ one has $q(a_i) = 1$ and hence $h(a_i) = 1$. By co-density, it follows $a_i = 1$. Thus $S = L$ — i.e., $q = 1_L$ is the identity and $g = k$. ◻

Corollary 2.7.2. *Codense frame homomorphisms are monomorphisms in the category of $\mathcal{F}$-separated frames.*

Remark 2.7.3. In particular, we see that codense frame homomorphisms are monomorphisms in the category of fit (or regular) frames. But this is not interesting, as it is well known that the following stronger property holds (cf. [91, Corollary V 1.6.2]):

> *a frame is subfit if and only if every frame homomorphism $f\colon L \to M$ is one-to-one (i.e., it is a monomorphism in the whole* Frm*).*

2.8 Singly generated extensions

Recall that $\mathcal{P}$-separation with respect to a closure operator is closed under monomorphisms (Theorem 2.2.5). In frame language, we have in particular that for a strongly Hausdorff (resp. $\mathcal{F}$-separated) frame L, if $L \to M$ is an epimorphism then M is also strongly Hausdorff (resp. $\mathcal{F}$-separated). It is therefore a natural question to determine when the converse implication holds; and in the strong Haudorff case it was shown in [24, Proposition 4.10] (originally due to Xiangdong) that for the class of singly generated frame extensions it does.

A frame M is a *singly generated extension* of a frame L if L is a subframe of M and M can be generated by L together with some $c \in M$. In this case, one usually denotes $M = L[c]$. Observe that for each $x \in M$, there are $a_x, b_x \in L$ with $x = a_x \vee (c \wedge b_x)$.

Proposition 2.8.1. *Let L be $\mathcal{F}$-separated and let $j\colon L \rightarrowtail M = L[c]$ be the subframe embedding corresponding to the singly generated extension by $c \in M$. Then M is $\mathcal{F}$-separated if and only if j is epic.*

Proof. As we have already noted, the "if" part always holds. Assume now that M is $\mathcal{F}$-separated and let $h,k\colon M \to N$ with $h \circ j = k \circ j$. We want to check that $h = k$. For that, we shall prove that the pair (h,k) respects covers (Theorem 2.3.12). Let C be a cover of M. For each $x \in C$, there are $a_x, b_x \in L$ with $x = a_x \vee (c \wedge b_x)$. Set $a := \bigvee_{x\in C} a_x$ and $b := \bigvee_{x \in C} b_x$. Then $a \vee (c \wedge b) = \bigvee C = 1$. Now, since $h \circ j = k \circ j$, one has $h(a_x) = k(a_x)$ and $h(b_x) = k(b_x)$ for all $x \in C$, and so

$$\begin{aligned}
\bigvee_{x\in C} h(x) \wedge k(x) &= \bigvee_{x\in C} \Big(h(a_x) \vee (h(c) \wedge h(b_x))\Big) \wedge \Big(k(a_x) \vee (k(c) \wedge k(b_x))\Big) \\
&= \bigvee_{x\in C} h(a_x) \vee (h(c) \wedge k(c) \wedge h(b_x)) = h(a) \vee (h(c) \wedge k(c) \wedge h(b)) \\
&= \Big(h(a) \vee (h(c) \wedge h(b))\Big) \wedge \Big(h(a) \vee (k(c) \wedge h(b))\Big) \\
&= h(a \vee (c \wedge b)) \wedge k(a \vee (c \wedge b)) = h(1) \wedge k(1) = 1.
\end{aligned}$$

Thus (h,k) respects covers and so $h = k$. □

2.9 A new first-order separation formula, and some of its properties

Besides Isbell's strong Hausdorff axiom, there a number of weaker variants of the Hausdorff axiom in the category of locales — for a comprehensive account on the topic we refer to [97, Chapter III].

Among those variants, there is one which deserves particular attention (for some justification about this, we refer to [97, p. 44] or [81, p. 192]). Following the terminology from [97], a frame L is *Hausdorff* (or, briefly, L has property (H)) if

$$1 \neq a \nleq b \implies \exists u, v \in L \text{ such that } u \nleq a,\ v \nleq b \text{ and } u \wedge v = 0. \tag{H}$$

Every strongly Hausdorff frame can be shown to be Hausdorff. Among the advantages of property (H), we have the fact that it is conservative (i.e., a space X is Hausdorff if and only if $\Omega(X)$ has property (H)) and well-behaved categorically (inherited by sublocales and closed under products).

Following the main idea of the present chapter, we might seek to find a first-order property implied by $\mathcal{F}$-separatedness which would be somehow "dual" to (H). The following theorem gives a solution to this problem:

Theorem 2.9.1. *For a frame L, the following conditions are equivalent and are all implied by $\mathcal{F}$-separatedness*:

(i) *For every $a,b \in L$ such that $1 \neq a \nleq b$, there exist $u,v \in L$ such that $u \nleq a$, $v \nleq b$ and $(u \to a) \vee (v \to b) = 1$;*

(ii) *For every $a,b \in L$ such that $1 \neq a \nleq b$, there exist $u,v \in L$ such that $a < u$, $b < v$ and $(u \to a) \vee (v \to b) = 1$;*

(iii) *For every $a,b \in L$ such that $1 \neq a \nleq b$, there exist $u,v \in L$ such that $v \leq a < u$, $v \nleq b$ and $(u \to a) \vee (v \to b) = 1$;*

(iv) *For every $a,b \in L$ such that $1 \neq a \nleq b$, there exist $u,v \in L$ such that $u \to a \neq a$, $v \to b \neq b$ and $u \vee v = 1$;*

(v) *For every $a,b \in L$ such that $1 \neq a \nleq b$, there exist $u,v \in L$ such that $a \leq u$, $b \leq v$, $u \to a \neq a$, $v \to b \neq b$ and $u \vee v = 1$;*

(vi) *For every $a,b \in L$ such that $1 \neq a \nleq b$, there exist $u,v \in L$ such that $a \leq u$, $u \to a \neq a$, $a \wedge (v \to b) \nleq b$ and $u \vee v = 1$.*

Proof. Let us start by showing that $\mathcal{F}$-separatedness implies (i). Let $1 \neq a \nleq b$. Then

$$a \parr b = \{(x,y) \in L \times L \mid x \leq a \text{ or } y \leq b\}$$

and since $(a,1) \in a \parr b$ and $(1,a) \notin a \parr b$, it follows from Lemma 1.5.2 that $a \parr b \notin D_L$. Hence $a \parr b \notin D_L = \bigcap_{D_L \subseteq \mathfrak{o}(U)} \mathfrak{o}(U)$ because L is $\mathcal{F}$-separated and so there exists a $U \in L \oplus U$ such that $D_L \subseteq \mathfrak{o}(U)$ and $a \parr b \notin \mathfrak{o}(U)$ — i.e., $\bigcap_{(x,y)\in U}((x \oplus y) \to a \parr b) \nsubseteq a \parr b$. Therefore, there is a pair $(u,v) \in L \times L$ such that for all $(x,y) \in U$, one has $(u,v) \in (x \oplus y) \to a \parr b$ but $(u,v) \notin a \parr b$. The latter means $u \nleq a$ and $v \nleq b$; while the former means that for all $(x,y) \in U$ one has $(u \wedge x) \oplus (v \wedge y) \subseteq a \parr b$, or equivalently $(u \wedge x, v \wedge y) \in a \parr b$. Hence, for each $(x,y) \in U$, one has either $u \wedge x \leq a$ or $v \wedge y \leq b$ and so $x \wedge y \leq (u \to a) \vee (v \to b)$. By Lemma 2.3.7 (1), the system $\widehat{U} = \{x \wedge y \mid (x,y) \in U\}$ is a cover of L, hence $(u \to a) \vee (v \to b) = 1$.

We now check that all the conditions are equivalent:

(i) $\Longrightarrow$ (ii): Let $1 \neq a \nleq b$. Then there are $u,v \in L$ with $u \nleq a$, $v \nleq b$ and $(u \to a) \vee (v \to b) = 1$. Set $u' := u \vee a$ and $v' := v \vee b$. Then $a < u'$, $b < v'$ and $(u' \to a) \vee (v' \to b) = (u \to a) \vee (v \to b) = 1$.

(ii) $\Longrightarrow$ (iii): Let $1 \neq a \nleq b$. Then one has $1 \neq a \nleq a \to b$ and hence there exist $u,v \in L$ such that $a < u$, $a \to b < v$ and $(u \to a) \vee (v \to (a \to b)) = 1$. Let $v' := v \wedge a$. Since $v \nleq a \to b$, one has $v' \nleq b$ (and $v' \leq a$). Moreover, by (H7), it follows that $v \to (a \to b) = (v \wedge a) \to b = v' \to b$. Hence the pair u,v' satisfies the required conditions.

(iii) $\Longrightarrow$ (iv): Let $1 \neq a \nleq b$. Then there exist $u,v \in L$ such that $v \leq a < u$, $v \nleq b$ and $(u \to a) \vee (v \to b) = 1$. Let $u' := u \to a$ and $v' := v \to b$. Then $u' \vee v' = 1$. Moreover, if $u' \to a \leq a$, then $u \leq (u \to a) \to a \leq a$ by (H9), a contradiction. Hence $u' \to a \neq a$ and similarly, $v' \to b \neq b$.

(iv) $\Longrightarrow$ (v) follows easily because $u \to a = (u \vee a) \to a$ and $v \to b = (v \vee b) \to b$; thus we may replace u (resp. v) by $u \vee a$ (resp. $v \vee b$).

(v) $\Longrightarrow$ (vi): Let $1 \neq a \nleq b$. Then $1 \neq a \nleq a \to b$ and hence there exist $u,v \in L$ such that $a \leq u$, $a \to b \leq v$, $u \to a \neq a$, $v \to (a \to b) \neq a \to b$ and $u \vee v = 1$. By (H5) and (H7) one has $a \wedge (v \to b) = a \wedge (a \to (v \to b)) = a \wedge (v \to (a \to b)) \nleq b$.

(vi) $\Longrightarrow$ (i): Let $1 \neq a \nleq b$. Then there exist $u,v \in L$ such that $a \leq u$, $u \to a \neq a$, $a \wedge (v \to b) \nleq b$ and $u \vee v = 1$. Let $u' := u \to a$ and $v' := v \to b$. Then $u' \nleq a$, $v' \nleq b$ and $(u' \to a) \vee (v' \to b) \geq u \vee v = 1$ by (H9). ◻

A frame satisfying one (and hence all) of the equivalent conditions above will be said to satisfy property (F), that is, L satisfies *property* (F) if

$$1 \neq a \nleq b \quad \Longrightarrow \quad \exists u,v \in L \text{ such that } u \nleq a,\ v \nleq b \text{ and } (u \to a) \vee (v \to b) = 1. \tag{F}$$

In what follows we study some of its basic properties.

Proposition 2.9.2. *Property* (F) *implies property* (T_1).

Proof. Let L be a frame and let $p \in L$ be a prime. Assume by contradiction that p is not maximal, i.e., $p \leq a \leq 1$ with $a \nleq p$ and $a \neq 1$. By hypothesis there exist $u,v \in L$ such that $u \nleq a$, $v \nleq p$ and $(u \to a) \vee (v \to p) = 1$. Now, since p is prime and $v \nleq p$, it follows that $v \to p = p$. Thus $1 = (u \to a) \vee p = u \to a$ (because $p \leq a \leq u \to a$) — i.e., $u \leq a$, a contradiction. ◻

Remarks 2.9.3. (1) Since subfitness does not imply property (T_1), it follows that subfitness does not imply property (F) either.

(2) Property (F) has a certain formal similarity with the strong De Morgan law ([74])

$$(a \to b) \vee (b \to a) = 1 \text{ for all } a,b \in L.$$

However, we point out that the latter does not imply property (F) (as the strong De Morgan law is equivalent to hereditary extremal disconnectedness and the latter does not imply (T_1)).

(3) Since fitness implies $\mathcal{F}$-separatedness, it also implies property (F). Now, there are a number of well-known and relatively well-understood separation properties implied by

fitness (notably, subfitness (sfit), but also prefitness (pfit) and weak subfitness (wsfit) — see [92, 97]). However, (F) is hereditary (see the following proposition) whereas none of the other properties is; hence none of them is equivalent to (F).

Proposition 2.9.4. *Property* (F) *is hereditary.*

Proof. Let L be a frame which has property (F) and let $S \subseteq L$ be a sublocale with corresponding surjection $\nu_S \colon L \twoheadrightarrow S$. We denote by $\vee^S$ (resp. $\vee$) the join in S (resp. L). Let $a, b \in S$ such that $1 \neq a \nleq b$. Since L satisfies property (F), there exist $u, v \in L$ such that $u \nleq a$, $v \nleq b$ and $(u \to a) \vee (v \to b) = 1$. Let $u' := \nu_S(u)$ and $v' := \nu_S(v)$. Then $u' \nleq a$, $v' \nleq b$ and $1 = \nu_S(u \to a) \vee^S \nu_S(v \to b) \leq (u' \to a) \vee^S (v' \to b)$. □

Remark 2.9.5. As a consequence of Proposition 2.9.4, it follows that (F) does not imply subfitness (otherwise, every sublocale of a locale satisfying (F) would be subfit; and hence the locale itself would be fit — but even $\mathcal{F}$-separation does not imply fitness, cf. Theorem 2.6.7). Combining this observation with Remarks 2.9.3 (1), we conclude that

> *subfitness and property* (F) *are not comparable.*

Finally, we move on to showing that property (F) behaves well with respect to products.

Proposition 2.9.6. *Arbitrary products of locales with property* (F) *also have property* (F).

Proof. The first part of the proof follows the same lines of that of [90, Lemma 1.9], cf. also [97, p. 45]. Let $\{L_i\}_{i\in I}$ be a family of frames satisfying property (F). Let $1 \neq V \nsubseteq W$ in $\bigoplus_{i\in I} L_i$. Pick $\boldsymbol{a} = (a_i)_{i\in I} \in V - W$. Let $\{i_1, \ldots, i_n\}$ be the set of indices such that $a_{i_j} \neq 1$ for all $j = 1, \ldots, n$. Let $\boldsymbol{a}^{(0)} := \boldsymbol{a}$ and for each $j = 1, \ldots, n$, let $\boldsymbol{a}^{(j)}$ be the element $\boldsymbol{a}$ but with all the entries in $i_1, \ldots, i_j$ replaced by 1. Since $\boldsymbol{a}^{(0)} = \boldsymbol{a} \in V$ and $\boldsymbol{a}^{(n)} = (1)_{i\in I} \notin V$, there is an $j_0 \in \{1, \ldots, n\}$ such that $\boldsymbol{a}^{(j_0-1)} \in V$ but $\boldsymbol{a}^{(j_0)} \notin V$.

For each $x \in L_{i_{j_0}}$ let $\boldsymbol{x}$ be $\boldsymbol{a}^{(j_0)}$ but with the 1 in position i_{j_0} replaced by x. Further, let $v := \bigvee\{x \in L_{i_{j_0}} \mid \boldsymbol{x} \in V\}$ and $w := \bigvee\{x \in L_{i_{j_0}} \mid \boldsymbol{x} \in W\}$. Because V and W are *cp*-ideals, one has $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$. If $v = 1$, then $\boldsymbol{a}^{(j_0)} = \boldsymbol{v} \in V$, a contradiction. Thus $v \neq 1$. Assume $v \leq w$. Now, since $\boldsymbol{a_{i_{j_0}}} = \boldsymbol{a}^{(j_0-1)} \in V$, it follows that $a_{i_{j_0}} \leq v \leq w$, and so $\boldsymbol{a} \leq \boldsymbol{w} \in W$. Since W is a downset, it follows that $\boldsymbol{a} \in W$, a contradiction. Hence $1 \neq v \nleq w$.

By condition (iv) in Theorem 2.9.1, there are $x, y \in L_{i_{j_0}}$, with $x \to v \nleq v$, $y \to w \nleq w$ and $x \vee y = 1$.

Let $x_i = 1 = y_i \in L_i$ for each $i \neq i_{j_0}$ and let $x_{i_{j_0}} = x$, $y_{i_{j_0}} = y$. Then obviously $(\oplus_i x_i) \vee (\oplus_i y_i) = 1$.

We claim that $(\oplus_i x_i) \to V \neq V$. Assume otherwise, by contradiction. Since $x \to v \nleq v$, there is a $c \in L_{i_{j_0}}$ such that $c \leq x \to v$ (i.e., $c \wedge x \leq v$) and $c \nleq v$. One obviously has $\boldsymbol{c} \wedge (x_i)_i = \boldsymbol{c \wedge x} \leq \boldsymbol{v} \in V$ and since V is a downset, we deduce that $\boldsymbol{c} \wedge (x_i)_i \in V$. It follows that $\boldsymbol{c} \in (\oplus_i x_i) \to V = V$. But if $\boldsymbol{c} \in V$, one has by definition of v that $c \leq v$, a contradiction. The fact that $(\oplus y_i) \to W \neq W$ may be shown similarly.

We have thus verified that $\bigoplus_{i\in I} L_i$ satisfies condition (iv) in Theorem 2.9.1 and so it satisfies property (F). □

By a standard category theory result (see e.g. [1, Theorem 16.8]), Propositions 2.9.4 and 2.9.6 imply the following:

Corollary 2.9.7. *Locales satisfying property* (F) *are epireflective in the category of locales.*

In conclusion, we have shown that there is a separation property (F) that is given by a first-order formula, that is implied by fitness and that implies property (T_1), but is not comparable with subfitness. Moreover, it is hereditary and has a good categorical behaviour. Therefore, it seems to deserve some further investigation. The following diagram shows the parallel situation between properties (H) and (F):

$$\begin{tikzcd}
(sH) \arrow[d, Rightarrow] & & (\mathcal{F}\text{-sep}) \arrow[d, Rightarrow] \\
(\mathrm{H}) \arrow[dr, Rightarrow] & & (\mathrm{F}) \arrow[dl, Rightarrow] \\
& (T_1) &
\end{tikzcd}$$

We end up the section with some sparse questions for future work:

Questions 2.9.8. (1) What does one get by combining subfitness and (F)?

(2) What does one need to add to (F) in order to reach fitness?

(3) What are the spaces with property (F)?

(4) Does compactness together with property (F) imply something stronger?

2.10 Semifitted diagonals and questions for future work

A sublocale S of a locale L is said to be *semifitted* if it is the intersection of a fitted sublocale with a closed sublocale. It was suggested to us by Graham Manuell to study locales with semifitted diagonal, as these are a natural common generalization of both (sH) and ($\mathcal{F}$-sep) locales. In this section we shall briefly discuss some aspects of this new class of locales.

Recall the usual closure $\overline{S}$ and the "other" closure $\mathring{S}$ of a sublocale S from Subsection 2.2.3. Now, the sublocale

$$\overline{S} \cap \mathring{S} = \bigcap_{S \subseteq \mathfrak{c}(a) \cap \mathfrak{o}(b)} \mathfrak{c}(a) \cap \mathfrak{o}(b)$$

clearly defines the least semifitted sublocale containing S and the assignment $S \mapsto \overline{S} \cap \mathring{S}$ defines another closure operator in Loc (this is a general fact: closure operators are ordered pointwisely and infima/suprema of closure operators exist and are computed pointwisely to the extent that these pointwise infima/suprema exist — see [42, 2.5]).

We now consider separation with respect to this closure operator:

Definition 2.10.1. A locale L is $\mathcal{S}$*-separated* (or it satisfies property ($\mathcal{S}$-sep)) if its diagonal is semifitted in $L\oplus L$.

As we noted earlier, we have the following trivial observation:

Lemma 2.10.2. *Every strongly Hausdorff locale and every $\mathcal{F}$-separated locale is $\mathcal{S}$-separated.*

It now makes sense to study similar phenomena to that described earlier in this chapter e.g. the existence of a Dowker-Strauss characterization, the relation with the T_U axiom or the existence of a characterization in terms of weakened morphisms.

2.10.1 Dowker-Strauss-type characterization of $\mathcal{S}$-separatedness

First, we have the following technical characterization (we omit the proof since it is similar to that of Theorem 2.3.8).

Theorem 2.10.3. *A frame L is $\mathcal{S}$-separated if and only if for any localic maps $f,g\colon M\to L$, the equalities $\langle f,g\rangle^*(d_L)=0$ and $\langle f,g\rangle^*(U)=1$ hold for all $U\in L\oplus L$ with $D_L\subseteq \mathfrak{o}(U)$ only if $f=g$.*

Lemma 2.10.4. *Let $f,g\colon M\to L$ be localic maps. Then the pair (f^*,g^*) respects covers and disjoint pairs if and only if $\langle f,g\rangle^*(d_L)=0$ and $\langle f,g\rangle^*(U)=1$ for all $U\in L\oplus L$ such that $D_L\subseteq \mathfrak{o}(U)$.*

Proof. It follows trivially from Lemma 2.3.10 and the obvious fact that the pair (f^*,g^*) respects disjoint pairs if and only if $\langle f,g\rangle^*(d_L)=0$ (recall that $d_L=\bigvee_{a\wedge b=0}a\oplus b$). □

From the two previous results we obtain a Dowker-Strauss-type theorem for $\mathcal{S}$-separatedness:

Theorem 2.10.5. *A frame L is $\mathcal{S}$-separated if and only if no pair of distinct frame homomorphisms with domain L respects covers and disjoint pairs.*

Corollary 2.10.6. *An $\mathcal{S}$-separated frame is T_U.*

Proof. By Lemmas 2.3.5 and 2.3.11, if a pair of frame homomorphisms is bounded above and below, then it respects covers and disjoint pairs. Now, if $h,k\colon L\to M$ are frame homomorphisms such that $h\le k$, then the pair (h,k) is obviously bounded above and below so it respects covers and disjoint pairs. By Theorem 2.10.5 it follows that $h=k$. □

2.10.2 A few notes on relaxed morphisms

The task of finding a characterization in terms of weakened frame homomorphisms seems to be more difficult. We provide a partial solution by introducing two new classes of functions:

- a *B-homomorphism* is a weak homomorphism $h\colon L\to M$ which additionally preserves meets of covering pairs (i.e., $h(a\wedge b)=h(a)\wedge h(b)$ whenever $a\vee b=1$).
- a *C-homomorphism* is an almost homomorphism $h\colon L\to M$ which additionally preserves joins of disjoint pairs (i.e., $h(a\vee b)=h(a)\vee h(b)$ whenever $a\wedge b=0$).

Accordingly, we will say that a frame L satisfies *property* (B) if

$$\text{each } B\text{-homomorphism } h\colon L \to M \text{ is a frame homomorphism.} \tag{B}$$

Similarly, a frame L satisfies *property* (C) if

$$\text{each } C\text{-homomorphism } h\colon L \to M \text{ is a frame homomorphism.} \tag{C}$$

Clearly, property (W) implies property (B) and property (A) implies property (C).

Proposition 2.10.7. *If L is a frame satisfying property* (B) (*resp.* (C))*, a pair of frame homomorphisms with domain L is bounded above* (*resp. below*) *if it respects covers and disjoint pairs.*

Proof. We shall only prove that under property (C), being bounded below is implied by respecting covers and disjoint pairs (the proof of the other statement is similar).

Let $h,k\colon L \to M$ be frame homomorphisms and suppose that the pair (h,k) respects covers. By the argument in the proof of Proposition 2.4.2, the "pointwise meet" mapping $h \wedge k$ is an almost homomorphism. Let us check that is also preserves joins of disjoint pairs whenever (h,k) respects disjoint pairs. Let $a,b \in L$ with $a \wedge b = 0$. One has

$$\begin{aligned}(h\wedge k)(a\vee b) &= h(a\vee b)\wedge k(a\vee b) = ((h(a)\vee h(b))\wedge (k(a)\vee k(b))\\ &= (h\wedge k)(a)\vee (h(a)\wedge k(b))\vee (h(b)\wedge k(a))\vee (h\wedge k)(b) = (h\wedge k)(a)\vee (h\wedge k)(b),\end{aligned}$$

and thus $h \wedge k$ is a C-homomorphism. Since L satisfies property (C), it follows that $h \wedge k$ is a frame homomorphism. As $h \geq h \wedge k \leq k$, the pair (h,k) is bounded below. $\square$

Combining Lemma 2.3.4, Theorem 2.10.5 and Proposition 2.10.7 we obtain the following:

Corollary 2.10.8. *A T_U-frame with property* (B) (*resp. property* (C)) *is $\mathcal{S}$-separated.*

The following diagram describes the relations between the properties discussed in this section.

$$\begin{tikzcd}
(sH) \arrow[d, Leftrightarrow] & & (\mathcal{F}\text{-sep}) \arrow[d, Leftrightarrow] \\
(\mathrm{W})+(T_U) \arrow[d, Rightarrow] & & (\mathrm{A})+(T_U) \arrow[d, Rightarrow] \\
(\mathrm{B})+(T_U) \arrow[dr, Rightarrow] & & (\mathrm{C})+(T_U) \arrow[dl, Rightarrow] \\
& (\mathcal{S}\text{-sep}) \arrow[d, Rightarrow] & \\
& (T_U) &
\end{tikzcd}$$

Questions 2.10.9. (1) Can the implications in Corollary 2.10.8 be reversed? If not, how could one combine B-homomorphisms and C-homomorphisms into a single notion in order to obtain a characterization of $\mathcal{S}$-separation in terms of relaxed frame homomorphisms?

(2) Find a counterexample to the implication ($\mathcal{S}$-sep) $\implies$ (T_U) (or prove they are equivalent). This seems to be a hard question: as far as we know examples of T_U frames which are neither fit nor strongly Hausdorff are not known.

2.11 Summary

The following table summarizes the parallel between ($\mathcal{F}$-sep) and (sH) studied throughout this chapter.

Table 2.3 Several parallels

	Banaschewski, Pultr	Arrieta, Picado, Pultr
Relaxed morphisms	Weak homomorphisms: (1) Morphism in Sup (2) Preserve $\top$ (3) Preserve disjoint pairs	Almost homomorphisms: (1) Morphism in PreFrm (2) Preserve $\perp$ (3) Preserve covers
Every relaxed homomorphism is a frame homomorphism	Property (W)	Property (A)
Sufficient condition	Hereditary normality (HN) implies property (W)	Hereditary extremal disconnectedness (HED) implies property (A)
Downset frames	The following are equivalent: (1) X is a forest, (2) $\mathsf{Dwn}(X)$ is (HN) (3) $\mathsf{Dwn}(X)$ has (W)	The following are equivalent: (1) X is a coforest, (2) $\mathsf{Dwn}(X)$ is (HED) (3) $\mathsf{Dwn}(X)$ is hereditarily (IED) (4) $\mathsf{Dwn}(X)$ has (A)
Diagonal separation	$(sH) \equiv (T_U) + (W)$	$(\mathcal{F}sep) \equiv (T_U) + (A)$
Dowker-Strauss-type separation	(sH) if and only if no distinct homomorphisms respect disjoint pairs	$(\mathcal{F}sep)$ if and only if no distinct homomorphisms respect covers
(Co)density	Every dense map in $\mathsf{HausLoc}$ is epic in $\mathsf{HausLoc}$	Every codense map in $\mathcal{F}\mathsf{SepLoc}$ is epic in $\mathcal{F}\mathsf{SepLoc}$
Associated first order property	Property (H)	Property (F)

Chapter 3

Cardinal generalizations of normality and their duals

The first goal of the present chapter is to study two cardinal generalizations of normality in the theory of locales:

- *κ-collectionwise normality* has been widely studied both classically (cf. [52]) as well as point-freely (cf. [102]). This property is strongly related to the *metric* hedgehog (see [65] for details).
- The class of *κ-totally collectionwise normal* locales will be introduced. We shall discuss its main properties and establish the relations with collectionwise normality, with an eye towards Chapter 5, where this property will be further studied in the context of the *compact* hedgehog.

Recall now the parallel between normality and extremal disconnectedness (we have already discussed some of its aspects in Subsection 2.4.3). The source of this duality in point-free topology was investigated in [68] where the authors introduced the relative approach of sublocale selections as a tool for formalizing the parallel. Given a normality-type property, this approach allows one to obtain the dual extremal disconnectedness-type property (and moreover, to produce further variants by varying the sublocale selection). Furthermore, upper and lower semicontinuity are dual notions is this setting, hence providing a convenient framework to unify and generalize typical insertion and extension theorems.

In light of this, the second goal of the chapter is to exploit the relative approach in order to identify the duals of collectionwise normality and total collectionwise normality. It will turn out that there is no real cardinal generalization, as both dual notions collapse to the base cases $\kappa = 2$ and $\kappa = 1$: extremal disconnectedness and the *Oz* property, respectively.

More precisely, this chapter is organized as follows. In Section 3.1 we review the basics of sublocale selections. In Section 3.2 we illustrate the relative technique by proving the dual half of a Tietze-type extension theorem missing in [68]. Sections 3.3 and 3.4 are devoted to study the two mentioned point-free cardinal generalizations of normality and their duals. A number of preliminary concepts and results are needed, in particular we introduce and discuss a cardinal generalization of z-embeddings in Subsection 3.4.1. Finally, in Section 3.5 our cardinal generalization is compared to a different one introduced by Blair for topological spaces, thereby providing some results about cardinal generalizations of *Oz* locales.

Most of the results in this chapter are a collaborative work with Javier Gutiérrez García and Jorge Picado and they cover a part of the following article:

[7] I. Arrieta, J. Gutiérrez García, and J. Picado, Frame presentations of compact hedgehogs and their properties, *Quaestiones Mathematicae*, accepted for publication.

Other results of [7] may be found in Chapters 5 and 6. A number of further results are unpublished work (e.g. Propositions 3.3.4 and 3.4.9 and the results in Subsection 3.3.2 and Section 3.5).

3.1 Basic concepts on sublocale selections

In what follows, we introduce a convenient setting for studying normality and extremal disconnectedness in parallel. It was introduced by Gutiérrez García and Picado in [68].

An object function $\mathbb{F}$ on the category of locales will be called a *sublocale selection* if $\mathbb{F}(L)$ is a class of complemented sublocales of L for every locale L. We shall denote by $\mathbb{F}^*$ the sublocale selection defined by

$$\mathbb{F}^*(L) = \{\, S^{\#} \mid S \in \mathbb{F}(L)\,\}$$

and we shall speak of it as the *dual selection* of $\mathbb{F}$.

Moreover, we shall say that $\mathbb{F}$ is closed under (binary, countable, arbitrary) joins (resp. meets) if $\mathbb{F}(L)$ is closed under (binary, countable, arbitrary) joins (resp. meets), taken in $\mathsf{S}(L)$, for every locale L.

We now have a relative notion of normality with respect to a sublocale selection.

Definition 3.1.1. Let $\mathbb{F}$ be a sublocale selection. A locale L is called $\mathbb{F}$-*normal* if for any $S,T \in \mathbb{F}(L)$

$$S \cap T = \mathsf{O} \quad \Longrightarrow \quad \exists\, A,B \in \mathbb{F}(L) \text{ such that } S \cap A = \mathsf{O} = T \cap V \text{ and } A \vee B = L.$$

3.1.1 The parallel between normality and extremal disconnectedness

The motivating example of a sublocale selection is the selection $\mathbb{F}_{\mathfrak{c}}$ given by all closed sublocales:

$$\mathbb{F}_{\mathfrak{c}}(L) = \{\, \mathfrak{c}(a) \mid a \in L\,\}.$$

It is straightforward to check that $\mathbb{F}_{\mathfrak{c}}$-normality is equivalent to standard normality.

Now, as an advantage of the relative context, one may consider the corresponding dual notions. In particular, one readily sees that $\mathbb{F}_{\mathfrak{c}}^*$-normality is precisely extremal disconnectedness.

3.1.2 Other important examples

Besides the standard example of the selection $\mathbb{F}_{\mathfrak{c}}$ consisting of all closed sublocales, other important selections are given by choosing regular closed sublocales, zero sublocales and δ-regular closed sublocales (recall Subsection 1.6.2). In the following, these will be denoted

$$\mathbb{F}_{\mathrm{reg}},\ \mathbb{F}_{\mathrm{z}},\ \text{and}\ \mathbb{F}_{\delta\mathrm{reg}}$$

respectively.

These and their duals yield the following relative notions of normality:

Table 3.1 Examples of $\mathbb{F}$-normality and their duals

Selection	$\mathbb{F}$-normality	$\mathbb{F}^*$-normality
$\mathbb{F}_{\mathfrak{c}}$	Normal	Extremally disconnected
$\mathbb{F}_{\mathrm{reg}}$	Mildly normal	Extremally disconnected
$\mathbb{F}_{\mathrm{z}}$	Always satisfied	F-frame
$\mathbb{F}_{\delta\mathrm{reg}}$	δ-normal	Extremally δ-disconnected

For more examples and a detailed account of the topic we refer the reader to [68]. We remark that δ-normal locales are a point-free generalization of the δ-normal spaces in the sense of [56] (see [56, Example 3.4 (a)] for a discussion on terminology). In Appendix A the reader may consult a brief discussion of continuity in the relative setting.

3.2 A dual extension theorem

We begin by introducing some terminology that will be convenient for the rest of the thesis. We shall say that a sublocale selection $\mathbb{F}$ is *hereditary* (resp. *weakly hereditary*) *on a locale L* if for each $S \in \mathbb{F}(L)$ the equality

$$\mathbb{F}(S) = \{\, S \cap T \mid T \in \mathbb{F}(L) \,\}$$

(resp. the inclusion $\{\, S \cap T \mid T \in \mathbb{F}(L) \,\} \subseteq \mathbb{F}(S)$) holds.

Clearly, the standard examples $\mathbb{F}_{\mathfrak{c}}$, $\mathbb{F}_{\mathrm{z}}$ and $\mathbb{F}_{\delta\mathrm{reg}}$ and their duals are weakly hereditary, and moreover, $\mathbb{F}_{\mathrm{reg}}^*$ is hereditary. Note that, on the other hand, the selections $\mathbb{F}_{\mathrm{z}}$ and $\mathbb{F}_{\delta\mathrm{reg}}$ are not hereditary in general. For example, $\mathbb{F}_{\mathrm{z}}$ is hereditary on L if and only if L has the property that every zero sublocale is z-embedded. According to [59, 8.20 and 8.J (3)], the Tychonoff plank is a non-normal space whose zero-sets are all z-embedded; hence this property is strictly weaker than normality.

For technical reasons, we shall frequently restrict ourselves to Katětov sublocale selections; for a precise definition we refer to Section A.2. A general extension theorem for hereditary Katětov selections closed under countable meets was proved in [68] (see also Theorem A.3.1). However, none of the selections $\mathbb{F}^*_{\mathfrak{c}}$, $\mathbb{F}^*_{\mathrm{z}}$ and $\mathbb{F}^*_{\delta\mathrm{reg}}$ are closed under countable meets; so our aim in this section is to illustrate the technique by providing a "dual" result where the selections $\mathbb{F}^*_{\mathfrak{c}}$, $\mathbb{F}^*_{\mathrm{z}}$ and $\mathbb{F}^*_{\delta\mathrm{reg}}$ can fit.

We shall say that a sublocale selection $\mathbb{F}$ is *co-hereditary on a locale* L if for each $S \in \mathbb{F}(L)$ the equality

$$\mathbb{F}^*(S) = \{ S \cap T \mid T \in \mathbb{F}^*(L) \}$$

holds. $\mathbb{F}$ is *co-hereditary* if it is co-hereditary on any locale.

Lemma 3.2.1. *Each of the sublocale selections* $\mathbb{F} = \mathbb{F}^*_{\mathfrak{c}}, \mathbb{F}^*_{\mathrm{z}}, \mathbb{F}^*_{\delta\mathrm{reg}}$ *is co-hereditary.*

Proof. $\mathbb{F}^*_{\mathfrak{c}}$: It is well known that closed sublocales of any sublocale S are of the form $S \cap T$ where T is closed in L.

$\mathbb{F}^*_{\mathrm{z}}$: It amounts to showing that for every cozero element a in L and b in $\mathfrak{o}(a)$, there exists a cozero element c in L with $\nu_{\mathfrak{o}(a)}(c) = b$. But in this situation one has that $a \wedge b$ is a cozero element in L (see for example [69, Corollary 5.6.2] or [50, Corollary 3.2.11]).

$\mathbb{F}^*_{\delta\mathrm{reg}}$: We have to show that for each δ-regular a in L and each δ-regular b in $\mathfrak{o}(a)$, there exists a δ-regular c in L with $\nu_{\mathfrak{o}(a)}(c) = b$. It is of course enough to show that $c := a \wedge b$ is δ-regular in L. By the isomorphism ${\downarrow}a \cong \mathfrak{o}(a)$, this is equivalent to show that if b is δ-regular in the frame ${\downarrow}a$, then it is δ-regular in L.

Since a is δ-regular in L, one can write $a = \bigvee_n a_n$ where $a_n \prec a$ for all $n \in \mathbb{N}$ (i.e., for each n there is a c_n with $c_n \wedge a_n = 0$ and $c_n \vee a = 1$). Since $x \prec a$ and $y \prec a$ imply $x \vee y \prec a$, we may assume that $\{a_n\}_{n\in\mathbb{N}}$ is increasing. Moreover, b is δ-regular in ${\downarrow}a$, so one can write $b = \bigvee_n b_n$ where $\{b_n\}_{n\in\mathbb{N}}$ is increasing and for each $n \in \mathbb{N}$ there is a d_n with $d_n \wedge b_n = 0$ and $d_n \vee b = a$. Let $x_n = c_n \vee d_n$. Then

$$x_n \wedge (a_n \wedge b_n) = (c_n \wedge a_n \wedge b_n) \vee (d_n \wedge a_n \wedge b_n) = 0,$$
$$x_n \vee b = c_n \vee (d_n \vee b) = c_n \vee a = 1.$$

Finally, $b \leq \bigvee_n a_n \wedge b_n$ because $b \leq a$ and $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are increasing. Hence $b = \bigvee_n a_n \wedge b_n$ with $a_n \wedge b_n \prec b$ in L. ◻

This is our claimed extension theorem (for the relative notions of continuity, we refer to Appendix A):

Theorem 3.2.2. *Let $\mathbb{F}$ be such that $\mathbb{F}^*$ is closed under countable meets and finite joins. The following are equivalent for a locale L on which $\mathbb{F}$ is co-hereditary and Katětov:*

(i) *L is $\mathbb{F}$-normal;*

(ii) *For each $S \in \mathbb{F}(L)$, every $f \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ has an extension $\overline{f} \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$.*

Proof. (i) $\Longrightarrow$ (ii): Let $S \in \mathbb{F}(L)$, $f \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ and $r \in \mathbb{Q}$. Since f is $\mathbb{F}$-continuous and $\mathbb{F}^*$ is closed under countable meets, both $f(r,\text{—})$ and $f(\text{—},r)$ belong to $\mathbb{F}^*(S)$ (cf. Lemma A.1.3). Since S is complemented then so are $f(r,\text{—})$ and $f(\text{—},r)$ and the maps $\sigma_1,\sigma_2\colon \mathbb{Q} \to L$ given by

$$\sigma_1(r) = S^\# \vee f(r,\text{—}) \quad \text{and} \quad \sigma_2(r) = S \cap f(\text{—},r)^\#$$

are extended scales in $\mathsf{S}(L)^{op}$; denote by f_1 and f_2 the corresponding functions in $\overline{\mathsf{F}}(L)$.

Since $f(r,\text{—})$ and $f(\text{—},r)$ belong to $\mathbb{F}^*(S)$, by co-heredity there exist $U_r, V_r \in \mathbb{F}^*(L)$ such that $f(r,\text{—}) = U_r \cap S$ and $f(\text{—},r) = V_r \cap S$. Then one has (recall (1.6.3))

$$f_1(r,\text{—}) = \bigcap_{s>r} \sigma_1(s) = S^\# \vee f(r,\text{—}) = S^\# \vee (U_r \cap S) = S^\# \vee U_r \in \mathbb{F}^*(L)$$

and

$$f_2(\text{—},r) = \bigcap_{s<r} \sigma_2(s)^\# = S^\# \vee f(\text{—},r) = S^\# \vee (V_r \cap S) = S^\# \vee V_r \in \mathbb{F}^*(L)$$

(as $\mathbb{F}^*(L)$ is closed under binary joins). It follows that f_1 is lower $\mathbb{F}^*$-continuous while f_2 is upper $\mathbb{F}^*$-continuous. This means that f_1 is upper $\mathbb{F}$-continuous and f_2 is lower $\mathbb{F}$-continuous. Moreover, by (1.6.4) we have $f_1 \le f_2$ because $f(\text{—},s)^\# \subseteq f(r,\text{—})$, and therefore $\sigma_1(r) \le \sigma_2(s)$, for any $s < r$. Since $\mathbb{F}$ is Katětov on L, by [68, Theorem 7.1] (or Theorem A.2.2) there is an $\mathbb{F}$-continuous $h \in \overline{\mathsf{F}}(L)$ such that $f_1 \le h \le f_2$. Let $h_S\colon \mathsf{S}(L) \to \mathsf{S}(S)$ be the coframe homomorphism $T \mapsto S \cap T$. One readily checks that

$$h_S(f_1(r,\text{—})) = f(r,\text{—}) \quad \text{and} \quad h_S(f_2(\text{—},s)) = f(\text{—},s)$$

and thus (recall the partial order (1.6.2)) $h_S \circ f_1 = f$ and $h_S \circ f_2 = f$. Finally note that

$$f = h_S \circ f_1 \le h_S \circ h \le h_S \circ f_2 = f.$$

It follows that $h_S \circ h = f$ and thus h is the desired $\mathbb{F}$-continuous extension of the given f.

(ii) $\Longrightarrow$ (i): Let $S, T \in \mathbb{F}(L)$ such that $S \cap T = \mathsf{O}$. Then $S \in \mathbb{F}(S \vee T) \cap \mathbb{F}^*(S \vee T)$. Indeed, $S \vee T \in \mathbb{F}(L)$ (because $\mathbb{F}^*$ is in particular closed under finite meets) and $T^\# \in \mathbb{F}^*(L)$. Hence by co-heredity one has $S = T^\# \cap (S \vee T) \in \mathbb{F}^*(S \vee T)$. Exchanging the roles of S and T we obtain $T \in \mathbb{F}^*(S \vee T)$ and hence $S = (S \vee T) \smallsetminus T \in \mathbb{F}(S \vee T)$. Then, by Proposition A.1.5 one has $\chi_S \in \overline{\mathsf{C}}^{\mathbb{F}}(S \vee T)$. Since $S \vee T \in \mathbb{F}(L)$, there is an extension $f \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$. Choose $A, B \in \mathbb{F}(L)$ with

$$f(1,\text{—}) \subseteq A \subseteq f(2,\text{—}) \quad \text{and} \quad f(\text{—},1) \subseteq B \subseteq f(\text{—},0).$$

Then

$$A \vee B \supseteq f(1,—) \vee f(—,1) = f((1,—) \wedge (—,1)) = f(0) = L$$

and $S \cap A = \chi_S(—,3) \cap A \subseteq f(—,3) \cap f(2,—) = \mathsf{O}$. Similarly, $T \cap B = \mathsf{O}$. □

The previous theorem generalizes and unifies extension results that appear in the literature. Our guiding examples $\mathbb{F}^*_{\mathfrak{c}}$, $\mathbb{F}^*_{\mathrm{z}}$ and $\mathbb{F}^*_{\delta\mathrm{reg}}$ satisfy the conditions of the theorem (cf. Lemma 3.2.1 and A.2).

The case $\mathbb{F} = \mathbb{F}^*_{\mathfrak{c}}$ yields the point-free counterpart of the extension result for extremally disconnected spaces of Gillman and Jerison [59]. In the localic setting, it was proved by Gutiérrez García, Kubiak and Picado [63, Theorem 5.5] by using a rather different argument to ours (their proof does not rely on an insertion theorem, ours follows the more classical path of deducing the extension theorem as a corollary of the insertion result).

Corollary 3.2.3. *The following are equivalent for a locale L:*

(i) *L is extremally disconnected;*

(ii) *For each $a \in L$, every $f \in \overline{\mathsf{C}}(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \overline{\mathsf{C}}(L)$.*

Specializing to the case $\mathbb{F} = \mathbb{F}^*_{\mathrm{z}}$, we obtain a result proved by Ball and Walters-Wayland in [20, Proposition 8.4.10]:

Corollary 3.2.4. *The following are equivalent for a locale L:*

(i) *L is an F-frame;*

(ii) *For each cozero element $a \in L$, every $f \in \overline{\mathsf{C}}(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \overline{\mathsf{C}}(L)$.*

Furthermore, $\mathbb{F} = \mathbb{F}^*_{\delta\mathrm{reg}}$ appears to produce a new result which we have not been able to find in the literature.

Corollary 3.2.5. *The following are equivalent for a locale L:*

(i) *L is extremally δ-disconnected;*

(ii) *For each δ-regular element $a \in L$, every $f \in \overline{\mathsf{C}}(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \overline{\mathsf{C}}(L)$.*

3.3 Collectionwise normality

3.3.1 Background

In this section we briefly consider the first cardinal generalization of normality that we shall meet, the so-called collectionwise normality. It has a remarkable role in the study of the point-free *metric* hedgehog (see [65] or Section 5.1 of Chapter 5). Our interest in this concept relies on the fact that the point-free *compact* hedgehog is related to a stronger variant of collectionwise normality that we will call *total* collectionwise normality (cf. Subsection 3.4.2).

Recall from [65] that a family $\{a_i\}_{i\in I}$ of elements of L is said to be *disjoint* if $a_i \wedge a_j = 0$ for every $i \neq j$. It is *discrete* (resp. *co-discrete*) if there is a cover C of L such that for any $c \in C$, $c \wedge a_i = 0$ (resp. $c \leq a_i$) for all i with at most one exception. Note, in particular, that any discrete family is clearly disjoint, and that a pair $\{a,b\}$ is co-discrete if and only if $a \vee b = 1$. Trivially, if a finite $\{a_1,a_2,\ldots,a_n\}$ is co-discrete then $a_1 \vee a_2 \vee \cdots \vee a_n = 1$, but not conversely for $n \geq 3$.

Recall also from [102] (see also [65, 97] for more information) that a frame is *κ-collectionwise normal* if for every co-discrete κ-family $\{a_i\}_{i\in I}$, there is a discrete $\{b_i\}_{i\in I}$ with $b_i \vee a_i = 1$ for all $i \in I$. Moreover a frame is *collectionwise normal* if it is κ-collectionwise normal for every cardinal κ.

Observe that the definition is trivially satisfied for $\kappa = 1$, hence throughout this section we shall assume that $\kappa \geq 2$. The following may be found in [65]:

Properties 3.3.1. (1) For every $\kappa \geq 2$, κ-collectionwise normality implies normality, hence κ-collectionwise normality is a cardinal generalization of normality;

(2) For every $2 \leq \kappa \leq \aleph_0$, κ-collectionwise normality is equivalent to normality;

(3) κ-collectionwise normality is hereditary with respect to closed sublocales.

We take this opportunity to rectify a slip in [65]: we can replace discrete families by disjoint families in the definition of *κ-collectionwise normal* frames, as we shall show in Proposition 3.3.3 below.

Lemma 3.3.2 ([102, Lemma 1.13]). *For any co-discrete $\{a_i\}_{i\in I} \subseteq L$ and $b \in L$, $b \vee \bigwedge_{i\in I} a_i = \bigwedge_{i\in I}(b \vee a_i)$.*

Proposition 3.3.3. *A frame L is κ-collectionwise normal if and only if for any co-discrete κ-family $\{a_i\}_{i\in I}$, there is a disjoint $\{b_i\}_{i\in I}$ such that $a_i \vee b_i = 1$ for all $i \in I$.*

Proof. The implication '$\Rightarrow$' is obvious since any discrete family is disjoint.

Conversely, let $\{a_i\}_{i\in I}$ be a co-discrete family. Then there is a disjoint family $\{b_i\}_{i\in I}$ such that $b_i \vee a_i = 1$ for all $i \in I$. Set

$$D := \{a \in L \mid a \wedge b_i \neq 0 \text{ for at most one } i\}$$

and $\overline{d} := \bigvee D$. Clearly $b_i \in D$, and hence $b_i \leq \overline{d}$, for each i. Then, by the previous lemma, $\overline{d} \vee \bigwedge_{i\in I} a_i = \bigwedge_{i\in I}(\overline{d} \vee a_i) \geq \bigwedge_{i\in I}(b_i \vee a_i) = 1$. Moreover, since κ-collectionwise normality implies normality, there are $u, v \in L$ such that $u \vee \bigwedge_{i\in I} a_i = 1 = v \vee \overline{d}$ and $u \wedge v = 0$. The family

$$\{u_i := b_i \wedge u\}_{i\in I}$$

is then the required discrete system. Indeed, $C := D \cup \{v\}$ is a cover of L (since $\bigvee C = \overline{d} \vee v = 1$), each $c \in C$ meets at most one u_i (since $u_i \wedge v \leq u \wedge v = 0$) and $u_i \vee a_i = (b_i \vee a_i) \wedge (u \vee a_i) = u \vee a_i \geq u \vee \bigwedge_{i\in I} a_i = 1$ for every i. □

We shall need the following result later (cf. [52, Exercise 5.5.1]).

Proposition 3.3.4. *The following conditions are equivalent for a frame L:*

(i) *L is hereditarily κ-collectionwise normal;*

(ii) *Every open sublocale of L is κ-collectionwise normal;*

(iii) *For every sublocale $S \subseteq L$ and every co-discrete κ-family $\{a_i\}_{i\in I}$ in S, there is a disjoint family $\{b_i\}_{i\in I}$ in L such that $\nu_S(a_i \vee b_i) = 1$ for all $i \in I$.*

Proof. (i) $\Longrightarrow$ (ii) is trivial and the implication (iii) $\Longrightarrow$ (i) follows from Proposition 3.3.3. Let us prove (ii) $\Longrightarrow$ (iii). Let S be a sublocale and let $\{a_i\}_{i\in I}$ be a co-discrete κ-family in S. For each $i \in I$, the frame $\downarrow(a_i \vee \bigwedge_{j\neq i} a_j)$ is normal (because it is κ-collectionwise normal). Hence there are $u_i, v_i \in \downarrow(a_i \vee \bigwedge_{j\neq i} a_j)$ such that $u_i \wedge v_i = 0$, $a_i \vee \bigwedge_{j\neq i} a_j = u_i \vee a_i$ and $a_i \vee \bigwedge_{j\neq i} a_j = v_i \vee \bigwedge_{j\neq i} a_j$. In particular, observe that

$$\nu_S(u_i \vee a_i) = \nu_S(a_i \vee \bigwedge_{j\neq i} a_j) = a_i \vee^S \bigwedge_{j\neq i} a_j = \bigwedge_{j\neq i} a_i \vee^S a_j = 1 \tag{3.3.1}$$

by an application of Lemma 3.3.2 and the obvious fact that a subfamily of a co-discrete family is co-discrete. Furthermore, taking meets with u_i in the equality $a_i \vee \bigwedge_{j\neq i} a_j = v_i \vee \bigwedge_{j\neq i} a_j$ we see that $u_i \wedge a_i \leq \bigwedge_{j\neq i} a_j$, and therefore, since $u_i \in \downarrow(a_i \vee \bigwedge_{j\neq i} a_j)$, we conclude that

$$u_i = u_i \wedge (a_i \vee \bigwedge_{j\neq i} a_j) = (u_i \wedge a_i) \vee (u_i \wedge \bigwedge_{j\neq i} a_j) \leq \bigwedge_{j\neq i} a_j. \tag{3.3.2}$$

Let now $u := \bigvee_{i\in I} u_i$. Then (3.3.2) shows that the family $\{u \wedge a_i\}_{i\in I}$ is co-discrete in $\downarrow u$ (just take $\{u_i\}_{i\in I}$ as the cover in the definition of co-discrete). Since $\downarrow u$ is κ-collectionwise normal, there is a disjoint $\{b_i\}_{i\in I}$ in L with $(u \wedge a_i) \vee b_i = u$ for all $i \in I$, that is, $u \leq a_i \vee b_i$ for all $i \in I$. Hence, $u_i \vee a_i \leq u \vee a_i \leq a_i \vee b_i$ and applying the nucleus ν_S and using (3.3.1) it follows that $\nu_S(a_i \vee b_i) = 1$ for all $i \in I$. □

Discreteness has an obvious translation to the language of sublocales. Indeed, we say that a family $\{S_i\}_{i\in I}$ of sublocales is *discrete* if there is a cover C of L such that for any $c \in C$, $\mathfrak{o}(c) \cap S_i = \mathsf{O}$ for all i with at most one exception. Then, a family $\{a_i\}_{i\in I}$ is discrete (resp. co-discrete) if and only if $\{\mathfrak{o}(a_i)\}_{i\in I}$ (resp $\{\mathfrak{c}(a_i)\}_{i\in I}$) is discrete. We can therefore recast the notion of collectionwise normality in terms of sublocales:

Lemma 3.3.5. *The following conditions are equivalent for a locale L:*

(i) *L is κ-collectionwise normal;*

(ii) *For every discrete κ-family $\{\mathfrak{c}(a_i)\}_{i\in i}$, there is a discrete $\{\mathfrak{o}(b_i)\}_{i\in I}$ such that $\mathfrak{c}(a_i) \subseteq \mathfrak{o}(b_i)$ for every $i \in I$;*

(iii) *For every discrete κ-family $\{\mathfrak{c}(a_i)\}_{i\in i}$, there is a pairwise disjoint $\{\mathfrak{o}(b_i)\}_{i\in I}$ such that $\mathfrak{c}(a_i) \subseteq \mathfrak{o}(b_i)$ for every $i \in I$.*

The formulation in the lemma is entirely in terms of sublocales and therefore we can now use the relative approach in order to study the dual property.

3.3.2 Relative notions of collectionwise normality

Let $\mathbb{F}$ be a sublocale selection. A family $\{S_i\}_{i\in I}$ of complemented sublocales of a locale L will be said to be $\mathbb{F}$-*discrete* if there is a cover $\mathcal{C} \subseteq \mathbb{F}^*(L)$ of $\mathsf{S}(L)$ (i.e., $\bigvee_{C\in\mathcal{C}} C = L$ in $\mathsf{S}(L)$) and such that for any $C \in \mathcal{C}$ one has $C \cap S_i = \mathsf{O}$ for all but at most one i.

Clearly, when $\mathbb{F} = \mathbb{F}_c$, the notion of discreteness reduces to that of Subsection 3.3.1.

Lemma 3.3.6. *Let L be a locale and $\mathbb{F}$ a sublocale selection. The following assertions hold*:

(1) *An $\mathbb{F}$-discrete family is pairwise disjoint*;

(2) *If S, T belong to $\mathbb{F}(L)$ and they are disjoint, then $\{S, T\}$ is $\mathbb{F}$-discrete.*

Proof. (1) Let $\{S_i\}_{i\in I}$ be an $\mathbb{F}$-discrete family of complemented sublocales. Then there is a $\mathcal{C} \subseteq \mathbb{F}^*(L)$ with $\bigvee_{C\in\mathcal{C}} C = L$ such that for any $C \in \mathcal{C}$ one has $C \cap S_i = \mathsf{O}$ for all but at most one i. For each $i \neq j$ and each $C \in \mathcal{C}$, one has $C \cap S_i \cap S_j = \mathsf{O}$, and so $S_i \cap S_j = S_i \cap S_j \cap L = S_i \cap S_j \cap \bigvee_{C\in\mathcal{C}} C = \bigvee_{C\in\mathcal{C}}(S_i \cap S_j \cap C) = \mathsf{O}$ because $S_i \cap S_j$ is complemented (cf. (1.2.3)).

(2) The cover $\mathcal{C} = \{S^\#, T^\#\} \subseteq \mathbb{F}^*(L)$ will do the job. □

Definition 3.3.7. Let L be a locale and $\mathbb{F}$ a sublocale selection. We shall say that

(1) L is *weakly κ-$\mathbb{F}$-collectionwise normal* if for every $\mathbb{F}$-discrete κ-family $\{S_i\}_{i\in I} \subseteq \mathbb{F}(L)$ there is a pairwise disjoint $\{T_i\}_{i\in I} \subseteq \mathbb{F}^*(L)$ with $S_i \subseteq T_i$ for all $i \in I$.

(2) L is κ-$\mathbb{F}$- *collectionwise normal* if for every $\mathbb{F}$-discrete κ-family $\{S_i\}_{i\in I} \subseteq \mathbb{F}(L)$ there is an $\mathbb{F}$-discrete $\{T_i\}_{i\in I} \subseteq \mathbb{F}^*(L)$ with $S_i \subseteq T_i$ for all $i \in I$.

It follows immediately from Lemma 3.3.6 (1) that κ-$\mathbb{F}$-collectionwise normality implies weak κ-$\mathbb{F}$-collectionwise normality.

Moreover, when $\mathbb{F} = \mathbb{F}_c$, both notions coincide with κ-collectionwise normality because of Lemma 3.3.5. Now, by Lemma 3.3.6 (2) we have the following:

Corollary 3.3.8. *Let L be a locale and $\mathbb{F}$ a sublocale selection. Then L is 2-$\mathbb{F}$-collectionwise normal if and only if it is $\mathbb{F}$-normal. Hence for every $\kappa \geq 2$, κ-$\mathbb{F}$-collectionwise normality implies $\mathbb{F}$-normality.*

3.3.3 Collectionwise extremal disconnectedness is just extremal disconnectedness

Our main interest now is to study the dual of collectionwise normality. We start with a few preliminary lemmas.

Let L be a locale and $\mathbb{F}$ a sublocale selection. Given $S \in \mathsf{S}(L)$, we denote

$$\overline{S}^{\mathbb{F}} = \bigcap\{T \in \mathbb{F}(L) \mid S \subseteq T\}$$

and we will speak of it as the $\mathbb{F}$-*closure* of S (especially whenever $\mathbb{F}$ is closed under meets, because in such case one has that $\overline{S}^{\mathbb{F}} \in \mathbb{F}(L)$).

Remark 3.3.9. Besides the usual localic closure, recall the "other" closure introduced in Subsection 1.2.1 of Chapter 1. Then, $\overline{S} = \overline{S}^{\mathbb{F}_c}$ and $\mathring{S} = \overline{S}^{\mathbb{F}_c^*}$, that is, the usual closure and the "other" closure are just the closure of S with respect to the sublocale selections $\mathbb{F}_c$ and $\mathbb{F}_c^*$, respectively. In general, if a sublocale selection $\mathbb{F}$ is *pullback stable* (that is, for every localic map $f\colon L \to M$ and $S \in \mathbb{F}(M)$, one has $f_{-1}[S] \in \mathbb{F}(L)$), then the operator $S \mapsto \overline{S}^{\mathbb{F}}$ is a closure operator in the sense of Definition 2.2.1.

Lemma 3.3.10. *Let L be a locale and $\mathbb{F}$ a sublocale selection such that $\mathbb{F}^*$ is closed under arbitrary meets. Then the following are equivalent*:

(i) *L is $\mathbb{F}$-normal*;

(ii) *For each $S \in \mathbb{F}(L)$, its $\mathbb{F}^*$-closure belongs to $\mathbb{F}(L)$;*

(iii) *If S and T are disjoint sublocales contained in $\mathbb{F}(L)$, then their $\mathbb{F}^*$-closures are also disjoint.*

Proof. (i) $\Longrightarrow$ (ii): Let $S \in \mathbb{F}(L)$. For simplicity we write $\overline{S} := \overline{S}^{\mathbb{F}^*}$. Since $\mathbb{F}^*$ is closed under intersections, we have $\overline{S} \in \mathbb{F}^*(L)$. Now, $S \cap \overline{S}^{\#} = \mathsf{O}$, and since $S, \overline{S}^{\#} \in \mathbb{F}(L)$, by $\mathbb{F}$-normality there are disjoint $M, N \in \mathbb{F}^*(L)$ with $S \subseteq M$ and $\overline{S}^{\#} \subseteq N$. But $S \subseteq M \in \mathbb{F}^*(L)$ implies $\overline{S} \subseteq M \subseteq N^{\#}$. Hence $\overline{S} = N^{\#} \in \mathbb{F}(L)$.

(ii) $\Longrightarrow$ (iii): Let $S, T \in \mathbb{F}(L)$ be disjoint and for simplicity write $\overline{S} = \overline{S}^{\mathbb{F}^*}$ and $\overline{T} = \overline{T}^{\mathbb{F}^*}$. Observe that we have the following chain of equivalences:

$$\overline{S} \cap \overline{T} = \mathsf{O} \iff \overline{S} \subseteq \overline{T}^{\#} \iff S \subseteq \overline{T}^{\#} \iff \overline{T} \subseteq S^{\#} \iff T \subseteq S^{\#} \iff T \cap S = \mathsf{O}$$

(in the second and fourth implication we use that $\overline{S}^{\#} \in \mathbb{F}^*(L)$ and $\overline{T}^{\#} \in \mathbb{F}^*(L)$ respectively).

(iii) $\Longrightarrow$ (i): This implication is trivial. □

Lemma 3.3.11. *Let L be a locale and $\mathbb{F}$ a sublocale selection such that $\mathbb{F}^*$ is closed under arbitrary meets. Then a family $\{S_i\}_{i \in I} \subseteq \mathbb{F}(L)$ is $\mathbb{F}$-discrete if and only if it is pairwise disjoint.*

Proof. The "only if" part always holds by Lemma 3.3.6 (1), so let us show the converse. For each $k \in I$ one has $S_k^{\#} \vee \bigvee_{i \in I} \bigcap_{j \neq i} S_j^{\#} \supseteq \bigcap_{j \neq k} S_j^{\#} \vee S_k^{\#} = L$ by pairwise disjointness and so

$$L = \bigcap_{k \in I} \Big(S_k^{\#} \vee \bigvee_{i \in i} \bigcap_{j \neq i} S_j^{\#} \Big) = \Big(\bigcap_{k \in I} S_k^{\#} \Big) \vee \Big(\bigvee_{i \in i} \bigcap_{j \neq i} S_j^{\#} \Big)$$

by coframe distributivity. But, clearly, $\bigcap_{k \in I} S_k^{\#} \subseteq \bigvee_{i \in i} \bigcap_{j \neq i} S_j^{\#}$, and therefore $\bigvee_{i \in i} \bigcap_{j \neq i} S_j^{\#} = L$. In other words, the family $C = \{ \bigcap_{j \neq i} S_j^{\#} \mid i \in I \}$ is a cover of $\mathsf{S}(L)$. Since $\mathbb{F}^*$ is closed under intersections, one has $C \subseteq \mathbb{F}^*(L)$. Clearly, C satisfies the required condition in the definition of $\mathbb{F}$-discreteness. □

Theorem 3.3.12. *Let L be a locale and $\mathbb{F}$ a sublocale selection such that $\mathbb{F}^*$ is closed under arbitrary meets. The following are equivalent for a cardinal κ:*

(i) *L is $\mathbb{F}$-normal;*

(ii) *For every pairwise disjoint family $\{S_i\}_{i\in I} \subseteq \mathbb{F}(L)$ there is a pairwise disjoint family $\{T_i\}_{i\in I} \subseteq \mathbb{F}^*(L)$ such that $S_i \subseteq T_i$ for each $i \in I$;*

(iii) *L is weakly κ-$\mathbb{F}$-collectionwise normal;*

(iv) *L is κ-$\mathbb{F}$-collectionwise normal.*

Proof. The implications (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are clear. Let us prove that (i) implies (iv). Let $\{S_i\}_{i\in I} \subseteq \mathbb{F}(L)$ be an $\mathbb{F}$-discrete family. In particular, by Lemma 3.3.6 (1) it is pairwise disjoint. Now, for simplicity denote $T_i = \overline{S_i}^{\mathbb{F}^*}$. By Lemma 3.3.10 one has that $\{T_i\}_{i\in I} \subseteq \mathbb{F}$ is a pairwise disjoint family. Therefore, by Lemma 3.3.11 we may conclude that the family $\{T_i\}_{i\in I}$ is $\mathbb{F}$-discrete, and hence it is the desired family that shows that L is κ-$\mathbb{F}$-collectionwise normal. □

In particular, for $\mathbb{F} = \mathbb{F}_{\mathfrak{c}}^*$ (which satisfies the conditions of the theorem), one obtains that (weak) κ-$\mathbb{F}_{\mathfrak{c}}^*$-collectionwise normality is just extremal disconnectedness. Hence, there is no real cardinal generalization of extremal disconnectedness in this setting.

3.4 Total collectionwise normality

3.4.1 z_κ^c-embeddings

We now introduce a cardinal generalization of the notion of z-embedding (cf. [14]), instrumental for the notion of total collectionwise normality:

Definition 3.4.1. Let κ be a cardinal. A sublocale S of L is *z_κ^c-embedded* in L if for every disjoint κ-family $\{a_i\}_{i\in I}$ of cozero elements of S, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $\nu_S(b_i) = a_i$ (that is, such that $\mathfrak{o}_S(a_i) = S \cap \mathfrak{o}(b_i)$) for every $i \in I$.

Remarks 3.4.2. (1) Clearly, the notion of a z_1^c-embedding coincides with the usual notion of z-embedding (recall Subsection 1.6.4). It is therefore clear that z_κ^c-embeddings are a cardinal generalization of z-embeddings.

(2) Moreover, by [50, Proposition 3.3] it follows at once that the notion of a z_2^c-embedding is also equivalent to that of a z-embedding.

(3) The counterpart of this concept for topological spaces was introduced by Gutiérrez García, Kubiak and de Prada Vicente in [61], under the different name of *κ-total z-embedding*. However, that terminology may lead to confusion as there is a further cardinal generalization of the notion of a z-embedding introduced by Blair [33] in the eighties called *z_κ-embedding*. However,

z^c_κ-embeddings do not appear to be generally comparable with Blair's z_κ-embeddings (see Theorem 3.4.19 and Corollary 3.5.4 below for a result indicating that the "*total*" terminology is indeed misleading). For a more precise formulation of Blair's notion see Section 3.5 below.

(4) The letter "c" in z^c_κ stands for *compact*. Indeed, as we will see in Chapter 5, z^c_κ-embedding can be characterized as a property about appropriate compact hedgehog-valued functions. In a parallel way, by [33, Theorem 3.8] the notion of z_κ-embedding is precisely what one gets by replacing the compact hedgehog by the metric hedgehog (cf. [65]).

We now investigate the case $\kappa = \aleph_0$.

Proposition 3.4.3. *A sublocale $S \subseteq L$ is $z^c_{\aleph_0}$-embedded if and only if it is z-embedded.*

Proof. The "only if" part is obvious. Conversely, assume that S is z-embedded and let $\{a_n\}_{n\in\mathbb{N}}$ be a countable disjoint family of cozero elements of S. For each $n \in \mathbb{N}$, let b_n be the join in S of the family $\{a_m\}_{m\neq n}$. Note that $\{a_n, b_n\}$ is a disjoint pair of cozero elements of S (since a countable join of cozero elements is again a cozero element). Then, by [50, Proposition 3.3], there is a disjoint pair $\{c_n, d_n\}$ of cozero elements of L such that $\nu_S(c_n) = a_n$ and $\nu_S(d_n) = b_n$. Take

$$u_1 = c_1 \quad \text{and} \quad u_n = c_n \wedge d_1 \wedge \cdots \wedge d_{n-1} \quad (n > 1).$$

Then, $\{u_n\}_{n\in\mathbb{N}}$ is the required disjoint family of cozero elements of L that extends $\{a_n\}_{n\in\mathbb{N}}$. Indeed, each u_n is a cozero element (because cozero elements are closed under finite meets); for disjointness, let $n < m$ and observe that $u_n \wedge u_m \leq c_n \wedge d_n = 0$. Finally, $\nu_S(u_1) = \nu_S(c_1) = a_1$ and, for $n > 1$,

$$\nu_S(u_n) = \nu_S(c_n) \wedge \nu_S(d_1) \wedge \cdots \wedge \nu_S(d_{n-1}) = a_n \wedge b_1 \wedge \cdots \wedge b_{n-1}.$$

Note that $a_n \leq b_m$ for $m = 1, \ldots, n-1$, hence $\nu_S(u_n) = a_n$, as claimed. □

3.4.2 Total collectionwise normality

We are now ready to introduce the following:

Definition 3.4.4. A locale is *totally κ-collectionwise normal* if all its closed sublocales are z^c_κ-embedded. A locale is *totally collectionwise normal* if it is totally κ-collectionwise normal for every cardinal κ.

As an obvious consequence of Theorem 1.6.2 and Proposition 3.4.3 one sees that total collectionwise normality is indeed a cardinal generalization of normality:

Proposition 3.4.5. *For $1 \leq \kappa \leq \aleph_0$, total κ-collectionwise normality is equivalent to normality.*

As suggested by the name, total collectionwise normality implies collectionwise normality. Before proving this, we recall the following point-free version of the pasting lemma:

Proposition 3.4.6 ([94, Proposition 4.4]). *Let L and M be frames, $a_1, a_2 \in M$ and $h_i \colon L \to \mathfrak{c}(a_i)$ $(i = 1,2)$ frame homomorphisms such that $h_1(x) \vee a_2 = h_2(x) \vee a_1$ for every $x \in L$. Then the map $h \colon L \to \mathfrak{c}(a_1 \wedge a_2)$ given by $h(x) = h_1(x) \wedge h_2(x)$ is a frame homomorphism such that the triangle*

$$\begin{tikzcd} L \arrow[d, dashed, "h"'] \arrow[dr, "h_i"] & \\ \mathfrak{c}(a_1 \wedge a_2) \arrow[r, two heads, "\nu_{\mathfrak{c}(a_i)}"'] & \mathfrak{c}(a_2) \end{tikzcd}$$

commutes for $i = 1,2$.

Proposition 3.4.7. *For every cardinal κ, total κ-collectionwise normality implies κ-collectionwise normality.*

Proof. Let $\{a_i\}_{i\in I}$ be a co-discrete κ-family in a frame L. Fix some $i \in I$ and consider constant extended real valued functions $h_1^{(i)} \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a_i)$ and $h_2^{(i)} \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(\bigwedge_{j\neq i} a_j)$, given by

$$h_1^{(i)}(r,\text{—}) = 1, \quad h_1^{(i)}(\text{—},r) = a_i \quad \text{and} \quad h_2^{(i)}(r,\text{—}) = \bigwedge_{j\neq i} a_j, \quad h_2^{(i)}(\text{—},r) = 1.$$

One has $a_i \vee \bigwedge_{j\neq i} a_j = \bigwedge_{j\neq i}(a_i \vee a_j) = 1$ (the first equality follows from Lemma 3.3.2 and the obvious fact that any subfamily of a co-discrete family is co-discrete, whereas the second equality holds because the family $\{\mathfrak{c}(a_i)\}_{i\in I}$ is pairwise disjoint whenever $\{a_i\}_{i\in I}$ is co-discrete). Then, by Proposition 3.4.6, there is a frame homomorphism

$$h^{(i)} \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a_i) \vee \mathfrak{c}(\bigwedge_{j\neq i} a_j) = \mathfrak{c}(\bigwedge_{j\in I} a_j)$$

given by $h^{(i)}(x) = h_1^{(i)}(x) \wedge h_2^{(i)}(x)$. But $\bigvee_{r\in\mathbb{Q}} h^{(i)}(r,\text{—}) = \bigwedge_{j\neq i} a_j$, thus $\bigwedge_{j\neq i} a_j$ is a cozero element in $\mathfrak{c}(\bigwedge_{j\in I} a_j)$. Hence the family $\{\bigwedge_{j\neq i} a_j\}_{i\in I}$ is a disjoint family of cozero elements in the closed sublocale $\mathfrak{c}(\bigwedge_{j\in I} a_j)$. Finally, by assumption, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $b_i \vee \bigwedge_{j\in I} a_j = \bigwedge_{j\neq i} a_j$; in particular,

$$b_i \vee a_i = b_i \vee \Big(\bigwedge_{j\in I} a_j\Big) \vee a_i = \Big(\bigwedge_{j\neq i} a_j\Big) \vee a_i = \bigwedge_{j\neq i}(a_j \vee a_i) = 1. \qquad \square$$

Corollary 3.4.8. *For every cardinal κ, total κ-collectionwise normality implies normality.*

Finally, a sufficient condition for total collectionwise normality can be given as follows.

Proposition 3.4.9. *For every cardinal κ, hereditary κ-collectionwise normality implies total κ-collectionwise normality.*

Proof. Let L be hereditarily κ-collectionwise normal, $a \in L$ and $\{a_i\}_{i\in I}$ a disjoint κ-family of cozero elements in $\mathfrak{c}(a)$. Set $c = \bigvee_{i\in I} a_i$, and for each $i \in I$, let $c_i := \bigvee_{j\neq i} a_j$. Then, the family $\{c_i\}_{i\in I}$ is co-discrete in

$$S = {\downarrow}^{\mathfrak{c}(a)} c = \{\, b \in L \mid a \le b \le c \,\}$$

(simply take the cover $\{a_i\}_{i\in I}$ in the definition of co-discrete). By Proposition 3.3.4 (iii), there is a disjoint $\{b_i\}_{i\in I}$ in L with $c = \nu_S(b_i \vee c_i) = (b_i \vee c_i \vee a) \wedge c \le b_i \vee c_i$ for all $i \in I$. Thus, for each $i \in I$,

$$a_i = a_i \wedge c \le (a_i \wedge b_i) \vee (c_i \wedge a_i) \le b_i \vee a$$

because $\{a_i\}_{i\in I}$ is disjoint in $\mathfrak{c}(a)$. For each $i \in I$, a_i is a cozero element of $\mathfrak{c}(a)$, so pick an $f_i \in \overline{\mathsf{C}}(\mathfrak{c}(a))$ with $a_i = f_i(\bigvee_{r\in\mathbb{Q}}(r,—))$. Let $g_i \in \overline{\mathsf{C}}(\mathfrak{c}(b_i))$ be the constant extended real valued function determined by $g_i(r,—) = b_i$. Then for all $r \in \mathbb{Q}$ one has $b_i \vee a = f_i(r,—) \vee b_i$. The inequality $\le$ holds because $f_i(r,—) \in \mathfrak{c}(a)$. For the reverse inequality, one has $f_i(r,—) \le f(\bigvee_{s\in\mathbb{Q}}(s,—)) = a_i \le b_i \vee a$. Therefore, for all $r \in \mathbb{Q}$ we have $g_i(r,—) \vee a = f_i(r,—) \vee b_i$. By Proposition 3.4.6, there is an $h_i \in \overline{\mathsf{C}}(L)$ with $h_i(r,—) \vee a = f_i(r,—)$ and $h_i(r,—) \vee b_i = g_i(r,—)$ for all $r \in \mathbb{Q}$. Let $d_i = h_i(\bigvee_{r\in\mathbb{Q}}(r,—))$. Then d_i is a cozero element in L with $d_i \vee a = a_i$ for each $i \in I$. Finally, $\{d_i\}_{i\in I}$ is also disjoint because $d_i \wedge d_j \le (d_i \vee b_i) \wedge (d_j \vee b_j) = b_i \wedge b_j = 0$ by disjointness of $\{b_i\}_{i\in I}$ in L. □

In Chapter 5 we shall also need the following observation.

Lemma 3.4.10. *Total κ-collectionwise normality is hereditary with respect to closed sublocales.*

Proof. Let $S = \mathfrak{c}(a)$ be a closed sublocale of a totally κ-collectionwise normal frame L. Let $\mathfrak{c}_S(b)$ be a closed sublocale of S and let $\{a_i\}_{i\in I}$ be a disjoint κ-family of cozero elements of $\mathfrak{c}_S(b)$. Since $\mathfrak{c}_S(b) = \mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathfrak{c}(a \vee b)$, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $b_i \vee a \vee b = a_i$ for every $i \in I$. Then $\{b_i \vee a\}_{i\in I} \subseteq S$ is the desired disjoint family extending $\{a_i\}_{i\in I}$. □

3.4.3 Relative versions

Our goal is now to study the dual of total collectionwise normality. We begin by considering the relative counterpart of the notion of zero sublocale:

Definition 3.4.11. Let L be a locale and $\mathbb{F}$ a sublocale selection. A sublocale T of L is an $\mathbb{F}$-*zero sublocale of* L if there is some $\mathbb{F}$-continuous $f \in \overline{\mathsf{F}}(L)$ such that $T = f(\bigvee_{r\in\mathbb{Q}}(r,—))$.

We denote by $\mathsf{Z}_{\mathbb{F}}(L)$ the set of all $\mathbb{F}$-zero sublocales of L. Since $\mathbb{F}$-continuity is self-dual (see Corollary A.1.2) it follows that $\mathbb{F}$-zero and $\mathbb{F}^*$-zero sublocales coincide, i.e., $\mathsf{Z}_{\mathbb{F}}(L) = \mathsf{Z}_{\mathbb{F}^*}(L)$.

For our five guiding examples, one obtains the usual notion of zero sublocale (see Table A.1).

Definition 3.4.12. Let L be a locale and $\mathbb{F}$ a sublocale selection. A sublocale S of L is $\mathbb{F}$-z^c_κ-*embedded* in L if for every κ-family $\{S_i\}_{i\in I}$ consisting of $\mathbb{F}$-zero sublocales of S such that $S_i \vee S_j = S$ for every $i \ne j$, there is a family $\{T_i\}_{i\in I}$ of $\mathbb{F}$-zero sublocales of L such that $T_i \vee T_j = L$ for every $i \ne j$ and $T_i \cap S = S_i$ for every $i \in I$.

Once again, we note that this is a self-dual notion, and for our guiding examples (Table A.1), it coincides with that of z^c_κ-embedding.

After this preparation, we are now ready to define the relative version of total κ-collectionwise normality:

Definition 3.4.13. Let $\mathbb{F}$ be a sublocale selection. A frame L is an $\mathbb{F}$-z^c_κ *frame* if every $S \in \mathbb{F}(L)$ is $\mathbb{F}$-z^c_κ-embedded.

For $\mathbb{F} = \mathbb{F}_c$ it coincides with the notion of total κ-collectionwise normality. More generally, for any of the guiding examples $\mathbb{F}$ in Table A.1 (and their duals), it amounts to requiring that every sublocale contained in $\mathbb{F}(L)$ is z^c_κ-embedded.

3.4.4 Relation with $\mathbb{F}$-normality

Recall that total κ-collectionwise normality implies normality for $\kappa \geq 1$. With an eye towards Chapter 6 we now generalize this result to a general sublocale selection closed under countable meets and finite joins.

Proposition 3.4.14. *Let $\mathbb{F}$ be a sublocale selection closed under countable meets and finite joins and let $\kappa \geq 2$. If L is a $\mathbb{F}$-z^c_κ frame and $\mathbb{F}$ is weakly hereditary on L then L is $\mathbb{F}$-normal.*

Proof. It is of course enough to show it for $\kappa = 2$. Let L be an $\mathbb{F}$-z^c_2 frame and consider $S, T \in \mathbb{F}(L)$ such that $S \cap T = \mathsf{O}$. Then $S, T \in \mathbb{F}(S \vee T)$ (because $S = S \cap (S \vee T)$ and $T = T \cap (S \vee T)$ and $\mathbb{F}$ is weakly hereditary on L), hence $S = (S \vee T) \smallsetminus T \in \mathbb{F}^*(S \vee T)$. Therefore, $\chi_S \in \overline{\mathsf{C}}^{\mathbb{F}}(S \vee T)$ by virtue of Proposition A.1.5.

Observe that $\chi_S(\bigvee_{r \in \mathbb{Q}}(r, \text{—})) = T$ and so T is an $\mathbb{F}$-zero sublocale of $S \vee T$. Exchanging the roles of S and T, we see that S is an $\mathbb{F}$-zero sublocale of $S \vee T$ as well. Now, $S \vee T \in \mathbb{F}(L)$ because $\mathbb{F}$ is closed under finite joins, and therefore it is $\mathbb{F}$-z^c_2-embedded in L. Since $\{S, T\}$ is a disjoint family in $S \vee T$ consisting of $\mathbb{F}$-zero sublocales, there exist $\mathbb{F}$-zero sublocales A, B of L such that $A \vee B = L$, $A \cap (S \vee T) = T$ and $B \cap (S \vee T) = S$. Accordingly, $A \cap S = \mathsf{O}$ and $B \cap T = \mathsf{O}$. Finally, by Lemma A.1.3, A and B belong to $\mathbb{F}(L)$, and so L is $\mathbb{F}$-normal. □

It remains to be proved that, in this context, $\mathbb{F}$-z^c_1 frames are $\mathbb{F}$-z^c_2. Assume that $\mathbb{F}(L)$ is closed under countable meets and finite joins in $\mathsf{S}(L)$. Then $\mathbb{F}(L)$ may be regarded as a sub-σ-frame of $\mathsf{S}(L)^{op}$; and in particular as a σ-frame in its own right. Hence, we may use the theory of cozero elements in σ-frames (recall Subsection 1.6.3).

Lemma 3.4.15. *Let $\mathbb{F}$ be a sublocale selection and L a locale such that $\mathbb{F}(L)$ is a sub-σ-frame of $\mathsf{S}(L)^{op}$. Then* $\mathrm{Coz}\,\mathbb{F}(L) = \mathsf{Z}_{\mathbb{F}}(L)$.

Proof. Let $S \in \mathrm{Coz}\,\mathbb{F}(L)$ — i.e., there is a σ-frame homomorphism $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathbb{F}(L)$ with $S = f(\bigvee_{r \in \mathbb{Q}}(r, \text{—}))$. Consider the composite $\mathfrak{L}(\overline{\mathbb{R}}) \xrightarrow{f} \mathbb{F}(L) \xrightarrow{\iota} \mathsf{S}(L)^{op}$. It is clearly a frame homomorphism because it sends relations in $\mathfrak{L}(\overline{\mathbb{R}})$ to identities in $\mathsf{S}(L)^{op}$ (note that the relations only involve countable joins). Hence $S = (\iota \circ f)(\bigvee_{r \in \mathbb{Q}}(r, \text{—}))$ and $\iota \circ f$ is $\mathbb{F}$-continuous by Lemma A.1.3. Thus $S \in \mathsf{Z}_{\mathbb{F}}(L)$. The reverse inclusion follows trivially from Lemma A.1.3. □

The next lemma generalizes [50, Proposition 3.3].

Lemma 3.4.16. *Let $\mathbb{F}$ be a sublocale selection closed under countable meets and finite joins and let U be an arbitrary sublocale of L. If S and T are $\mathbb{F}$-zero sublocales of L such that $U = (U \cap S) \vee (U \cap T)$, then there are $\mathbb{F}$-zero sublocales S' and T' of L such that $S' \vee T' = L$, $U \cap S' = U \cap S$ and $U \cap T' = U \cap T$.*

Proof. By the previous lemma and Theorem 1.6.1 it follows that $\mathsf{Z}_{\mathbb{F}}(L)$ is a regular sub-σ-frame of $\mathbb{F}(L)$. Therefore, there are $\{S_n\}_{n\in\mathbb{N}}, \{T_n\}_{n\in\mathbb{N}} \subseteq \mathsf{Z}_{\mathbb{F}}(L)$ such that $S_n \prec S$ and $T_n \prec T$ in $\mathsf{Z}_{\mathbb{F}}(L)$ for each $n \in \mathbb{N}$, $S = \bigsqcup_n S_n$ and $T = \bigsqcup_n T_n$. By substituting S_n by $S_1 \sqcup \cdots \sqcup S_n$ (that satisfies $S_1 \sqcup \cdots \sqcup S_n \prec S$ in $\mathsf{Z}_{\mathbb{F}}(L)$), we may assume that $\{S_n\}_{n\in\mathbb{N}}$ (and also $\{T_n\}_{n\in\mathbb{N}}$) is increasing. Since $S_n \prec S$ and $T_n \prec T$ in $\mathsf{Z}_{\mathbb{F}}(L)$ for each $n \in \mathbb{N}$, there are C_n, D_n in $\mathsf{Z}_{\mathbb{F}}(L)$ such that $S_n \sqcap C_n = T_n \sqcap D_n = L$ and $S \sqcup C_n = T \sqcup D_n = \mathsf{O}$. Set $S' := \bigcap_{n\in\mathbb{N}} S_n \vee D_n \in \mathsf{Z}_{\mathbb{F}}(L)$ and $T' := \bigcap_{n\in\mathbb{N}} T_n \vee C_n \in \mathsf{Z}_{\mathbb{F}}(L)$. Clearly, $S' \vee T' = L$ (since $S_n \vee C_n = T_n \vee D_n = L$ and $\{S_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$ are increasing). Now, since $T \cap D_n = \mathsf{O}$, we observe that $T \cap S' = \bigcap_n (T \cap S_n) \vee (T \cap D_n) = \bigcap_n T \cap S_n = T \cap S$. Hence,

$$U \cap S' = (U \cap S \cap S') \vee (U \cap T \cap S') = (U \cap S \cap S') \vee (U \cap T \cap S) \subseteq U \cap S$$

while $U \cap S \subseteq U \cap S'$ is trivial from the definition of S'. Hence $U \cap S' = U \cap S$. The other identity follows by symmetry. □

Corollary 3.4.17. *Let $\mathbb{F}$ be a sublocale selection such that either $\mathbb{F}$ or $\mathbb{F}^*$ is closed under countable meets and finite joins. Then a frame is $\mathbb{F}$-z_1^c if and only if it is $\mathbb{F}$-z_2^c.*

Combining the last corollary with Proposition 3.4.14 we get the following:

Corollary 3.4.18. *Let $\mathbb{F}$ be a sublocale selection closed under countable meets and finite joins. Then, for each cardinal $\kappa \geq 1$, any $\mathbb{F}$-z_κ^c frame on which $\mathbb{F}$ is weakly hereditary is $\mathbb{F}$-normal.*

3.4.5 $\mathbb{F}_{\mathfrak{c}}^*$-$z_\kappa^c$ frames are just *Oz* frames

We can now dualize total collectionwise normality by picking $\mathbb{F} = \mathbb{F}_{\mathfrak{c}}^*$. Explicitly, a frame is $\mathbb{F}_{\mathfrak{c}}^*$-$z_\kappa^c$ if for each $a \in L$ and every κ-family $\{a_i\}_{i\in I}$ of cozero elements of $\mathfrak{o}(a)$ which is disjoint in $\mathfrak{o}(a)$ (i.e., $a_i \wedge a_j = a^*$ for all $i \neq j$), there is a disjoint κ-family $\{b_i\}_{i\in I}$ of cozero elements of L such that $\nu_{\mathfrak{o}(a)}(b_i) = a \to b_i = a_i$ for all $i \in I$.

In particular, we note that $\mathbb{F}_{\mathfrak{c}}^*$-$z_1^c$ frames are those in which every open sublocale is z-embedded — i.e., what in the literature are usually called *Oz frames* [25]. The *Oz* property is strictly weaker than extremal disconnectedness (cf. [25, Proposition 4.2]). Hence for a general sublocale selection $\mathbb{F}$, property $\mathbb{F}$-z_κ^c does not imply $\mathbb{F}$-normality (cf. Corollary 3.4.18).

Now, the main result of this section states that, as it happened in Theorem 3.3.12, the new notion collapses to the base case $\kappa = 1$:

Theorem 3.4.19. *An Oz frame is $\mathbb{F}_{\mathfrak{c}}^*$-$z_\kappa^c$ for every cardinal κ.*

Proof. Let $a \in L$ and let $\{a_i\}_{i\in I}$ be a κ-family of cozero elements in $\mathfrak{o}(a)$ with $a_i \wedge a_j = a^*$ for every $i \neq j$. Since L is *Oz*, for each $i \in I$ there is a cozero element b_i in L with $a \to b_i = a_i$. Note that the family $\{b_i\}_{i\in I}$ is not generally disjoint. Set now

$$c_i = b_i \wedge (a \wedge a_i)^{**}.$$

Then, $a \to c_i = (a \to b_i) \wedge (a \to (a \wedge a_i)^{**}) = a_i \wedge (a \to (a \wedge a_i)^{**}) = a_i$ (because $a \wedge a_i \leq (a \wedge a_i)^{**}$ — i.e., $a_i \leq a \to (a \wedge a_i)^{**}$).

Moreover, $\{c_i\}_{i\in I}$ is disjoint. Indeed, recall that double pseudocomplementation commutes with finite meets, and hence

$$c_i \wedge c_j \leq (a \wedge a_i)^{**} \wedge (a \wedge a_j)^{**} = (a \wedge a_i \wedge a_j)^{**} = (a \wedge a^*)^{**} = 0$$

for all $i \neq j$. Finally, each c_i is a cozero element in L because regular elements in Oz frames are cozero (cf. [25, Proposition 2.2]), and because cozero elements are closed under finite meets. □

Remark 3.4.20. This characterization of the Oz property extends that given in [50, Proposition 3.3] (for the open quotient case) from $\kappa = 2$ to an arbitrary cardinal.

3.5 Blair's cardinal generalization

As mentioned in Remarks 3.4.2, Blair introduced a different cardinal generalization of the notion of z-embedding (which he called z_κ-*embedding* for topological spaces). First, following [65] we shall say that a disjoint family of cozero elements $\{a_i\}_{i\in I}$ in a locale L is a *join cozero family* if $\bigvee_{i\in I} a_i$ is also a cozero element of L.

Now we propose the following point-free extension of Blair's notion.

Definition 3.5.1. Let κ be a cardinal. A sublocale S of L is z_κ-*embedded* in L if for every join cozero κ-family $\{a_i\}_{i\in I}$ in S, there is a join cozero family $\{b_i\}_{i\in I}$ in L such that $\nu_S(b_i) = a_i$ (that is, such that $\mathfrak{o}_S(a_i) = S \cap \mathfrak{o}(b_i)$) for every $i \in I$.

Remark 3.5.2. Blair's original definition of z_κ-embeddings for spaces is formally different to the one just given (join cozero families were not used in [33]). However, it follows immediately from [33, Theorem 3.8 (2)] and [65, Proposition 4.4] that the concept just defined is a conservative extension of the original topological notion.

By [65, Proposition 4.4 and Theorem 7.3] it follows easily that a locale is κ-collectionwise normal if and only if all its closed sublocales are z_κ-embedded (recall that analogously a locale is totally κ-collectionwise normal if and only if all its closed sublocales are z_κ^c-embedded).

On the dual side, we conservatively extend Blair's notion of Oz_κ space by saying that a frame is Oz_κ if all its open sublocales are z_κ-embedded (once again, for $\kappa = 1$ one simply gets Oz frames). Unlike the case of $\mathbb{F}_c^*$-z_κ^c frames, Oz_κ does not collapse to Oz (cf. [33, Proposition 8.14]; there are even extremally disconnected spaces which are not Oz_κ for suitable κ).

We conclude this section by giving a new characterization of Oz_κ frames in terms of join cozero κ-families.

Lemma 3.5.3. *Let L be an Oz_κ frame. Then a disjoint κ-family $\{a_i\}_{i\in I}$ of cozero elements of L is a join cozero family.*

Proof. We have to show that the join $\bigvee_{i\in I} a_i$ is a cozero element in L. For every $i \in I$, we may write $a_i = \bigvee_{n\in\mathbb{N}} a_n^i$ with a_n^i regular in L (as $a_i = \bigvee_{a_n^i \prec a} a_n^i$ and $b \prec a_i$ implies $b^{**} \prec a_i$). Obviously, for each $n \in \mathbb{N}$, the family $\{a_n^i\}_{i\in I}$ is disjoint (because so is $\{a_i\}_{i\in I}$ and $a_n^i \leq a_i$).

For each $n \in \mathbb{N}$, set

$$b_n = \bigvee_{i\in I} a_n^i.$$

Let $n \in \mathbb{N}$. Clearly, a_n^i is complemented (and hence cozero) in $\downarrow(b_n \vee b_n^*)$ for all $i \in I$ (observe that its complement is $b_n^* \vee \bigvee_{j\neq i} a_n^j$). Moreover, $\{a_n^i\}_{i\in I}$ is discrete in $\downarrow(b_n \vee b_n^*)$. The family $C = \{a_n^i \mid i \in I\} \cup \{b_n^*\}$ is obviously a cover of $\downarrow(b_n \vee b_n^*)$ and each of its members meets at most one of the a_n^i's. By Remark 4.5(4) in [65], $\{a_n^i\}_{i\in I}$ is a join cozero κ-family in $\downarrow(b_n \vee b_n^*)$.

Since L is Oz_κ, there is a join cozero κ-family $\{c_n^i\}_{i\in I}$ in L such that for all $i \in I$,

$$c_n^i \wedge (b_n \vee b_n^*) = a_n^i. \tag{3.5.1}$$

Taking double pseudocomplements, using the fact that double pseudocomplementation commutes with finite meets, and by an application of the first De Morgan law (FDM), one has

$$c_n^{i**} \wedge (b_n^* \wedge b_n^{**})^* = a_n^{i**} = a_n^i.$$

by the regularity of a_n^i. Since $b_n^* \wedge b_b^{**} = 0$, it follows that $c_n^{i**} = a_n^i$. In particular, $c_n^i \leq a_n^i$. But by (3.5.1) we also have $a_n^i \leq c_n^i$. Therefore, $a_n^i = c_n^i$ for all $i \in I$. But $\{c_n^i\}_{i\in i}$ is a join cozero family in L, hence $\bigvee_{i\in I} c_n^i = \bigvee_{i\in I} a_n^i = b_n$ is a cozero element in L. Since countable joins of cozero elements are cozero, $\bigvee_{n\in\mathbb{N}} b_n = \bigvee_{i\in i} a_i$ is a cozero element of L. □

Corollary 3.5.4. *The following are equivalent for a frame L and a cardinal κ:*

(i) *L is Oz_κ;*

(ii) *L is Oz and every disjoint κ-family consisting of cozero elements of L is a join cozero family.*

Proof. (i) $\Longrightarrow$ (ii): That Oz_κ implies Oz is trivial and the other half follows from Lemma 3.5.3.

(ii) $\Longrightarrow$ (i): This implication is a trivial consequence of Theorem 3.4.19 and the definitions of z_κ-embedding resp. z_κ^c-embedding. □

Chapter 4

Other infinite generalizations of extremal disconnectedness

In Chapter 3 we studied several variants of extremal disconnectedness together with their cardinal generalizations. As is well known, extremal disconnectedness can also be characterized by the finite second De Morgan law. More precisely, L is extremally disconnected if and only if

$$(a \wedge b)^* = a^* \vee b^*, \qquad \text{for every } a,b \in L, \tag{ED}$$

or, equivalently,

$$(a \vee b)^{**} = a^{**} \vee b^{**}, \qquad \text{for every } a,b \in L. \tag{ED$'$}$$

In light of the expressions above, it also makes sense to consider other cardinal generalizations of extremal disconnectedness, namely the infinite analogues of the formulas (ED) and (ED)′.

In this chapter, we will show that when considered in the infinite case, these conditions are no longer equivalent, and they define two different properties strictly between Booleaness — denoted by (CB) — and extremal disconnectedness.

The stronger one corresponds to the infinite second De Morgan law (IDM), and it can be expressed as the conjunction of the weaker one (which we call infinite extremal disconnectedness, (IED) for the sake of brevity) and a weak scatteredness condition. It is the goal of the present chapter to study these new conditions and present some classes of interesting examples.

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[4] I. Arrieta, On infinite variants of De Morgan law in locale theory, *Journal of Pure and Applied Algebra*, vol. 225 (1), art. no. 106460, 2021.

4.1 Preliminaries

We start by recalling the following facts (see [15, Theorem 1.2]):

Proposition 4.1.1. *Let L be a frame. Then the following hold*:

(1) $\bigwedge_{i=1}^{n} a_i^{**} = (\bigwedge_{i=1}^{n} a_i)^{**}$ *for every* $\{a_i\}_{i=1}^{n} \subseteq L$;

(2) $(a \to b)^{**} = a^{**} \to b^{**}$ *for every* $a, b \in L$.

Note that the infinite version of (1) is not true in general. Therefore, we shall say that a frame L is *$\perp$-scattered* if the nucleus $(-)^{**}\colon L \longrightarrow L$ preserves arbitrary meets — i.e., if

$$\bigwedge_{i\in I} a_i^{**} = \Big(\bigwedge_{i\in I} a_i\Big)^{**}, \qquad \text{for every } \{a_i\}_{i\in I} \subseteq L. \qquad (\perp\text{-scattered})$$

Remarks 4.1.2. (1) This terminology was introduced in [53] in the broader topos-theoretic setting. The topos theoretic notion is a faithful extension of the present notion — i.e., a locale L is $\perp$-scatered if and only if its topos of sheaves is $\perp$-scattered. It is a property strictly weaker than scatteredness, in the sense introduced by Plewe in [100], where a locale is said to be *scattered* if for each sublocale S of L, the Booleanization B_S is an open sublocale in S. More precisely, scatteredness is just the hereditary variant of $\perp$-scatteredness (see Proposition 4.1.3 (viii) below).

(2) This notion was also considered by Dube and Sarpoushi under the name of *near Booleaness* (cf. [49, Theorem 4.9]).

We have the following easy characterization:

Proposition 4.1.3 ([4, Proposition 4.1])**.** *The following conditions are equivalent for a frame L*:

(i) *L is $\perp$-scattered*;

(ii) *The nucleus $(-)^{**}\colon L \to L$ preserves arbitrary meets*;

(iii) *The frame homomorphism $(-)^{**}\colon L \to B_L$ preserves arbitrary meets*;

(iv) *$(-)^{**}\colon L \to B_L$ is a complete Heyting homomorphism;*

(v) $(\bigwedge_{i\in I} a_i)^{*} = (\bigvee_{i\in I} a_i^{*})^{**}$ *for every* $\{a_i\}_{i\in I} \subseteq L$;

(vi) *If $\{a_i\}_{i\in I} \subseteq L$ satisfies $\bigwedge_{i\in I} a_i = 0$, then $\bigwedge_{i\in I} a_i^{**} = 0$*;

(vii) *There exists an open $\perp$-scattered dense sublocale*;

(viii) *The Booleanization B_L is an open sublocale*;

(ix) *The interior of a dense sublocale is dense*.

We shall need the following lemmas later on:

Lemma 4.1.4. *The property of being $\perp$-scattered is inherited*

(1) *by open sublocales;*

(2) *by dense sublocales.*

Proof. (1) Let $a \in L$. It is easily seen that the pseudocomplement b^{*a} of an element b in $\downarrow a$ is given by $b^{*a} = b^* \wedge a$ and that $(b^{*a})^{*a} = b^{**} \wedge a$. Since nonempty meets in $\downarrow a$ are computed as in L, it is clear that the assertion holds.

(2) Pseudocomplements in dense sublocales are the same as in the ambient frame; and sublocales are always closed under meets. It is then obvious that being $\perp$-scattered is inherited by dense sublocales. □

Lemma 4.1.5. *If $f : L \longrightarrow M$ is an open localic map and L is $\perp$-scattered, then so is $f[L]$.*

Proof. First, since a localic map is open if and only if both halves of its surjection-embedding factorization are open, we can assume that f is also a surjection. Then its left adjoint f^* corresponds to an open subframe embedding, and these are well known to be closed under arbitrary meets and under pseudocomplementation [91, Proposition III 7.2]. □

We turn now our attention to extremal disconnectedness. In the following, we gather several well-known formulations (see [74, 91]):

Proposition 4.1.6. *The following conditions are equivalent for a frame L:*

(i) *L is extremally disconnected;*

(ii) $(\bigwedge_{i=1}^n a_i)^* = \bigvee_{i=1}^n a_i^*$ *for every* $\{a_i\}_{i=1}^n \subseteq L$; (*Second De Morgan law*)

(iii) *If* $\{a_i\}_{i=1}^n \subseteq L$ *satisfies* $\bigwedge_{i=1}^n a_i = 0$, *then* $\bigvee_{i=1}^n a_i^* = 1$;

(iv) $(\bigvee_{i=1}^n a_i)^{**} = \bigvee_{i=1}^n a_i^{**}$ *for every* $\{a_i\}_{i=1}^n \subseteq L$;

(v) *The nucleus* $(-)^{**} : L \longrightarrow L$ *preserves finite joins;*

(vi) *The nucleus* $(-)^{**} : L \longrightarrow L$ *is a lattice homomorphism;*

(vii) $(\bigwedge_{i=1}^n a_i)^* = \bigvee_{i=1}^n a_i^*$ *for every* $\{a_i\}_{i=1}^n \subseteq B_L$;

(viii) *If* $\{a_i\}_{i=1}^n \subseteq L$ *satisfies* $(\bigvee_{i=1}^n a_i)^* = 0$, *then* $\bigvee_{i=1}^n a_i^{**} = 1$.

It is easy to check that (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (vi) $\Longleftrightarrow$ (vii) $\Longleftrightarrow$ (viii), and (viii) $\Longrightarrow$ (iii) is true because of Lemma 4.1.1 (1).

4.2 Infinite versions of extremal disconnectedness

4.2.1 Infinitely De Morgan frames

We shall say that a frame L is *infinitely De Morgan* if it satisfies the *infinite* second De Morgan law — i.e., if

$$\Big(\bigwedge_{i\in I} a_i\Big)^* = \bigvee_{i\in I} a_i^*, \qquad \text{for every } \{a_i\}_{i\in I} \subseteq L. \tag{IDM}$$

For brevity such a frame will be referred to as an *IDM* frame.

We have the following characterization (cf. Proposition 4.1.6 (ii)–(iii)):

Proposition 4.2.1. *The following conditions are equivalent for a frame L:*

(i) *L is an IDM frame;*

(ii) *If $\{a_i\}_{i\in I} \subseteq L$ satisfies $\bigwedge_{i\in I} a_i = 0$, then $\bigvee_{i\in I} a_i^* = 1$.*

Proof. (i) $\Longrightarrow$ (ii) is obvious. (ii) $\Longrightarrow$ (i): Let $\{a_i\}_{i\in I} \subseteq L$ and $a = (\bigwedge_i a_i)^*$. Since $(\bigwedge_i a_i) \wedge a = 0$, it follows that $(\bigvee_i a_i^*) \vee a^* = 1$. Hence $a \leq \bigvee_i a_i^*$. The reverse inequality is trivial. □

From Proposition 4.2.1 (ii), we obtain the following characterization:

Corollary 4.2.2. *A locale L is IDM if and only if for each family of closed sublocales with dense join, the family of their interiors covers L.*

Remarks 4.2.3. (1) A frame which is also a coframe does not necessarily satisfy (IDM), even if it is extremally disconnected. Note that in a coframe one has an infinite second De Morgan law for supplements, but these need not coincide with pseudocomplements. For instance the frame $L = [0, 1]$ is extremally disconnected and totally ordered (and thus a coframe) but not IDM, see (2) below.

(2) It follows immediately from Proposition 4.2.1 (ii) that if 0 is a covered prime in a frame L then L is an IDM frame. The converse is true if L is totally ordered. Indeed, if $\{a_i\}_{i\in I} \subseteq L$ is such that $\bigwedge_i a_i = 0$ then, by (IDM), we have $\bigvee_i a_i^* = 1$. Since $a^* = 0$ whenever $a \neq 0$ it follows that there is some $i \in I$ with $a_i = 0$.

(3) Let L be *any* frame and define L^* to be the poset obtained by adding a new bottom element $\perp$ to L. It is easily seen that L^* is also a frame and that $\perp$ is a covered prime. Accordingly, one has that this new frame is IDM.

(4) Any complete Boolean algebra is an IDM frame, but there are non Boolean IDM frames. For instance, any non Boolean frame L such that 0 is a covered prime. An easy such example is the totally ordered frame $L = \mathbb{N} \cup \{+\infty\}$, or any frame constructed as in (3) above.

IDM frames are very close to being Boolean; in fact, under the very weak separation axiom of weak subfitness, both concepts coincide. Recall from Chapter 2 that a frame is weakly subfit if for each $a \neq 0$ there is some $c \neq 1$ with $c \vee a = 1$. Somewhat surprisingly, this property can also be characterized by the following formula for pseudocomplements:

Lemma 4.2.4. ([92, Theorem 5.2]) *Let L be a frame. The formula*

$$a^* = \bigwedge\{c \in L \mid c \vee a = 1\}$$

is valid for every $a \in L$ if and only if L is weakly subfit.

Any Boolean algebra is trivially weakly subfit. Moreover:

Lemma 4.2.5. *Let L be a frame. Then L is Boolean if and only if it is a weakly subfit and IDM frame.*

Proof. We only need to prove sufficiency. Let L be a weakly subfit and IDM frame and $a \in L$. By the previous lemma and the infinite second De Morgan law we get

$$a^{**} = (\bigwedge\{c \in L \mid c \vee a = 1\})^* = \bigvee\{c^* \mid c \vee a = 1\}.$$

Now if $c \vee a = 1$ it follows that $c^* \leq a$, hence $a^{**} \leq a$ for all $a \in L$. Thus L is Boolean. □

Remark 4.2.6. IDM does not imply weak subfitness and conversely. Indeed, the frame $L = \mathbb{N} \cup \{+\infty\}$ is IDM but not weakly subfit, and the cofinite topology on an infinite set is weakly subfit but not IDM.

4.2.2 Infinitely extremally disconnected frames

We shall say that a frame L is *infinitely extremally disconnected* if the nucleus $(-)^{**}\colon L \longrightarrow L$ preserves arbitrary joins — i.e., if

$$\Big(\bigvee_{i\in I} a_i\Big)^{**} = \bigvee_{i\in I} a_i^{**}, \quad \text{for every } \{a_i\}_{i\in I} \subseteq L. \tag{IED}$$

For brevity such a frame will be referred to as an *IED* frame. We have the following characterization (cf. Proposition 4.1.6 (iv)–(viii)):

Proposition 4.2.7. *The following conditions are equivalent for a frame L*:

(i) *L is an IED frame;*

(ii) *The nucleus $(-)^{**}\colon L \longrightarrow L$ preserves arbitrary joins;*

(iii) *The nucleus $(-)^{**}\colon L \longrightarrow L$ is a frame homomorphism;*

(iv) $(\bigwedge_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$ *for every $\{a_i\}_{i\in I} \subseteq B_L$;*

(v) *If $\{a_i\}_{i\in I} \subseteq L$ satisfies $(\bigvee_{i\in I} a_i)^* = 0$, then $\bigvee_{i\in I} a_i^{**} = 1$.*

Proof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious.

(iv) $\Longrightarrow$ (v): Let $\{a_i\}_{i\in I} \subseteq L$ such that $(\bigvee_i a_i)^* = 0$. Then $\{a_i^*\}_{i\in I} \subseteq B_L$ and so $1 = (\bigvee_i a_i)^{**} = (\bigwedge_i a_i^*)^* \leq \bigvee_i a_i^{**}$.

(v) $\Longrightarrow$ (i): Let $\{a_i\}_{i\in I} \subseteq L$ and $a = (\bigvee_i a_i)^*$. Since $(a \vee (\bigvee_i a_i))^* = 0$, it follows that $a^{**} \vee (\bigvee_i a_i^{**}) = 1$. Hence, $a^* = (\bigvee_i a_i)^{**} \leq \bigvee_i a_i^{**}$. The reverse inequality is trivial. □

From Proposition 4.2.7 (v), we obtain the following characterization:

Corollary 4.2.8. *A locale L is IED if and only if for each family of open sublocales with dense join, the family of the interiors of their closures covers L.*

Remarks 4.2.9. (1) In any irreducible frame L, i.e., such that $B_L = \{0, 1\}$, or equivalently such that 0 is prime (cf. [47]) condition (iv) in Proposition 4.2.7 is trivially satisfied. Consequently, any irreducible frame L is IED. In particular, totally ordered frames are clearly irreducible, and hence IED.

(2) Since (IDM) trivially implies condition (iv) in Proposition 4.2.7, it follows that any IDM frame is IED, but there are IED frames which fail to be IDM. An easy such example is the cofinite topology on an infinite set.

(3) Any IED frame is obviously extremally disconnected. However the converse is false, as any non-Boolean regular extremally disconnected frame shows (see Lemma 4.2.11 below). An easy such example is the Stone-Čech compactification of the frame of natural numbers.

(4) In any semi-irreducible frame L (i.e., such that B_L is finite), condition (iv) in Proposition 4.2.7 is clearly satisfied if the frame is extremally disconnected. Consequently, any semi-irreducible extremally disconnected frame L is IED.

Consequently we have the following chain of implications

$$\text{(CB)} \implies \text{(IDM)} \implies \text{(IED)} \implies \text{(ED)},$$

and none of them can be reversed. Frames satisfying (IED) arise quite naturally. We have seen that every irreducible frame is IED. Another interesting class of examples is given in the following:

Proposition 4.2.10. *Let L be a frame which is also a coframe. Then the following are equivalent*:

(i) *L is extremally disconnected;*

(ii) *L is an IED frame.*

Proof. We only need to prove (i) $\Longrightarrow$ (ii). Let L be an extremally disconnected frame which is also a coframe and let $\{a_i\}_{i\in I} \subseteq L$. We have $(\bigvee_i a_i)^* \vee \bigvee_j a_j^{**} = (\bigwedge_i a_i^*) \vee \bigvee_j a_j^{**} = \bigwedge_i (a_i^* \vee \bigvee_j a_j^{**})$ by coframe distributivity. Now, for each $i \in I$, one has $a_i^* \vee \bigvee_j a_j^{**} \geq a_i^* \vee a_i^{**} = 1$ by extremal disconnectedness. Hence $(\bigvee_i a_i)^* \vee \bigvee_j a_j^{**} = 1$, which implies $(\bigvee_i a_i)^{**} \leq \bigvee_j a_j^{**}$, the non-trivial inequality of (IED). □

Note that an analogous result for (IDM) is not true by Remark 4.2.3 (1). Moreover, we see in particular that the topology of every Alexandroff extremally disconnected space is IED (cf. also Corollary 4.4.8 below); and these spaces appear to arise in other areas (see for example [21]).

Finally, we identify a condition which together with property (IED) implies Booleaness. We recall that a frame L is said to be *semiregular* if every element is a join of regular elements. It is easily seen that every regular frame is semiregular.

Lemma 4.2.11. *Let L be a frame. Then L is Boolean if and only if it is a semiregular IED frame.*

Proof. We only need to prove sufficiency. Semiregularity means that B_L generates L by joins. But (IED) is equivalent to B_L being closed under joins. Hence $B_L = L$. □

4.2.3 The relation between IED frames and IDM frames

Recall that an IDM frame is IED. Furthermore, any IDM frame is trivially $\perp$-scattered (cf. Proposition 4.2.1 and Proposition 4.1.3 (vi)). Moreover:

Proposition 4.2.12. *Let L be a frame. Then L is IDM if and only if it is $\perp$-scattered and IED.*

Proof. We only need to prove sufficiency. Let L be a $\perp$-scattered and IED frame and consider $\{a_i\}_{i\in I} \subseteq L$ such that $\bigwedge_i a_i = 0$. By $\perp$-scatteredness one has $(\bigvee_i a_i^*)^* = \bigwedge_i a_i^{**} \leq (\bigwedge_i a_i)^{**} = 0$ and hence Proposition 4.2.7 (v) implies that $\bigvee_i a_i^* = \bigvee_i a_i^{***} = 1$. By Proposition 4.2.1 (ii) it follows that L is IDM. □

Now, if we combine this characterization with Propositions 4.1.3 and 4.2.7 we obtain:

Corollary 4.2.13. *A frame is IDM if and only if the nucleus $(-)^{**} \colon L \to L$ is a complete Heyting homomorphism.*

4.3 Properties of IDM and IED frames

We now state some further equivalent formulations of these properties in terms of the Booleanization:

Proposition 4.3.1. *The following conditions are equivalent for a frame L:*

(i) *L is an IED frame;*

(ii) *The Booleanization B_L is a subframe of L.*

Proof. The result follows immediately from Proposition 4.2.7 and the fact that a nucleus preserves arbitrary joins if and only if its associated sublocale is closed under arbitrary joins. □

The following is immediate from Propositions 4.1.3 and 4.3.2 and Proposition 4.2.12.

Proposition 4.3.2. *The following conditions are equivalent for a frame L:*

(i) *L is an IDM frame;*

(ii) *The Booleanization B_L is an open sublocale and a subframe of L;*

(iii) *The Booleanization B_L is open and a complete sublattice of L.*

It is a well-known fact that extremal disconnectedness is preserved under taking open or dense sublocales [64] and taking images under open localic morphisms [68]. We have also proved in Lemmas 4.1.4 and 4.1.5 that the same applies for $\perp$-scatteredness. In what follows we extend these results to the properties (IED) and (IDM).

Proposition 4.3.3. *Both properties IED and IDM are inherited*

(1) *by open sublocales;*

(2) *by dense sublocales.*

Proof. Clearly, the assertion for IDM frames will follow from the one for IED frames combined with Lemma 4.1.4 and Proposition 4.2.12. Now, that IED is inherited by open sublocales can be proved as in Lemma 4.1.4 (1). Let us finally show that IED is hereditary with respect to dense sublocales. Let S be a dense sublocale of an IED frame L and denote the joins in S by $\bigsqcup$. Note that in any dense sublocale one has $(\bigvee_i s_i)^* = (\bigsqcup_i s_i)^*$ for each $\{s_i\}_{i\in I} \subseteq S$. Indeed, by the first de Morgan law (FDM) in L (resp. in S) and the fact that pseudocomplements and meets are the same in S and L, both sides are equal to $\bigwedge_i s_i^*$. Since S is dense, we have that $B_L \subseteq S$, and therefore by the (IED) law in L, one has $\bigvee_i s_i^{**} = (\bigvee_i s_i)^{**} \in S$. Thus the join of $\{s_i^{**}\}_{i\in I} \subseteq S$ in S coincides with the one in L. It follows that $\bigsqcup_i s_i^{**} = \bigvee_i s_i^{**} = (\bigvee_i s_i)^{**} = (\bigsqcup_i s_i)^{**}$. □

We now have the following trivial observation:

Lemma 4.3.4. *Let M be a subframe of L and assume that M is closed under pseudocomplementation in L. If L is IED, then so is M.*

Corollary 4.3.5. *If $f\colon L \longrightarrow M$ is an open localic map and L is IED (resp. IDM), then so is $f[L]$.*

Proof. In view of Lemma 4.1.5 and Proposition 4.2.12, it suffices to show the assertion for IED. Moreover, since a localic map is open if and only if both halves of its surjection-embedding factorization are open, one can assume without loss of generality that f is surjective. Now, the left adjoint f^* of f corresponds to an open subframe inclusion, and these are closed under pseudocomplements. Therefore, the result follows from the previous lemma. □

4.4 Hereditary variants

Recall that given a property $\mathcal{P}$ of locales, a locale L is said to be hereditarily $\mathcal{P}$ if each sublocale of L satisfies $\mathcal{P}$. Our main interest in this section is to study hereditarily IDM and hereditarily IED locales. We first note the following:

Proposition 4.4.1. *Let $\mathcal{P}$ be a property of locales such that each dense sublocale of a locale satisfying $\mathcal{P}$ also satisfies $\mathcal{P}$. Then a locale L is hereditarily $\mathcal{P}$ if and only if each closed sublocale of L satisfies $\mathcal{P}$.*

Proof. We only need to prove sufficiency. Let L be a locale such that each closed sublocale of L satisfies $\mathcal{P}$ and let S be an arbitrary sublocale of L. Then $\overline{S}$ is closed and so it has property $\mathcal{P}$. Now S is dense in $\overline{S}$ and since $\mathcal{P}$ is hereditary with respect to dense sublocales, it follows that S also has property $\mathcal{P}$. □

From Lemma 4.1.4 and Proposition 4.3.3 we get then the following (note that scatteredness is precisely hereditary $\perp$-scatteredness — cf. [100]):

Corollary 4.4.2. *Let L be a locale. Then:*

(1) *L is scattered if and only if each closed sublocale of L is $\perp$-scattered.*

(2) *L is hereditarily extremally disconnected if and only if each closed sublocale of L is extremally disconnected.*

(3) *L is hereditarily IDM if and only if each closed sublocale of L is IDM.*

(4) *L is hereditarily IED if and only if each closed sublocale of L is IED.*

Remarks 4.4.3. (1) By Proposition 4.2.12 we now have that a locale L is hereditarily IDM if and only if it is scattered and hereditarily IED.

(2) Since a locale of the form $\Omega(X)$ can have more sublocales than the induced ones, it is not clear from the definition whether hereditary IED and IDM are conservative properties. But they are, in view of (3) and (4) above.

We met hereditarily extremally disconnected locales in Subsection 2.4.3. The following proposition provides some further well-known characterizations, see for example [64, 74].

Proposition 4.4.4. *The following conditions are equivalent for a frame L:*

(i) *L is hereditarily extremally disconnected;*

(ii) $(a \to b) \vee (b \to a) = 1$ *for every $a, b \in L$;* (*Strong De Morgan law*)

(iii) $(\bigwedge_{i=1}^n a_i) \to b = \bigvee_{i=1}^n (a_i \to b)$ *for every $b \in L$ and every $\{a_i\}_{i=1}^n \subseteq L$;*

(iv) $(\bigwedge_{i=1}^n a_i) \to b = \bigvee_{i=1}^n (a_i \to b)$ *for every $b \in L$ and every $\{a_i\}_{i=1}^n \subseteq \mathfrak{c}(b)$;*

(v) $((\bigvee_{i=1}^n a_i) \to b) \to b = \bigvee_{i=1}^n ((a_i \to b) \to b)$ *for every $b \in L$ and every $\{a_i\}_{i=1}^n \subseteq L$;*

(vi) $((\bigvee_{i=1}^n a_i) \to b) \to b = \bigvee_{i=1}^n ((a_i \to b) \to b)$ *for every $b \in L$ and every $\{a_i\}_{i=1}^n \subseteq \mathfrak{c}(b)$.*

We now have the following characterizations of hereditarily IDM and IED frames:

Proposition 4.4.5. *The following conditions are equivalent for a frame L:*

(i) *L is hereditarily IDM;*

(ii) $(\bigwedge_{i\in I} a_i) \to b = \bigvee_{i\in I}(a_i \to b)$ *for every $b \in L$ and every $\{a_i\}_{i\in I} \subseteq L$;* ($\bigwedge\bigvee$-*distr*)

(iii) $(\bigwedge_{i\in I} a_i) \to b = \bigvee_{i\in I}(a_i \to b)$ *for every $b \in L$ and every $\{a_i\}_{i\in I} \subseteq \mathfrak{c}(b)$.*

Proof. (i) $\Longrightarrow$ (ii): Let $b \in L$, $\{a_i\}_{i\in I} \subseteq L$ and $d = (\bigwedge_i a_i) \wedge b$. By hypothesis $\mathfrak{c}(d)$ is IDM, and since pseudocomplementation in $\mathfrak{c}(d)$ is given by $x^{*\mathfrak{c}(d)} = x \to d$ and $\mathfrak{c}(d)$ is closed under meets and nonempty joins in L it follows that $(\bigwedge_i a_i) \to d = (\bigwedge_i a_i)^{*\mathfrak{c}(d)} \leq \bigvee_i a_i^{*\mathfrak{c}(d)} = \bigvee_i (a_i \to d)$. Consequently $(\bigwedge_i a_i) \to b = (\bigwedge_i a_i) \to d \leq \bigvee_i (a_i \to d) \leq \bigvee_i (a_i \to b)$, whereas the reverse inequality is trivial.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): By Corollary 4.4.2, it is enough to prove that each closed sublocale is IDM. Let $b \in L$ and $\{a_i\}_{i\in I} \subseteq \mathfrak{c}(b)$ such that $\bigwedge_i a_i = b$. Then $\bigvee_i a_i^{*\mathfrak{c}(b)} = \bigvee_i (a_i \to b) = (\bigwedge_i a_i) \to b = 1$. By Proposition 4.2.1 (ii) it follows that $\mathfrak{c}(b)$ is IDM. □

Similarly one has the following.

Proposition 4.4.6. *The following conditions are equivalent for a frame L:*

(i) *L is hereditarily IED;*

(ii) $((\bigvee_{i\in I} a_i) \to b) \to b = \bigvee_{i\in I}((a_i \to b) \to b)$ *for every $b \in L$ and all $\{a_i\}_{i\in I} \subseteq L$;*

(iii) $((\bigvee_{i\in I} a_i) \to b) \to b = \bigvee_{i\in I}((a_i \to b) \to b)$ *for every $b \in L$ and every $\{a_i\}_{i\in I} \subseteq \mathfrak{c}(b)$.*

Example 4.4.7. (1) Every totally ordered frame is hereditarily IED by Remark 4.2.9 (1).

(2) For any frame L, the IDM frame L^* constructed in Remark 4.2.3 (3) is not, in general, hereditarily IDM nor hereditarily IED (in fact, L^* is hereditarily IDM, resp. IED, if and only if so is L, because proper closed sublocales of L^* and closed sublocales of L coincide).

Note that if a frame is aditionally a coframe, then all its closed sublocales are also coframes. Therefore, by Proposition 4.2.10, we can now improve Proposition 2.5.2 and obtain a further supply of hereditarily IED frames:

Corollary 4.4.8. *The following are equivalent for a poset X:*

(i) *X is a coforest;*

(ii) $\mathsf{Dwn}(X)$ *is hereditarily extremally disconnected;*

(iii) $\mathsf{Dwn}(X)$ *is hereditarily* IED;

(iv) $\mathsf{Dwn}(X)$ *has property* (A).

4.5 The largest dense IED sublocale

We conclude this chapter by exploring further the category of IED locales. The following proposition (together with the results thereafter) provides some evidence of the fact that the IED condition itself is actually a better behaved strengthening of extremal disconnectedness compared to the IDM condition. Furthermore, in view of the DeMorganization construction (namely, the existence of the largest dense De Morgan sublocale) established in [38, Theorem 2.10] it also seems to share a stronger parallel with extremal disconnectedness.

Proposition 4.5.1. *Any locale has a largest dense IED sublocale.*

Proof. We define the following binary relation (in the sense of Section 1.4):

$$R = \{ (\bigvee_i a_i^{**}, (\bigvee_i a_i)^{**}) \mid \{a_i\}_{i\in I} \subseteq L \} \subseteq L \times L.$$

We set $S := L/R$. As explained in Section 1.4, S is a sublocale of L; and an application of the first De Morgan law (FDM) shows that 0 is R-saturated — i.e., S is a dense sublocale. If $T \subseteq L$ is an arbitrary dense IED sublocale of L, we want to show that $T \subseteq S$. By density, pseudocomplements coincide in each of the frames T, S and L. Let ν_T denote the left adjoint to the sublocale embedding and denote the joins in T by $\sqcup$. Since ν_T is a dense surjection, it preserves pseudocomplements (i.e., it is nearly open, see [76, p. 227] and recall Subsection 1.2.3). Let $t \in T$. Our goal is to show that t is R-saturated, that is, $(\bigvee_i a_i)^{**} \to t = (\bigvee_i a_i^{**}) \to t$ for any family $\{a_i\}_{i\in I} \subseteq L$. Since T satisfies (IED), one obtains

$$\nu_T(\bigvee_i a_i^{**}) = \bigsqcup_i \nu_T(a_i)^{**} = (\bigsqcup_i \nu_T(a_i))^{**} = \nu_T(\bigvee_i a_i)^{**} = \nu_T((\bigvee_i a_i)^{**}).$$

Then, since $t \in T$, we have

$$(\bigvee_i a_i)^{**} \to t = \nu_T((\bigvee_i a_i)^{**}) \to t = \nu_T(\bigvee_i a_i^{**}) \to t = (\bigvee_i a_i^{**}) \to t,$$

as required. The only point remaining is to show that S is IED. Let $\{s_i\}_{i\in I} \subseteq S$. Then

$$(\bigsqcup_i s_i)^{**} = \nu_S(\bigvee_i s_i)^{**} = \nu_S((\bigvee_i s_i)^{**}) = \nu_S(\bigvee_i s_i^{**}) = \bigsqcup_i s_i^{**},$$

where $\bigsqcup$ now denotes join in S and we have used again the fact that ν_S is nearly open as it is a dense surjection. This proves the result. □

Remark 4.5.2. The construction of the largest IED sublocale is not generally functorial (this should not be a surprise because neither of the Booleanization or the DeMorganization construction [38] are normally functorial). Nevertheless, there are certain morphisms for which it is. We do not know how to characterize the class of those morphisms which restrict to the largest IED sublocales but it notably includes all the nearly open frame homomorphisms (this is easily seen by an application of Theorem 1.4.1).

Lemma 4.5.3. *If L is totally ordered, there exists the largest dense IDM sublocale if and only if L is IDM.*

Proof. The "if" part is trivial. Conversely, if L is a totally ordered frame, $a \to b$ is either equal to 1 or b for each $a, b \in L$, and hence a sublocale of L is just a subset closed under meets. Moreover, since a sublocale is in particular a subposet, it is also a chain, and hence the characterization in Remark 4.2.3 (2) still applies. Denote by S the largest dense IDM sublocale of L. By contradiction, if $S \neq L$, pick an $a \in L$ such that $a \notin S$. Then $S \cup \{a\}$ is obviously closed under meets and hence a (dense) sublocale. Furthermore, 0 is a covered prime in $S \cup \{a\}$, for if $a \wedge \bigwedge_i a_i = 0$ for some $\{a_i\}_{i \in I} \subseteq S$, since $a \neq 0$ and 0 is always prime in a chain, it follows that $\bigwedge_i a_i = 0$ and so $a_i = 0$ for some $i \in I$. This contradicts the maximality of S. ◻

The previous lemma yields examples of locales that do not possess the largest dense IDM sublocale (for instance, $L = [0, 1]$).

Remark 4.5.4. Originally the DeMorganization construction was proved more generally for toposes, cf. [38, 39]. Therefore, it seems natural to consider Proposition 4.5.1 in that context. We are not going to do so in this dissertation, except to say that one would need to define the IED property for toposes appropriately. It is not sensible to define an IED topos to be one in which double negation $\neg\neg\colon \Omega \longrightarrow \Omega$ has an internal right adjoint, since an easy modification of the proof of Theorem 6 in [104] shows that in that case the topos would be necessarily Boolean.

Chapter 5

Frame presentations of compact hedgehogs and their properties

5.1 Introduction

The usual topology on the (extended) reals can be naturally introduced in two completely different ways:

- It is the metric topology induced by the euclidean metric.
- It is the Lawson topology induced by the linear order.

The first approach is probably the best known. In this case the topology, being a metric topology, is generated by the basis of all open balls, i.e. the open intervals $\langle a,b\rangle$ with $a<b$ in $\mathbb{R}$ (or just with $a<b$ in $\mathbb{Q}$).

The second approach is of particular interest when one is interested in notions like lower and upper semicontinuity. In this case one first generates two topologies:

(1) The *Scott topology*, that is, the smallest topology in which the sets

$$\Uparrow a = \{x \in \mathbb{R} \mid a < x\}$$

are *open* for all a in $\mathbb{R}$ (or, equivalently, with a just in $\mathbb{Q}$).

(2) The *lower topology*, that is, the smallest topology in which the principal filters

$$\uparrow a = \{x \in \mathbb{R} \mid a \le x\}$$

are *closed* for all a in $\mathbb{R}$ (or, equivalently, with $a \in \mathbb{Q}$).

Then the usual euclidean topology is the Lawson topology, that is, the common refinement of the Scott and the lower topologies ([58, Chapter III]).

Now, the hedgehog can be described as a set of spines identified at a single point. More precisely, given a cardinal κ and a set I of cardinality κ, the *hedgehog with κ spines* $J(\kappa)$ is the disjoint union $\bigcup_{i\in I}\left(\overline{\mathbb{R}}\times\{i\}\right)$ of κ copies (the spines) of the extended real line identified at $-\infty$:

$$J(\kappa)=\{-\infty\}\cup\bigcup_{i\in I}((-\infty,+\infty]\times\{i\}).$$

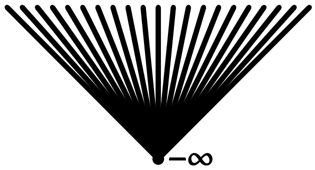


Fig. 5.1 The hedgehog.

The metric topology on $J(\kappa)$ is precisely the cardinal generalization of the metric topology on the unit real interval (see [3] for a description of this topology). Point-freely, it is modelled by the *frame of the metric hedgehog with κ spines* [65], namely the frame $\mathfrak{L}(J(\kappa))$ generated by abstract symbols $(r,\!-\!)_i$ and $(-,r)$, $r\in\mathbb{Q}$ and $i\in I$, subject to the following relations (see Figure 5.2):

(h0) $(r,\!-\!)_i\wedge(s,\!-\!)_j=0$ whenever $i\neq j$;

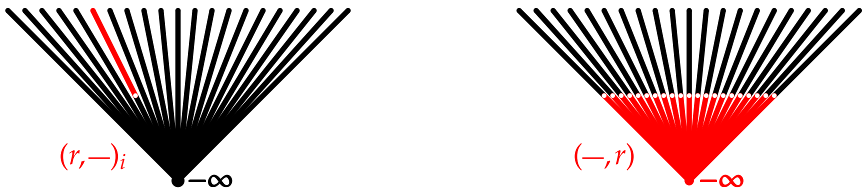
(h1) $(r,\!-\!)_i\wedge(-,s)=0$ whenever $r\geq s$ and $i\in I$;

(h2) $\bigvee_{i\in I}(r_i,\!-\!)_i\vee(-,s)=1$ whenever $r_i<s$ for every $i\in I$;

(h3) $(r,\!-\!)_i=\bigvee_{s>r}(s,\!-\!)_i$ for every $r\in\mathbb{Q}$ and $i\in I$;

(h4) $(-,r)=\bigvee_{s<r}(-,s)$ for every $r\in\mathbb{Q}$.

Fig. 5.2 The metric hedgehog generators.



We can also consider an extension on $J(\kappa)$ of the Lawson topology. For that purpose, introduce first the following (partial) order on $J(\kappa)$:

$$(t,i)\leq(s,j)\;\equiv\;t=-\infty\quad\text{or}\quad i=j,\ t\leq s.$$

The poset $(J(\kappa),\leq)$ is evidently a cardinal generalization of $(\overline{\mathbb{R}},\leq)$, being $(J(1),\leq)$ precisely $(\overline{\mathbb{R}},\leq)$. In general, for an arbitrary cardinal κ, it fails to be a complete lattice (but it is still a bounded complete domain [58]). We can still generate two topologies:

(1) The *Scott topology*, that is, the smallest topology in which the sets

$$\Uparrow(r,i) = \{\,(t,j) \in J(\kappa) \mid (r,i) \ll (t,j)\,\} = (r,+\infty] \times \{i\}$$

are *open* for all $r \in \mathbb{Q}$ and $i \in I$.

(2) The *lower topology*, that is, the smallest topology in which the principal filters

$$\uparrow(r,i) = \{\,(t,j) \in J(\kappa) \mid (r,i) \leq (t,j)\,\} = [r,+\infty] \times \{i\}$$

are *closed* for all $r \in \mathbb{Q}$ and $i \in I$.

The Lawson topology is the common refinement of the Scott and the lower topologies. This is a compact Hausdorff topology on $J(\kappa)$, referred to as the *compact hedgehog space* and denoted by

$$\Lambda J(\kappa)$$

(see [58, Exercise III-3.2 and Theorem III-5.8]). It yields a separable metrizable space if and only if $\kappa \leq \aleph_0$ (see [58, Corollary III-4.6.] and [3, Properties 6.7 (6) and (7)]). A subbasis of $\Lambda(J(\kappa))$ is given by

$$\{\,(r,\text{—})_i \mid r \in \mathbb{Q}, i \in I\,\} \cup \{\,(\text{—},r)_i \mid r \in \mathbb{Q}, i \in I\,\}$$

where $(r,\text{—})_i := (r,+\infty] \times \{i\}$ and $(\text{—},r)_i := J(\kappa) - [r,+\infty] \times \{i\}$ (see Figure 5.3).

With this topology, natural notions of upper and lower semicontinuity arise. We recall from [61] that a function f defined on a topological space X with values in the hedgehog $J(\kappa)$ is said to be *lower semicontinuous* if it is continuous with respect to the Scott topology — i.e., $f^{-1}((r,+\infty] \times \{i\})$ is open in X for every $r \in \mathbb{Q}$ and $i \in I$ (this notion should not be confused with the one of Blair and Swardson [34]); similarly, it is *upper semicontinuous* if it is continuous with respect to the lower topology — i.e., $f^{-1}(J(\kappa) - [r,+\infty] \times \{i\})$ is open in X for every $r \in \mathbb{Q}$ and $i \in I$. It is said to be *continuous* if it is continuous with respect to the Lawson topology, i.e. if it is both lower and upper semicontinuous.

Our aim in this chapter is to study the compact topology of the hedgehog via frame presentations by generators and relations (cf. Subsection 1.1.1). The main focus will be on the point-free version of continuous and semicontinuous functions with values in the compact hedgehog that arise from it, and their relation with variants and generalizations of normality introduced in Chapter 3.

The material presented in this chapter is part of a joint work with Javier Gutiérrez García and Jorge Picado:

[7] I. Arrieta, J. Gutiérrez García, and J. Picado, Frame presentations of compact hedgehogs and their properties, *Quaestiones Mathematicae*, accepted for publication.

5.2 The compact localic hedgehog and its basic properties

We define the *frame of the compact hedgehog with κ spines* to be the frame $\mathfrak{L}(cJ(\kappa))$ presented by generators $(r,\text{—})_i$ and $(\text{—},r)_i$, $r \in \mathbb{Q}$ and $i \in I$, subject to the following relations (cf. Figure 5.3):

(ch0) $(r,\text{—})_i \wedge (s,\text{—})_j = 0$ whenever $i \neq j$;

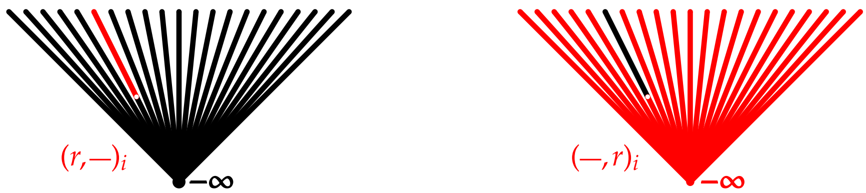
(ch1) $(r,\text{—})_i \wedge (\text{—},s)_i = 0$ whenever $r \geq s$ for every $i \in I$;

(ch2) $(r,\text{—})_i \vee (\text{—},s)_i = 1$ whenever $r < s$ for every $i \in I$;

(ch3) $(r,\text{—})_i = \bigvee_{s>r} (s,\text{—})_i$ for every $r \in \mathbb{Q}$ and $i \in I$;

(ch4) $(\text{—},r)_i = \bigvee_{s<r} (\text{—},s)_i$ for every $r \in \mathbb{Q}$ and $i \in I$.

Fig. 5.3 The compact hedgehog generators.



Remark 5.2.1. There is an alternative presentation for the frame $\mathfrak{L}(cJ(\kappa))$. Indeed, we define $\mathfrak{L}_c(J(\kappa))$ to be the subframe of the frame of the metric hedgehog $\mathfrak{L}(J(\kappa))$ generated by the elements

$$(r,\text{—})_i \quad \text{and} \quad (r,\text{—})_i^* = \bigvee_{j \neq i} (r-1,\text{—})_j \vee (\text{—},r), \qquad r \in \mathbb{Q},\ i \in I.$$

It is then a straightforward (but tedious) exercise to check that $\mathfrak{L}_c(J(\kappa)) \cong \mathfrak{L}(cJ(\kappa))$. We shall omit the details.

The following proposition indicates that the compact hedgehog and the metric hedgehog coincide when κ is a finite cardinal. Since the latter was studied in [65], we shall mostly be interested in the infinite case.

Proposition 5.2.2. *$\mathfrak{L}_c(J(\kappa))$ is a proper subframe of $\mathfrak{L}(J(\kappa))$ if and only if κ is infinite.*

Proof. If κ is finite, then $\bigwedge_{i\in I} (r,\text{—})_i^* = \left(\bigwedge_{i\in I} \bigvee_{j\neq i} (r-1,\text{—})_j\right) \vee (\text{—},r) = (\text{—},r)$, hence $\mathfrak{L}_c(J(\kappa)) = \mathfrak{L}(J(\kappa))$.

Otherwise, if κ is infinite, then the frame $\mathfrak{L}(J(\kappa))$ is not compact (this is a consequence of the defining relation (h2), see [65, Remark 3.1]). But, as we shall see in Theorem 5.2.5 below, $\mathfrak{L}(cJ(\kappa))$, and hence $\mathfrak{L}_c(J(\kappa))$, is a compact frame. $\square$

We start with a few basic properties of the frame of the compact hedgehog. First, it is a straightforward exercise to check that for each $i \in I$ the assignments from $\mathfrak{L}(cJ(\kappa))$ into $\mathfrak{L}(\overline{\mathbb{R}})$

given by

$$(r,\text{—})_j \longmapsto \begin{cases} (r,\text{—}), & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \quad \text{and} \quad (\text{—},r)_j \longmapsto \begin{cases} (\text{—},r), & \text{if } j = i, \\ 1, & \text{if } j \neq i, \end{cases}$$

turn the defining relations (ch0)–(ch4) into identities in $\mathfrak{L}(\overline{\mathbb{R}})$ and thus determine a surjective frame homomorphism $h_i \colon \mathfrak{L}(cJ(\kappa)) \to \mathfrak{L}(\overline{\mathbb{R}})$. Observe that similarly we can define surjective frame homomorphisms $h'_i \colon \mathfrak{L}(cJ(\kappa)) \to \Omega(\overline{\mathbb{Q}})$. The following properties follow easily from the existence of these frame homomorphisms:

Properties 5.2.3. (1) $(r,\text{—})_i \wedge (s,\text{—})_j = 0$ if and only if $i \neq j$;

(2) $(r,\text{—})_i \wedge (\text{—},s)_i = 0$ if and only if $r \geq s$;

(3) $(r,\text{—})_i \vee (\text{—},s)_j = 1$ if and only if $r < s$ and $i = j$;

(4) $(\text{—},r)_i \vee (\text{—},s)_j = 1$ if and only if $i \neq j$;

(5) $\bigvee_{i\in I} \bigvee_{r\in\mathbb{Q}} (r,\text{—})_i \neq 1$;

(6) For each $i \in I$, $\bigvee_{r\in\mathbb{Q}} (\text{—},r)_i \neq 1$.

Next, we introduce another important family of frame homomorphisms:

Proposition 5.2.4. *For each $i \in I$, there is a frame homomorphism $\pi_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(cJ(\kappa))$ given by*

$$\pi_i(r,\text{—}) = (r,\text{—})_i \quad \textit{and} \quad \pi_i(\text{—},r) = (\text{—},r)_i$$

for all $r \in \mathbb{Q}$.

Proof. Let us confirm that it sends the relations (r0)–(r3) into identities in $\mathfrak{L}(cJ(\kappa))$:

(r1) Let $r \geq s$ in $\mathbb{Q}$. Then $\pi_i(r,\text{—}) \wedge \pi_i(\text{—},s) = (r,\text{—})_i \wedge (\text{—},s)_i = 0$ by (ch1).

(r2) Let $r < s$ in $\mathbb{Q}$. Then $\pi_i(r,\text{—}) \vee \pi_i(\text{—},s) = (r,\text{—})_i \vee (\text{—},s)_i = 1$ by (ch2).

(r3) Let $r \in \mathbb{Q}$. We have $\pi_i(r,\text{—}) = (r,\text{—})_i = \bigvee_{s>r}(s,\text{—})_i = \bigvee_{s>r} \pi_i(s,\text{—})$ by (ch3).

(r4) Let $r \in \mathbb{Q}$. We have $\pi_i(\text{—},r) = (\text{—},r)_i = \bigvee_{s<r}(\text{—},s)_i = \bigvee_{s<r} \pi_i(\text{—},s)$ by (ch4). □

We shall refer to π_i as the *i-th projection*. Observe that $h_i \circ \pi_i$ is the identity in $\mathfrak{L}(\overline{\mathbb{R}})$, and so π_i is injective.

We can now prove the main result of this section.

Theorem 5.2.5. *$\mathfrak{L}(cJ(\kappa))$ is a compact regular frame.*

Proof. Consider the unique frame homomorphism f, given by the coproduct universal property, for which the following diagram commutes:

$$\begin{tikzcd} \mathfrak{L}(\overline{\mathbb{R}}) \arrow[r, "\iota_i"] \arrow[dr, "\pi_i"'] & \bigoplus_{i\in I} \mathfrak{L}(\overline{\mathbb{R}}) \arrow[d, dashed, "f"] \\ & \mathfrak{L}(cJ(\kappa)) \end{tikzcd}$$

Let

$$a=\bigvee_{i\neq j}\iota_i(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))\wedge\iota_j(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))\in\bigoplus_{i\in I}\mathfrak{L}(\overline{\mathbb{R}}).$$

By (ch0) we have

$$f(a)=\bigvee_{i\neq j}f(\iota_i(\bigvee_{r\in\mathbb{Q}}(r,\text{—})))\wedge f(\iota_j(\bigvee_{s\in\mathbb{Q}}(s,\text{—})))=\bigvee_{i\neq j}\bigvee_{r\in\mathbb{Q}}\bigvee_{s\in\mathbb{Q}}(r,\text{—})_i\wedge(s,\text{—})_j=0.$$

Moreover, $f(a\vee\iota_i(r,\text{—}))=f(\iota_i(r,\text{—}))=\pi_i(r,\text{—})=(r,\text{—})_i$ and $f(a\vee\iota_i(\text{—},s))=f(\iota_i(\text{—},s))=\pi_i(\text{—},s)=(\text{—},s)_i$ for every $i\in I$ and $r,s\in\mathbb{Q}$. Hence the map $k\colon\mathfrak{c}(a)\to\mathfrak{L}(cJ(\kappa))$ given by $k(x)=f(x)$ for each $x\in\mathfrak{c}(a)$ is a surjective frame homomorphism making the following triangle

$$\begin{tikzcd}
\bigoplus_{i\in I}\mathfrak{L}(\overline{\mathbb{R}}) \arrow[r, "f"] \arrow[d, "(-)\vee a"'] & \mathfrak{L}(cJ(\kappa)) \\
\mathfrak{c}(a) \arrow[ur, "k"'] &
\end{tikzcd}$$

commute. On the other hand, the assignments

$$(\text{—},r)_i\longmapsto\iota_i(r,\text{—})\vee a\quad\text{ and }\quad(s,\text{—})_i\longmapsto\iota_i(\text{—},s)\vee a$$

for each $r,s\in\mathbb{Q}$ and $i\in I$ determine a frame homomorphism

$$g\colon\mathfrak{L}(cJ(\kappa))\to\mathfrak{c}(a)$$

(the fact that they turn the relations (ch0)–(ch4) into identities in $\mathfrak{c}(a)$ follows easily from the fact that each ι_i is a frame homomorphism and the fact that the relations (r0)–(r3) are satisfied in $\mathfrak{L}(\overline{\mathbb{R}})$). Thus g is the unique frame homomorphism that makes the triangle

$$\begin{tikzcd}
\bigoplus_{i\in I}\mathfrak{L}(\overline{\mathbb{R}}) \arrow[r, "f"] \arrow[d, "(-)\vee a"'] & \mathfrak{L}(cJ(\kappa)) \arrow[dl, "g"] \\
\mathfrak{c}(a) &
\end{tikzcd}$$

commutative (the fact that it commutes obviously follows from the fact that the coproduct injections are jointly epimorphic). Consequently, $\mathfrak{L}(cJ(\kappa))$ and $\mathfrak{c}(a)$ are isomorphic frames, and the latter is regular and compact because it is a closed sublocale of a regular and compact frame (by Tychonoff's Theorem for frames [91]). □

Remark 5.2.6. Since the localic Tychonoff's Theorem [82] and compactness of $\mathfrak{L}(\overline{\mathbb{R}})$ [22] are constructively valid, the proof above is also constructively valid provided the index set I has decidable equality (i.e., for all $i,j\in I$, one has either $i=j$ or $i\neq j$). Implicitly, we had already assumed this in the defining relation (ch0).

Regarding metrizability, we have the following:

Proposition 5.2.7. *$\mathfrak{L}(cJ(\kappa))$ is metrizable if and only if $\kappa \leq \aleph_0$.*

Proof. The coproduct of countably many metrizable frames is metrizable by virtue of [71, p. 31]. Hence, if $\kappa \leq \aleph_0$, then, for $|I| = \kappa$, $\bigoplus_{i\in I} \mathfrak{L}(\overline{\mathbb{R}})$ is metrizable, and so is any of its frame quotients, thus $\mathfrak{L}(cJ(\kappa))$ is metrizable.

Conversely, if $\mathfrak{L}(cJ(\kappa))$ is metrizable, since it is also compact, then it must have a countable $\bigvee$-base ([29, 4.3]). Let $B = \{b_n\}_{n\in\mathbb{N}}$ be such a base. Then, for each $i \in I$, there is some $n_i \in \mathbb{N}$ such that $0 \neq b_{n_i} \leq (0,\text{—})_i$. Consequently, $\{b_{n_i}\}_{i\in I}$ is a disjoint family of nonzero elements contained in B, hence $\kappa = |I| \leq \aleph_0$. ◻

Hence, by [46, Proposition 3], we have:

Corollary 5.2.8. *For $\kappa \leq \aleph_0$, any regular subframe of $\mathfrak{L}(cJ(\kappa))$ is metrizable.*

5.2.1 The spectrum of $\mathfrak{L}(cJ(\kappa))$

In what follows, we shall show that the spectrum of $\mathfrak{L}(cJ(\kappa))$ is homeomorphic to the hedgehog $J(\kappa)$ endowed with the compact topology (see [61]). First we need a few lemmas about primes in $\mathfrak{L}(cJ(\kappa))$.

Lemma 5.2.9. *All the following elements of $\mathfrak{L}(cJ(\kappa))$ are prime (hence maximal):*

(1) $\bigvee_{r>t} (r,\text{—})_i \vee \bigvee_{r<t} (\text{—},r)_i$ *for any $t \in \mathbb{R}$ and $i \in I$;*

(2) $\bigvee_{i\in I} \bigvee_{r\in\mathbb{Q}} (r,\text{—})_i$;

(3) $\bigvee_{r\in\mathbb{Q}} (\text{—},r)_i$ *for any $i \in I$.*

Proof. First note that since $\mathfrak{L}(cJ(\kappa))$ is a regular frame, any prime element is maximal by Property 1.2.3 (4). We only show the case for (2), the others may be proved similarly.

By Property 5.2.3 (5), the element $p = \bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} (r,\text{—})_i$ is not the top element. Clearly, p is prime if and only if the map

$$h\colon \mathfrak{L}(cJ(\kappa)) \to \{0 < 1\},$$

given by $h(a) = 0$ if $a \leq p$ and $h(a) = 1$ otherwise, is a frame homomorphism. For that it suffices to show that the assignments $h(r,\text{—})_i = 0$ if and only if $(r,\text{—})_i \leq p$, and $h(\text{—},r)_i = 0$ if and only if $(\text{—},r)_i \leq p$, send the defining relations into identities. But $(r,\text{—})_i \leq p$ for any $r \in \mathbb{Q}$ and $i \in I$. Hence $h(r,\text{—})_i = 0$ for all $r \in \mathbb{Q}$ and $i \in I$. Moreover, $(\text{—},r)_i \leq p$ together with (ch2) would imply $p = 1$, hence $h(\text{—},r)_i = 1$ for all $r \in \mathbb{Q}$ and $i \in I$. Now it is clear that h turns relations (ch0)–(ch4) into identities in the two-element frame $\{0 < 1\}$. ◻

Lemma 5.2.10. *For each $p \in \mathsf{pt}(\mathfrak{L}(cJ(\kappa)))$ let*

$$\alpha(p) = \bigvee\Big\{ r \in \mathbb{Q} \mid \bigvee_{i\in I} (r,\text{—})_i \not\leq p \Big\} \in \overline{\mathbb{R}}.$$

We have:

(1) $\alpha(p) = -\infty$ *if and only if* $p = \bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} (r,\text{—})_i$;

(2) *If* $\alpha(p) \neq -\infty$, *then there is a unique* $i_p \in I$ *such that* $(r,\text{—})_{i_p} \not\leq p$ *for some* $r \in \mathbb{Q}$;

(3) *If* $\alpha(p) \neq -\infty$, *then* $\alpha(p) = \bigwedge\{s \in \mathbb{Q} \mid (\text{—},s)_{i_p} \not\leq p\}$;

(4) *If* $\alpha(p) \in \mathbb{R}$, *then* $p = \left(\bigvee_{r>\alpha(p)} (r,\text{—})_{i_p}\right) \vee \left(\bigvee_{s<\alpha(p)} (\text{—},s)_{i_p}\right)$;

(5) *If* $\alpha(p) = +\infty$, *then* $p = \bigvee_{r\in\mathbb{Q}}(\text{—},r)_{i_p}$.

Proof. (1) Clearly $\alpha(p) = -\infty$ if and only if $\bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} (r,\text{—})_i \leq p$. The conclusion follows from Lemma 5.2.9 (2).
(2) The existence is obvious from the definition of $\alpha(p)$. For uniqueness, assume that there are distinct $i_p, j_p \in I$ such that $(r,\text{—})_{i_p} \not\leq p$ and $(s,\text{—})_{j_p} \not\leq p$. Then $(r,\text{—})_{i_p} \wedge (s,\text{—})_{j_p} \not\leq p$ since p is prime, which contradicts (ch0).
(3) Let $r \in \mathbb{Q}$ such that $\bigvee_{i\in I} (r,\text{—})_i \not\leq p$. Then there is an $i \in I$ satisfying $(r,\text{—})_i \not\leq p$. By uniqueness of i_p, $i = i_p$. Let $s \in \mathbb{Q}$ such that $(\text{—},s)_{i_p} \not\leq p$. Then $r \leq s$ (otherwise, by (ch0), $s < r$ would imply $0 = (r,\text{—})_{i_p} \wedge (\text{—},s)_{i_p} \not\leq p$). Hence $\alpha(p) \leq \bigwedge\{s \in \mathbb{Q} \mid (\text{—},s)_{i_p} \not\leq p\}$. The inequality cannot be strict, for otherwise there would exist $r_1, s_1 \in \mathbb{Q}$ such that

$$\alpha(p) < r_1 < s_1 < \bigwedge\{s \in \mathbb{Q} \mid (\text{—},s)_{i_p} \not\leq p\},$$

and then $(r_1,\text{—})_{i_p} \leq p$ and $(\text{—},s_1)_{i_p} \leq p$, a contradiction (since $1 = (r_1,\text{—})_{i_p} \vee (\text{—},s_1)_{i_p}$ by (ch2)).
(4) Suppose $\alpha(p) \in \mathbb{R}$. By Lemma 5.2.9 (1), it is enough to show that for every $r > \alpha(p)$ and every $s < \alpha(p)$ one has $(r,\text{—})_{i_p} \leq p$ and $(\text{—},s)_{i_p} \leq p$. Now, the former inequality follows from the definition of $\alpha(p)$ while the latter follows from (3).
(5) It follows from (3) that $\bigvee_{r\in\mathbb{Q}}(\text{—},r)_{i_p} \leq p$. The equality follows then from Lemma 5.2.9 (3). □

Proposition 5.2.11. *The spectrum of* $\mathfrak{L}(cJ(\kappa))$ *is homeomorphic to the compact hedgehog space* $\Lambda J(\kappa)$.

Proof. Consider the map $\pi\colon \Sigma(\mathfrak{L}(cJ(\kappa))) \longrightarrow \Lambda J(\kappa)$ given by

$$\pi(p) = \begin{cases} (\alpha(p), i_p) & \text{if } \alpha(p) \neq -\infty; \\ \boldsymbol{-\infty} & \text{otherwise.} \end{cases}$$

It readily follows from Lemma 5.2.10 (1), (4) and (5) that π is one-to-one. Let us show that π is also onto.

By Lemma 5.2.10 (1), $\pi\left(\bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} (r,\text{—})_i\right) = \boldsymbol{-\infty}$, and, by Lemma 5.2.10 (5), $\pi\left(\bigvee_{r\in\mathbb{Q}} (\text{—},r)_i\right) = (+\infty, i)$. For each $t \in \mathbb{R}$ and $i \in I$ set

$$p_{(t,i)} = \Big(\bigvee_{r>t} (r,\text{—})_i\Big) \vee \Big(\bigvee_{r<t} (\text{—},r)_i\Big).$$

It is straightforward to check that $\bigvee_{j\in I}(s,\text{—})_j \leq p_{(t,i)}$ if and only if $s \geq t$. Hence $\alpha(p_{(t,i)}) = \bigvee\{s \mid s < t\} = t$. Moreover, if we select $s < t$, then we have $(s,\text{—})_i \not\leq p_{(t,i)}$ (as otherwise $p_{(t,i)} = (s,\text{—})_i \vee p_{(t,i)} = 1$ by (ch2), contradicting maximality). Therefore $i_{p_{(t,i)}} = i$ and so $\pi(p_{(t,i)}) = (t,i)$.

Furthermore, π is lower semicontinuous, since

$$\pi^{-1}((r,+\infty]\times\{i\}) = \{p\in\Sigma(\mathfrak{L}(cJ(\kappa)))\mid (r,\text{—})_i\not\leq p\} = \Sigma_{(r,\text{—})_i}$$

is open for every $r\in\mathbb{Q}$ and $i\in I$, and upper semicontinuous, since

$$\pi^{-1}(J(\kappa)-[r,+\infty]\times\{i\}) = \{p\in\Sigma(\mathfrak{L}(cJ(\kappa)))\mid (\text{—},r)_i\not\leq p\} = \Sigma_{(\text{—},r)_i}$$

is open for every $r\in\mathbb{Q}$ and $i\in I$. Hence π is continuous.

Finally, let us prove that π is an open map. Note that, since $\mathfrak{L}(cJ(\kappa)))$ is generated by $\{(r,\text{—})_i,(\text{—},r)_i\mid r\in\mathbb{Q}, i\in I\}$ and π is a bijection, it suffices to show that the sets $\pi(\Sigma_{(r,\text{—})_i})$ and $\pi(\Sigma_{(\text{—},r)_i})$ are open for every $r\in\mathbb{Q}$ and $i\in I$. We have

$$\pi(\Sigma_{(r,\text{—})_i}) = \{\pi(p)\mid (r,\text{—})_i\not\leq p\} = \{(t,i)\mid t>r\} = (r,1]\times\{i\} \qquad \text{and}$$
$$\pi(\Sigma_{(\text{—},r)_i}) = \{\pi(p)\mid (\text{—},r)_i\not\leq p\} = J(\kappa)-[r,1]\times\{i\}.$$

Hence π is a homeomorphism. □

5.3 Semicontinuities

We are now in position to start developing the theory of semicontinuities of compact hedgehog-valued functions. First, a *compact hedgehog-valued continuous function will be a frame homomorphism* $f\colon\mathfrak{L}(cJ(\kappa))\to L$. The family of all compact hedgehog-valued continuous functions will be denoted by $\mathsf{C}_\kappa(L)$. Furthermore, a *compact hedgehog-valued*

- *function* on L will be a frame homomorphism $f\colon\mathfrak{L}(cJ(\kappa))\to\mathsf{S}(L)^{op}$;
- *lower semicontinuous function* on L will be a frame homomorphism $f\colon\mathfrak{L}(cJ(\kappa))\to\mathsf{S}(L)^{op}$ such that $f((r,\text{—})_i)$ is closed for every $r\in\mathbb{Q}$ and $i\in I$;
- *upper semicontinuous function* on L will be a frame homomorphism $f\colon\mathfrak{L}(cJ(\kappa))\to\mathsf{S}(L)^{op}$ such that $f((\text{—},r)_i)$ is closed for every $r\in\mathbb{Q}$ and $i\in I$.

The corresponding classes of compact hedgehog-valued functions will be denoted by, respectively,

$$\mathsf{F}_\kappa(L),\ \mathsf{LSC}_\kappa(L),\ \text{ and }\ \mathsf{USC}_\kappa(L).$$

By the isomorphism $L\cong\mathfrak{c}_L[L]$ of Subsection 1.2.1, we may regard compact hedgehog-valued continuous functions on L as frame homomorphisms $f\colon\mathfrak{L}(cJ(\kappa))\to\mathsf{S}(L)^{op}$ such that $f((\text{—},r)_i)$ and $f((r,\text{—})_i)$ are closed for every $r\in\mathbb{Q}$ and $i\in I$. Under this identification, we note that $\mathsf{C}_\kappa(L)=\mathsf{LSC}_\kappa(L)\cap\mathsf{USC}_\kappa(L)$.

The following lemma is an immediate consequence of the definition of the i-th projection π_i.

Lemma 5.3.1. *Let L be a locale and $f \in \mathsf{F}_\kappa(L)$. Then:*

(1) $f \in \mathsf{LSC}_\kappa(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{LSC}}(L)$ *for all* $i \in I$;

(2) $f \in \mathsf{USC}_\kappa(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{USC}}(L)$ *for all* $i \in I$;

(3) $f \in \mathsf{C}_\kappa(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{C}}(L)$ *for all* $i \in I$.

5.3.1 Disjoint families of extended real-valued functions

In this subsection we shall show that disjoint families of extended real-valued functions are closely related to certain compact hedgehog-valued functions. Given a topological space X, a family of functions $\left\{f_i \colon : X \to \overline{\mathbb{R}}\right\}_{i\in I}$ is said to be *disjoint* if for each $i \neq j$ and $x \in X$, either $f_i(x) = -\infty$ or $f_j(x) = -\infty$. The following is the point-free extension of this notion:

Definition 5.3.2. Let
$$\mathcal{H} = \left\{h_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathsf{S}(L)^{op}\right\}_{i\in I} \subseteq \overline{\mathsf{F}}(L)$$
be a family of extended real-valued functions on L and set $S_i = h_i(\bigvee_{r\in\mathbb{Q}}(r,\text{—}))$. We say that $\mathcal{H}$ is *disjoint* if $S_i \sqcap S_j = 0$ (i.e., $S_i \vee S_j = L$) for every $i \neq j$.

Remark 5.3.3. In the particular case of $\mathcal{H} \subseteq \overline{\mathsf{C}}(L)$, if we regard functions in $\mathcal{H}$ simply as frame homomorphisms $\mathfrak{L}(\overline{\mathbb{R}}) \to L$, the disjointness condition is equivalent to the family $\{a_i\}_{i\in I}$ being disjoint, where $a_i = \bigvee_{r\in\mathbb{Q}} h_i(r,\text{—})$ is the cozero element associated to h_i.

We now have the following fundamental result:

Proposition 5.3.4. *If $\mathcal{H} = \{h_i \mid i \in I\}$ is a disjoint κ-family of extended real-valued functions on L, then there is a unique $h_{\mathcal{H}} \in \mathsf{F}_\kappa(L)$ such that the diagram*

$$\begin{tikzcd}
\mathfrak{L}(\overline{\mathbb{R}}) \arrow[r, "\pi_i"] \arrow[dr, "h_i"'] & \mathfrak{L}(cJ(\kappa)) \arrow[d, dashed, "h_{\mathcal{H}}"] \\
& \mathsf{S}(L)^{op}
\end{tikzcd}$$

commutes for all $i \in I$. Conversely, given $h \in \mathsf{F}_\kappa(L)$, the κ-family $\{h \circ \pi_i\}_{i\in I}$ is disjoint.

Proof. Let us first show the uniqueness part. If $h_{\mathcal{H}} \circ \pi_i = h_i$ for all i, then $h_{\mathcal{H}}((r,\text{—})_i) = h_{\mathcal{H}}(\pi_i(r,\text{—})) = h_i(r,\text{—})$ and $h_{\mathcal{H}}((\text{—},r)_i) = h_{\mathcal{H}}(\pi_i(\text{—},r)) = h_i(\text{—},r)$, and thus $h_{\mathcal{H}}$ is uniquely determined.

For the existence, we define $h_{\mathcal{H}} \colon \mathfrak{L}(cJ(\kappa)) \to \mathsf{S}(L)^{op}$ by the assignments $h_{\mathcal{H}}((r,\text{—})_i) = h_i(r,\text{—})$ and $h_{\mathcal{H}}((\text{—},r)_i) = h_i(\text{—},r)$. Let us confirm that it turns the relations (ch0)–(ch4) into identities in $\mathsf{S}(L)^{op}$:

(ch0) Let $i \neq j$. Then $h_{\mathcal{H}}((r,\text{—})_i) \sqcap h((s,\text{—})_j) = h_i(r,\text{—}) \sqcap h_j(s,\text{—}) \leq S_i \sqcap S_j = 0$.

(ch1) Let $r \geq s$. Then $h_{\mathcal{H}}((r,\text{—})_i) \sqcap h_{\mathcal{H}}((\text{—},s)_i) = h_i(r,\text{—}) \sqcap h_i(\text{—},s) = h_i((r,\text{—}) \wedge (\text{—},s)) = h_i(0) = 0$.

(ch2) Let $r < s$. Then $h_{\mathcal{H}}((r,\text{—})_i) \sqcup h_{\mathcal{H}}((\text{—},s)_i) = h_i(r,\text{—}) \sqcup h_i(\text{—},s) = h_i((r,\text{—}) \vee (\text{—},s)) = h_i(1) = 1$.

(ch3) $h_{\mathcal{H}}((r,\text{—})_i) = h_i(r,\text{—}) = h_i\left(\bigvee_{s>r}(s,\text{—})\right) = \bigsqcup_{s>r} h((s,\text{—})_i)$.

(ch4) Similar to (ch3).

Trivially, $h_{\mathcal{H}} \circ \pi_i = h_i$ for all $i \in I$. The converse statement is an easy consequence of (ch0) and the frame distributive law. □

We conclude with some immediate corollaries of Lemma 5.3.1, Proposition 5.3.4 and Remark 5.3.3.

Corollary 5.3.5. *Let L be a locale, $\mathcal{H} \subseteq \overline{\mathsf{F}}(L)$ a disjoint κ-family and $h_{\mathcal{H}}$ be the compact hedgehog-valued function provided by Proposition 5.3.4. Then:*

(1) *$h_{\mathcal{H}} \in \mathsf{LSC}_\kappa(L)$ if and only if $h \in \overline{\mathsf{LSC}}(L)$ for all $h \in \mathcal{H}$;*

(2) *$h_{\mathcal{H}} \in \mathsf{USC}_\kappa(L)$ if and only if $h \in \overline{\mathsf{USC}}(L)$ for all $h \in \mathcal{H}$;*

(3) *$h_{\mathcal{H}} \in \mathsf{C}_\kappa(L)$ if and only if $h \in \overline{\mathsf{C}}(L)$ for all $h \in \mathcal{H}$.*

Corollary 5.3.6. *A κ-family $\{a_i\}_{i\in I} \subseteq L$ is disjoint and consists of cozero elements of L if and only if there is an $f \in \mathsf{C}_\kappa(L)$ such that $a_i = \bigvee_{r\in\mathbb{Q}} f((r,\text{—})_i)$ for all $i \in I$.*

It is convenient to specialize the previous general results to the case of (extended) characteristic functions. We recall that given a complemented sublocale S of a locale L, the *extended characteristic function* $\chi_S \in \overline{\mathsf{F}}(L)$ (see [26, Example 2]) is defined by

$$\chi_S(r,\text{—}) = S^{\#} \quad \text{and} \quad \chi_S(\text{—},r) = S, \quad r \in \mathbb{Q}.$$

Obviously, $\chi_S \in \overline{\mathsf{LSC}}(L)$ (resp. $\chi_S \in \overline{\mathsf{USC}}(L)$) if and only if S is an open (resp. closed) sublocale.

Remark 5.3.7. A κ-family $C = \{S_i\}_{i\in I}$ of complemented sublocales of L is pairwise disjoint in $\mathsf{S}(L)$ if and only if the corresponding κ-family $\{\chi_{S_i}\}_{i\in I}$ of extended characteristic functions is disjoint in the sense of Definition 5.3.2. Hence, by Proposition 5.3.4, such a family induces an $h \in \mathsf{F}_\kappa(L)$ such that $h \circ \pi_i = \chi_{S_i}$ for all $i \in I$. This h will be denoted by χ_C and we shall refer to it as the *characteristic function* of the family C.

Finally, from Corollary 5.3.5 we obtain:

Corollary 5.3.8. *Let L be a locale and $C = \{S_i\}_{i\in I}$ a pairwise disjoint κ-family of complemented sublocales of L. Then:*

(1) *$\chi_C \in \mathsf{LSC}_\kappa(L)$ if and only if each S_i is open;*

(2) *$\chi_C \in \mathsf{USC}_\kappa(L)$ if and only if each S_i is closed;*

(3) *$\chi_C \in \mathsf{C}_\kappa(L)$ if and only if each S_i is clopen.*

5.4 Extension results

We say that a disjoint κ-family $\mathcal{H}_S \subseteq \overline{\mathsf{C}}(S)$ *can be disjointly extended* to L if there is a disjoint κ-family $\mathcal{H} = \left\{\overline{h} \mid h \in \mathcal{H}_S\right\} \subseteq \overline{\mathsf{C}}(L)$ in which each $\overline{h}$ is a continuous extension of h. Further, a locale L will be said to have the *κ-disjoint extension property* if for each $a \in L$ every disjoint κ-family $\mathcal{H}_{\mathfrak{c}(a)} \subseteq \overline{\mathsf{C}}(\mathfrak{c}(a))$ can be disjointly extended to L. The following characterization of the κ-disjoint extension property is a straightforward consequence of Lemma 5.3.1 (3) and Proposition 5.3.4 (but see also Theorem 5.4.3 below for Tietze-type result containing a different characterization).

Proposition 5.4.1. *The following are equivalent for a locale L:*

(i) *L has the κ-disjoint extension property;*

(ii) *For each $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ has an extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

Recall the notion of z_κ^c-embedding from Subsection 3.4.1 of Chapter 3. We are now ready to characterize it as a property about appropriate compact hedgehog-valued functions (see Remark 3.4.2 (4)).

Lemma 5.4.2. *The following are equivalent for a sublocale $S \subseteq L$:*

(i) *S is z_κ^c-embedded in L;*

(ii) *For each $f \in \mathsf{C}_\kappa(S)$, there is a $g \in \mathsf{C}_\kappa(L)$ such that $\nu_S\big(g(\bigvee_{r\in\mathbb{Q}}(r,\text{—})_i)\big) = f(\bigvee_{r\in\mathbb{Q}}(r,\text{—})_i)$ for every $i \in I$.*

Proof. (i) $\Longrightarrow$ (ii): Let $f \in \mathsf{C}_\kappa(S)$. For each $i \in I$ set

$$a_i := \bigvee_{r\in\mathbb{Q}} (f \circ \pi_i)(r,\text{—}) = \bigvee_{r\in\mathbb{Q}} f((r,\text{—})_i).$$

By Corollary 5.3.6, $\{a_i\}_{i\in I}$ is a disjoint κ-family of cozero elements of S. Then, by assumption, there is a disjoint family $\{b_i\}_{i\in I}$ of cozero elements of L such that $\nu_S(b_i) = a_i$ for every $i \in I$. Applying Corollary 5.3.6 to $\{b_i\}_{i\in I}$ we get a $g \in \mathsf{C}_\kappa(L)$ such that $b_i = \bigvee_{r\in\mathbb{Q}} g((r,\text{—})_i)$. Finally, $\nu_S\big(\bigvee_{r\in\mathbb{Q}} g((r,\text{—})_i)\big) = \nu_S(b_i) = \bigvee_{r\in\mathbb{Q}} f((r,\text{—})_i)$ for every $i \in I$.

(ii) $\Longrightarrow$ (i): Let $\{a_i\}_{i\in I}$ be a disjoint κ-family of cozero elements of S, and take the $f \in \mathsf{C}_\kappa(S)$, provided by Corollary 5.3.6, that satisfies $a_i = \bigvee_{r\in\mathbb{Q}} f((r,\text{—})_i)$ for all $i \in I$. By hypothesis, there is a $g \in \mathsf{C}_\kappa(L)$ such that $\nu_S\left(\bigvee_{r\in\mathbb{Q}} g((r,\text{—})_i)\right) = a_i$ for all $i \in I$. Set $b_i := \bigvee_{r\in\mathbb{Q}} g((r,\text{—})_i)$ for each $i \in I$. Clearly, $\{b_i\}_{i\in I}$ is the claimed disjoint family. $\square$

The following is the main result of this section and provides a Tietze-type theorem for compact hedgehog valued functions that characterizes total κ-collectionwise normality (recall Section 3.4 in Chapter 3).

Theorem 5.4.3 (Tietze-type theorem for total κ-collectionwise normality)**.** *The following are equivalent for a locale L:*

(i) *L is totally κ-collectionwise normal;*

(ii) *For each $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ has an extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

Proof. (i) $\Longrightarrow$ (ii): Let $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ and set $a_i := \bigvee_{r\in\mathbb{Q}} f((r,\text{—})_i)$. By the previous lemma, there is a $g \in \mathsf{C}_\kappa(L)$ such that $a \vee b_i = a_i$ for every $i \in I$, where $b_i := \bigvee_{r\in\mathbb{Q}} g((r,\text{—})_i)$. For each $i \in I$, consider

$$h_1^{(i)} = f \circ \pi_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a) \quad \text{and} \quad h_2^{(i)} = \mathbf{0} \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(b_i)$$

(the latter defined by $h_2^{(i)}(r,\text{—}) = b_i$ and $h_2^{(i)}(\text{—},r) = 1$ for every $r \in \mathbb{Q}$). Let us show that $h_1^{(i)}(x) \vee b_i = h_2^{(i)}(x) \vee a$ for every $x \in \mathfrak{L}(\overline{\mathbb{R}})$ by checking it for the generators of $\mathfrak{L}(\overline{\mathbb{R}})$. For each $r \in \mathbb{Q}$ we have

$$a \vee b_i \le h_1^{(i)}(r,\text{—}) \vee b_i = f((r,\text{—})_i) \vee b_i \le a_i \vee b_i = a \vee b_i.$$

Hence $h_1^{(i)}(r,\text{—}) \vee b_i = a \vee b_i = h_2^{(i)}(r,\text{—}) \vee a$. On the other hand, pick some rational $t < r$ and conclude that

$$\begin{aligned} h_1^{(i)}(\text{—},r) \vee b_i &= (h^{(i)}(\text{—},r) \vee a) \vee b_i = f((\text{—},r)_i) \vee (a \vee b_i) = f((\text{—},r)_i) \vee a_i \\ &\ge f((\text{—},r)_i) \vee a_i \ge f((\text{—},r)_i) \vee f((t,\text{—})_i) = 1. \end{aligned}$$

Hence $h_1^{(i)}(\text{—},r) \vee b_i = 1 = h_2^{(i)}(\text{—},r) \vee a$.

Consequently, by Proposition 3.4.6, there is a frame homomorphism $h_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a \wedge b_i)$, such that $\nu_1 \circ h_i = h_1^{(i)}$ and $\nu_2 \circ h_i = h_2^{(i)}$, where $\nu_1 \colon \mathfrak{c}(a \wedge b_i) \to \mathfrak{c}(a)$ and $\nu_2 \colon \mathfrak{c}(a \wedge b_i) \to \mathfrak{c}(b_i)$ are the associated surjections. Since L is normal (by Corollary 3.4.8), the standard point-free version of Tietze's extension theorem [91, Corollary XIV 7.6.1] yields a frame homomorphism $g_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $\nu \circ g_i = h_i$, where $\nu \colon L \to \mathfrak{c}(a \wedge b_i)$ is the corresponding surjection. Observe that the family $\{g_i\}_{i\in I}$ is disjoint since

$$g_i(\bigvee_{r\in\mathbb{Q}} (r,\text{—})) \le h_i(\bigvee_{r\in\mathbb{Q}} (r,\text{—})) \le h_2^{(i)}(\bigvee_{r\in\mathbb{Q}} (r,\text{—})) = b_i$$

and $\{b_i\}_{i\in I}$ is disjoint. Therefore, consider the continuous hedgehog-valued function $h \colon \mathfrak{L}(cJ(\kappa)) \to L$ defined by $h \circ \pi_i = g_i$ for all $i \in I$ (provided by Proposition 5.3.4 and Corollary 5.3.5). This is the claimed extension. Indeed, denote by $\nu_{\mathfrak{c}(a)} \colon L \to \mathfrak{c}(a)$ the surjection associated to $\mathfrak{c}(a)$. Then

$$\nu_{\mathfrak{c}(a)} \circ h \circ \pi_i = \nu_{\mathfrak{c}(a)} \circ g_i = \nu_1 \circ \nu \circ g_i = \nu_1 \circ h_i = h_1^{(i)} = f \circ \pi_i,$$

and thus $\nu_{\mathfrak{c}(a)} \circ h = f$ follows from the uniqueness of Proposition 5.3.4.

(ii) $\Longrightarrow$ (i) is an immediate consequence of the implication (ii) $\Longrightarrow$ (i) in the previous lemma. $\square$

Corollary 5.4.4. *The following are equivalent for a locale L:*

(i) *L is totally collectionwise normal;*

(ii) *For every $\kappa \geq 1$ and $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ has an extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

We can give a new characterization of normality by combining the previous theorem with Proposition 3.4.5:

Corollary 5.4.5. *The following are equivalent for a locale L:*

(i) *L is normal;*

(ii) *For each $a \in L$, every $f \in \mathsf{C}_{\aleph_0}(\mathfrak{c}(a))$ has an extension $\overline{f} \in \mathsf{C}_{\aleph_0}(L)$.*

5.5 Insertion results

We close this chapter with the corresponding Katětov-Tong-type insertion results for compact hedgehog-valued functions.

Recall the partial order in $\overline{\mathsf{C}}(L)$ in (1.6.1). We may extend it to $\mathsf{C}_\kappa(L)$ by defining, for any $f, g \in \mathsf{C}_\kappa(L)$,

$$f \leq g \iff f \circ \pi_i \leq g \circ \pi_i \text{ for every } i \in I. \tag{5.5.1}$$

Since $\mathsf{F}_\kappa(L) = \mathsf{C}_\kappa(\mathsf{S}(L)^{op})$, equation (5.5.1) describes in particular a partial order in $\mathsf{F}_\kappa(L)$, explicitly given by

$$f \leq g \iff f(\text{—}, r)_i \subseteq g(\text{—}, r)_i \text{ for all } r \in \mathbb{Q},\, i \in I \iff g(r, \text{—})_i \subseteq f(r, \text{—})_i \text{ for all } r \in \mathbb{Q},\, i \in I. \tag{5.5.2}$$

This is our cardinal generalization of the Katětov-Tong insertion theorem:

Theorem 5.5.1. *The following are equivalent for a locale L:*

(i) *L is normal;*

(ii) *For every $\kappa \geq 1$, and every $f \in \mathsf{USC}_\kappa(L)$ and $g \in \mathsf{LSC}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(iii) *There is a $\kappa \geq 1$ such that for every $f \in \mathsf{USC}_\kappa(L)$ and $g \in \mathsf{LSC}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

Proof. (i) $\Longrightarrow$ (ii): Let $\kappa \geq 1$, $|I| = \kappa$, $f \in \mathsf{USC}_\kappa(L)$ and $g \in \mathsf{LSC}_\kappa(L)$ with $f \leq g$ and $i \in I$. Then $f \circ \pi_i \leq g \circ \pi_i$ in $\overline{\mathsf{F}}(L)$, and $f \circ \pi_i \in \overline{\mathsf{USC}}(L)$ and $g \circ \pi_i \in \overline{\mathsf{LSC}}(L)$ (by Corollary 5.3.1). By the standard point-free version of Katětov-Tong insertion theorem [62], there is an $h_i \in \overline{\mathsf{C}}(L)$ such that $f \circ \pi_i \leq h_i \leq g \circ \pi_i$. Since $\{g \circ \pi_i\}_{i \in I}$ is a disjoint family, then so is $\{h_i\}_{i \in I}$. Let $h \in \mathsf{C}_\kappa(L)$ be the function defined by $h \circ \pi_i = h_i$ for all $i \in I$ (provided by Proposition 5.3.4 and Corollary 5.3.5). Obviously, $f \leq h \leq g$.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): Let $|I| = \kappa$ and fix some $i_0 \in I$. Let $a, b \in L$ such that $a \vee b = 1$. Consider the pairwise disjoint κ-families $\mathcal{C} = \{S_i\}_{i \in I}$ and $\mathcal{D} = \{T_i\}_{i \in I}$ defined by $S_{i_0} = \mathfrak{c}(a)$, $T_{i_0} = \mathfrak{o}(b)$ and $S_i = \mathsf{O} = T_i$ for every $i \neq i_0$. By Corollary 5.3.8, $\chi_{\mathcal{C}} \in \mathsf{USC}_\kappa(L)$ and $\chi_{\mathcal{D}} \in \mathsf{LSC}_\kappa(L)$. Moreover, since $a \vee b = 1$ is equivalent to $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ in $\mathsf{S}(L)$, it follows that $\chi_{\mathcal{C}} \leq \chi_{\mathcal{D}}$. Hence, there is an $h \in \mathsf{C}_\kappa(L)$ such that $\chi_{\mathcal{C}} \leq h \leq \chi_{\mathcal{D}}$, from which it follows that $\chi_{\mathfrak{c}(a)} \leq h \circ \pi_{i_0} \leq \chi_{\mathfrak{o}(b)}$. The normality of L follows then from the standard point-free version of Urysohn's Lemma (as e.g. in the formulation of [62, Corollary 8.2]). □

We also have the following modified version which characterizes total κ-collectionwise normality rather than normality.

Theorem 5.5.2. *The following are equivalent for a locale L:*

(i) *L is totally κ-collectionwise normal;*

(ii) *For each $a \in L$ and every $f \in \mathsf{USC}_\kappa(\mathfrak{c}(a))$ and $g \in \mathsf{LSC}_\kappa(\mathfrak{c}(a))$ such that $f \leq g$, there exists an $\overline{h} \in \mathsf{C}_\kappa(L)$ such that $f \leq \nu_{\mathfrak{c}(a)} \circ \overline{h} \leq g$.*

Proof. (i) $\Longrightarrow$ (ii): Suppose L is totally κ-collectionwise normal and consider $a \in L$. By Corollary 3.4.8, L is normal, and so is $\mathfrak{c}(a)$ (because normality is a closed-hereditary property). Let $f \in \mathsf{USC}_\kappa(\mathfrak{c}(a))$ and $g \in \mathsf{LSC}_\kappa(\mathfrak{c}(a))$ such that $f \leq g$. By Theorem 5.5.1, there is an $h \in C_\kappa(\mathfrak{c}(a))$ such that $f \leq h \leq g$. Then, since $\mathfrak{c}(a)$ is also totally κ-collectionwise normal (by Lemma 3.4.10), the conclusion follows readily from Theorem 5.4.3.

(ii) $\Longrightarrow$ (i): Let $\mathfrak{c}(a)$ be a closed sublocale of L and let $h \in \mathsf{C}_\kappa(\mathfrak{c}(a))$. Applying the insertion condition to $h \leq h$ we get an $\bar{h} \in \mathsf{C}_\kappa(L)$ such that $h \leq \nu_{\mathfrak{c}(a)} \circ \bar{h} \leq h$. Of course, $h = \nu_{\mathfrak{c}(a)} \circ \bar{h}$ is an extension of h and the conclusion follows then from Theorem 5.4.3. □

Chapter 6

A relative view on the theory of compact hedgehog-valued functions

As described in Chapter 3, the theory of sublocale selections is useful for unifying several variants of normality and their duals. In this setting, upper and lower semicontinuous (extended) real-valued functions are also dual to each other (see Appendix A). Now, in view of Chapter 5, it is also natural to pursue a relative view on compact hedgehog-valued functions. Observe that, unlike in the case of the extended reals, the generators $(r,\text{—})_i$ and $(\text{—},r)_i$ do not play a symmetric role in the presentation of the frame of the compact hedgehog. In the present chapter, we shall show that it is still possible to introduce relative variants of upper and lower semicontinuity of compact hedgehog-valued functions in such a way that they are dual to each other. Moreover, extension and insertion results generalizing those of Chapter 5 and multiple corollaries are provided.

6.1 Relative semicontinuities

We now introduce relative counterparts of the semicontinuous and continuous compact hedgehog-valued functions introduced in Section 5.3. Throughout this chapter, $\mathbb{F}$ will always denote a sublocale selection (recall Section 3.1).

An $f \in \mathsf{F}_\kappa(L)$ will be called

- *lower* $\mathbb{F}$*-semicontinuous* if for every $r < s$ in $\mathbb{Q}$ and $i \in I$, there is an $F^i_{r,s} \in \mathbb{F}(L)$ such that $f((s,\text{—})_i) \leq F^i_{r,s} \leq f((r,\text{—})_i)$;
- *upper* $\mathbb{F}$*-semicontinuous* if for every $r < s$ in $\mathbb{Q}$ and $i \in I$, there is a $G^i_{r,s} \in \mathbb{F}(L)$ such that $f((\text{—},r)_i) \leq G^i_{r,s} \leq f((\text{—},s)_i)$;
- $\mathbb{F}$*-continuous* if it is lower and upper $\mathbb{F}$-semicontinuous.

This defines the following subclasses of $\mathsf{F}_\kappa(L)$, respectively:

$$\mathsf{LSC}_\kappa^{\mathbb{F}}(L), \quad \mathsf{USC}_\kappa^{\mathbb{F}}(L), \quad \text{and} \quad \mathsf{C}_\kappa^{\mathbb{F}}(L) = \mathsf{LSC}_\kappa^{\mathbb{F}}(L) \cap \mathsf{USC}_\kappa^{\mathbb{F}}(L).$$

Recall that $\mathsf{F}_\kappa(L)$ is partially ordered (cf. (5.5.2)), hence so are $\mathsf{LSC}_\kappa^{\mathbb{F}}(L)$, $\mathsf{USC}_\kappa^{\mathbb{F}}(L)$ and $\mathsf{C}_\kappa^{\mathbb{F}}(L)$. The following results are all very easy to prove (cf. Lemma 5.3.1 and Corollaries 5.3.5 and 5.3.8); we omit the proofs:

Proposition 6.1.1. *Let L be a locale and $f \in \mathsf{F}_\kappa(L)$. Then*:

(1) $f \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ *for all* $i \in I$;

(2) $f \in \mathsf{USC}_\kappa^{\mathbb{F}}(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ *for all* $i \in I$;

(3) $f \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ *if and only if* $f \circ \pi_i \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ *for all* $i \in I$.

In particular, for $\mathbb{F} = \mathbb{F}_\mathfrak{c}$ one recovers the notions from Section 5.3. Combining Proposition 6.1.1 with Proposition A.1.1 we have that upper and lower semicontinuity are dual notions:

Corollary 6.1.2. *Let L be a locale and $f \in \mathsf{F}_\kappa(L)$. Then*:

(1) $f \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$ *if and only if* $f \in \mathsf{USC}_\kappa^{\mathbb{F}^*}(L)$;

(2) $f \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ *if and only if* $f \in \mathsf{C}_\kappa^{\mathbb{F}^*}(L)$.

Corollary 6.1.3. *Let L be a locale, $\mathcal{H} \subseteq \overline{\mathsf{F}}(L)$ a disjoint κ-family and $h_\mathcal{H}$ the compact hedgehog-valued function provided by Proposition 5.3.4. Then*:

(1) $h_\mathcal{H} \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$ *if and only if* $h \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ *for all* $h \in \mathcal{H}$;

(2) $h_\mathcal{H} \in \mathsf{USC}_\kappa^{\mathbb{F}}(L)$ *if and only if* $h \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ *for all* $h \in \mathcal{H}$;

(3) $h_\mathcal{H} \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ *if and only if* $h \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ *for all* $h \in \mathcal{H}$.

Proposition 6.1.4. *Let L be a locale and $C = \{S_i\}_{i\in I}$ a pairwise disjoint κ-family of complemented sublocales. Then*:

(1) $\chi_C \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$ *if and only if* $S_i \in \mathbb{F}^*(L)$ *for all* $i \in I$;

(2) $\chi_C \in \mathsf{USC}_\kappa^{\mathbb{F}}(L)$ *if and only if* $S_i \in \mathbb{F}(L)$ *for all* $i \in I$;

(3) $\chi_C \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ *if and only if* $S_i \in \mathbb{F}(L) \cap \mathbb{F}^*(L)$ *for all* $i \in I$.

6.2 Relative extension results

After this preparation, we may now prove a generalized extension result for selections closed under countable meets and finite joins — i.e., a cardinal extension of Theorem A.3.1.

Theorem 6.2.1. *Let $\mathbb{F}$ be closed under countable meets and finite joins. The following are equivalent for a cardinal κ and a locale L on which $\mathbb{F}$ is hereditary and Katětov:*

(i) *L is an $\mathbb{F}$-z^c_κ frame;*

(ii) *For each $S \in \mathbb{F}(L)$, every $f \in \mathsf{C}^{\mathbb{F}}_\kappa(S)$ has an extension $\overline{f} \in \mathsf{C}^{\mathbb{F}}_\kappa(L)$.*

Proof. (i) $\Longrightarrow$ (ii): Let $f \in \mathsf{C}^{\mathbb{F}}_\kappa(S)$ and for each $i \in I$ set $S_i := \bigcap_{r\in\mathbb{Q}}(f \circ \pi_i)(r,\text{—})$. Then $\{S_i\}_{i\in I}$ is a disjoint family of $\mathbb{F}$-zero sublocales of S. Since $S \in \mathbb{F}(L)$ is $\mathbb{F}$-z^c_κ-embedded, there is a disjoint family $\{T_i\}_{i\in I}$ of $\mathbb{F}$-zero sublocales of L such that $T_i \cap S = S_i$ for every $i \in I$. For each $i \in I$, set $h^{(i)}_1 := f \circ \pi_i \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ and consider the extended real valued function $h^{(i)}_2 \in \overline{\mathsf{F}}(T_i)$ defined by

$$h^{(i)}_2(r,\text{—}) = T_i = 0_{\mathsf{S}(T_i)^{op}}, \quad \text{and} \quad h^{(i)}_2(\text{—},r) = \mathsf{O} = 1_{\mathsf{S}(T_i)^{op}}$$

for every $r \in \mathbb{Q}$. Trivially, $h^{(i)}_2 \in \overline{\mathsf{C}}^{\mathbb{F}}(T_i)$ (as $1_{\mathsf{S}(T_i)^{op}}, 0_{\mathsf{S}(T_i)^{op}} \in \mathbb{F}(T_i)$). Let us show that $h^{(i)}_1(x) \cap T_i = h^{(i)}_2(x) \cap S$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$ by showing it for the generators of $\mathfrak{L}(\overline{\mathbb{R}})$. For any $(r,\text{—})$ we have $S_i \subseteq (f \circ \pi_i)(r,\text{—}) \subseteq S$ and thus $S \cap T_i = S_i \cap T_i \subseteq (f \circ \pi_i)(r,\text{—}) \cap T_i \subseteq S \cap T_i$. Hence

$$h^{(i)}_1(r,\text{—}) \cap T_i = S \cap T_i = h^{(i)}_2(r,\text{—}) \cap S.$$

Further, for any $(\text{—},r)$ select $t \in \mathbb{Q}$ such that $t < r$; we then have

$$h^{(i)}_1(\text{—},r) \cap T_i = h^{(i)}_1(\text{—},r) \cap S \cap T_i = h^{(i)}_1(\text{—},r) \cap S_i \subseteq h^{(i)}_1(\text{—},r) \cap h^{(i)}_1(t,\text{—}) = h^{(i)}_1((\text{—},r) \vee (t,\text{—})) = \mathsf{O}.$$

Hence $h^{(i)}_1(\text{—},r) \cap T_i = \mathsf{O} = h^{(i)}_2(\text{—},r) \cap S$. We may therefore apply Lemma 3.4.6 to conclude that for each $i \in I$ there is a $h_i \in \overline{\mathsf{F}}(S \vee T_i)$ given by $h_i(x) = h^{(i)}_1(x) \vee h^{(i)}_2(x)$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$. Now, because of the hereditary property and the fact that $\mathbb{F}$ is closed under finite joins, it follows easily that h_i is $\mathbb{F}$-continuous.

Since L is $\mathbb{F}$-normal (by Corollary 3.4.18) and $\mathbb{F}$ is a Katětov and hereditary selection on L, closed under countable meets and finite joins, we may apply [68, Theorem 8.6] (see also Theorem A.3.1) and obtain frame homomorphisms $g_i \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ such that $g_i(x) \cap (S \vee T_i) = h_i(x)$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$. Note that $\{g_i\}_{i\in I}$ is a disjoint family. Indeed, for each $i \neq j$,

$$\begin{aligned} g_i\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big) \vee g_j\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big) &\supseteq h_i\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big) \vee h_j\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big) \\ &\supseteq h^{(i)}_2\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big) \vee h^{(j)}_2\Big(\bigvee_{r\in\mathbb{Q}}(r,\text{—})\Big) = T_i \vee T_j = L. \end{aligned}$$

Now consider the $h \in \mathsf{F}_\kappa(L)$ defined by $h \circ \pi_i = g_i$ for all $i \in I$ (recall Proposition 5.3.4), which is $\mathbb{F}$-continuous by virtue of Corollary 6.1.3. We claim that h is the desired extension.

We need to check that $h(x) \cap S = f(x)$ for all $x \in \mathfrak{L}(cJ(\kappa))$ by showing it for the generators of $\mathfrak{L}(cJ(\kappa))$. For any $(r,\text{—})_i$ we have

$$\begin{aligned} S \cap g_i(r,\text{—}) &= S \cap (S \vee T_i) \cap g_i(r,\text{—}) = S \cap h_i(r,\text{—}) = h_1^{(i)}(r,\text{—}) \vee (S \cap h_2^{(i)}(r,\text{—})) \\ &= (f \circ \pi_i)(r,\text{—}) \vee (S \cap T_i) = (f \circ \pi_i)(r,\text{—}) \vee S_i = (f \circ \pi_i)(r,\text{—}). \end{aligned}$$

Hence $h((r,\text{—})_i) \cap S = g_i(r,\text{—}) \cap S = f((r,\text{—})_i)$. Furthermore, for any $(\text{—},r)_i$,

$$\begin{aligned} S \cap g_i(\text{—},r) &= S \cap (S \vee T_i) \cap g_i(\text{—},r) = S \cap h_i(\text{—},r) = h_1^{(i)}(\text{—},r) \vee (S \cap h_2^{(i)}(\text{—},r)) \\ &= (f \circ \pi_i)(\text{—},r) \vee (S \cap \mathsf{O}) = (f \circ \pi_i)(\text{—},r) \vee \mathsf{O} = (f \circ \pi_i)(\text{—},r). \end{aligned}$$

Hence $h((\text{—},r)_i) \cap S = g_i(\text{—},r) \cap S = f((\text{—},r)_i)$.

(ii) $\Longrightarrow$ (i): Let $S \in \mathbb{F}(L)$ and let $\{S_i\}_{i \in I}$ be a disjoint family of $\mathbb{F}$-zero sublocales of S. Then for each $i \in I$ there is an $f_i \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ such that $f_i(\bigvee_{r \in \mathbb{Q}}(r,\text{—})) = S_i$. Let $f \in \mathsf{F}_\kappa(S)$ be the unique frame homomorphism such that $f \circ \pi_i = f_i$ for all $i \in I$. By Corollary 6.1.3, $f \in \mathsf{C}_\kappa^{\mathbb{F}}(S)$. Then, by assumption, there is an $\mathbb{F}$-continuous extension $\overline{f} \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ of f. Set $T_i := \overline{f}(\bigvee_{r \in \mathbb{Q}}(r,\text{—})_i)$ for each $i \in I$. It is clear that $\{T_i\}_{i \in I}$ is the desired disjoint family of $\mathbb{F}$-zero sublocales. □

As a particular case, we obtain Theorem 5.4.3. Other sublocale selections are not generally hereditary, but the theorem may still be applicable. For instance, considering the family $\mathbb{F}_{\mathrm{Z}}$ one obtains a cardinal extension of the well-known classical fact that zero subspaces are z-embedded if and only if they are C^*-embedded (see [2, Corollary 7.5]):

Corollary 6.2.2. *The following are equivalent for a locale L:*

(i) *L is an $\mathbb{F}_{\mathrm{Z}}$-z_κ^c frame (i.e., every zero sublocale is z_κ^c-embedded);*

(ii) *For every cozero element $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{c}(a))$ has a continuous extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

Proof. The proof of the implication (ii) $\Longrightarrow$ (i) in Theorem 6.2.1 does not require the selection to be hereditary, hence (ii) $\Longrightarrow$ (i) follows. Now, assume (i) holds. In particular, every zero sublocale of L is z-embedded; but, as we noted in Section 3.2, this is equivalent to the family $\mathbb{F}_{\mathrm{Z}}$ being hereditary on L. Thus we may apply Theorem 6.2.1. □

One may also attempt a "dual" extension theorem — i.e., a cardinal generalization of Theorem 3.2.2. Since we shall need specific results for the selection $\mathbb{F}_{\mathfrak{c}}^*$ (cf. Subsection 3.4.5) we content ourselves with the case $\mathbb{F}_{\mathfrak{c}}^*$:

Theorem 6.2.3. *The following are equivalent for a locale L:*

(i) *L is extremally disconnected;*

(ii) *For every $\kappa \geq 1$ and $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \mathsf{C}_\kappa(L)$;*

(iii) *There is a $\kappa \geq 1$ such that for every $a \in L$, every $f \in \mathsf{C}_\kappa(\mathfrak{o}(a))$ has a continuous extension $\overline{f} \in \mathsf{C}_\kappa(L)$.*

Proof. (i) $\Longrightarrow$ (ii): For each $i \in I$, set $h_1^{(i)} := f \circ \pi_i \in \overline{\mathsf{C}}(\mathfrak{o}(a))$ and consider the extended real valued function $h_2^{(i)} \in \overline{\mathsf{C}}(\mathfrak{o}(b_i^*))$ defined by

$$h_2^{(i)}(r,\text{—}) = 0_{\mathfrak{o}(b_i^*)} = b_i^{**} \quad \text{and} \quad h_2^{(i)}(\text{—},r) = 1$$

for every $r \in \mathbb{Q}$. Let us show that $h_1^{(i)}(x) \wedge a \wedge b_i^* = h_2^{(i)}(x) \wedge a \wedge b_i^*$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$ by showing it for the generators of $\mathfrak{L}(\overline{\mathbb{R}})$. For each $r \in \mathbb{Q}$ we have

$$h_1^{(i)}(r,\text{—}) \wedge a \wedge b_i^* \leq a_i \wedge a \wedge b_i^* = 0 = h_2^{(i)}(r,\text{—}) \wedge a \wedge b_i^*.$$

Moreover, by (r1) one has $(f \circ \pi_i)(\text{—},r) \vee a_i = 1$ and therefore

$$a \wedge b_i^* \leq (f \circ \pi_i)(\text{—},r) \vee (a \wedge b_i^* \wedge a_i) = h_1^{(i)}(\text{—},r).$$

Hence

$$h_1^{(i)}(\text{—},r) \wedge a \wedge b_i^* = a \wedge b_i^* = h_2^{(i)}(\text{—},r) \wedge a \wedge b_i^*.$$

Consequently, by [94, 3.2, 3.3] there is an $h_i \in \overline{\mathsf{C}}(\mathfrak{o}(a \vee b_i^*))$ given by

$$h_i(x) = (h_1^{(i)}(x) \wedge a) \vee (h_2^{(i)}(x) \wedge b_i^*)$$

— i.e., $h_i(r,\text{—}) = (f \circ \pi_i)(r,\text{—}) \wedge a$ and $h_i(\text{—},r) = ((f \circ \pi_i)(\text{—},r) \wedge a) \vee b_i^*$ for every $r \in \mathbb{Q}$, which extends $h_1^{(i)}$ and $h_2^{(i)}$. By Corollary 3.2.3, for each $i \in I$ there is a $g_i \in \overline{\mathsf{C}}(L)$ which extends h_i (i.e., satisfying $\nu_{\mathfrak{o}(a \vee b_i^*)} \circ g_i = h_i$). Let us check that the family $\{g_i\}_{i \in I}$ is disjoint:

$$\begin{aligned} g_i\Big(\bigvee_{r \in \mathbb{Q}} (r,\text{—})\Big) \wedge g_j\Big(\bigvee_{s \in \mathbb{Q}} (s,\text{—})\Big) &\leq h_i\Big(\bigvee_{r \in \mathbb{Q}} (r,\text{—})\Big) \wedge h_j\Big(\bigvee_{s \in \mathbb{Q}} (s,\text{—})\Big) \\ &= (f \circ \pi_i)\Big(\bigvee_{r \in \mathbb{Q}} (r,\text{—})\Big) \wedge (f \circ \pi_j)\Big(\bigvee_{s \in \mathbb{Q}} (s,\text{—})\Big) \wedge a \\ &= a_i \wedge a_j \wedge a = a^* \wedge a = 0. \end{aligned}$$

For each $i \in I$, g_i extends h_i, and the latter extends $h_1^{(i)} = f \circ \pi_i$, hence g_i extends $f \circ \pi_i$. Therefore the function $\overline{f} \in \mathsf{C}_\kappa(L)$ given by $\overline{f} \circ \pi_i = g_i$ extends f: indeed, note that $\nu_{\mathfrak{o}(a)} \circ \overline{f} \circ \pi_i = \nu_{\mathfrak{o}(a)} \circ g_i = f \circ \pi_i$ and use the uniqueness clause of Proposition 5.3.4.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i) is similar to (ii) $\Longrightarrow$ (i) in Theorem 3.2.2. □

6.3 Relative insertion results

We may now easily prove a generalized insertion result for compact hedgehog-valued functions (cf. Theorem 5.5.1):

Theorem 6.3.1. *The following are equivalent for any locale L such that $\mathbb{F}$ is a Katětov selection on L and $L \in \mathbb{F}(L) \cap \mathbb{F}^*(L)$:*

(i) *L is $\mathbb{F}$-normal;*

(ii) *For every $\kappa \geq 1$, and every $f \in \mathsf{USC}_\kappa^{\mathbb{F}}(L)$ and $g \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ such that $f \leq h \leq g$;*

(iii) *There is a $\kappa \geq 1$ such that for every $f \in \mathsf{USC}_\kappa^{\mathbb{F}}(L)$ and $g \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ such that $f \leq h \leq g$.*

Proof. (i) $\Longrightarrow$ (ii): Let $\kappa \geq 1$, $|I| = \kappa$, $f \in \mathsf{USC}_\kappa^{\mathbb{F}}(L)$ and $g \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$ with $f \leq g$ and $i \in I$. Then, by Corollary 6.1.1, $f \circ \pi_i \leq g \circ \pi_i$ in $\overline{\mathsf{F}}(L)$ with $f \circ \pi_i \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ and $g \circ \pi_i \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$. By [68, Theorem 7.1] (or Theorem A.2.2), there is an $h_i \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ such that $f \circ \pi_i \leq h_i \leq g \circ \pi_i$. Since $\{g \circ \pi_i\}_{i \in I}$ is a disjoint family, then so is $\{h_i\}_{i \in I}$. Let $h \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ be the function given by Corollary 6.1.3 (defined by $h \circ \pi_i = h_i$ for every $i \in I$). It satisfies $f \leq h \leq g$, as claimed.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): Let $|I| = \kappa$ and fix some $i_0 \in I$. Let $S, T \in \mathbb{F}(L)$ such that $S \cap T = \mathsf{O}$. Define pairwise disjoint κ-families $\mathcal{C} = \{S_i\}_{i \in I}$ and $\mathcal{D} = \{T_i\}_{i \in I}$ by $S_{i_0} = S$, $T_{i_0} = T^\#$ and $S_i = \mathsf{O} = T_i$ for $i \neq i_0$. By Proposition 6.1.4 (and the fact that O and L belong to $\mathbb{F}(L)$), one has $\chi_{\mathcal{C}} \in \mathsf{USC}_\kappa^{\mathbb{F}}(L)$ and $\chi_{\mathcal{D}} \in \mathsf{LSC}_\kappa^{\mathbb{F}}(L)$. Moreover, $\chi_{\mathcal{C}} \leq \chi_{\mathcal{D}}$, since $S \subseteq T^\#$. Hence, there exists $h \in \mathsf{C}_\kappa^{\mathbb{F}}(L)$ such that $\chi_{\mathcal{C}} \leq h \leq \chi_{\mathcal{D}}$, from which it follows in particular that $\chi_S \leq h \circ \pi_{i_0} \leq \chi_{T^\#}$. Let $A, B \in \mathbb{F}(L)$ such that

$$(h \circ \pi_{i_0})(1,\text{—}) \subseteq A \subseteq (h \circ \pi_{i_0})(2,\text{—}) \quad \text{and} \quad (h \circ \pi_{i_0})(\text{—},1) \subseteq B \subseteq (h \circ \pi_{i_0})(\text{—},0).$$

Then

$$A \vee B \supseteq (h \circ \pi_{i_0})((1,\text{—}) \wedge (\text{—},1)) = L \quad \text{and} \quad S \cap A = \chi_S(\text{—},3) \cap A \subseteq (h \circ \pi_{i_0})((\text{—},3) \vee (2,\text{—})) = \mathsf{O}.$$

Similarly, $T \cap B = \mathsf{O}$. □

Note that the particular case $\mathbb{F} = \mathbb{F}_\mathfrak{c}$ yields immediately Theorem 5.5.1. In the case $\mathbb{F} = \mathbb{F}_\mathsf{Z}$ we obtain a point-free cardinal generalization of a classical result of Blatter and Seever [35].

Corollary 6.3.2. *Let L be a locale and $\kappa \geq 1$. Then for every upper zero-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every lower zero-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

For $\mathbb{F} = \mathbb{F}_{\delta\mathrm{reg}}$ we obtain a cardinal generalization of a result whose classical counterpart we have not found in the literature.

Corollary 6.3.3. *The following are equivalent for a locale L:*

(i) *L is δ-normal;*

(ii) *For every $\kappa \geq 1$, and every upper regular-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every lower regular-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(iii) *There is a $\kappa \geq 1$ such that for every upper regular-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and lower regular-semicontinuous $g \in \mathsf{F}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

Similarly, for $\mathbb{F} = \mathbb{F}_{\mathrm{reg}}$ we obtain a cardinal extension of a classical result by Lane [84].

Corollary 6.3.4. *The following are equivalent for a locale L:*

(i) *L is mildly normal;*

(ii) *For every $\kappa \geq 1$, and every upper normal-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every lower normal-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(iii) *There is a $\kappa \geq 1$ such that for every upper normal-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and lower normal-semicontinuous $g \in \mathsf{F}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

Proof. Recall that $\mathbb{F}_{\mathrm{reg}}$ is a Katětov selection on any mildly normal locale (see Example A.2.1). Moreover, note that the proof of implication (iii) $\Longrightarrow$ (i) in Theorem 6.3.1 does not need $\mathbb{F}$ to be Katětov. □

We may also apply Theorem 6.3.1 to the dual sublocale selections (cf. Proposition 6.1.2). Combining the cases $\mathbb{F} = \mathbb{F}^*_{\mathfrak{c}}$ and $\mathbb{F} = \mathbb{F}^*_{\mathrm{reg}}$ we have the following cardinal generalization of a classical result by Lane [85] and of the point-free version of Stone's insertion theorem [63].

Corollary 6.3.5. *The following are equivalent for a locale L:*

(i) *L is extremally disconnected;*

(ii) *For every $\kappa \geq 1$, and every $f \in \mathsf{LSC}_\kappa(L)$ and $g \in \mathsf{USC}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(iii) *There is a $\kappa \geq 1$ such that for every $f \in \mathsf{LSC}_\kappa(L)$ and $g \in \mathsf{USC}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(iv) *For every $\kappa \geq 1$, and every lower normal-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and upper normal-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(v) *There is a $\kappa \geq 1$ such that for every lower normal-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and upper normal-semicontinuous $g \in \mathsf{F}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

For $\mathbb{F} = \mathbb{F}_{\mathcal{Z}}^*$, we obtain a point-free cardinal generalization of a classical result by Seever [106].

Corollary 6.3.6. *The following are equivalent for a locale L:*

(i) *L is an F-frame;*

(ii) *For every $\kappa \geq 1$, and every lower zero-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every upper zero-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(iii) *There is a $\kappa \geq 1$ such that for every lower zero-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every upper zero-semicontinuous $g \in \mathsf{F}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

Finally, for $\mathbb{F} = \mathbb{F}_{\delta\mathrm{reg}}^*$ we obtain a cardinal extension of a result whose classical counterpart we have not been able to find in the literature:

Corollary 6.3.7. *The following are equivalent for a locale L:*

(i) *L is an extremally δ-disconnected frame;*

(ii) *For every $\kappa \geq 1$, and every lower regular-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every upper regular-semicontinuous $g \in \mathsf{F}_\kappa(L)$ such that $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$;*

(iii) *There is a $\kappa \geq 1$ such that for every lower regular-semicontinuous $f \in \mathsf{F}_\kappa(L)$ and every upper regular-semicontinuous $g \in \mathsf{F}_\kappa(L)$ satisfying $f \leq g$, there exists an $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.*

Chapter 7

The Boolean algebra of smooth sublocales

7.1 Introduction

A major difference between the category of topological spaces and the category of locales is the nature of their lattices of regular subobjects. Whereas in the former they constitute very special lattices — complete and atomic Boolean algebras — in the latter they are coframes. Hence, subobject lattices in Loc are more complicated objects and they comprise one of the fundamental areas of study in locale theory. Accordingly, if L is a locale, the locale $\mathsf{S}(L)^{op}$ can be regarded from two quite opposing points of view:

(1) As a sort of "discrete version" of L. Quoting Johnstone [80]:

> *We may think of X_d as playing a role in locale theory analogous to that of the discrete modification of a space X (that is, the space obtained by retopologizing the underlying set of X with the discrete topology).*

This idea has been extensively and successfully used for obtaining a pleasant theory of real-valued functions in point-free topology (see e.g. [62, 55, 60, 66, 26, 67, 68, 14]).

(2) As a locale in its own right which will contain some amount of information of L itself (cf. for example the well-known fact that a locale L is totally spatial if and only if $\mathsf{S}(L)^{op}$ is spatial [89]).[1]

Of course, these two viewpoints can arise simultaneously. For example, we shall see that the system $\mathsf{S}_b(L)$ consisting of smooth sublocales is useful for detecting whether L is T_D-spatial; but, on the other hand, $\mathsf{S}_b(L)$ can also be proved to be a discretization of L (and, in fact, typically "more discrete" than $\mathsf{S}(L)^{op}$), see Section 7.6 below. Actually, the system $\mathsf{S}_b(L)$

[1] Clearly, we cannot expect the coframe $\mathsf{S}(L)$ to contain all of the information of L: the non-Boolean Sierpinski locale and the 4-element Boolean algebra have isomorphic coframes of sublocales.

will allow us to present an alternative description of arbitrary compact hedgehog-valued functions without affecting the treatment of semicontinuities from Chapter 5.

In particular, following viewpoint (2), we may study subcolocales of $\mathsf{S}(L)$. A systematic study of some of these families of sublocales will hopefully lead to a better understanding of the complex lattice $\mathsf{S}(L)$. Chapters 7 and 8 are devoted to the study of two important subcolocales of $\mathsf{S}(L)$: the system of smooth sublocales (which, in the subfit case, has enjoyed some attention recently) and the system of D-sublocales. Moreover, this study will be largely carried out in parallel: e.g. their connection with the T_D axiom, or functoriality properties.

After that, in Chapter 9, we relate these subcolocales to other known ones; and we will see that inclusions between these systems actually characterize well-known and interesting properties of the locale in question (among others, spatiality, total spatiality, T_D-spatiality, total T_D-spatiality, being a T_1-locale, etc.).

Most of the material presented in this chapter is contained in the article [5], but the contents of Subsection 7.7.4 are a joint work with Javier Gutiérrez García [6]:

[5] I. Arrieta, On joins of complemented sublocales, *Algebra Universalis*, vol. 83, art. no. 1, 2022.

[6] I. Arrieta and J. Gutiérrez García, On the categorical behaviour of locales and D-localic maps, *Quaestiones Mathematicae*, accepted for publication.

However, some other additional results have also been included in this thesis (for example Subsection 7.6.1 is new and so are some of the results in Section 7.7).

7.2 Preliminaries about $\mathsf{S}_c(L)$

If L is a locale, denote by $\mathsf{S}_c(L)$ the subset of $\mathsf{S}(L)$ consisting of joins of closed sublocales — i.e.,

$$\mathsf{S}_c(L) = \Big\{ \bigvee_{a\in A} \mathfrak{c}(a) \mid A \subseteq L \Big\},$$

endowed with the inclusion order inherited from $\mathsf{S}(L)$. Clearly, the subset $\mathsf{S}_c(L)$ is closed under the formation of arbitrary joins in $\mathsf{S}(L)$ and is therefore a complete lattice (cf. [99]). Actually, Picado, Pultr and Tozzi showed in [99, 2.3] that, somewhat surprisingly, $\mathsf{S}_c(L)$ is *always* a frame. Moreover, for a large class of frames, $\mathsf{S}_c(L)$ turns out to be Boolean. Indeed, one of the main results from [99] reads as follows:

Theorem 7.2.1 ([99, Theorem 3.5]). *The following are equivalent for a locale L:*

(i) *L is subfit;*

(ii) $\mathsf{S}_c(L)$ *is a Boolean algebra;*

(iii) $\mathsf{S}_c(L)$ *is the Booleanization of* $\mathsf{S}(L)$*;*

(iv) $\mathsf{S}_c(L)$ *is a subcolocale of* $\mathsf{S}(L)$.

For an $L = \Omega(X)$, the coframe $\mathsf{S}(L)$ contains typically more sublocales than those induced by X (cf. Section 1.2). However, for the T_1-spatial case, the smaller $\mathsf{S}_c(L)$ consists precisely of the induced sublocales:

Theorem 7.2.2 ([99, Theorem 4.5])**.** *If X is a T_1-space, then $\mathsf{S}_c(\Omega(X)) \cong \mathcal{P}(X)$.*

In light of the last two results, the frame $\mathsf{S}_c(L)$ has subsequently attracted attention in point-free topology. Let us mention for example the following aspects:

- the naturality of the construction as a maximal essential extension in the category of frames [19];
- its role as a discretization of L for modeling not necessarily continuous real-valued functions (conservatively in the class of T_1-spaces) [93];
- its (non-) functoriality properties [17];
- as a useful tool for studying several (conservative) point-free extensions of classical topological properties [48].

All of the applications listed above have been carried out under the assumption that L is subfit (i.e., that $\mathsf{S}_c(L)$ is Boolean). Throughout this chapter, we shall systematically investigate the Booleanization of $\mathsf{S}(L)$ for an arbitrary L and extend the results listed above to this more general setting. In particular, we shall emphasize in Subsection 7.6.1 its role as a discrete cover of locales thus yielding a conservative theory of (not necessarily continuous) localic hedgehog-valued functions .

7.3 The Boolean algebra of smooth sublocales

7.3.1 Smooth sublocales

We start by recalling that a sublocale S of L is said to be locally closed if it is of the form $S = \mathfrak{o}(a) \cap \mathfrak{c}(b)$ for some $a, b \in L$. Moreover, following Isbell [71], a sublocale S is said to be *smooth* if it is a join of complemented sublocales in L. Moreover, any complemented sublocale is a join of locally closed sublocales. Indeed, if S is a complemented sublocale, then $S^{\#} = \bigcap_i \mathfrak{c}(a_i) \vee \mathfrak{o}(b_i)$ for some $a_i, b_i \in L$. Thus

$$S = S^{\#\#} = \bigvee_i (\mathfrak{c}(a_i) \vee \mathfrak{o}(b_i))^{\#} = \bigvee_i \mathfrak{o}(a_i) \cap \mathfrak{c}(b_i).$$

Hence,

a sublocale is smooth if and only if it is a join of locally closed sublocales.

We denote by $\mathsf{S}_b(L)$ the subset of $\mathsf{S}(L)$ consisting of smooth sublocales — i.e.,

$$\mathsf{S}_b(L) = \Big\{ \bigvee_{a \in A, b \in B} \mathfrak{c}(a) \cap \mathfrak{o}(b) \mid A, B \subseteq L \Big\}.$$

The system $\mathsf{S}_b(L)$ will be the main object of study of this chapter, and we shall always consider it endowed with the ordering inherited from $\mathsf{S}(L)$, that is, inclusion between sublocales.

7.3.2 The Boolean algebra $\mathsf{S}_b(L)$

Observe that complemented elements of a locale are always regular, and meets of regular elements are regular, consequently any meet of complemented elements is regular. Moreover:

Lemma 7.3.1. *Any regular element of a zero-dimensional frame is a meet of complemented elements.*

Proof. Let L be a zero-dimensional frame and $a = b^*$ for some $b \in L$. Then there exists a family $\{c_i\}_{i\in I}$ of complemented elements such that $b = \bigvee_i c_i$. By the first De Morgan law (FDM), $a = b^* = (\bigvee_i c_i)^* = \bigwedge_i c_i^* = \bigwedge_i c_i^c$. □

Corollary 7.3.2. *Let L be a zero-dimensional frame. Then the Booleanization B_L of L is precisely the set of all meets of complemented elements.*

Since $\mathsf{S}(L)^{op}$ is a zero-dimensional frame, we immediately obtain the following:

Corollary 7.3.3. *The Booleanization subcolocale of $\mathsf{S}(L)$ is precisely $\mathsf{S}_b(L)$ — i.e., one has*

$$\mathsf{S}_b(L) = \{ S \in \mathsf{S}(L) \mid S = S^{\#\#} \}.$$

Recall from Theorem 7.2.1 that L is subfit if and only if the Booleanization of $\mathsf{S}(L)$ is $\mathsf{S}_c(L)$. Combining this with the previous corollary yields the following:

Corollary 7.3.4. *L is subfit if and only if $\mathsf{S}_c(L) = \mathsf{S}_b(L)$.*

Remark 7.3.5. Since $\mathsf{S}_b(L)$ is a subcolocale of $\mathsf{S}(L)$, it is closed under arbitrary joins and the co-Heyting operator in $\mathsf{S}(L)$. However, it is generally not closed under meets. Consequently, in this chapter we shall denote by $\wedge^{\mathsf{S}_b(L)}$ (or just by $\wedge$, when there is no danger of confusion) the meet operation in $\mathsf{S}_b(L)$.

Nevertheless, in one important case meets in $\mathsf{S}_b(L)$ are just intersections: if C is a complemented sublocale and $S \in \mathsf{S}_b(L)$, we have $C \wedge S = C \cap S$. Indeed, if $S = \bigvee_i C_i$ with each C_i complemented, then one has $C \cap S = C \cap \bigvee_i C_i = \bigvee_i C \cap C_i \in \mathsf{S}_b(L)$ because of the linearity from (1.2.3).

7.3.3 Categorical characterization of smoothness

Perhaps somewhat surprisingly, smooth sublocales can be characterized purely in categorical terms: they are precisely the sublocales (=regular subobjects in Loc) S of L such that surjections (=epimorphisms in Loc) are stable under pulling back along the inclusion $S \hookrightarrow L$. Since we have not found this result in the literature, we devote the present subsection to proving it.

Firstly, we need to recall the following Frobenius-type formula which holds in the category of locales.

Proposition 7.3.6 ([112, Proposition 1.4]). *Let $f\colon L \to M$ be a localic map, let C be a complemented sublocale of M and let S be an arbitrary sublocale of L. Then, $f[S \cap f_{-1}[C]] = f[S] \cap C$. In particular, if f is a surjection, $f[f_{-1}[C]] = C$ for every complemented $C \subseteq M$ — that is, arbitrary surjections are pullback-stable along complemented inclusions.*

We can now give our characterization of smooth sublocales:

Proposition 7.3.7. *Let L be a locale and $S \in \mathsf{S}(L)$. Then the following are equivalent:*

(1) *S is smooth;*

(2) *for every surjection $f\colon M \twoheadrightarrow L$ in Loc, $f[f_{-1}[S]] = S$.*

Proof. (1) $\Longrightarrow$ (2): The inclusion $\subseteq$ follows by the adjunction $f[-] \dashv f_{-1}[-]$. Let us now show the inclusion $S \subseteq f[f_{-1}[S]]$. By zero-dimensionality of $\mathsf{S}(L)^{op}$ write $f[f_{-1}[S]] = \bigcap_i C_i$ for a suitable family $\{C_i\}_{i\in I}$ of complemented sublocales, and by smoothness write $S = \bigvee_j D_j$ for a suitable family $\{D_j\}_{j\in J}$ of complemented sublocales. Let $i \in I$ and $j \in J$. One has $f[f_{-1}[D_j]] \subseteq f[f_{-1}[S]] \subseteq C_i$ and by Proposition 7.3.6 the left-hand side equals D_j. Hence we have $D_j \subseteq C_i$ for each $i \in I$ and $j \in J$, that is, $S = \bigvee_j D_j \subseteq \bigcap_j C_i = f[f_{-1}[S]]$.

(2) $\Longrightarrow$ (1): Let S be a sublocale satisfying the property. By looking at the left adjoint frame homomorphisms, we see that the composite localic map

$$p\colon \mathsf{S}_b(L)^{op} \xhookrightarrow{\ \iota\ } \mathsf{S}(L)^{op} \xrightarrow{\ \mathfrak{c}_*\ }\!\!\!\!\rightarrow L$$

is a surjection (observe that $\iota^*(S) = S^{\#\#}$, see Subsection 1.2.4). Now, we have

$$\begin{aligned} p_{-1}[S] &= \iota_{-1}[\mathfrak{c}_{*-1}[S]] = \iota_{-1}[\mathfrak{c}(S)] = \mathsf{S}_b(L)^{op} \cap \mathfrak{c}(S) = \{\,T \in \mathsf{S}_b(L) \mid T \subseteq S\,\} \\ &= \{\,T \in \mathsf{S}_b(L) \mid T \subseteq S^{\#\#}\,\} = \mathsf{S}_b(L)^{op} \cap \mathfrak{c}(S^{\#\#}) = p_{-1}[S^{\#\#}]. \end{aligned}$$

By adjunction we therefore have $p[p_{-1}[S]] \subseteq S^{\#\#}$ and by assumption the left-hand side equals S. Therefore $S \subseteq S^{\#\#}$ and S is smooth. □

7.4 Relating $\mathsf{S}_b(L)$ and $\mathsf{S}_c(L)$

As we mentioned in Section 7.2 the system $\mathsf{S}_c(L)$ is always a frame. However, the rather technical proof from [99] may somehow obscure the geometric intuition. Here we present an alternative and more direct proof based on $\mathsf{S}_b(L)$.

First, note that we have the inclusion $\mathsf{S}_c(L) \subseteq \mathsf{S}_b(L)$. Moreover this inclusion preserves arbitrary joins, since joins in both of them are just joins is $\mathsf{S}(L)$ — i.e., we have the following chain of suplattice embeddings:

$$\mathsf{S}_c(L) \subseteq \mathsf{S}_b(L) \subseteq \mathsf{S}(L).$$

In fact we have more:

Theorem 7.4.1. $\mathsf{S}_c(L)$ *is a subframe of* $\mathsf{S}_b(L)$. *In particular, it is a frame.*

Proof. The only point remaining is to show that $\mathsf{S}_c(L)$ is closed under binary meets in $\mathsf{S}_b(L)$. Let $S = \bigvee_i \mathfrak{c}(a_i)$ and $T = \bigvee_j \mathfrak{c}(b_j)$ in $\mathsf{S}_c(L)$. Then

$$S = (\bigcap_i \mathfrak{o}(a_i))^{\#} \quad \text{and} \quad T = (\bigcap_j \mathfrak{o}(b_j))^{\#}$$

and their meet in $\mathsf{S}_b(L)$ is given by the formula $S \wedge^{\mathsf{S}_b(L)} T = (S \cap T)^{\#\#} = (S^{\#} \vee T^{\#})^{\#}$ (see Subsection 1.2.4 for the dual of this general fact on the Booleanization for frames). Thus

$$S \wedge^{\mathsf{S}_b(L)} T = (S^{\#} \vee T^{\#})^{\#} = ((\textstyle\bigcap_i \mathfrak{o}(a_i))^{\#\#} \vee (\bigcap_j \mathfrak{o}(b_j))^{\#\#})^{\#} = (\bigcap_{i,j} \mathfrak{o}(a_i \vee b_j))^{\#\#\#} = \bigvee_{i,j} \mathfrak{c}(a_i \vee b_j),$$

where the third equality follows from the fact that in a coframe double supplements commute with finite joins (cf. Proposition 4.1.1 (1)) and from the coframe distributivity of $\mathsf{S}(L)$. Hence the meet of S and T in $\mathsf{S}_b(L)$ lies in $\mathsf{S}_c(L)$ and so it is also their meet in $\mathsf{S}_c(L)$. □

7.5 Connections with T_D-spatiality

7.5.1 Computing $\mathsf{S}_b(\Omega(X))$ for a T_D-space X

For a T_1-space X, we have an isomorphism $\mathsf{S}_c(\Omega(X)) \cong \mathcal{P}(X)$ (see [99]). With our $\mathsf{S}_b(L)$, we can extend the result to the T_D case.

If A is a subspace of a topological space X, let us denote by $\widetilde{A}$ the corresponding induced sublocale in $\mathsf{S}(\Omega(X))$ (recall Section 1.2).

Lemma 7.5.1. *Let X be a T_D-space. The map* $\pi\colon \mathcal{P}(X) \longrightarrow \mathsf{S}(\Omega(X))$ *given by* $\pi(A) = \widetilde{A}$ *restricts to an isomorphism* $\mathcal{P}(X) \longrightarrow \mathsf{S}_b(\Omega(X))$ *— i.e., classical subspaces correspond precisely to smooth sublocales.*

Proof. It is well known that π is injective if and only if X is T_D (see [91, VI 1.2]) and that π always preserves joins (see e.g. [98, p. 66]). The only task remaining is to show that the image of π coincides with $\mathsf{S}_b(\Omega(X))$.

Let $A \subseteq X$. Then we have

$$\pi(A) = \widetilde{A} = \widetilde{\bigcup_{x \in A} \{x\}} = \bigvee_{x \in A} \widetilde{\{x\}} = \bigvee_{x \in A} \mathfrak{b}(X - \overline{\{x\}}),$$

so by Lemmas 1.3.4 and 1.3.3, one has $\pi(A) \in \mathsf{S}_b(\Omega(X))$. Finally, if $S \in \mathsf{S}_b(\Omega(X))$, we have $S = \bigvee_i C_i$ with C_i complemented. But it is well known that complemented sublocales are always induced — i.e., $C_i = \pi(A_i)$ for suitable $A_i \subseteq X$. Hence $S = \pi(\bigcup_i A_i)$. □

7.5.2 More on T_D-spatiality

More interestingly, the converse of the previous lemma also holds.

Theorem 7.5.2. *The following are equivalent for a locale L:*

(i) *L is T_D-spatial;*

(ii) *The map $\mathfrak{m}\colon \mathcal{P}(\mathsf{pt}_D(L)) \to \mathsf{S}_b(L)$ which sends $Y \subseteq \mathsf{pt}_D(L)$ to $\bigvee_{p\in Y} \mathfrak{b}(p)$ is an isomorphism (whose inverse $\mathsf{pt}_D\colon \mathsf{S}_b(L) \to \mathcal{P}(\mathsf{pt}_D(L))$ sends $S \in \mathsf{S}_b(L)$ to $\mathsf{pt}_D(S)$);*

(iii) *There exists an isomorphism $\mathsf{S}_b(L) \cong \mathcal{P}(\mathsf{pt}_D(L))$;*

(iv) *$\mathsf{S}_b(L)$ is atomic (i.e., it is spatial).*

Proof. (i) $\Longrightarrow$ (ii): Let $L = \Omega(X)$ with X a T_D-space. The map $X \to \mathsf{pt}_D(\Omega(X)) = \{X - \overline{\{x\}} \mid x \in X\}$ given by $x \mapsto X - \overline{\{x\}}$ is a bijection because it is the unit of the adjunction (1.3.1). Hence this implication is just a re-statement of Lemma 7.5.1.

(ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv): These implications are trivial.

(iv) $\Longrightarrow$ (i): Because of Properties 1.2.3 (2) and (5), the set of prime elements of $\mathsf{S}_b(L)^{op}$ is precisely the set of sublocales $\mathfrak{b}(p)$ (with p prime in L) contained in $\mathsf{S}_b(L)$. But a two-element sublocale is complemented as soon as it belongs to $\mathsf{S}_b(L)$ (indeed, if $\{1,p\} = \bigvee_i C_i$ with each C_i complemented, then there is a $C_i \neq \mathsf{O}$ and hence $C_i = \{1,p\}$). Accordingly, by virtue of Lemma 1.3.3 we have that prime elements of $\mathsf{S}_b(L)^{op}$ are precisely the $\mathfrak{b}(p)$ with p covered in L.

Since $\mathsf{S}_b(L)^{op}$ is spatial, each smooth sublocale is a meet of primes in $\mathsf{S}_b(L)^{op}$. In particular, for each $a \in L$,

$$\mathfrak{c}(a) = \bigwedge_i^{\mathsf{S}_b(L)^{op}} \mathfrak{b}(p_i) = \bigvee_i \mathfrak{b}(p_i)$$

for a suitable family $\{p_i\}_{i\in I}$ of covered primes in L. Hence $a = \bigwedge_i x_i$ with $x_i \in \mathfrak{b}(p_i) = \{1,p_i\}$ for each $i \in I$. We have therefore shown that every element of L is a meet of covered primes. The result now follows from Lemma 1.3.6. $\square$

7.6 $\mathsf{S}_b(L)$ as a discrete cover of L

First, we note that L is always embedded as a subframe in $\mathsf{S}_b(L)$

Lemma 7.6.1. *There is an injective frame homomorphism $\mathfrak{o}_L\colon L \to \mathsf{S}_b(L)$ which sends $a \in L$ to $\mathfrak{o}_L(a)$. Moreover, this map $\mathfrak{o}_L$ is an epimorphism in* Frm.

Proof. Clearly, $\mathfrak{o}_L$ is a frame homomorphism since it is the composite of the canonical embedding of L into $\mathsf{S}(L)^{op}$ followed by the quotient of $\mathsf{S}(L)^{op}$ onto its Booleanization and by complementation in the Boolean algebra $\mathsf{S}_b(L)$:

$$L \xrightarrow{\ \mathfrak{c}_L\ } \mathsf{S}(L)^{op} \xrightarrow{(-)^{\#\#}} \mathsf{S}_b(L)^{op} \xrightarrow{(-)^{\#}(\sim)} \mathsf{S}_b(L)$$

The fact that it is an epimorphism follows easily from the fact that frame homomorphisms commute with complements. $\square$

The map $\mathfrak{o}_L\colon L \rightarrowtail \mathsf{S}_b(L)$ is in fact of a quite canonical nature: it can be characterized as the maximal essential extension of L in the category of frames (this follows readily by an application of [19, Proposition 4.3]). For a more detailed account on this notion we refer to [19].

If instead we look at the localic side, the right adjoint $(\mathfrak{o}_L)_*\colon \mathsf{S}_b(L)^{op} \twoheadrightarrow L$ is a surjection of locales; and can thus be regarded as a "discrete cover" of the locale L; similar to the canonical surjection $(\mathfrak{c}_L)_*\colon \mathsf{S}(L)^{op} \twoheadrightarrow L$, but this time more discrete.

Following the spirit in [93], the viewpoint of $\mathsf{S}_b(L)$ as a "discrete cover" can be usefully exploited for dealing with an alternative representation of real-valued (or more generally hedgehog-valued) localic maps, as we show in the next subsection.

7.6.1 An application: another representation of hedgehog-valued functions

Let

$$\mathsf{F}'_\kappa(L) = \mathsf{Frm}(\mathfrak{L}(J(\kappa)), \mathsf{S}_b(L)).$$

Since $\mathsf{F}'_\kappa(L) = \mathsf{C}_\kappa(\mathsf{S}_b(L))$, it is partially ordered with the ordering from (5.5.1). Explicitly, for $f, g \in \mathsf{F}'_\kappa(L)$ we have

$$f \le g \iff f((r,\text{—})_i) \subseteq g((r,\text{—})_i) \text{ for all } i \in I, r \in \mathbb{Q} \iff g((\text{—},r)_i) \subseteq f((\text{—},r)_i) \text{ for all } i \in I, r \in \mathbb{Q}.$$

Frame homomorphisms in $\mathsf{F}'_\kappa(L)$ can be regarded as an alternative representation of localic hedgehog-valued maps on L.

Moreover, in view of this representation, it is natural to formulate also alternative notions of lower resp. upper semicontinuity. More precisely, set

$$\mathsf{LSC}'_\kappa(L) = \{ f \in \mathsf{F}'_\kappa(L) \mid f((r,\text{—})_i) \text{ is an open sublocale for all } r \in \mathbb{Q} \text{ and } i \in I \}$$

and

$$\mathsf{USC}'_\kappa(L) = \{ f \in \mathsf{F}'_\kappa(L) \mid f((\text{—},r)_i) \text{ is an open sublocale for all } r \in \mathbb{Q} \text{ and } i \in I \}.$$

The following theorem ensures that the notions of lower (resp. upper semicontinuity) that arise from the alternative representation discussed above do not alter the theory of semicontinuity from Section 5.3 in Chapter 5:

Theorem 7.6.2. *For every locale L, there are order-preserving bijections*

$$\mathsf{LSC}_\kappa(L) \cong \mathsf{LSC}'_\kappa(L) \quad \textit{and} \quad \mathsf{USC}_\kappa(L) \cong \mathsf{USC}'_\kappa(L).$$

Proof. Lower semicontinuity. On one direction, if $f \in \mathsf{LSC}_\kappa(L)$, we consider the composite

$$\varphi(f)\colon \mathfrak{L}(J(\kappa)) \xrightarrow{\;f\;} \mathsf{S}(L)^{op} \xrightarrow{(-)^{\#\#}} \mathsf{S}_b(L)^{op} \xrightarrow{(-)^{\#}(\sim)} \mathsf{S}_b(L).$$

If $r \in \mathbb{Q}$ and $i \in I$, we have that $f((r,\text{—})_i)$ is closed and thus $\varphi(f)((r,\text{—})_i) = (f((r,\text{—})_i))^{\#}$ is open.

On the other way around, let $g \in \mathsf{LSC}'_\kappa(L)$. For each $i \in I$ the family $\{g((r,\text{—})_i)^\#\}_{r\in\mathbb{Q}}$ is an antitone family in $\mathsf{S}(L)^{op}$ consisting of closed sublocales, and therefore it defines an $h_i \in \overline{\mathsf{F}}(L)$ given by

$$\begin{aligned} h_i(r,\text{—}) &= \bigcap_{s>r} g((s,\text{—})_i)^\# = \Big(\bigcap_{s>r} g((s,\text{—})_i)^\#\Big)^{\#\#} = \Big(\bigvee_{s>r} g((s,\text{—})_i)^{\#\#}\Big)^\# \\ &= \Big(\bigvee_{s>r} g((s,\text{—})_i)\Big)^\# = g\Big(\bigvee_{s>r}(s,\text{—})_i\Big)^\# = g((r,\text{—})_i)^\# \end{aligned}$$

where the last equality follows from (ch3) and we have used the fact that $\bigcap_{s>r} g((s,\text{—})_i)^\#$ is closed and thus smooth. Clearly, $h_i \in \overline{\mathsf{LSC}}(L)$. Moreover, for $i \neq j$, one has

$$h_i(r,\text{—}) \vee h_j(s,\text{—}) = g((r,\text{—})_i)^\# \vee g((s,\text{—})_j)^\# = \Big(g((r,\text{—})_i) \wedge^{\mathsf{S}_b(L)} g((s,\text{—})_j)\Big)^\# = \mathsf{O}^\# = L$$

by (ch0) and so $\{h_i\}_{i\in I}$ is a disjoint family. Hence there is a unique $\psi(g) \in \mathsf{LSC}_\kappa(L)$ satisfying $\psi(g)\circ\pi_i = h_i$ for all $i \in I$. An application of the antitone $(-)^\#$ easily shows that the two maps $\varphi\colon \mathsf{LSC}_\kappa(L) \to \mathsf{LSC}'_\kappa(L)$ and $\psi\colon \mathsf{LSC}'_\kappa(L) \to \mathsf{LSC}_\kappa(L)$ are monotone. Moreover, we have

$$\psi(\varphi(f))\circ\pi_i(r,\text{—}) = f\circ\pi_i(r,\text{—}) \quad \text{and} \quad \varphi(\psi(g))\circ\pi_i(r,\text{—}) = g\circ\pi_i(r,\text{—})$$

for all $i \in i$. Indeed, $\psi(\varphi(f))\circ\pi_i(r,\text{—}) = \varphi(f)((r,\text{—})_i)^\# = f((r,\text{—})_i)^{\#\#} = f((r,\text{—})_i)$ where the last equality follows because $f((r,\text{—})_i)$ is closed and hence complemented. The second identity is equally straightforward. Hence $\psi(\varphi(f))\circ\pi_i = f\circ\pi_i$ and $\varphi(\psi(g))\circ\pi_i = g\circ\pi_i$ for each $i \in I$, which by the uniqueness clause in Proposition 5.3.4 imply $\psi(\varphi(f)) = f$ and $\varphi(\psi(g)) = g$, that is, φ and ψ are mutually inverse isomorphisms.

Upper semicontinuity. The case of upper semicontinuity is similar. We only sketch the proof of how to build the function $\psi\colon \mathsf{USC}'_\kappa(L) \to \mathsf{USC}_\kappa(L)$. If $g \in \mathsf{USC}'_\kappa(L)$, then for each $i \in I$, the collection $\{g((\text{—},r)_i)^\#\}_{r\in\mathbb{Q}}$ is a monotone family in $\mathsf{S}(L)^{op}$ consisting of closed sublocales, hence it defines an $h_i \in \overline{\mathsf{F}}(L)$ satisfying $h_i(r,\text{—}) = \bigcap_{s>r} g((\text{—},s)_i)$ and

$$\begin{aligned} h_i(\text{—},r) &= \bigcap_{s<r} g((\text{—},s)_i)^\# = \Big(\bigcap_{s<r} g((\text{—},s)_i)^\#\Big)^{\#\#} = \Big(\bigvee_{s<r} g((\text{—},s)_i)^{\#\#}\Big)^\# \\ &= \Big(\bigvee_{s<r} g((\text{—},s)_i)\Big)^\# = g(\bigvee_{s<r}(\text{—},s)_i)^\# = g((\text{—},r)_i)^\#. \end{aligned}$$

Now the family $\{h_i\}_{i\in I}$ is disjoint: if $i \neq j$ then

$$h_i(r,\text{—}) \vee h_j(r',\text{—}) = \bigcap_{s>r,s'>r'} g((\text{—},s)_i) \vee g((\text{—},s')_j) = \bigcap_{s>r,s'>r'} g((\text{—},s)_i \vee (\text{—},s')_j) = L$$

because of Property 5.2.3 (4). Hence there is a unique $\psi(g) \in \overline{\mathsf{F}}(L)$ with $\psi(g)\circ\pi_i = h_i$ for all $i \in I$. It is straightforward to verify that it fullfils the required properties. □

For the spatial case, we have the following

Proposition 7.6.3. *If X is a T_D-space, we have an isomorphism $\mathsf{F}'_\kappa(\Omega(X)) \cong \mathsf{Set}(X, \Lambda J(\kappa))$ — i.e., frame homomorphisms in $\mathsf{F}'_\kappa(\Omega(X))$ correspond bijectively to functions $X \to \Lambda J(\kappa)$.*

Proof. By Lemma 7.5.1, in the T_D-case we have $\mathsf{S}_b(\Omega(X)) \cong \mathcal{P}(X)$. Then, if we denote by X_{dis} the discrete topology on X, one has

$$\begin{aligned}\mathsf{F}'_\kappa(\Omega(X)) &= \mathsf{Frm}(\mathfrak{L}(J(\kappa)), \mathsf{S}_b(\Omega(X))) \cong \mathsf{Frm}(\mathfrak{L}(J(\kappa)), \mathcal{P}(X)) = \mathsf{Frm}(\mathfrak{L}(J(\kappa)), \Omega(X_{dis}))\\ &\cong \mathsf{Top}(X_{dis}, \Sigma(\mathfrak{L}(J(\kappa)))) \cong \mathsf{Top}(X_{dis}, \Lambda J(\kappa)) \cong \mathsf{Set}(X, \Lambda J(\kappa)),\end{aligned}$$

by using the isomorphism $\Sigma(\mathfrak{L}(J(\kappa))) \cong \Lambda J(\kappa)$ from Proposition 5.2.11. □

Theorem 7.6.2 and Proposition 7.6.3 generalize and improve the results in [93] from several points of view:

- Firstly, it is a cardinal generalization, that is, for $\kappa = 1$ we obtain results for $\mathfrak{L}(\overline{\mathbb{R}})$. A similar argument can be used in order to obtain results for $\mathfrak{L}(\mathbb{R})$.
- The standing assumption in [93] is subfitness. Here Theorem 7.6.2 holds without any separation hypothesis.
- Moreover, in the spatial case one has $\mathsf{F}'_\kappa(\Omega(X)) \cong \mathsf{Set}(X, \Lambda J(\kappa))$ for every T_D-space X. The analogous result (for $\kappa = 1$) corresponding to $\mathsf{S}_c(L)$ requires the space in question to be T_1 (cf. [93]).

7.7 Functoriality

We have seen that $\mathsf{S}_b(L)$ is a useful discretization of a locale L which possesses a number of advantages with respect to the larger system $\mathsf{S}(L)^{op}$; in particular it allows a conservative treatment of the notion of real- or hedgehog-valued function.

There is, however, one aspect of $\mathsf{S}_b(L)$ which shows a worse behaviour than that of its counterpart $\mathsf{S}(L)^{op}$, namely the fact that the assignment $L \mapsto \mathsf{S}_b(L)$ is generally not functorial (in fact, this does not come as a surprise regarding the non-functoriality of the Booleanization [28]).

It is therefore a fundamental problem in the theory to study the class of morphisms for which $L \mapsto \mathsf{S}_b(L)$ is actually a functorial construction.

More precisely, if $f : L \to M$ is a frame homomorphism, the map $\mathsf{S}_b(f) : \mathsf{S}_b(L) \to \mathsf{S}_b(M)$ given by

$$\mathsf{S}_b(f)(S) = \bigvee \{\, \mathfrak{o}_M(f(a)) \cap \mathfrak{c}_M(f(b)) \mid \mathfrak{o}_L(a) \cap \mathfrak{c}_L(b) \subseteq S \,\} \tag{7.7.1}$$

is the only possible candidate for obtaining a commutative square

$$\begin{tikzcd}
\mathsf{S}_b(L) \arrow[r, "\mathsf{S}_b(f)"] & \mathsf{S}_b(M) \\
L \arrow[u, tail, "\mathfrak{o}_L"] \arrow[r, "f"] & M \arrow[u, tail, "\mathfrak{o}_M"']
\end{tikzcd} \tag{7.7.2}$$

in Frm. We shall say that f has an S_b-*lift* (or that it S_b-*lifts*) if the $\mathsf{S}_b(f)$ defined above is indeed a frame homomorphism (clearly, the square always commutes). In that case, we will speak of $\mathsf{S}_b(f)$ as the S_b-*lift* of f.

Some functoriality results (in the subfit case) were proved in [17].

In this section we provide further S_b-lifting results, both positive and negative, which can be summarized as follows:

- A frame homomorphism S_b-lifts if and only if both halves of its (RegEpi, Mono) factorization S_b-lift (Proposition 7.7.2).
- A surjection S_b-lifts if and only if it corresponds to a smooth quotient (Corollary 7.7.3). In particular, a quotient onto a point S_b-lifts if and only if it corresponds to a covered prime.
- The canonical monomorphism $\mathfrak{c}\colon L \rightarrowtail \mathsf{S}(L)^{op}$ seldom S_b-lifts (Proposition 7.7.7).
- Under subfitness, all closed maps of locales S_b-lift (Theorem 7.7.10).
- Étale maps of locales S_b-lift (Theorem 7.7.10).

7.7.1 General results

The map $\mathsf{S}_b(f)$ from above is generally not a frame homomorphism but it is more than just a plain set map:

Lemma 7.7.1. *If $f\colon L \to M$ is a frame homomorphism then $\mathsf{S}_b(f)$ preserves finite meets.*

Proof. Let $S,T \in \mathsf{S}_b(L)$. Using the frame distributivity in $\mathsf{S}_b(L)$ and (7.7.1), we have

$$\begin{aligned}
\mathsf{S}_b(f)(S) \wedge \mathsf{S}_b(f)(T) &= \Big(\bigvee_{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq S} \mathfrak{o}_M(f(a)) \cap \mathfrak{c}_M(f(b))\Big) \wedge \Big(\bigvee_{\mathfrak{o}_L(a')\cap\mathfrak{c}_L(b')\subseteq T} \mathfrak{o}_M(f(a')) \cap \mathfrak{c}_M(f(b'))\Big) \\
&= \bigvee_{\substack{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq S \\ \mathfrak{o}_L(a')\cap\mathfrak{c}_L(b')\subseteq T}} \mathfrak{o}_M(f(a)) \cap \mathfrak{c}_M(f(b)) \cap \mathfrak{o}_M(f(a')) \cap \mathfrak{c}_M(f(b')) \\
&= \bigvee_{\substack{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq S \\ \mathfrak{o}_L(a')\cap\mathfrak{c}_L(b')\subseteq T}} \mathfrak{o}_M(f(a\wedge a')) \cap \mathfrak{c}_M(f(b\vee b')) \\
&\subseteq \bigvee_{\mathfrak{o}_L(a\wedge a')\cap\mathfrak{c}_L(b\vee b')\subseteq S\wedge T} \mathfrak{o}_M(f(a\wedge a')) \cap \mathfrak{c}_M(f(b\vee b')) \\
&\subseteq \bigvee_{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq S\wedge T} \mathfrak{o}_M(f(a)) \cap \mathfrak{c}_M(f(b)) = \mathsf{S}_b(f)(S\wedge T).
\end{aligned}$$

The other inequality is trivial. □

The following result indicates that the problem of studying S_b-lifts of frame homomorphisms amounts to studying S_b-lifts of surjections and S_b-lifts of subframe embeddings.

Proposition 7.7.2. *A frame homomorphism S_b-lifts if and only if both halves of its (Regular Epi, Mono) factorization S_b-lift. Moreover, in case f has an S_b-lift,*

(1) *if f is surjective, then so is $\mathsf{S}_b(f)$;*

(2) *if f is injective, then so is $\mathsf{S}_b(f)$.*

Proof. The "if" part is clear, so let us show the "only if". Assume that a frame homomorphism $f\colon L \to M$ has a lift $\mathsf{S}_b(f)$. We factor $\mathsf{S}_b(f)$ through its image, say

$$\mathsf{S}_b(L) \xrightarrow{e} \mathsf{S}_b(f)[\mathsf{S}_b(L)] \xrightarrow{m} \mathsf{S}_b(M)$$

— i.e., $\mathsf{S}_b(f) = m \circ e$ where e is surjective and m is injective. The surjection $e\colon \mathsf{S}_b(L) \twoheadrightarrow \mathsf{S}_b(f)[\mathsf{S}_b(L)]$ corresponds to a sublocale of the Boolean locale $\mathsf{S}_b(L)$, and every sublocale of a Boolean locale is open, hence e corresponds to an open sublocale — i.e., there is a $T \in \mathsf{S}_b(L)$ and an isomorphism $k\colon {\downarrow}T \to \mathsf{S}_b(f)[\mathsf{S}_b(L)]$ making the following diagram commutative:

$$\begin{tikzcd} \mathsf{S}_b(L) \arrow[r, two heads, "e"] \arrow[dr, "(-)\wedge T"'] & \mathsf{S}_b(f)[\mathsf{S}_b(L)] \\ & {\downarrow}T \arrow[u, "k"'] \end{tikzcd}$$

Therefore, we have that $M = \mathsf{S}_b(f)(L) = e(L) = e(T) = \mathsf{S}_b(f)(T)$. Now, since $T \in \mathsf{S}_b(L)$, we can write $T = \bigvee_{\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq T} \mathfrak{o}(a) \cap \mathfrak{c}(b)$, and so (7.7.1) and (7.7.2) yield

$$\begin{aligned} M = \mathsf{S}_b(f)(T) &= \bigvee_{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq T} \mathsf{S}_b(f)(\mathfrak{o}_L(a)) \cap \mathsf{S}_b(f)(\mathfrak{c}_L(b)) \\ &= \bigvee_{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq T} \mathfrak{o}_M(f(a)) \cap \mathfrak{c}_M(f(b)) = \bigvee_{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq T} (f_*)_{-1}[\mathfrak{o}_L(a) \cap \mathfrak{c}_L(b)]. \end{aligned}$$

The colocalic map $f_*[-]\colon \mathsf{S}(M) \longrightarrow \mathsf{S}(L)$ preserves joins, and hence we obtain

$$f_*[M] = \bigvee_{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq T} f_*[(f_*)_{-1}[\mathfrak{o}_L(a) \cap \mathfrak{c}_L(b)]] \subseteq \bigvee_{\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)\subseteq T} \mathfrak{o}_L(a) \cap \mathfrak{c}_L(b) = T$$

where the inclusion follows because of the adjunction $f_*[-] \dashv (f_*)_{-1}[-]$. Hence we have inclusions of sublocales $f_*[M] \subseteq T \subseteq L$. Now, let $i\colon L \twoheadrightarrow T$ and $j\colon T \twoheadrightarrow f_*[M]$ be the corresponding frame surjections. Then f factors as

$$L \xrightarrow{i} T \xrightarrow{j} f_*[M] \xrightarrow{n} M$$

(note that $(j \circ i, n)$ is the (Regular Epi, Mono) factorization of f). We also observe that since $T \in \mathsf{S}_b(L)$, then $\mathsf{S}_b(T) = {\downarrow}T$. For the remainder of the proof it is convenient to consider the diagram

$$\begin{tikzcd} \mathsf{S}_b(L) \arrow[r, two heads, "(-)\wedge T"] & \mathsf{S}_b(T) \arrow[rr, tail, "m\circ k"] & & \mathsf{S}_b(M) \\ L \arrow[u, tail, "\mathfrak{o}_L"] \arrow[r, two heads, "i"] & T \arrow[u, tail, "\mathfrak{o}_T"'] \arrow[r, two heads, "j"] & f_*[M] \arrow[r, tail, "n"] & M \arrow[u, tail, "\mathfrak{o}_M"'] \end{tikzcd} \tag{7.7.3}$$

The left hand side square commutes because for all $a \in L$ one has $\mathfrak{o}_T(i(a)) = \mathfrak{o}_L(a) \cap T$ (cf. [91, III 6.2.1]). Commutativity of the right hand side square follows from the commutativity of the left hand side square, commutativity of the outer square (i.e., (7.7.2) above) and the fact that i is an epimorphism. Now, since the composite $\mathfrak{o}_M \circ n \circ j$ is monic, so is j, thus it is an isomorphism — i.e., $f_*[M] = T$. Diagram (7.7.3) therefore displays the desired S_b-lifts. Observe that diagram (7.7.3) also proves that (1) and (2) of the statement hold. □

7.7.2 Surjections

It is now easy to fully characterize surjections which S_b-lift:

Corollary 7.7.3. *Let $f\colon L \twoheadrightarrow S$ be a frame surjection onto a sublocale S of L. Then f has an S_b-lift if and only if S is smooth.*

Proof. $\Longrightarrow$: On the way of proving the Proposition 7.7.2 we showed that the image of the map which is S_b-lifted always corresponds to a join of complemented sublocales of its domain, hence this implication follows.

$\Longleftarrow$: If $S \in \mathsf{S}_b(L)$, it is immediate to check that $\mathsf{S}_b(S) = {\downarrow}S \subseteq \mathsf{S}_b(L)$. Hence there is a surjection $\mathsf{S}_b(L) \to \mathsf{S}_b(S)$ which maps T to $S \wedge T$. The fact that $\mathfrak{o}_S(f(a)) = S \cap \mathfrak{o}_L(a) = S \wedge \mathfrak{o}_L(a)$ ensures that the relevant diagram commutes. □

Remark 7.7.4. The authors proved in [17] that, in the regular case, an open frame homomorphism with Boolean codomain S_b-lifts. However, in view of Proposition 7.7.2, this result is equivalent to the statement that every open surjection with Boolean codomain S_b-lifts (which is, in turn, a very special case of the last corollary). Indeed, a frame homomorphism is open if and only if both halves of its (RegEpi, Mono) factorization are open, so an open $f\colon L \to B$ (with B Boolean) S_b-lifts if and only if the open surjection $L \twoheadrightarrow f[L]$ and the open subframe embedding $f[L] \rightarrowtail B$ have S_b-lifts. But an open subframe of a Boolean frame is necessarily Boolean, hence the embedding $f[L] \rightarrowtail B$ trivially S_b-lifts.

In particular, we can strengthen [17, Theorem 4.5] as follows:

Corollary 7.7.5. *Let p be a prime in a frame L. Then the frame surjection associated to the sublocale $\mathfrak{b}(p)$ has an S_b-lift if and only if p is a covered prime.*

As an easy consequence, we also have the following necessary condition for a map to S_b-lift:

Corollary 7.7.6. *If a frame homomorphism S_b-lifts, it is a D-homomorphism.*

Proof. Let $f\colon L \to M$ have an S_b-lift and let p be a covered prime in M. By Corollary 7.7.5 it follows that the left adjoint of the inclusion $\mathfrak{b}(p) \hookrightarrow M$ has an S_b-lift. Since f also S_b-lifts, the left adjoint of the upper-left composite in the square

$$\begin{tikzcd} M \arrow[r, "f_*"] & L \\ \mathfrak{b}(p) \arrow[u, hook] \arrow[r, two heads, "f_*"] & \mathfrak{b}(f_*(p)) \arrow[u, hook] \end{tikzcd}$$

has an S_b-lift. Now, the bottom-right composite consists of an isomorphism followed by $\mathfrak{b}(f_*(p)) \hookrightarrow L$, hence the left adjoint of $\mathfrak{b}(f_*(p)) \hookrightarrow L$ has an S_b-lift, which by another application of Corollary 7.7.5 implies that $f_*(p)$ is covered. □

7.7.3 Monomorphisms

Dealing with monomorphisms is a more difficult task. Here we will not be able to provide a full characterization but instead give some negative and positive results — including a large class of monomorphisms which indeed S_b-lift.

We start with the following result:

Proposition 7.7.7. *Let L be fit. The canonical monomorphism $\mathfrak{c}_L\colon L \rightarrowtail \mathsf{S}(L)^{op}$ has an S_b-lift if and only if $\mathsf{S}(L)$ is Boolean.*

Proof. The "if" implication is trivial, so let us assume that $\mathfrak{c}_L\colon L \rightarrowtail \mathsf{S}(L)^{op}$ has an S_b-lift. Consider the surjective coframe homomorphism $f\colon S(L) \to \mathsf{S}_b(L)^{op}$ sending S to $S^\#$. Let us show that it is also injective. Let $S, T \in \mathsf{S}(L)$ with $S^\# = T^\#$. Since L is fit, we can write $S = \bigcap_i \mathfrak{o}_L(a_i)$ and $T = \bigcap_j \mathfrak{o}_L(b_j)$ for appropriate families $\{a_i\}_{i\in I}, \{b_j\}_{j\in J} \subseteq L$. Now, from $S^\# = T^\#$, we obtain $\bigvee_i \mathfrak{c}_L(a_i) = \bigvee_j \mathfrak{c}_L(b_j)$. Since $\mathsf{S}_b(\mathfrak{c}_L)(\mathfrak{c}_L(a)) = \mathfrak{c}_{\mathsf{S}(L)^{op}}(\mathfrak{c}_L(a))$ for each $a \in L$ and the S_b-lift $\mathsf{S}_b(\mathfrak{c}_L)\colon \mathsf{S}_b(L) \to \mathsf{S}_b(\mathsf{S}(L)^{op})$ preserves arbitrary joins, and it follows that

$$\bigvee_i \mathfrak{c}_{\mathsf{S}(L)^{op}}(\mathfrak{c}_L(a_i)) = \mathsf{S}_b(\mathfrak{c}_L)(\bigvee_i \mathfrak{c}_L(a_i)) = \mathsf{S}_b(\mathfrak{c}_L)(\bigvee_j \mathfrak{c}_L(b_j)) = \bigvee_j \mathfrak{c}_{\mathsf{S}(L)^{op}}(\mathfrak{c}_L(b_j)).$$

But the $\mathfrak{c}_L(a_i)$ and $\mathfrak{c}_L(b_j)$ are complemented in $\mathsf{S}(L)^{op}$, so we can write the last equality as $\bigvee_i \mathfrak{o}_{\mathsf{S}(L)^{op}}(\mathfrak{o}_L(a_i)) = \bigvee_j \mathfrak{o}_{\mathsf{S}(L)^{op}}(\mathfrak{o}_L(b_j))$, and therefore we have

$$\mathfrak{o}_{\mathsf{S}(L)^{op}}(\bigvee_i \mathfrak{o}_L(a_i)) = \mathfrak{o}_{\mathsf{S}(L)^{op}}(\bigvee_j \mathfrak{o}_L(b_j)).$$

Accordingly, $\bigvee_i \mathfrak{o}_L(a_i) = \bigvee_j \mathfrak{o}_L(b_j)$ in $\mathsf{S}(L)^{op}$ — i.e., $S = \bigcap_i \mathfrak{o}_L(a_i) = \bigcap_j \mathfrak{o}_L(b_j) = T$. Therefore $\mathsf{S}(L)$ is Boolean. □

Remark 7.7.8. Recall that locales L such that $\mathsf{S}(L)$ is Boolean are known as scattered (see [100, Theorem 11]). It is a very restrictive property, and therefore the previous proposition shows that the canonical monomorphism $\mathfrak{c}_L\colon L \rightarrowtail \mathsf{S}(L)^{op}$ seldom lifts.

In what follows, we shall use localic techniques (rather than working in the category of frames as in the preceding results) as it seems much simpler because we will have to deal with localic images and preimages.

Recall that a morphism $f\colon L \to M$ in Loc is said to be an *étale map* or *local homeomorphism* [36, 1.7] if there is a cover $\{a_i\}_{i\in I}$ of L such that $f[\mathfrak{o}_L(a_i)]$ is an open sublocale in M and the restriction $f|_{\mathfrak{o}_L(a_i)}\colon \mathfrak{o}_L(a_i) \to f[\mathfrak{o}_L(a_i)]$ is an isomorphism for every $i \in I$.

Lemma 7.7.9. *Let $f\colon L \to M$ be a localic map. Then the following assertions hold*:

(1) *If f is an étale map and $S \in \mathsf{S}_b(L)$, then $f[S] \in \mathsf{S}_b(M)$;*

(2) *If L is subfit, f is a closed map and $S \in \mathsf{S}_b(L)$, then $f[S] \in \mathsf{S}_b(M)$;*

(3) *If L is T_D-spatial, f^* is a D-homomorphism and $S \in \mathsf{S}_b(L)$, then $f[S] \in \mathsf{S}_b(M)$.*

Proof. (1) Write $S = \bigvee_{j\in J} \mathfrak{c}_L(x_j) \cap \mathfrak{o}_L(y_j) \in \mathsf{S}_b(L)$ and let $\{a_i\}_{i\in I}$ be the cover of L as in the definition of local homeomorphisms. Since complemented sublocales are linear (cf. (1.2.3)) and $f[-]$ preserves joins, we have

$$f[S] = f[S \cap \textstyle\bigvee_i \mathfrak{o}_L(a_i)] = f[\bigvee_{i,j} \mathfrak{c}_L(x_j) \cap \mathfrak{o}_L(y_j) \cap \mathfrak{o}_L(a_i)] = \bigvee_{i,j} f[\mathfrak{c}_L(x_j) \cap \mathfrak{o}_L(y_j \wedge a_i)].$$

Now for each $i \in I$ and $j \in J$, the sublocale $\mathfrak{c}_L(x_j) \cap \mathfrak{o}_L(y_j \wedge a_i)$ is locally closed in $\mathfrak{o}_L(a_i)$ — note that it equals $\mathfrak{c}_{\mathfrak{o}(a_i)}(\nu_{\mathfrak{o}(a_i)}(x_j)) \cap \mathfrak{o}_{\mathfrak{o}_L(a_i)}(\nu_{\mathfrak{o}_L(a_i)}(y_j))$. Since $f|_{\mathfrak{o}_L(a_i)}\colon \mathfrak{o}_L(a_i) \to f[\mathfrak{o}_L(a_i)]$ is an isomorphism, $f[\mathfrak{c}_L(x_j) \cap \mathfrak{o}_L(y_j \wedge a_i)]$ is locally closed in $f[\mathfrak{o}(a_i)]$. But $f[\mathfrak{o}_L(a_i)]$ is open in M, so it follows easily that $f[\mathfrak{c}_L(x_j) \cap \mathfrak{o}_L(y_j \wedge a_i)]$ is locally closed in M. Hence $f[S] \in \mathsf{S}_b(M)$.

(2) By subfitness, write $S = \bigvee_{j\in J} \mathfrak{c}_L(x_j) \in \mathsf{S}_c(L) = \mathsf{S}_b(L)$ (see Corollary 7.3.4). Then $f[S] = \bigvee_j f[\mathfrak{c}_L(x_j)]$ and each $f[\mathfrak{c}_L(x_j)]$ is closed in M. Hence the result follows.

(3) Since L is T_D-spatial and $S \in \mathsf{S}_b(L)$, by Lemma 1.3.4 and Lemma 7.5.1 we deduce that $S = \bigvee_{i\in I} \mathfrak{b}_L(p_i)$, with p_i covered primes in L. Again, $f[S] = \bigvee_{i\in i} f[\mathfrak{b}_L(p_i)] = \bigvee_{i\in i} \mathfrak{b}_M(f(p_i))$ and each $\mathfrak{b}_M(f(p_i))$ is complemented because f^* is a D-homomorphism. □

Theorem 7.7.10. *Let $f\colon L \to M$ be a localic map. Then the following assertions hold*:

(1) *If f is an étale map, then f^* has an S_b-lift;*

(2) *If L is subfit and f is a closed map, then f^* has an S_b-lift;*

(3) *If L is T_D-spatial and f^* is a D-homomorphism, then f^* has an S_b-lift.*

Proof. Let f and L be as in (1), (2) or (3). The coframe homomorphism $f_{-1}[-]$ is coweakly open (i.e., the frame homomorphism $f_{-1}[-]^{op}$ is weakly open, cf. Subsection 1.2.3) if and only if $f_{-1}[S]^{\#\#} \subseteq f_{-1}[S^{\#\#}]$ — i.e., if and only if $f[f_{-1}[S]^{\#\#}] \subseteq S^{\#\#}$. In view of the previous lemma, since $f_{-1}[S]^{\#\#} \in \mathsf{S}_b(L)$ it follows that $f[f_{-1}[S]^{\#\#}] \in \mathsf{S}_b(M)$. Hence $f[f_{-1}[S]^{\#\#}] \subseteq S^{\#\#}$ if and only if $f[f_{-1}[S]^{\#\#}] \subseteq S$. But the latter holds trivially because $f_{-1}[S]^{\#\#} \subseteq f_{-1}[S]$. Then, by [28, Proposition 1.1], there is a frame homomorphism h making the square

$$\begin{tikzcd} \mathsf{S}_b(M)^{op} \arrow[r, "h"] & \mathsf{S}_b(L)^{op} \\ \mathsf{S}(M)^{op} \arrow[r, "f_{-1}[-]^{op}"] \arrow[u, two heads, "(-)^{\#\#}"] & \mathsf{S}(L)^{op} \arrow[u, two heads, "(-)^{\#\#}"'] \end{tikzcd}$$

commutative. By the usual functoriality of the assembly and by complementation in the Boolean algebras $\mathsf{S}_b(M)$ and $\mathsf{S}_b(L)$ we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{S}_b(M) \arrow[r] & \mathsf{S}_b(L) \\ \mathsf{S}_b(M)^{op} \arrow[r, "h"] \arrow[u, "\sim"] & \mathsf{S}_b(L)^{op} \arrow[u, "\sim"'] \\ \mathsf{S}(M)^{op} \arrow[r, "f_{-1}[-]^{op}"] \arrow[u, two heads, "(-)^{\#\#}"] & \mathsf{S}(L)^{op} \arrow[u, two heads, "(-)^{\#\#}"'] \\ M \arrow[r, "f^*"] \arrow[u, "\mathfrak{c}_M"] & L \arrow[u, "\mathfrak{c}_L"'] \end{tikzcd}$$

whose outer square displays the desired S_b-lift. □

Remark 7.7.11. Using Proposition 7.7.2 and the fact that a localic map is closed if and only if both halves of its (RegEpi, Mono) factorization are closed, it follows that item (2) of the previous theorem is equivalent to the statement that, under subfitness,

> *closed surjections of locales and closed sublocale embeddings have* S_b*-lifts*

— but the latter holds by Corollary 7.7.3 as closed embeddings are smooth. Hence, what we *really* get is that closed surjections of locales have S_b-lifts. This is the largest class of monomorphisms of frames which we know to have S_b-lifts (the open case proved by Ball, Picado and Pultr is much more restrictive as the codomain has to be Boolean, cf. [17] and Remark 7.7.4).

Corollary 7.7.12. *If M is T_D-spatial, a frame homomorphism $f\colon L \to M$ has an* S_b*-lift if and only if it is a D-homomorphism.*

Proof. Necessity follows by Corollary 7.7.6 and sufficiency follows by Theorem 7.7.10. □

7.7.4 Interaction with localic products

As an application of the previously proved results, we can now shed some light over a question posed to us by Martin Hyland concerning how the assignment $L \mapsto \mathsf{S}_b(L)$ interacts with localic products.

We start with a few properties about prime elements in localic products. Recall that if L and M are frames and $a \in L$ and $b \in M$, we denote $a \parr b = (a \oplus 1) \vee (1 \oplus b) \in L \oplus M$ (see Subsection 2.2.5).

Lemma 7.7.13. *Let $L \oplus M$ be the localic product of L and M, $\pi_1 \colon L \oplus M \to L$ and $\pi_2 \colon L \oplus M \to M$ the projections, $a \in L$ and $b \in M$, and $\{a_i\}_{i \in I} \subseteq L$ and $\{b_j\}_{j \in J} \subseteq M$. Then:*

(1) $a \parr b = \{(x, y) \in L \times M \mid x \leq a \text{ or } y \leq b\}$.

(2) $\left(\bigwedge_{i \in I} a_i\right) \parr \left(\bigwedge_{j \in J} b_j\right) = \bigwedge_{i \in I, j \in J} a_i \parr b_j$.

(3) *If $b \neq 1$, then $\pi_1(a \parr b) = a$, and if $a \neq 1$, then $\pi_2(a \parr b) = b$.*

(4) *If a is prime in L and b is prime in M, then $a \parr b$ is prime in $L \oplus M$.*

(5) *If a is a covered prime in L with cover a^+ and b is a covered prime in M with cover b^+, then $a \parr b$ is covered in $L \oplus M$ with cover $(a \parr b)^+ = (a^+ \parr b) \wedge (a \parr b^+)$.*

Proof. (1) follows easily from the fact that $\{(x, y) \in L \times M \mid x \leq a \text{ or } y \leq b\}$ is a *cp*-ideal.

The inequality $\leq$ in (2) is trivial so let us show the reverse one. Let $(x, y) \in \bigwedge_{i \in i, j \in J} a_i \parr b_j$. By (1), it suffices to show that $x \leq \bigwedge_{i \in i} a_i$ or $y \leq \bigwedge_{j \in J} b_j$. Assume that $x \not\leq \bigwedge_{i \in I} a_i$. Then there is an $i_0 \in I$ with $x \not\leq a_{i_0}$. But $(x, y) \in \bigwedge_{i \in i, j \in J} a_i \parr b_j$ and so $(x, y) \in a_{i_0} \parr b_j$ for all $j \in J$. By (1), we have $y \leq b_j$ for all $j \in J$ — i.e., $y \leq \bigwedge_{j \in J} b_j$.

For (3), we use the adjunction $\iota_1 \dashv \pi_1$ to compute

$$\pi_1(a \parr b) = \bigvee\{x \in L \mid x \oplus 1 \leq a \parr b\} = \bigvee\{x \in L \mid x \leq a \text{ or } b = 1\}.$$

Similarly $\pi_2(a \parr b) = \bigvee\{y \in M \mid a = 1 \text{ or } y \leq b\}$.

(4) can be shown using the fact that $\Sigma \colon \mathsf{Loc} \to \mathsf{Top}$ is a right adjoint and hence it preserves limits. For the sake of completeness, we give a direct proof. First, let $a \in \mathsf{pt}(L)$ and $b \in \mathsf{pt}(M)$. Let $U_1, U_2 \in L \oplus M$ with $U_1 \wedge U_2 \leq a \parr b$ and suppose that $U_1 \not\leq a \parr b$. Then there is an $(x_1, y_1) \in U_1$ with $x_1 \not\leq a$ and $y_1 \not\leq b$. For each $(x_2, y_2) \in U_2$, one has $(x_1 \wedge x_2, y_1 \wedge y_2) \in U_1 \wedge U_2 \leq a \parr b$, and so either $x_1 \wedge x_2 \leq a$ or $y_1 \wedge y_2 \leq b$. By primality of a and b, it follows that either $x_2 \leq a$ or $y_2 \leq b$. Thus $U_2 \leq a \parr b$ and $a \parr b \in \mathsf{pt}(L \oplus M)$.

For (5), let $a \in \mathsf{pt}_D(L)$ and $b \in \mathsf{pt}_D(M)$. Since $\{x \oplus y \mid x \in L,\ y \in M\}$ is a $\bigvee$-base of $L \oplus M$, we shall use Proposition 1.3.2 (iii) for proving that $a \parr b$ is covered with

$$(a \parr b)^+ := (a^+ \parr b) \wedge (a \parr b^+) = (a^+ \oplus b^+) \vee (a \parr b).$$

Obviously, $a \parr b < (a \parr b)^+$ (if the equality holds then we would have $(a^+, b^+) \in (a^+ \parr b) \wedge (a \parr b^+) = a \parr b$ and so either $a^+ \leq a$ or $b^+ \leq b$, a contradiction). Now let $x \in L$ and $y \in M$ with

$x \oplus y \le (a \invamp b)^+$. If $x \le a$ or $y \le b$, then $x \oplus y \le a \invamp b$ and we are done. Hence suppose that $x \not\le a$ and $y \not\le b$. Then $a^+ \le x \vee a$ and $b^+ \le y \vee b$ and so $(a \invamp b)^+ \le ((x \vee a) \oplus (y \vee b)) \vee (a \invamp b) = (x \oplus y) \vee (a \invamp b)$, as required. Hence $a \invamp b \in \mathsf{pt}_D(L \oplus M)$. □

Corollary 7.7.14. *The map* $\varphi_{L,M} \colon \mathsf{pt}(L) \times \mathsf{pt}(M) \to \mathsf{pt}(L \oplus M)$ *given by* $\varphi_{L,M}(p,q) = p \invamp q$ *is a bijection.*

Proof. $\varphi_{L,M}$ is well-defined by Lemma 7.7.13 (4) and it is obviously injective. Moreover, given a prime $U \in \mathsf{pt}(L \oplus M)$, since localic maps send primes into primes, one has $p = \pi_1(U) \in \mathsf{pt}(L)$ and $q = \pi_2(U) \in \mathsf{pt}(M)$ and clearly $p \invamp q \le U$. On the other hand, if $(a,b) \in U$ then $a \oplus b = (a \invamp 0) \wedge (0 \invamp b) \le U$, and since U is prime, either $a \invamp 0 \le U$ or $0 \invamp b \le U$. Assume without loss of generality the former. Then $\iota_1(a) = a \oplus 1 = a \invamp 0 \le U$, i.e., $a \le \pi_1(U) = p$, and thus $(a,b) \in p \invamp q$. Consequently, $p \invamp q = U$ and $\varphi_{L,M}$ is surjective. □

We can now show the following property of coproduct injections:

Lemma 7.7.15. *Let L and M be frames. The coproduct injections $\iota_1 \colon L \to L \oplus M$ and $\iota_2 \colon M \to L \oplus M$ are D-homomorphisms. Equivalently, the projections $\pi_1 \colon L \oplus M \to L$ and $\pi_2 \colon L \oplus M \to M$ are D-localic maps.*

Proof. Let $U \in \mathsf{pt}_D(L \oplus M)$. Since in particular U is a prime in $L \oplus M$, by Corollary 7.7.14 there are $p \in \mathsf{pt}(L)$ and $q \in \mathsf{pt}(M)$ with $U = p \invamp q$. By Lemma 7.7.13 (3) we have to show that $p \in \mathsf{pt}_D(L)$ and $q \in \mathsf{pt}_D(M)$. We shall only show that $p \in \mathsf{pt}_D(L)$ since the other case is similar. Assume that $p = \bigwedge_i a_i$ with $\{a_i\}_{i \in I} \subseteq L$ and let $(a,b) \in \bigwedge_i (a_i \invamp q)$. If $b \le q$, then obviously $(a,b) \in p \invamp q$. On the other hand, if $b \not\le q$, then $a \le a_i$ for all $i \in I$, and so $a \le \bigwedge_i a_i = p$. Thus $(a,b) \in p \invamp q$. This shows that $\bigwedge_i (a_i \invamp q) \le p \invamp q$, whereas the reverse inequality is trivial. Since $U = p \invamp q \in \mathsf{pt}_D(L \oplus M)$, there is an $i_0 \in I$ with $a_{i_0} \invamp q = p \invamp q$. Since $q \ne 1$, it follows that $p = a_{i_0}$. □

Remark 7.7.16. (1) By the previous lemma and Theorem 7.7.10 (3), it follows that if $L \oplus M$ is T_D-spatial, then the coproduct injections $\iota_1 \colon L \to L \oplus M$ and $\iota_2 \colon M \to L \oplus M$ have S_b-lifts. Hence there is a canonical comparison frame homomorphism

$$\langle \mathsf{S}_b(\iota_1), \mathsf{S}_b(\iota_2) \rangle \colon \mathsf{S}_b(L) \oplus \mathsf{S}_b(M) \to \mathsf{S}_b(L \oplus M).$$

For more information on this comparison map, see Theorem 7.7.23 below.

(2) The coproduct $L \oplus M$ being T_D-spatial is not a necessary condition in (1). A sublocale is *pointless* if it does not contain any prime element. Now, if B is a nontrivial pointless Boolean algebra, then $B \oplus B$ is not spatial and $\iota_1, \iota_2 \colon B \to B \oplus B$ have S_b-lifts (for trivial reasons, as the domain is Boolean).

Corollary 7.7.17. *Let L and M be frames. The map $\psi_{L,M} \colon \mathsf{pt}_D(L) \times \mathsf{pt}_D(M) \to \mathsf{pt}_D(L \oplus M)$ given by $\psi_{L,M}(p,q) = p \invamp q$ is a bijection.*

Proof. $\psi_{L,M}$ is well-defined by Lemma 7.7.13 (5) and it is obviously injective. Moreover, it is surjective by the proof of Lemma 7.7.15, because we showed that if $U \in \mathsf{pt}_D(L \oplus M)$, then $U = p \invamp q$ with $p \in \mathsf{pt}_D(L)$ and $q \in \mathsf{pt}_D(M)$. □

Corollary 7.7.18. *Let L and M be frames. Then the following are equivalent*:

(i) *$L\oplus M$ is T_D-spatial;*

(ii) *$L\oplus M$ is spatial and both L and M are T_D-spatial.*

Proof. (i) $\Longrightarrow$ (ii) follows immediately from Lemma 7.7.15 and Corollary 1.3.7.

(ii) $\Longrightarrow$ (i): Let $U\in L\oplus M$. Since $L\oplus M$ is spatial, then one can write $U=\bigwedge_{i\in I} p_i \parr q_i$ with $\{p_i\}_{i\in I}\subseteq \mathsf{pt}(L)$ and $\{q_i\}_{i\in I}\subseteq \mathsf{pt}(M)$ by Corollary 7.7.14. Now, L is T_D-spatial, so by Lemma 1.3.6, for each $i\in I$ there is a family $\{p^i_j\}_{j\in I_i}\subseteq \mathsf{pt}_D(L)$ with $p_i=\bigwedge_{j\in I_i} p^i_j$. Similarly, for each $i\in I$ there is a family $\{q^i_k\}_{k\in J_i}\subseteq \mathsf{pt}_D(M)$ with $q_i=\bigwedge_{k\in J_i} q^i_k$. By Lemma 7.7.13 (2) and (5), it follows that $U=\bigwedge_{i\in I,\, j\in I_i,\, k\in J_i} p^i_j \parr q^i_k$ with each $p^i_j\parr q^i_k$ being covered in $L\oplus M$, so the assertion now follows from Lemma 1.3.6. ◻

Recall the category Frm_D of Subsection 1.3.2. As a consequence of the previous results we obtain the following:

Corollary 7.7.19. *Let L and M be frames. Then the system $(L\oplus M,\iota_1,\iota_2)$ is a coproduct in Frm_D. Consequently, the category Frm_D has finite coproducts and the inclusion functor $I\colon \mathsf{Frm}_D\hookrightarrow \mathsf{Frm}$ preserves them.*

Proof. The coproduct injections ι_1 and ι_2 are D-homomorphisms by Lemma 7.7.15. It remains to be proved that if $h\colon L\to P$ and $k\colon M\to P$ are D-homomorphisms, then the induced map $\langle h,k\rangle\colon L\oplus M\to P$ is also a D-homomorphism. Hence let $p\in \mathsf{pt}_D(P)$. Then

$$\begin{aligned}\langle h,k\rangle_*(p) &= \bigvee\{a\oplus b \mid h(a)\wedge k(b)\le p\} = \bigvee\{a\oplus b\mid h(a)\le p \text{ or } k(b)\le p\}\\ &= \bigvee\{a\oplus b\mid a\le h_*(p) \text{ or } b\le k_*(p)\} = \bigvee\{a\oplus b\mid a\le h_*(p)\}\vee\bigvee\{a\oplus b\mid b\le k_*(p)\}\\ &= (h_*(p)\oplus 1)\vee(1\oplus k_*(p)) = h_*(p)\parr k_*(p).\end{aligned}$$

Since h and k are D-homomorphisms, $h_*(p)\in \mathsf{pt}_D(L)$ and $h_*(q)\in\mathsf{pt}_D(M)$, so the conclusion now follows from Lemma 7.7.13 (5). ◻

Since $\Sigma'\colon \mathsf{Loc}_D\to\mathsf{Top}_D$ is a right adjoint (see (1.3.1)), it preserves products, so by the previous corollary we can improve the bijection in Corollary 7.7.17 to a homeomorphism (observe that finite products of T_D-spaces are T_D, hence finite products in Top_D are just products in Top):

Corollary 7.7.20. *Let L and M be locales. Then the canonical map*

$$(\Sigma'(\pi_1),\Sigma'(\pi_2))\colon \Sigma'(L\oplus M)\to\Sigma'(L)\times\Sigma'(M)$$

is a homeomorphism.

As an application of the above, we obtain a *finite* T_D-analogue of a well-known result for the classical spectrum, namely the fact that for *sober* spaces X_i, if $\oplus_{i\in I}\Omega(X_i)$ is spatial, then $\oplus_{i\in I}\Omega(X_i)\cong\Omega(\prod_{i\in I}X_i)$ (see [91, IV 5.4.2]).

Corollary 7.7.21. *Let X and Y be T_D-spaces. Then the the following are equivalent:*

(i) $\Omega(X)\oplus\Omega(Y)\cong\Omega(X\times Y)$;

(ii) $\Omega(X)\oplus\Omega(Y)$ *is spatial;*

(iii) $\Omega(X)\oplus\Omega(Y)$ *is T_D-spatial.*

Proof. (i) $\Longrightarrow$ (ii) is trivial and the equivalence between (ii) and (iii) follows from Corollary 7.7.18. Finally, assume that $\Omega(X)\oplus\Omega(Y)$ is T_D-spatial. Then, one has $\Omega(X)\oplus\Omega(Y)\cong\Omega(\Sigma'(\Omega(X)\oplus\Omega(Y))$ via the counit of the adjunction $\Omega\vdash\Sigma'$, which is an isomorphism by T_D-spatiality. Finally,

$$\Omega(\Sigma'(\Omega(X)\oplus\Omega(Y)))\cong\Omega(\Sigma'(\Omega(X))\times\Sigma'(\Omega(Y)))\cong\Omega(X\times Y),$$

where the first isomorphism follows by applying Ω to the homeomorphism in Corollary 7.7.20, and the second isomorphism follows from the fact that the unit of the adjunction $\Omega\vdash\Sigma'$ is always an isomorphism. Hence $\Omega(X)\oplus\Omega(Y)\cong\Omega(X\times Y)$. □

We shall need one last lemma concerning one-point sublocales and product projections.

Lemma 7.7.22. *Let L and M be frames and assume that $L\oplus M$ is spatial. If $p\in\mathsf{pt}_D(L)$ and $q\in\mathsf{pt}_D(M)$, then $\mathfrak{b}(p\parr q)=(\pi_1)_{-1}[\mathfrak{b}(p)]\cap(\pi_2)_{-1}[\mathfrak{b}(q)]$.*

Proof. The inclusion $\subseteq$ follows immediately from Lemma 7.7.13 (3) since $\pi_1[\mathfrak{b}(p\parr q)]=\mathfrak{b}(\pi_1(p\parr q))=\mathfrak{b}(p)$ and $\pi_2[\mathfrak{b}(p\parr q)]=\mathfrak{b}(\pi_2(p\parr q))=\mathfrak{b}(q)$.

Let us show the reverse inequality. Since $L\oplus M$ is spatial, we have $L\oplus M=\bigvee_i\mathfrak{b}(p_i\parr q_i)$ for $\{p_i\}_{i\in I}\subseteq\mathsf{pt}(L)$ and $\{q_i\}_{i\in I}\subseteq\mathsf{pt}(M)$, by Corollary 7.7.14. Now, by Lemma 1.3.3, $\mathfrak{b}(p)$ (resp. $\mathfrak{b}(q)$) is complemented in L (resp. M), hence the sublocale $S:=(\pi_1)_{-1}[\mathfrak{b}(p)]\cap(\pi_2)_{-1}[\mathfrak{b}(q)]$ is complemented in $L\oplus M$. By linearity (cf. (1.2.3)), it follows that

$$S=S\cap\bigvee_i\mathfrak{b}(p_i\parr q_i)=\bigvee_i S\cap\mathfrak{b}(p_i\parr q_i).$$

If $\mathfrak{b}(p_i\parr q_i)\subseteq S$ then by adjunction we have $\mathfrak{b}(p_i)=\pi_1[\mathfrak{b}(p_i\parr q_i)]\subseteq\mathfrak{b}(p)$, thus $p_i=p$. Similarly, $q=q_i$. Since either $S\cap\mathfrak{b}(p_i\parr q_i)=\mathsf{O}$ or $S\cap\mathfrak{b}(p_i\parr q_i)=\mathfrak{b}(p_i\parr q_i)$, it follows that $S\subseteq\mathfrak{b}(p\parr q)$. □

We are now in position to prove the main result, which connects preservation of certain localic products by $\mathsf{S}_b(-)$ and the T_D-spatiality of them.

Theorem 7.7.23. *The following are equivalent for a frame L:*

(i) *$L\oplus L$ is T_D-spatial;*

(ii) *The coproduct injections $\iota_1,\iota_2\colon L\to L\oplus L$ have S_b-lifts and the comparison frame homomorphism $\langle\mathsf{S}_b(\iota_1),\mathsf{S}_b(\iota_2)\rangle\colon\mathsf{S}_b(L)\oplus\mathsf{S}_b(L)\to\mathsf{S}_b(L\oplus L)$ is an isomorphism;*

(iii) *There exists an isomorphism $\mathsf{S}_b(L\oplus L)\cong\mathsf{S}_b(L)\oplus\mathsf{S}_b(L)$.*

Proof. (i) $\Longrightarrow$ (ii). By Remark 7.7.16 (1), the coproduct injections S_b-lift and hence the comparison map $\langle \mathsf{S}_b(\iota_1), \mathsf{S}_b(\iota_2)\rangle$ exists. For $p \in \mathsf{pt}_D(L)$ and $q \in \mathsf{pt}_D(M)$ we have $\mathfrak{b}(p) \in \mathsf{S}_b(L)$ and $\mathfrak{b}(q) \in \mathsf{S}_b(M)$. One has by (7.7.1)

$$\begin{aligned}\mathsf{S}_b(\iota_1)(\mathfrak{b}(p)) &= \bigvee\{\, \mathfrak{o}_{L\oplus L}(\iota_1(a)) \cap \mathfrak{c}_{L\oplus L}(\iota_1(b)) \mid \mathfrak{o}_L(a)\cap \mathfrak{c}_L(b) \subseteq \mathfrak{b}(p)\,\}\\ &= \bigvee\{\, (\pi_1)_{-1}[\mathfrak{o}_L(a)\cap \mathfrak{c}_L(b)] \mid \mathfrak{o}_L(a)\cap \mathfrak{c}_L(b) \subseteq \mathfrak{b}(p)\,\} = (\pi_1)_{-1}[\mathfrak{b}(p)]\end{aligned}$$

and similarly $\mathsf{S}_b(\iota_2)(\mathfrak{b}(q)) = (\pi_2)_{-1}[\mathfrak{b}(q)]$. It follows that

$$\langle \mathsf{S}_b(\iota_1), \mathsf{S}_b(\iota_2)\rangle(\mathfrak{b}(p)\oplus \mathfrak{b}(q)) = (\pi_1)_{-1}[\mathfrak{b}(p)] \cap (\pi_2)_{-1}[\mathfrak{b}(q)] = \mathfrak{b}(p \parr q),$$

where the last equality follows by Lemma 7.7.22.

Now, since $L\oplus L$ (and hence L) is T_D-spatial, it follows by Theorem 7.5.2 that the maps $\mathfrak{m}\colon \mathcal{P}(\mathsf{pt}_D(L\oplus L)) \to \mathsf{S}_b(L\oplus L)$ and $\mathsf{pt}_D \oplus \mathsf{pt}_D\colon \mathsf{S}_b(L)\oplus \mathsf{S}_b(L) \to \mathcal{P}(\mathsf{pt}_D(L))\oplus \mathcal{P}(\mathsf{pt}_D(L))$ are isomorphisms. On the other hand, the map $\psi_{L,L}\colon \mathsf{pt}_D(L)\times \mathsf{pt}_D(L) \to \mathsf{pt}_D(L\oplus L)$ from Corollary 7.7.17 is a bijection, and hence $\mathcal{P}(\psi_{L,L})$ is an isomorphism (where $\mathcal{P}$ is the covariant power set functor). Finally, it is well known (see for example [96, 1.6.4] for a direct proof) that for any set X, the map $\mathcal{P}(X)\oplus\mathcal{P}(X) \to \mathcal{P}(X\times X)$ that sends $A\oplus B \in \mathcal{P}(X)\oplus\mathcal{P}(X)$ to $A\times B$ is an isomorphism. Consequently, the composite

$$\begin{tikzcd}
\mathsf{S}_b(L)\oplus\mathsf{S}_b(L) \arrow[rr, dotted] \arrow[d, "\mathsf{pt}_D\oplus\mathsf{pt}_D"'] & & \mathsf{S}_b(L\oplus L) \\
\mathcal{P}(\mathsf{pt}_D(L))\oplus\mathcal{P}(\mathsf{pt}_D(L)) \arrow[r] & \mathcal{P}(\mathsf{pt}_D(L)\times\mathsf{pt}_D(L)) \arrow[r, "\mathcal{P}(\psi_{L,L})"] & \mathcal{P}(\mathsf{pt}_D(L\oplus L)) \arrow[u, "\mathfrak{m}"']
\end{tikzcd}$$

is an isomorphism which sends each $\mathfrak{b}(p)\oplus\mathfrak{b}(q) \in \mathsf{S}_b(L)\oplus\mathsf{S}_b(L)$ into $\mathfrak{b}(p\parr q)$. Hence it coincides with $\langle \mathsf{S}_b(\iota_1), \mathsf{S}_b(\iota_2)\rangle$.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i): Assume that $\mathsf{S}_b(L\oplus L) \cong \mathsf{S}_b(L)\oplus\mathsf{S}_b(L)$ holds. Since $\mathsf{S}_b(L\oplus L)$ is Boolean, so is $\mathsf{S}_b(L)\oplus\mathsf{S}_b(L)$. In particular, the localic diagonal in $\mathsf{S}_b(L)\oplus\mathsf{S}_b(L)$ is open and hence $\mathsf{S}_b(L)$ is atomic (see [83]). Now, since $\mathsf{S}_b(L)$ is atomic, the product $\mathsf{S}_b(L)\oplus\mathsf{S}_b(L)$ is atomic as well (as we mentioned above, $\mathcal{P}(X\times X)\cong\mathcal{P}(X)\oplus\mathcal{P}(X)$ for any set X). Thus $\mathsf{S}_b(L\oplus L)$ is atomic, and by Theorem 7.5.2 it follows that $L\oplus L$ is T_D-spatial. $\square$

Chapter 8

The coframe of D-sublocales

The symmetry between sobriety and the T_D property yields multiple parallel constructions and results for the classical spectrum and for the T_D-spectrum (we refer to Section 1.3 for a brief account, for more information see [103, 31]). In this chapter we will introduce a new class of sublocales, namely the family consisting of *D-sublocales*. As we will see, this turns out to be the appropriate restriction of the notion of sublocale in the duality of T_D-spaces due to Banaschewski and Pultr [31] (see Subsection 1.3.2). In fact, the concept of D-sublocale will be closely related to the T_D axiom and T_D-spatiality, and it will come in handy for answering questions such as *"when is every prime of a locale covered?"* or *"how can we characterize those locales whose sublocales are all T_D-spatial?"*. In particular we will provide a Niefield-Rosenthal type theorem (cf. [89]) for *total T_D-spatiality*.

The results in Sections 8.1, 8.2, 8.3 and 8.5 have been obtained in collaboration with Anna Laura Suarez, whereas Section 8.4 is a joint work with Javier Gutiérrez García. This material is contained respectively in the following papers:

[6] I. Arrieta and J. Gutiérrez García, On the categorical behaviour of locales and D-localic maps, *Quaestiones Mathematicae*, accepted for publication.

[9] I. Arrieta and A. L. Suarez, The coframe of D-sublocales of a locale and the T_D-duality, *Topology and its Applications*, vol. 291, art. no. 107614, 2021.

Finally, most of the material presented in Sections 8.6 and 8.7 is unpublished work.

8.1 D-sublocales

Let $S \subseteq L$ be a sublocale and recall Property 1.2.3 (2): a prime in S is precisely a prime in L contained in S — i.e., $\mathsf{pt}(S) = \mathsf{pt}(L) \cap S$.

However, the analogous assertion when *prime* is replaced by *covered prime* is no longer true in general — i.e., a covered prime in L contained in S is obviously covered in S, but we may have sublocales $S \subseteq L$ and covered primes p in S which are not covered in the whole of L. Indeed, observe that every prime, covered or not, is covered in a sufficiently small sublocale:

If p is a prime in L, then it is trivially a covered prime in the sublocale $\mathfrak{b}(p) = \{1, p\}$.

These considerations motivate the following definition:

Definition 8.1.1. A sublocale S of a locale L will be said to be a *D-sublocale* if covered primes in S are covered primes in L, that is, if $\mathsf{pt}_D(S) \subseteq \mathsf{pt}_D(L)$.

Remarks 8.1.2. (1) We note that S is a D-sublocale of L if and only if the embedding $S \hookrightarrow L$ is a D-localic map (i.e., if and only if the frame surjection $L \twoheadrightarrow S$ corresponding to S is a D-homomorphism).

(2) If S and T are sublocales of a locale L such that $S \subseteq T$, then $\mathsf{pt}_D(T) \cap S \subseteq \mathsf{pt}_D(S)$. Consequently, S is a D-sublocale if and only if $\mathsf{pt}_D(S) = \mathsf{pt}_D(T) \cap S$ for any sublocale T such that $S \subseteq T$.

(3) A one-point sublocale $\mathfrak{b}(p)$ is a D-sublocale if and only if p is a covered prime.

8.2 The zero-dimensional frame $\mathsf{S}_D(L)^{op}$

We are now in position to introduce a new subset of $\mathsf{S}(L)$: denote by $\mathsf{S}_D(L)$ the collection of D-sublocales of L — i.e.,

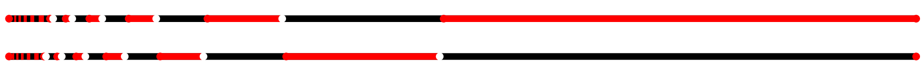
$$\mathsf{S}_D(L) = \{ S \in \mathsf{S}(L) \mid \mathsf{pt}_D(S) \subseteq \mathsf{pt}_D(L) \}$$

ordered under the inclusion inherited from $\mathsf{S}(L)$. It is the goal of this chapter to study systematically the structure of $\mathsf{S}_D(L)$ and to explore its various applications in point-free topology and connections with the Boolean algebra $\mathsf{S}_b(L)$ from Chapter 7.

We begin by observing that the system of D-sublocales is generally not closed under intersections:

Example 8.2.1. One of the simplest examples of an intersection of two D-sublocales which is not a D-sublocale seems to be the following: let $L = [0,1]$ (the unit interval with its usual total order). A subset of a totally ordered set is a sublocale if and only if it is closed under meets, so the following two subsets are indeed sublocales:

$$S = \{0,1\} \cup \bigcup_{n\in\mathbb{N}} \Big[\tfrac{1}{2n}, \tfrac{1}{2n-1}\Big) \quad \text{and} \quad T = \{0,1\} \cup \bigcup_{n\in\mathbb{N}} \Big[\tfrac{1}{2n+1}, \tfrac{1}{2n}\Big).$$



S

T

Obviously, for every $s \neq 1$ in S, one has $s = \bigwedge_{s<t\in S} t$, and this shows that $\mathsf{pt}_D(S) = \varnothing$, so S is a D-sublocale of L. Similarly, $\mathsf{pt}_D(T) = \varnothing$ and so it is also a D-sublocale of L. Now, $S \cap T = \{0,1\}$ and this is not a D-sublocale: 0 is obviously covered in $\{0,1\}$ but not in L.

We now move on to showing one of the main results of this chapter, namely that the system $\mathsf{S}_D(L)$ is a subcolocale of $\mathsf{S}(L)$. First we need the following observation (cf. Property 1.2.3 (3)):

Lemma 8.2.2. *If* $\{S_i\}_{i\in I} \subseteq \mathsf{S}_D(L)$ *is a family of D-sublocales, then* $\mathsf{pt}_D\big(\bigvee_{i\in I} S_i\big) = \bigcup_{i\in I} \mathsf{pt}_D(S_i)$.

Proof. Since S_i is a D-sublocale for each $i \in I$, by Remark 8.1.2 (2), we have $\bigcup_{i\in I} \mathsf{pt}_D(S_i) = \bigcup_{i\in I} \mathsf{pt}_D(\bigvee_{j\in I} S_j) \cap S_i \subseteq \mathsf{pt}_D(\bigvee_{i\in I} S_i)$. For the reverse inclusion, let $p \in \mathsf{pt}_D(\bigvee_i S_i)$. Since $p \in \bigvee_i S_i$, then $p = \bigwedge_i a_i$ for some $\{a_i\}_{i\in I} \subseteq L$ with $a_i \in S_i \subseteq \bigvee_i S_i$ for each $i \in I$, and since p is covered in $\bigvee_i S_i$, it follows that there is an $i_0 \in I$ such that $p = a_{i_0} \in S_{i_0}$. Hence $p \in \mathsf{pt}_D(S_{i_0}) \subseteq \bigcup_{i\in I} \mathsf{pt}_D(S_i)$. □

Proposition 8.2.3. *The system* $\mathsf{S}_D(L)$ *is closed under arbitrary joins in* $\mathsf{S}(L)$.

Proof. Let $\{S_i\}_{i\in I} \subseteq \mathsf{S}_D(L)$. Since S_i is a D-sublocale for each $i \in I$, we have that $\mathsf{pt}_D(S_i) \subseteq \mathsf{pt}_D(L)$ for each $i \in I$ and hence it follows by the previous lemma that $\mathsf{pt}_D\big(\bigvee_{i\in I} S_i\big) = \bigcup_{i\in I} \mathsf{pt}_D(S_i) \subseteq \mathsf{pt}_D(L)$ — i.e., $\bigvee_{i\in I} S_i$ is a D-sublocale. □

Lemma 8.2.4. *Let p be a prime element in a frame L and let S and T be sublocales of L. Then*:

(1) $p \in \mathsf{pt}_D(S)$ *if and only if* $p \notin S \smallsetminus \mathfrak{b}(p)$;

(2) *If* $p \in \mathsf{pt}_D(S \smallsetminus T)$ *then either* $p \in \mathsf{pt}_D(S)$ *or* $p \in T$.

Proof. (1) By Lemma 1.3.3 we see that $p \in \mathsf{pt}_D(S)$ if and only if $\mathfrak{b}(p)$ is complemented in S, that is, if and only if $\mathfrak{b}(p) \cap (S \smallsetminus \mathfrak{b}(p)) = \mathsf{O}$ in light of Property 1.2.1 (6). The later is clearly equivalent to $p \notin S \smallsetminus \mathfrak{b}(p)$.

(2) Let $p \in \mathsf{pt}_D(S \smallsetminus T)$ and suppose that $p \notin \mathsf{pt}_D(S)$. By two applications of (1) we then have $p \notin (S \smallsetminus T) \smallsetminus \mathfrak{b}(p)$ and $p \in S \smallsetminus \mathfrak{b}(p)$. By Property 1.2.1 (5), $p \notin (S \smallsetminus \mathfrak{b}(p)) \smallsetminus T$. Since $\mathfrak{b}(p) \subseteq S \smallsetminus \mathfrak{b}(p)$, it follows that $p \notin \mathfrak{b}(p) \smallsetminus T$, that is, $\mathfrak{b}(p) \smallsetminus T = \mathsf{O}$. Therefore, $p \in T$ by Property 1.2.1 (2). □

Proposition 8.2.5. *If* $S \in \mathsf{S}_D(L)$ *and* $T \in \mathsf{S}(L)$, *then* $S \smallsetminus T \in \mathsf{S}_D(L)$.

Proof. Let $p \in \mathsf{pt}_D(S \smallsetminus T)$. Since $\mathsf{S}(L)^{op}$ is zero-dimensional, there is a family $\{C_i\}_{i\in I}$ of complemented sublocales with $T = \bigcap_i C_i$. Then, by Properties 1.2.1 (3) and (4), $p \in S \smallsetminus T = \bigvee_i (S \smallsetminus C_i) = \bigvee_i (S \cap C_i^{\#})$ and so $p = \bigwedge_i a_i$ for some $\{a_i\}_{i\in I} \subseteq L$ such that $a_i \in S \cap C_i^{\#} \subseteq S \smallsetminus T$ for each $i \in I$. Since p is a covered prime in $S \smallsetminus T$, it follows that there is an $i_0 \in I$ with $p = a_{i_0} \in C_{i_0}^{\#}$, hence $p \notin T$ because $C_{i_0} \cap C_{i_0}^{\#} = \mathsf{O}$. By Lemma 8.2.4 (2) it follows that $p \in \mathsf{pt}_D(S) \subseteq \mathsf{pt}_D(L)$, as desired. □

We have therefore almost shown the following

Theorem 8.2.6. $\mathsf{S}_D(L)$ *is a dense D-subcolocale*[1] *of* $\mathsf{S}(L)$. *In particular, it is a coframe.*

Proof. The fact that $\mathsf{S}_D(L)^{op}$ is a sublocale of $\mathsf{S}(L)^{op}$ follows from the two previous propositions. Moreover, density follows from the obvious fact that L (i.e., the bottom element of $\mathsf{S}(L)^{op}$) belongs to $\mathsf{S}_D(L)$. Finally, since $\mathsf{S}(L)^{op}$ is a zero-dimensional frame, it is in particular fit; and by Property 1.2.3 (4) primes in any fit frame are maximal (thus covered). Hence, every sublocale of $\mathsf{S}(L)^{op}$ is a D-sublocale. □

Corollary 8.2.7. *The inclusion* $\mathsf{S}_b(L) \subseteq \mathsf{S}_D(L)$ *holds.*

[1]This has to be understood just as a shorthand for "$\mathsf{S}_D(L)^{op}$ is a dense D-sublocale of $\mathsf{S}(L)^{op}$"

Proof. $\mathsf{S}_b(L)^{op}$ is the Booleanization of $\mathsf{S}(L)^{op}$ and it can therefore be characterized as the least dense sublocale of $\mathsf{S}(L)^{op}$. □

Corollary 8.2.8. *The frame $\mathsf{S}_D(L)^{op}$ is zero-dimensional.*

Proof. If $S \in \mathsf{S}_D(L)$, by zero-dimensionality of $\mathsf{S}(L)^{op}$ there is a family $\{C_i\}_{i\in I}$ of complemented sublocales with $S = \bigcap_i C_i$. By Corollary 8.2.7 one has $C_i \in \mathsf{S}_b(L) \subseteq \mathsf{S}_D(L)$ for each $i \in I$, and so S is actually the meet of $\{C_i\}_{i\in I}$ in $\mathsf{S}_D(L)$. □

We are now in position to provide several examples of D-sublocales.

Examples 8.2.9. (1) In view of Corollary 8.2.7, every smooth sublocale is a D-sublocale. In particular, open, closed, locally closed and complemented sublocales are all D-sublocales.

(2) Recall that a sublocale is pointless if it does not contain any prime element. Every pointless sublocale S of a locale L is a D-sublocale, for trivial reasons: one simply has $\mathsf{pt}_D(S) \subseteq \mathsf{pt}(S) = \varnothing$.

(3) It is well known that joins of pointless sublocales may contain points (see for example [73, 2.3]). However, by (2) and Proposition 8.2.3 it follows that joins of pointless sublocales are also D-sublocales.

(4) We recall that a sublocale S of a locale L is *codense* (in L) [27, 43] if the associated frame surjection $\nu_S : L \twoheadrightarrow S$ satisfies the implication

$$\nu_S(a) = 1 \implies a = 1$$

for any $a \in L$ (cf. Subsection 2.7). Then we claim that every codense sublocale S of L such that covered primes in S are maximal in S is a D-sublocale (so, for instance, every T_1 codense sublocale is a D-sublocale). Indeed, let $p \in \mathsf{pt}_D(S)$ and $a \in L$ with $p \le a$. Then $p \le \nu_S(a)$ and since p is maximal in S, either $p = \nu_S(a)$ or $\nu_S(a) = 1$. In the former case, we have $a \le p$ by adjunction, and in the latter, by codensity it follows that $a = 1$. Thus p is maximal, and in particular covered, in L.

(5) Given any locale L, its diagonal sublocale $D_L \subseteq L \oplus L$ (cf. Chapter 2) is always a D-sublocale of $L \oplus L$. To see this, recall the isomorphism $(\delta_L)_* : L \to D_L$ from Subsection 1.5.4. Clearly, for any $p \in \mathsf{pt}(L)$ one has $(\delta_L)_*(p) = p \parr p$. Hence the covered primes in D_L are those of the form $p \parr p$ with $p \in \mathsf{pt}_D(L)$. But all such primes are covered in $L \oplus L$ by Lemma 7.7.13 (5). In passing, we note that consequently the property of being a D-sublocale cannot be pullback stable (cf. Subsection 2.2.4).

For a large class of locales we have that every sublocale is a D-sublocale. For the moment, we have the following (but we shall discuss further this class of locales in Section 8.7.1 below):

Proposition 8.2.10. *The following are equivalent for a locale L:*

(i) *Every prime of L is covered — i.e.,* $\mathsf{pt}(L) = \mathsf{pt}_D(L)$*;*

(ii) $\mathsf{sp}[\mathsf{S}(L)] \subseteq \mathsf{S}_b(L)$*;*

(iii) $\mathsf{sp}[\mathsf{S}(L)] \subseteq \mathsf{S}_D(L)$*;*

(iv) $\mathsf{S}_D(L) = \mathsf{S}(L)$*;*

(v) $\mathsf{S}_D(L)$ *is closed under arbitrary intersections in* $\mathsf{S}(L)$*.*

Proof. For (i) $\Longrightarrow$ (ii), recall that by Proposition 1.2.4 every $S \in \mathsf{sp}[\mathsf{S}(L)]$ is of the form $S = \mathsf{sp}(S) = \bigvee_{p\in \mathsf{pt}(S)} \mathfrak{b}(p)$. But by Lemma 1.3.3 each $\mathfrak{b}(p)$ is complemented, and thus it belongs to $\mathsf{S}_b(L)$. Thus $S \in \mathsf{S}_b(L)$. The implication (ii) $\Longrightarrow$ (iii) follows immediately from Corollary 8.2.7. Assume now that (iii) holds, let S be a sublocale and $p \in \mathsf{pt}_D(S)$. Then $\mathfrak{b}(p) \in \mathsf{sp}[\mathsf{S}(L)] \subseteq \mathsf{S}_D(L)$ and thus p is a covered prime in L by Remark 8.1.2 (3). Hence (iv) follows. The implication (iv) $\Longrightarrow$ (v) is trivial. Assume finally that (v) holds and let $p \in \mathsf{pt}(L)$. By zero-dimensionality of $\mathsf{S}(L)^{op}$ there is a family $\{C_i\}_{i\in I}$ of complemented sublocales with $\mathfrak{b}(p) = \bigcap_i C_i$. By Corollary 8.2.7, one has $C_i \in \mathsf{S}_b(L) \subseteq \mathsf{S}_D(L)$ and so by assumption $\mathfrak{b}(p) \in \mathsf{S}_D(L)$. Then $p \in \mathsf{pt}_D(L)$ by Remark 8.1.2 (3). □

8.3 Connections with T_D-spatiality

After having explored some basic examples and properties concerning D-sublocales, we now study their applications in the T_D-duality.

We start by providing a construction which reveals the close connection between T_D-spatiality and the coframe $\mathsf{S}_D(L)$. If S is a D-sublocale we have $\mathsf{pt}_D(S) \subseteq \mathsf{pt}_D(L)$ and thus there is a well-defined map

$$\mathsf{pt}_D \colon \mathsf{S}_D(L) \longrightarrow \mathcal{P}(\mathsf{pt}_D(L)).$$

Moreover, let

$$\mathfrak{M} \colon \mathcal{P}(\mathsf{pt}_D(L)) \longrightarrow \mathsf{S}_D(L)$$

be the map given by

$$\mathfrak{M}(Y) = \bigvee_{p\in Y} \mathfrak{b}(p) = \{ \textstyle\bigwedge M \mid M \subseteq Y \}$$

for each $Y \subseteq \mathsf{pt}_D(L)$ (the second equality clearly follows from the general formula for joins in $\mathsf{S}(L)$ —see (1.2.2)). Observe that $\mathfrak{M}$ is well-defined. Indeed, each $\mathfrak{b}(p)$ with $p \in Y \subseteq \mathsf{pt}_D(L)$ is obviously a D-sublocale by Remark 8.1.2 (3) and $\mathsf{S}_D(L)$ is closed under joins (by Proposition 8.2.3).

After this preparation, we have the following:

Proposition 8.3.1. *There is an adjunction*

$$\begin{tikzcd} \mathcal{P}(\mathsf{pt}_D(L)) \arrow[r, bend left, "\mathfrak{M}"] \arrow[r, phantom, "\perp"] & \mathsf{S}_D(L) \arrow[l, bend left, "\mathsf{pt}_D"] \end{tikzcd}$$

with the following properties:

(1) $\mathsf{pt}_D \circ \mathfrak{M}$ *is the identity;*

(2) *The fixpoints of* $\mathfrak{M} \circ \mathsf{pt}_D$ *are the D-sublocales which are* T_D*-spatial;*

(3) pt_D *is a complete lattice homomorphism.*

Proof. Clearly both pt_D and $\mathfrak{M}$ are monotone. For the adjunction, we have to check that $\mathfrak{M}(Y) \subseteq S$ if and only if $Y \subseteq \mathsf{pt}_D(S)$ for each $S \in \mathsf{S}_D(L)$ and $Y \subseteq \mathsf{pt}_D(L)$. The "only if" implication is trivial because $Y \subseteq \mathfrak{M}(Y)$, and the converse clearly follows from the fact that sublocales are closed under meets.

(1) Let $Y \subseteq \mathsf{pt}_D(L)$ and $p \in \mathsf{pt}_D(\mathfrak{M}(Y))$. Since $p \in \mathfrak{M}(Y)$, then $p = \bigwedge M$ for some $M \subseteq Y$, and since p is covered in Y, it follows that $p \in M \subseteq Y$. Hence $\mathsf{pt}_D(\mathfrak{M}(Y)) \subseteq Y$. The other inclusion follows by adjunction.

(2) Let $S \in \mathsf{S}_D(L)$. Then $\mathfrak{M}(\mathsf{pt}_D(S)) = S$ if and only if every element of S is a meet of a subset of $\mathsf{pt}_D(S)$, and, by Lemma 1.3.6 this is equivalent to S being T_D-spatial.

(3) pt_D preserves meets as it is a right adjoint, and it preserves joins by Lemma 8.2.2. □

Furthermore, as it was done for the (classical) spatialization in Subsection 1.2.5, it is useful to recast the T_D-spatialization $\epsilon'_L \colon \Omega(\Sigma'(L)) \to L$ from Subsection 1.3.2 as a concrete sublocale. Since ϵ'_L is injective, it corresponds to a sublocale of L. Now, as a consequence of Proposition 8.3.1 we can identify the concrete sublocale to which it corresponds:

Corollary 8.3.2. *The* T_D*-spatialization of L is given by* $\mathfrak{M}(\mathsf{pt}_D(L)) = \bigvee_{p \in \mathsf{pt}_D(L)} \mathfrak{b}(p)$.

Proof. It follows at once from the adjunction in Subsection 1.3.2 that for every D-localic map $f \colon M \to L$ with M a T_D-spatial frame, there is a unique $g \colon M \to \Omega(\Sigma'(L))$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & \Omega(\Sigma'(L)) \arrow[dl, hook', "\epsilon'_L"] \\ L & \end{tikzcd}$$

commutes. It follows from Remark 8.1.2 (1) that $\epsilon'_L \colon \Omega(\Sigma'(L)) \to L$ corresponds to the largest T_D-spatial D-sublocale of L. Moreover, $\mathfrak{M}(\mathsf{pt}_D(L))$ is T_D-spatial by Proposition 8.3.1 (2). Finally, suppose S is a further T_D-spatial D-sublocale of L. Then, by Proposition 8.3.1, $S = \mathfrak{M}(\mathsf{pt}_D(S)) \subseteq \mathfrak{M}(\mathsf{pt}_D(L))$. Hence, $\mathfrak{M}(\mathsf{pt}_D(L))$ is the largest T_D-spatial D-sublocale of L. □

It is now apparent that, as we claimed at the beginning of this chapter, D-sublocales in the T_D-duality take up the role of plain sublocales in the classical sober-spatial duality (some more justification is given in Section 8.4 below, see also further applications in Section 8.5).

From the last corollary, it follows that $\mathfrak{M} \circ \mathsf{pt}_D \colon \mathsf{S}_D(L) \to \mathsf{S}_D(L)$ sends every D-sublocale to its T_D-spatialization. Therefore, we will denote this map by

$$\mathsf{sp}_D := \mathfrak{M} \circ \mathsf{pt}_D \colon \mathsf{S}_D(L) \to \mathsf{S}_D(L)$$

and we will call it the *T_D-spatialization operator* on L.

In the following corollary we gather some properties concerning the operator sp_D and its fixset that will be useful at a later stage:

Corollary 8.3.3. *The following properties hold*:

1. *The operator* $\mathsf{sp}_D \colon \mathsf{S}_D(L) \to \mathsf{S}_D(L)$ *is a conucleus on* $\mathsf{S}_D(L)$*;*
2. *The fixset* $\mathsf{sp}_D[\mathsf{S}_D(L)]$ *of the conucleus* sp_D *is the ordered collection of T_D-spatial D-sublocales, and it is a subcolocale of* $\mathsf{S}_D(L)$*;*
3. $\mathsf{sp}_D[\mathsf{S}_D(L)]$ *is a complete and atomic Boolean algebra. Moreover, when we regard* $\mathsf{sp}_D[\mathsf{S}_D(L)]^{op}$ *as a Boolean sublocale of* $\mathsf{S}(L)^{op}$*, we have* $\mathsf{sp}_D[\mathsf{S}_D(L)]^{op} = \mathfrak{b}_{\mathsf{S}(L)^{op}}(\mathsf{sp}_D(L))$.

Proof. (1) By the adjunction in Proposition 8.3.1 it follows that $\mathsf{sp}_D \colon \mathsf{S}_D(L) \to \mathsf{S}_D(L)$ is an interior operator. Moreover, it preserves joins (indeed, $\mathfrak{M}$ preserves joins as a left adjoint and pt_D preserves joins by Lemma 8.2.2). Hence it is a conucleus on $\mathsf{S}_D(L)$.

(2) is trivial in view of Proposition 8.3.1 (2) and the familiar fact that the fixset of a nucleus is a sublocale set.

(3) An adjunction between posets restricts to an isomorphism between the corresponding fixsets, so in light of (1) and (2) of Proposition 8.3.1 it follows that $\mathsf{S}_D(L) \cong \mathcal{P}(\mathsf{pt}_D(L))$ and hence it is a complete and atomic Boolean algebra. For the last assertion, we note that if S is a Boolean sublocale of a locale L, then $S = \mathfrak{b}_L(\bigwedge S)$. Applying this observation, we get $\mathsf{sp}_D[\mathsf{S}_D(L)]^{op} = \mathfrak{b}_{\mathsf{S}(L)^{op}}(S)$ where S is the least element in $\mathsf{sp}_D[\mathsf{S}_D(L)]^{op}$ — i.e., the largest T_D-spatial D-sublocale of L, namely $\mathsf{sp}_D(L)$. □

8.3.1 Global versus local

We have just seen that the collection $\mathsf{sp}_D[\mathsf{S}_D(L)]$ consisting of T_D-spatial D-sublocales is a subcolocale of $\mathsf{S}_D(L)$. Throughout this section, we will say that this construction is *local* in order to emphasize that $\mathsf{sp}_D[\mathsf{S}_D(L)] = \{\mathfrak{M}(\mathsf{pt}_D(S)) \mid S \in \mathsf{S}_D(L)\}$ is obtained by applying the operator

$$\mathsf{sp}_D \colon \mathsf{S}_D(L) \to \mathsf{S}_D(L)$$

pointwisely. However, $\mathsf{S}_D(L)^{op}$ is a sublocale of $\mathsf{S}(L)^{op}$, and hence in particular a frame in its own right. Accordingly, we can also compute the T_D-spatialization of $\mathsf{S}_D(L)^{op}$, the *global*

T_D-spatialization, so to speak. Because of Corollary 8.3.2, the latter can be computed precisely as $\mathsf{sp}_D(\mathsf{S}_D(L)^{op}) = \mathfrak{M}(\mathsf{pt}_D(\mathsf{S}_D(L)^{op}))$ where

$$\mathsf{sp}_D \colon \mathsf{S}_D(\mathsf{S}_D(L)^{op}) \to \mathsf{S}_D(\mathsf{S}_D(L)^{op})$$

is the T_D-spatialization operator on $\mathsf{S}_D(L)^{op}$.

It is therefore a natural question to ask whether both the "local" and the "global" construction agree and whether we can use the notation $\mathsf{sp}_D[\mathsf{S}_D(L)]$ without risk of confusion. The following is an affirmative answer:

Proposition 8.3.4. *For every locale L, we have the equality $\mathsf{sp}_D[\mathsf{S}_D(L)]^{op} = \mathsf{sp}_D(\mathsf{S}_D(L)^{op})$ — i.e., the local and the global T_D-spatializations coincide.*

Proof. Since $\mathsf{S}(L)^{op}$ is zero-dimensional and $\mathsf{S}_D(L)^{op}$ is a sublocale of $\mathsf{S}(L)^{op}$, by Properties 1.2.3 (2) and (4) it follows that all the primes of $\mathsf{S}_D(L)^{op}$ are maximal (and in particular covered). Hence $\mathsf{pt}_D(\mathsf{S}_D(L)^{op}) = \mathsf{pt}(\mathsf{S}_D(L)^{op})$, and by Property 1.2.3 (5) and Remark 8.1.2 (3) we have that $\mathsf{pt}_D(\mathsf{S}_D(L)^{op}) = \{\, \mathfrak{b}(p) \mid p \in \mathsf{pt}_D(L) \,\}$ and so

$$\begin{aligned}\mathsf{sp}_D(\mathsf{S}_D(L)^{op}) = \mathfrak{M}(\mathsf{pt}_D(\mathsf{S}_D(L)^{op})) &= \Big\{ \bigwedge^{\mathsf{S}(L)^{op}} M \mid M \subseteq \mathsf{pt}_D(\mathsf{S}_D(L)^{op}) \Big\} = \Big\{ \bigvee_{p\in Y} \mathfrak{b}(p) \mid Y \subseteq \mathsf{pt}_D(L) \Big\} \\ &= \{\, \mathfrak{M}(Y) \mid Y \subseteq \mathsf{pt}_D(L) \,\} = \mathsf{sp}_D[\mathsf{S}_D(L)]^{op},\end{aligned}$$

where the last equality follows from the adjunction in Proposition 8.3.1. □

We state the following corollary for future reference:

Corollary 8.3.5. *The following are equivalent for a locale L:*

(i) *Every D-sublocale of L is T_D-spatial — i.e., $\mathsf{S}_D(L) = \mathsf{sp}_D[\mathsf{S}_D(L)]$;*

(ii) *$\mathsf{S}_D(L)^{op}$ is T_D-spatial;*

(iii) *$\mathsf{S}_D(L)^{op}$ is spatial.*

Proof. The equivalence between (i) and (ii) follows readily from the previous proposition. Finally, since all primes in $\mathsf{S}_D(L)^{op}$ are covered, T_D-spatiality is equivalent to spatiality (to see this, use Lemma 1.3.6 and the fact that a frame is spatial if and only if every element is a meet of primes). □

8.4 D-sublocales and regular monomorphisms

As is well known, regular subobjects in the category Loc are precisely the embeddings of sublocales. Now, we claimed above that the D-sublocales constitute the right restriction of the notion of sublocale to the T_D-duality, and it therefore seems a natural question whether the D-sublocales are the regular subobjects in Loc_D. It is the goal of this section to provide an affirmative answer to this problem.

For this purpose, we first need to study the existence of equalizers in Loc_D. We begin by noting the following easy fact:

Lemma 8.4.1. *Let $f, g\colon L \to M$ be D-localic maps such that their equalizer in* Loc *is a D-sublocale. Then, that equalizer is also the equalizer of f and g in* Loc_D*.*

Proof. Let $S \subseteq L$ be the equalizer of f and g in Loc. By hypothesis, the embedding $\iota\colon S \hookrightarrow L$ is a morphism in Loc_D. Let $h\colon N \to L$ be a D-localic map such that $f \circ h = g \circ h$. We only need to show that the unique localic map k such that $\iota \circ k = h$ is a D-localic map. Let $p \in \mathsf{pt}_D(N)$. Then $k(p) = h(p) \in S \cap \mathsf{pt}_D(L) \subseteq \mathsf{pt}_D(S)$ (cf. Remark 8.1.2 (2)). □

Further, it follows from Proposition 8.2.3 that there is always the largest D-sublocale contained in a given sublocale, and it can therefore be tempting to conjecture that the equalizer of any pair of D-localic maps is given by the largest D-sublocale contained in their Loc-equalizer. However, it does not satisfy the appropriate universal property because the embedding part of the factorization of a D-localic map is generally not a D-sublocale. In fact, equalizers in Loc_D may fail to exist at all.

We need the following:

Lemma 8.4.2. *Let L and M be locales and let $f, g\colon L \to M$ be localic maps. If $e\colon E \to L$ is an equalizer of f and g in* Loc_D *then $e[E]$ is a D-sublocale of L.*

Proof. Let $p \in \mathsf{pt}_D(e[E])$. By Lemma 1.3.3, $\mathfrak{b}(p)$ is a complemented sublocale of $e[E]$ and it follows that $e_{-1}[\mathfrak{b}(p)]$ is a complemented sublocale of E. Indeed, consider the factorization of e, namely

$$E \xrightarrow{\ j\ } \!\!\!\!\rightarrow e[E] \xhookrightarrow{\ \iota\ } L\ .$$

Then $e_{-1}[\mathfrak{b}(p)] = j_{-1}[\iota_{-1}[\mathfrak{b}(p)]] = j_{-1}[\mathfrak{b}(p) \cap e[E]] = j_{-1}[\mathfrak{b}(p)]$ and recall that coframe homomorphisms preserve complements.

We distinguish two cases:

(1) Suppose first that there is some $q \in \mathsf{pt}_D(e_{-1}[\mathfrak{b}(p)])$. Then $\mathfrak{b}(q) \subseteq e_{-1}[\mathfrak{b}(p)]$ so by adjunction $\mathfrak{b}(e(q)) = e[\mathfrak{b}(q)] \subseteq \mathfrak{b}(p)$ — i.e., $e(q) = p$. But $e_{-1}[\mathfrak{b}(p)]$ is a complemented sublocale of E, so in particular it is a D-sublocale by Corollary 8.2.7. Thus $q \in \mathsf{pt}_D(E)$, and since e is a D-localic map it follows that $p = e(q) \in \mathsf{pt}_D(L)$, as required.

(2) Assume now that $\mathsf{pt}_D(e_{-1}[\mathfrak{b}(p)]) = \varnothing$ and select a locale M such that $\mathsf{pt}(M) \neq \varnothing$ and $\mathsf{pt}_D(M) = \varnothing$ (as e.g. the totally ordered $M = [0,1]$) and let h be the composite

$$M \longrightarrow\!\!\!\!\rightarrow \mathfrak{b}(p) \hookrightarrow L$$

where the first map is the unique surjection onto the terminal locale given by $h(1) = 1$ and $h(a) = p$ for all $a < 1$. Since $\mathsf{pt}_D(M) = \varnothing$, h is a D-localic map, and it equalizes f and g because so does e and $p \in e[E]$. Hence there is a unique D-localic map $k\colon M \to E$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "k"] \arrow[dr, "h"'] & E \arrow[d, "e"] \\ & L \end{tikzcd}$$

commutes. Pick $p_0 \in \mathsf{pt}(M)$ and set $q_0 := k(p_0)$. Since localic maps send primes into primes, we have $q_0 \in \mathsf{pt}(E)$, and $e(q_0) = h(p_0) = p$.

Finally, let ℓ be the composite

$$e_{-1}[\mathfrak{b}(p)] \twoheadrightarrow \mathfrak{b}(q_0) \hookrightarrow E\ .$$

Then $e \circ \ell = e \circ \iota$ where $\iota\colon e_{-1}[\mathfrak{b}(p)] \hookrightarrow E$ is the inclusion. Since $\mathsf{pt}_D(e_{-1}[\mathfrak{b}(p)]) = \varnothing$, ℓ and ι are trivially D-localic maps and by the uniqueness clause of the equalizer we must then have $\ell = \iota$. But then $\mathfrak{b}(q_0) = e_{-1}[\mathfrak{b}(p)]$ is a complemented sublocale of E — i.e., $q_0 \in \mathsf{pt}_D(E)$. Since e is a D-localic map, $p = e(q_0) \in \mathsf{pt}_D(L)$, as required. □

Proposition 8.4.3. *Let L and M be locales and let $f, g\colon L \to M$ be D-localic maps. If the equalizer of f and g exists in Loc_D then their Loc-equalizer is a D-sublocale.*

Proof. Assume that

$$E \xrightarrow{\ e\ } L \underset{g}{\overset{f}{\rightrightarrows}} M$$

is an equalizer in Loc_D and let $S \subseteq L$ be the equalizer of f and g in Loc. Hence $e[E] \subseteq S$ by the universal property of the equalizer, and by the previous lemma we know that $e[E]$ is a D-sublocale of L. Let $p \in \mathsf{pt}_D(S)$. Select a nontrivial pointless Boolean algebra B and let h be the composite

$$B \twoheadrightarrow \mathfrak{b}(p) \hookrightarrow L$$

where the first map is the unique surjection onto the terminal locale given by $h(1) = 1$ and $h(a) = p$ for all $a < 1$. Since $\mathsf{pt}_D(B) \subseteq \mathsf{pt}(B) = \varnothing$, h is a D-localic map, and it equalizes f and g because $p \in S$. Hence there is a unique D-localic map $k\colon B \to E$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "k"] \arrow[dr, "h"'] & E \arrow[d, "e"] \\ & L \end{tikzcd}$$

commutes. Then $p = h(0) = e(k(0)) \in e[E] \subseteq S$ and since $e[E]$ is a D-sublocale, $p \in \mathsf{pt}_D(S) \cap e[E] \subseteq \mathsf{pt}_D(e[E]) \subseteq \mathsf{pt}_D(L)$ by Remark 8.1.2 (2). Hence S is a D-sublocale. □

Example 8.4.4. Let $L = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and let $f\colon L \to L$ be given by $f(0) = 0$, $f(1) = 1$ and $f(\frac{1}{n}) = \frac{1}{n+1}$. One readily verifies that f is a D-localic map. It follows that

$$\mathfrak{b}(0) = \{0,1\} \hookrightarrow L \underset{f}{\overset{1_L}{\rightrightarrows}} L$$

is the equalizer in Loc. Hence, the equalizer of f and 1_L in Loc_D does not exist because $\mathfrak{b}(0)$ is not a D-sublocale.

Combining the last proposition with Lemma 8.4.1 we are now in position to fully characterize the existence of equalizers in Loc_D.

Corollary 8.4.5. *Let L and M be locales and let $f,g\colon L \to M$ be D-localic maps. Then the equalizer of f and g in Loc_D exists if and only if the equalizer of f and g in Loc is a D-sublocale (and so their equalizers in Loc and Loc_D coincide).*

A sufficient condition for the situation in Proposition 8.4.3 is as follows:

Proposition 8.4.6. *If M has a complemented diagonal (e.g., if M is locally strongly Hausdorff), then, for every pair of morphisms $f,g\colon L \to M$ in Loc_D, their equalizer in Loc_D exists and is given by their equalizer in Loc.*

Proof. It follows from general category theory (see Subsection 2.2.4) that the equalizer of f and g in Loc can be computed as the preimage of the diagonal along the map $\langle f,g\rangle\colon L \to M \oplus M$. But the preimage operator is a coframe homomorphism and so it sends complemented sublocales into complemented sublocales. It follows that the equalizer in Loc is a complemented sublocale of L. But complemented sublocales are D-sublocales by Corollary 8.2.7. □

As we claimed that D-sublocales play the role of plain sublocales in the duality of T_D-spaces, we now provide some more evidence for this assertion by showing that they are precisely the regular monomorphisms in Loc_D:

Proposition 8.4.7. *Any D-sublocale is a regular monomorphism in Loc_D.*

Proof. Let $S \subseteq L$ be a D-sublocale. As is well known, sublocale embeddings are regular monomorphisms in Loc, and hence $S \hookrightarrow L$ is the equalizer of its cokernel pair in Loc — i.e., there is an equalizer diagram

$$S \hookrightarrow L \underset{g}{\overset{f}{\rightrightarrows}} P$$

in Loc, where $P = \{(x,y) \in L \times L \mid \nu_S(x) = \nu_S(y)\}$ (ν_S denotes the nucleus associated to the sublocale S),

$$f(a) = \bigvee\{(b,c) \in P \mid b \le a\} = (a, \nu_S(a)) \quad \text{and} \quad g(a) = \bigvee\{(b,c) \in P \mid c \le a\} = (\nu_S(a), a)$$

for each $a \in L$. Observe that f (resp. g) is the right adjoint of the coordinate projection $P \to L$ sending (b,c) to b (resp. (b,c) to c).

By Lemma 8.4.1, it suffices to show that f and g are D-localic maps, as then the equalizer diagram above will be a equalizer diagram in Loc_D. By symmetry, we shall just prove it for f. Let $p \in \mathsf{pt}_D(L)$ and denote by p^+ the cover of p in L. We distinguish two cases:

(1) If $p \in S$ then $\nu_S(p) = p$, and since $p < p^+$ it follows that $f(p) = (p,p) < (p^+,p^+)$. Let $(b,c) \in P$ with
$$f(p) = (p,p) \leq (b,c) \leq (p^+,p^+).$$
Then $p \leq b \wedge c \leq p^+$ and we again distinguish two cases:

(1.1) If $b \wedge c = p$ then $b \leq \nu_S(b) = \nu_S(b \wedge c) = \nu_S(p) = p$ and hence $(b,c) \leq f(p)$.

(1.2) If $b \wedge c = p^+$ then $(p^+,p^+) \leq (b,c)$.

Consequently (p^+,p^+) is the cover of $f(p)$ in P.

(2) If $p \notin S$ — i.e., $p < \nu_S(p)$ then since $p \leq \nu_S(p) \wedge p^+ \leq p^+$ and p is prime, we must have $\nu_S(p) \wedge p^+ = p^+$. It follows that $\nu_S(p^+) = \nu_S(p)$ and so we have $f(p) = (p, \nu_S(p)) = (p, \nu_S(p^+)) < (p^+, \nu_S(p^+)) = f(p^+)$. Let $(b,c) \in P$ with
$$f(p) = (p, \nu_S(p^+)) \leq (b,c) \leq (p^+, \nu_S(p^+)) = f(p^+).$$
Then $p \leq b \leq p^+$ and $c = \nu_S(p^+)$. If $b = p$ then clearly $(b,c) = f(p)$, and if $b = p^+$ then $(b,c) = f(p^+)$. Consequently $f(p^+)$ is the cover of $f(p)$ in P. □

Corollary 8.4.8. *The D-sublocales are precisely the regular subobjects in Loc_D.*

Proof. The "only if" implication follows from Proposition 8.4.7 whereas the "if" implication follows from Proposition 8.4.3. □

8.5 Total T_D-spatiality

Let us recall from Simmons [109] that a locale is called *totally spatial* if every sublocale is spatial. A classical and beautiful result in locale theory asserts that L is totally spatial if and only if $\mathsf{S}(L)^{op}$ is spatial (Niefield and Rosenthal [89]).

In this section we introduce and study the T_D-analogue of total spatiality and we shall show that D-sublocales play an important role.

Definition 8.5.1. A locale L is *totally T_D-spatial* if every sublocale of L is T_D-spatial.

This is our characterization of total T_D-spatiality:

Theorem 8.5.2. *The following are equivalent for a locale L:*

(i) *L is totally T_D-spatial;*

(ii) *Every D-sublocale of L is T_D-spatial;*

(iii) *$\mathfrak{M} \circ \mathsf{pt}_D$ is the identity on $\mathsf{S}_D(L)$;*

(iv) *There is an isomorphism $\mathsf{S}_D(L) \cong \mathcal{P}(\mathsf{pt}_D(L))$;*

(v) *$\mathsf{S}_D(L)^{op}$ is T_D-spatial;*

(vi) *$\mathsf{S}_D(L)^{op}$ is spatial;*

(vii) *$\mathsf{S}_D(L)^{op}$ is spatial and Boolean (i.e., a complete and atomic Boolean algebra);*

(viii) *$\mathsf{S}_D(L) = \mathsf{S}_b(L)$ and L is T_D-spatial;*

(ix) *Every nonzero sublocale of L contains a covered prime in itself.*

Proof. (i) obviously implies (ii) and (ii) is equivalent to (iii) by the adjunction in Proposition 8.3.1.

(iii) $\Longrightarrow$ (iv): The isomorphism is obtained by restricting the adjunction in Proposition 8.3.1 to the fixsets, which under the assumption are $\mathcal{P}(\mathsf{pt}_D(L))$ and $\mathsf{S}_D(L)$.

(iv) $\Longrightarrow$ (v) and (v) $\Longrightarrow$ (vi) are clear.

(vi) $\Longrightarrow$ (vii): From Corollary 8.3.5 we obtain that $\mathsf{sp}_D[\mathsf{S}_D(L)] = \mathsf{S}_D(L)$. Now, use Corollary 8.3.3 (3).

(vii) $\Longrightarrow$ (viii): Suppose that $\mathsf{S}_D(L)^{op}$ is spatial and Boolean. If $\mathsf{S}_D(L)^{op}$ is Boolean, since it is also dense, it must coincide with the unique Boolean dense sublocale of $\mathsf{S}(L)^{op}$, namely $\mathsf{S}_b(L)^{op}$. Hence $\mathsf{S}_D(L) = \mathsf{S}_b(L)$ and so $\mathsf{S}_b(L)^{op} \cong \mathsf{S}_b(L)$ is also spatial. By Theorem 7.5.2 we conclude that L is T_D-spatial.

(viii) $\Longrightarrow$ (ix): Notice that every non-smooth sublocale has a covered prime in itself (otherwise, we would have a non-smooth sublocale S with $\mathsf{pt}_D(S) = \varnothing$, then S would be obviously a D-sublocale so by assumption it would be smooth, a contradiction). But L being T_D-spatial, say $L = \Omega(X)$ with X a T_D-space, every (nonzero) smooth sublocale of it is induced by a subspace $A \subseteq X$ by Lemma 7.5.1. Hence, it contains a covered prime of L, in particular a covered prime in itself. Thus every nonzero sublocale contains a covered prime in itself.

(ix) $\Longrightarrow$ (i): Clearly, the condition (ix) is hereditary with respect to any sublocale. Hence, it suffices to show that L is T_D-spatial. If for any complemented sublocale $C \subseteq L$ such that $\mathsf{sp}_D(L) \subseteq C$ one has $C = L$, then by zero-dimensionality of $\mathsf{S}(L)^{op}$, it follows that $L = \mathsf{sp}_D(L)$ and so L is T_D-spatial. Hence assume by way of contradiction that there is a complemented sublocale $C \subseteq L$ such that $C \neq L$ and $\mathsf{sp}_D(L) \subseteq C$. Since $C \neq L$, then $C^\# \neq \mathsf{O}$, so by assumption there is a $p \in \mathsf{pt}_D(C^\#)$. But $C^\#$ is complemented; in particular it is a D-sublocale, hence $p \in \mathsf{pt}_D(L)$. But then $\mathfrak{b}(p) \subseteq \mathsf{sp}_D(L) \subseteq C$, whence $\mathfrak{b}(p) \subseteq C \cap C^\# = \mathsf{O}$, a contradiction. $\square$

Remarks 8.5.3. (1) The fact that T_D-spatiality of D-sublocales implies total T_D-spatiality (i.e., T_D-spatiality of *all* sublocales) may seem somewhat surprising: we emphasize that even under the conditions of Theorem 8.5.2 there may be strictly more sublocales than D-sublocales (for an example, see Example 8.5.6 below). If, additionally, every sublocale is a D-sublocale, we already reach strong T_D-spatiality (see Theorem 8.5.4: every sublocale of L is strongly T_D-spatial if and only if L is totally T_D-spatial and every sublocale is a D-sublocale).

(2) Regarding the equivalence (i) $\iff$ (v), we have achieved a counterpart of the aforementioned classical result of Niefield and Rosenthal which states that a frame is totally spatial if and only if $\mathsf{S}(L)^{op}$ is spatial. Here we have the parallel statement that

a frame is totally T_D-spatial if and only if $\mathsf{S}_D(L)^{op}$ is T_D-spatial.

(3) It is sometimes incorrectly stated in the literature that for a T_D-space X, $\mathsf{S}(\Omega(X))^{op}$ is spatial if and only if it is Boolean (see e.g. [89, p. 267]). The assertion is generally not true. Certainly if $\mathsf{S}(\Omega(X))^{op}$ is Boolean then it is spatial (every sublocale is complemented, and thus induced by a subspace of X, so $\Omega(X)$ is totally spatial). However, spatiality of $\mathsf{S}(\Omega(X))^{op}$ is strictly weaker. For the sake of clarification, we characterize the stronger condition in the corollary of the following theorem (the result is an easy combination of Theorem 7.5.2, Theorem 8.5.2 and Lemma 1.3.9 and we omit the details). We also give an example of the strict implication afterwards.

Theorem 8.5.4. *The following are equivalent for a locale L:*

(i) *Every sublocale of L is strongly T_D-spatial;*

(ii) *L is totally (T_D-)spatial and strongly T_D-spatial;*

(iii) *L is totally (T_D-)spatial and all its primes are covered;*

(iv) *There is an isomorphism $\mathsf{S}(L) \cong \mathcal{P}(\mathsf{pt}_D(L))$;*

(v) *$\mathsf{S}(L)^{op}$ is spatial and Boolean (i.e., a complete and atomic Boolean algebra);*

(vi) *$\mathsf{S}(L)^{op}$ is Boolean and L is (T_D-)spatial;*

(vii) *Every nonzero sublocale of L contains a covered prime of L.*

Combining the equivalence (iii) $\iff$ (vi) of last result with Lemma 1.3.4, we obtain the following (cf. also [108]):

Corollary 8.5.5. *If X is a T_0-space, $\mathsf{S}(\Omega(X))^{op}$ is Boolean if and only if $\mathsf{S}(\Omega(X))^{op}$ is spatial and X is sober and T_D.*

Example 8.5.6. Consider the totally ordered set of the natural numbers with the Alexandroff topology, that is, $\Omega(\mathbb{N}) = \{\varnothing\} \cup \{\uparrow n \mid n \in \mathbb{N}\}$. Observe that for any $n > 1$ the element $\uparrow n = \mathbb{N} - \overline{\{n-1\}}$ is a covered prime in $\Omega(\mathbb{N})$ and $\varnothing$ is a prime which is not covered. Hence condition (iii) in Theorem 8.5.4 is not satisfied. Moreover, it was pointed out in [12] that $\Omega(\mathbb{N})$ is totally spatial. In fact, one has slightly more:

every sublocale is T_D-spatial.

In order to see this, let S be a nonzero sublocale. If $S = \{\varnothing, \mathbb{N}\}$, obviously S contains a covered prime in itself (namely, $\varnothing$). Otherwise, S contains an element of the form $\uparrow n$ with $n > 1$, but all such elements are covered in $\Omega(\mathbb{N})$, and in particular they are covered in S. Thus condition (ix) in Theorem 8.5.2 holds. Of course, $\{\varnothing, \mathbb{N}\}$ is a sublocale but not a D-sublocale.

8.6 Functoriality

Recall the non-functoriality of $\mathsf{S}_b(L)$ from Section 7.7. Here we have a similar situation: the assignment $L \mapsto \mathsf{S}_D(L)^{op}$ is not functorial in the whole of the category of frames. Accordingly, we have to deal with individual lifts once again. Precisely, in the present section we will say that a frame homomorphism $f\colon L \to M$ has an S_D-*lift* (or that it S_D-*lifts*) if there is a coframe homomorphism $\mathsf{S}_D(f)\colon \mathsf{S}_D(L) \to \mathsf{S}_D(M)$ such that the associated naturality square

$$\begin{tikzcd} \mathsf{S}_D(L)^{op} \arrow[r, "\mathsf{S}_D(f)^{op}"] & \mathsf{S}_D(M)^{op} \\ L \arrow[u, tail, "\mathfrak{c}_L"] \arrow[r, "f"] & M \arrow[u, tail, "\mathfrak{c}_M"'] \end{tikzcd} \tag{8.6.1}$$

in the category of frames commutes.

Of course, the coframe of sublocales yields a functor $\mathsf{S}(-)\colon \mathsf{Frm} \to \mathsf{Frm}$ (cf. Subsection 1.2.2). Moreover, we have justified that the system $\mathsf{S}_D(L)$ takes up the role of the whole $\mathsf{S}(L)$ when switching from the classical duality to the T_D-duality. It is therefore a natural and important question whether if, after all, the subcategory of Frm consisting of maps which S_D-lift is just (or, better, it contains) the category Frm_D from Subsection 1.3.2 (because D-homomorphisms are the "good morphisms" in the T_D-duality). We left this question open when writing [9] and in this thesis we will give a negative solution to it.[2]. Before that, we prove a few more general results.

8.6.1 General results

We begin by an easy description of the candidate for a S_D-lift:

Lemma 8.6.1. *Let $f\colon L \to M$ be a frame homomorphism which S_D-lifts and S be a D-sublocale of L. Then $\mathsf{S}_D(f)(S)$ can be computed as the largest D-sublocale in M which is contained in $(f_*)_{-1}[S]$.*

Proof. If S is a sublocale of L, then one can write $S = \bigcap_{S \subseteq \mathfrak{o}_L(a) \vee \mathfrak{c}_L(b)} \mathfrak{o}_L(a) \vee \mathfrak{c}_L(b)$. Each of the sublocales $\mathfrak{o}_L(a) \vee \mathfrak{c}_L(b)$ belongs to $\mathsf{S}_D(L)$ (as complemented sublocales belong to $\mathsf{S}_D(L)$; see

[2]However, we will see that surjections which S_D-lift are precisely D-homomorphisms.

Corollary 8.2.7), and observe that if S happens to be a D-sublocale, then the intersection $\bigcap_{S\subseteq \mathfrak{o}_L(a)\vee \mathfrak{c}_L(b)} \mathfrak{o}_L(a)\vee \mathfrak{c}_L(b)$ coincides with the meet $\bigwedge_{S\subseteq \mathfrak{o}_L(a)\vee \mathfrak{c}_L(b)} \mathfrak{o}_L(a)\vee \mathfrak{c}_L(b)$ taken in $\mathsf{S}_D(L)$.

Now, applying the coframe homomorphism $\mathsf{S}_D(f)$ and using the fact that coframe homomorphisms commute with complements and that preimage is also a coframe homomorphism, we readily obtain

$$\begin{aligned}\mathsf{S}_D(f)(S) &= \bigwedge_{S\subseteq \mathfrak{o}_L(a)\vee \mathfrak{c}_L(b)} \mathfrak{c}_M(f(a))\vee \mathfrak{o}_M(f(b))\\ &= j_{\mathsf{S}_D(M)}\Big(\bigcap_{S\subseteq \mathfrak{o}_L(a)\vee \mathfrak{c}_L(b)} (f_*)_{-1}[\mathfrak{c}_L(a)\vee \mathfrak{o}_L(b)]\Big) = j_{\mathsf{S}_D(M)}\Big((f_*)_{-1}\Big[\bigcap_{S\subseteq \mathfrak{o}_L(a)\vee \mathfrak{c}_L(b)} \mathfrak{c}_L(a)\vee \mathfrak{o}_L(b)\Big]\Big)\\ &= j_{\mathsf{S}_D(M)}((f_*)_{-1}[S]),\end{aligned}$$

where $\bigwedge$ denotes meet in $\mathsf{S}_D(M)$ and $j_{\mathsf{S}_D(M)}$ is the conucleus associated to $\mathsf{S}_D(M)$ in $\mathsf{S}(M)$. Then, one has that $\mathsf{S}_D(f)(S)$ is the largest D-sublocale of M contained in $(f_*)_{-1}[S]$. $\square$

We shall need the following corollary later:

Corollary 8.6.2. *The following properties hold*:

(1) *Let $f\colon L\to M$ be a frame homomorphism which S_D-lifts, let S be a D-sublocale of L, and assume that every prime in M is covered. Then $\mathsf{S}_D(f)(S) = (f_*)_{-1}[S]$.*

(2) *If every prime of M is covered, a frame homomorphism $f\colon L\to M$ has an S_D-lift if and only if*

$$f_*\Big[(f_*)_{-1}\Big[\bigcap_{i\in I} S_i\Big]\Big] \subseteq \bigwedge_{i\in I}^{\mathsf{S}_D(L)} S_i$$

for every $\{S_i\}_{i\in I}\subseteq \mathsf{S}_D(L)$.

Proof. If every prime is covered, every sublocale is a D-sublocale so the first assertion follows from the previous lemma. Now, in view of this, f has an S_D-lift if and only if the usual preimage is a coframe homomorphism when regarded as a map $\mathsf{S}_D(L)\to\mathsf{S}_D(M)=\mathsf{S}(M)$; it clearly preserves finite joins so it S_D-lifts if and only if it preserves arbitrary meets. But for every $\{S_i\}_{i\in I}\subseteq \mathsf{S}_D(L)$, we have the chain of equivalences

$$\bigcap_i (f_*)_{-1}[S_i] = (f_*)_{-1}\Big[\bigwedge_i^{\mathsf{S}_D(L)} S_i\Big] \iff (f_*)_{-1}\Big[\bigcap_i S_i\Big] \subseteq (f_*)_{-1}\Big[\bigwedge_i^{\mathsf{S}_D(L)} S_i\Big] \iff f_*\Big[(f_*)_{-1}\Big[\bigcap_i S_i\Big]\Big] \subseteq \bigwedge_i^{\mathsf{S}_D(L)} S_i.$$

$\square$

8.6.2 Surjections

In parallel with the situation concerning $\mathsf{S}_b(L)$, surjections which S_D-lift are easily characterized. The following results are the counterparts of Corollaries 7.7.5, 7.7.6 and 7.7.3.

Corollary 8.6.3. *Let p be a prime in a locale L. Then the frame surjection associated to the sublocale $\mathfrak{b}(p)$ has an S_D-lift if and only if p is a covered prime.*

Proof. $\Rightarrow$: By Corollary 8.2.7 we have that $\mathcal{C} = \{C \in \mathsf{S}(L) \mid \mathfrak{b}(p) \subseteq C,\ C \text{ complemented}\} \subseteq \mathsf{S}_D(L)$. Moreover, because of the zero-dimensionality of $\mathsf{S}(L)^{op}$ and Corollary 8.6.2 (2) one clearly has $\bigwedge \mathcal{C} = \bigcap \mathcal{C} = \mathfrak{b}(p) \in \mathsf{S}_D(L)$. It follows that $p \in \mathsf{pt}_D(L)$.

$\Leftarrow$: Since $\mathfrak{b}(p)$ trivially satisfies the assumptions in Corollary 8.6.2 (2), it is enough to prove that $\mathfrak{b}(p) \cap \bigcap_i S_i \subseteq \bigwedge_i S_i$ for every family $\{S_i\}_{i\in I}$ of D-sublocales of L. Let $\{S_i\}_{i\in I} \subseteq \mathsf{S}_D(L)$. If $\mathfrak{b}(p) \cap \bigcap_i S_i = \mathsf{O}$, then obviously $\mathfrak{b}(p) \cap \bigcap_i S_i \subseteq \bigwedge_i S_i$. Otherwise $\mathfrak{b}(p) = \mathfrak{b}(p) \cap \bigcap_i S_i \subseteq \bigcap_i S_i$, and applying the conucleus $j_{\mathsf{S}_D(L)}$, we obtain $\mathfrak{b}(p) = j_{\mathsf{S}_D(L)}(\mathfrak{b}(p)) \subseteq \bigwedge_i S_i$. □

Now, exactly the same proof as the one of Corollary 7.7.6 yields the following necessary condition:

Corollary 8.6.4. *If a frame homomorphism S_D-lifts, it is a D-homomorphism.*

Corollary 8.6.5. *Let $f\colon L \twoheadrightarrow S$ be a frame surjection onto a sublocale S of L. Then f has an S_D-lift if and only if S is a D-sublocale.*

Proof. $\Rightarrow$: This implication follows by Corollary 8.6.4 and Remark 8.1.2 (1).

$\Leftarrow$: If S is a D-sublocale of L and $T \subseteq S$, it is clear that T is a D-sublocale of S if and only if T is a D-sublocale of L. It follows that $\mathsf{S}_D(S) = {\downarrow}^{\mathsf{S}_D(L)} S$. Hence there is a coframe homomorphism $\mathsf{S}_D(L) \to \mathsf{S}_D(S)$ given by $T \mapsto T \wedge S$. For showing that the relevant naturality square commutes it suffices to check that $S \cap \mathfrak{c}_L(a)$ is a D-sublocale of L. But this follows because $S \cap \mathfrak{c}_L(a) = \mathfrak{c}_S(\nu_S(a))$ is a closed (hence smooth) sublocale of S, so by Corollary 8.2.7 it is a D-sublocale of S. Therefore, $\mathsf{pt}_D(S \cap \mathfrak{c}_L(a)) \subseteq \mathsf{pt}_D(S) \subseteq \mathsf{pt}_D(L)$. □

8.6.3 Monomorphisms

As mentioned above, unfortunately being a D-homomorphism is not a sufficient condition for a monomorphism to S_D-lift. For showing this, we start by specializing Corollary 8.6.2 to a particular case:

Lemma 8.6.6. *Let $f\colon L \to M$ be a frame homomorphism such that f_* is a surjection stable under pullback along inclusions in* Loc[3]*, and assume that every prime of M is covered. Then f has an S_D-lift if and only if every prime of L is covered.*

Proof. By Corollary 8.6.2 (2), f has an S_D-lift if and only if $f_*[(f_*)_{-1}[\bigcap_{i\in I} S_i]] \subseteq \bigwedge_{i\in I}^{\mathsf{S}_D(L)} S_i$ for every $\{S_i\}_{i\in I} \subseteq \mathsf{S}_D(L)$. By hypothesis, $f_*[(f_*)_{-1}[\bigcap_{i\in I} S_i]] = \bigcap_{i\in I} S_i$ and therefore f has an S_D-lift if and only if $\bigcap_{i\in I} S_i = \bigwedge_{i\in I}^{\mathsf{S}_D(L)} S_i$ for every $\{S_i\}_{i\in I} \subseteq \mathsf{S}_D(L)$. But by Proposition 8.2.10 this holds if and only if every prime of L is covered. □

Example 8.6.7. Consider the natural numbers $\mathbb{N}$ with the cofinite topology[4]. It is a fact of general topology due to Shimrat [107] that every space can be expressed as the image of a

[3]That is, a *semi-stable epimorphism* in the terminology of Plewe [101].
[4]Or, more generally, any other T_D-space with a non-covered prime.

Hausdorff space under an open continuous map. So let X be Hausdorff and $f\colon X \to \mathbb{N}$ be an open continuous surjection. Let

$$h := \Omega(f)\colon \Omega(\mathbb{N}) \to \Omega(X)$$

be the associated frame homomorphism given by preimage. Then, h is injective, and moreover it is also an open frame homomorphism. Hence h_* is an open surjection in Loc, and in particular, by [75, Theorem 4.7] it is stable under pullback along inclusions. Now since X and $\mathbb{N}$ are both T_D, it follows that h is a D-homomorphism (recall Lemma 1.3.5).

Finally every prime in X is covered (as X is Hausdorff, so in particular it is sober and T_D), but $\varnothing$ is a non-covered prime in $\mathbb{N}$ (as $\varnothing = \bigwedge_{n\in\mathbb{N}} \mathbb{N} - \{n\}$). By the previous lemma, h does not S_D-lift.

Therefore, we have shown that

> *there is an open D-homomorphism between spatial frames which is a monomorphism and does not S_D-lift.*

Remark 8.6.8. In the spatial case, this example exhibits a major difference with the smooth case (compare it with Corollary 7.7.12).

8.7 Concluding remarks

Our analysis of T_D-spatiality and related topics relies heavily on a subset (actually subspace) of the prime spectrum, namely the subset consisting of covered primes. However, there is also the more restricted subset of maximal elements which are well known to be connected to "closed points" and thus to the T_1-axiom (recall e.g. the observation that p is maximal if and only if the one-point sublocale $\mathfrak{b}(p)$ is closed).

Accordingly, one may wonder how much of our theory can be developed when replacing "covered prime" by "maximal element" and, vice-versa, which familiar facts concerning maximal elements still hold in our setting concerning covered primes.

In this last section we present several examples of opposite behaviour between "T_D-points" and "T_1-points".

8.7.1 A categorical comment

We have met several point-free counterparts of the T_1 axiom from classical topology. Among others, there is the notion of T_1-frame (see Subsection 1.1.3) introduced by Rosický and Šmarda [105]. More precisely, we recall that a frame is T_1 if every prime is maximal (clearly, a sober space is T_1 if and only if the associated frame is T_1 in this sense).

Despite the fact that other point-free analogues of the T_1 axiom (especially subfitness or fitness) are arguably more relevant in point-free topology [97, p. xv], the T_1-frames have the following motivating property:

Theorem 8.7.1 ([105]). *The epireflective hull of the subcategory of* Loc *consisting of sober* T_1 *topologies is precisely the subcategory of* T_1*-locales.*

We have already implicitly met its T_D-analogous notion in Proposition 8.2.10. In light of the preceding discussion, we formulate it as follows:

Definition 8.7.2. A T_D*-frame* is one in which every prime is covered.

Unfortunately the counterpart of Theorem 8.7.1 is not true when "T_1" is replaced by "T_D". In fact,

the full subcategory of Loc *consisting of* T_D*-locales is not closed under products.*

Example 8.7.3. One of the simplest examples seems to be the following: the Sierpiński locale $\mathbb{S}$ is clearly a T_D-locale, but the infinite product $\oplus_{n\in\mathbb{N}}\Omega(\mathbb{S})$ is not T_D. Indeed, by way of contradiction suppose that in the product $\oplus_{n\in\mathbb{N}}\Omega(\mathbb{S})$ every prime is covered. Then

$$\Sigma' \oplus_{n\in\mathbb{N}} \Omega(\mathbb{S}) = \Sigma \oplus_{n\in\mathbb{N}} \Omega(\mathbb{S}) \cong \prod_{n\in\mathbb{N}} \Sigma\Omega(\mathbb{S}) = \prod_{n\in\mathbb{N}} \mathbb{S}.$$

But $\Sigma'(L)$ is always T_D (Subsection 1.3.2), and hence so is $\prod_{n\in\mathbb{N}}\mathbb{S}$, a contradiction with the well-known fact that an infinite product of Sierpiński spaces is not T_D (see e.g. [37]).

This last fact provides an example where T_1-locales behave better than T_D-locales.

8.7.2 The system of M-sublocales

An essential part of the theory developed so far exploited the subcolocale nature of $\mathsf{S}_D(L)$ (see Table 9.1 for a comprehensive account of the examples discussed so far).

According to the considerations above, it is a natural question whether one can build an analogue construction with "covered prime" replaced by "maximal element". The aim of the following is to show that, unfortunately, there is generally no such nice structure in the modified context of maximal elements.

Inded, let us call a sublocale S of L an *M-sublocale* if maximal elements in S are maximal in L — i.e., if $\mathsf{max}(S) \subseteq \mathsf{max}(L)$. Moreover let $\mathsf{S}_M(L)$ denote the family of M-sublocales of L. Then one has:

Remark 8.7.4. Let L have a non-maximal covered prime. We claim that $\mathsf{S}_M(L)$ is not a subcolocale of $\mathsf{S}(L)$. Indeed, let p be a covered prime which is not maximal and by way of contradiction suppose that $\mathsf{S}_M(L)$ is a subcolocale. Since $L \in \mathsf{S}_M(L)$, we actually have that $\mathsf{S}_M(L)$ is a dense subcolocale, and thus $\mathsf{S}_b(L) \subseteq \mathsf{S}_M(L)$. Since p is covered, by Lemma 1.3.3 it follows that $\mathfrak{b}(p)$ is complemented. Then $\mathfrak{b}(p) \in \mathsf{S}_b(L) \subseteq \mathsf{S}_M(L)$. In view of the fact that p is maximal in $\mathfrak{b}(p)$ and $\mathfrak{b}(p) \in \mathsf{S}_M(L)$, we deduce that p is maximal in L, a contradiction.

In particular, if X is a T_D-space which is not T_1, the system $\mathsf{S}_M(\Omega(X))$ is not a subcolocale; thus showing that in this context T_1-points lack a convenient property which T_D-points enjoy.

8.7.3 Total T_1-spatiality

Recall that every subspace of a T_D-space (resp. T_1-space) is also T_D (resp. T_1). Therefore, if L is T_D-spatial (resp. T_1-spatial), say $L = \Omega(X)$ with X a T_D-space (resp. T_1-space), then every sublocale induced by a subspace of X is trivially T_D-spatial (resp. T_1-spatial). It naturally arises the related question whether every *spatial* sublocale is T_D-spatial (resp. T_1-spatial). Certainly, if X is sober, the answer to both questions is positive (since spatial sublocales are induced). Furthermore, for the T_1-spatial case, Herrlich and Pultr proved that such a situation can happen only if X is sober:

Theorem 8.7.5 ([70, Theorem 3.4]). *If X is a T_1-space, then every spatial sublocale of $\Omega(X)$ is T_1-spatial if and only if X is sober.*

In the T_D-spatial case, sobriety is not a necessary condition, as Example 8.5.6 shows. Observe also that Theorem 8.7.5 shows that there is no interesting notion of total T_1-spatiality: it follows that every sublocale of L is T_1-spatial if and only if L is totally spatial and all its primes are maximal — i.e., in the T_1-case the conditions analogous to those of Theorem 8.5.2 collapse to the analogues of those of Theorem 8.5.4.

It still remains the question whether a T_D-spatial frame can have a spatial sublocale which is not T_D-spatial. It can, as the following example shows. Let $L = [0,1]$ denote the totally ordered frame of the unit interval and consider the frame surjection

$$h\colon \mathsf{Dwn}(L) \twoheadrightarrow L$$

given by $h(U) = \bigvee U$. Note that $(L, \mathsf{Dwn}(L))$ is a T_D-space and so $\mathsf{Dwn}(L)$ is T_D-spatial. Moreover, L is (isomorphic to) a sublocale of $\mathsf{Dwn}(L)$, and L is spatial (since every element is prime) but not T_D-spatial (actually it does not contain any covered prime).

Incidentally, this example also shows that a totally spatial T_D-spatial locale is not necessarily totally T_D-spatial. Indeed, $\mathsf{Dwn}(L)$ is totally ordered (it consists of $\downarrow t$ and $<t := \{s \in [0,1] \mid s < t\}$ for each $t \in [0,1]$) and so it follows from [13, Corollary 4.2] that $\mathsf{Dwn}(L)$ is totally spatial (and T_D-spatial, as we mentioned above).

Chapter 9

The big picture — relating subcolocales of $\mathsf{S}(L)$

We devoted Chapters 7 and 8 to the study of two subcolocales of $\mathsf{S}(L)$, namely the system of smooth sublocales and that of D-sublocales, denoted by

$$\mathsf{S}_b(L) \quad \text{and} \quad \mathsf{S}_D(L)$$

respectively. We also met the spatialization subcolocale

$$\mathsf{sp}[\mathsf{S}(L)]$$

from Subsection 1.2.5 and Picado, Pultr and Tozzi's family of joins of closed sublocales, the

$$\mathsf{S}_c(L)$$

from Section 7.2 (the latter is generally not a subcolocale but just a sub-suplattice).

Now, it turns out that relations between these subsets and $\mathsf{S}(L)$ often characterize interesting and well-known properties of the locale in question (recall for example the statement that $\mathsf{sp}[\mathsf{S}(L)] = \mathsf{S}(L)$ if and only if every sublocale of L is spatial [89], or its T_D-analogue in Theorem 8.5.2). In this chapter we shall study all the remaining possible inclusions between the aforementioned subsets. The material contained in this chapter is part of the joint paper [9] with Anna Laura Suarez:

[9] I. Arrieta and A. L. Suarez, The coframe of D-sublocales of a locale and the T_D-duality, *Topology and its Applications*, vol. 291, art. no. 107614, 2021.

9.1 A few more results

In what follows we prove some additional results in order to complete the missing gaps in Table 9.1 below. We start by observing the following easy result:

Lemma 9.1.1. *The following are equivalent for a locale L:*

(i) *L is spatial;*

(ii) $\mathsf{S}_b(L) \subseteq \mathsf{sp}[\mathsf{S}(L)]$;

(iii) $\mathsf{S}_c(L) \subseteq \mathsf{sp}[\mathsf{S}(L)]$.

Proof. If (i) holds, we have $L \in \mathsf{sp}[\mathsf{S}(L)]$ (recall Proposition 1.2.4). This means that $\mathsf{sp}[\mathsf{S}(L)]^{op}$ is a dense sublocale of $\mathsf{S}(L)^{op}$; hence it contains its least dense sublocale. Moreover, (ii) $\Longrightarrow$ (iii) is obvious because $\mathsf{S}_c(L) \subseteq \mathsf{S}_b(L)$ and (iii) $\Longrightarrow$ (i) follows immediately from Proposition 1.2.4. □

9.1.1 Total spatiality

Somewhat more surprisingly the family $\mathsf{S}_D(L)$ can also be used for characterizing "plain" total spatiality:

Proposition 9.1.2. *The following are equivalent for a locale L:*

(i) *L is totally spatial;*

(ii) $\mathsf{S}_D(L) \subseteq \mathsf{sp}[\mathsf{S}(L)]$.

Proof. If L is totally spatial then $\mathsf{S}_D(L) \subseteq \mathsf{sp}[\mathsf{S}(L)] = \mathsf{S}(L)$. Conversely, if $\mathsf{S}_D(L) \subseteq \mathsf{sp}[\mathsf{S}(L)]$ then every pointless sublocale is spatial (see Example 8.2.9 (2)). Thus every nontrivial sublocale of L has a point. By [89, p. 269], it follows that L is totally spatial. However, since no proof of the latter fact is provided therein, we give one for the sake of completeness:

Let S be an arbitrary sublocale and let C be a complemented sublocale such that $\mathsf{sp}(S) \subseteq C$. If $S \cap C^{\#} \neq \mathsf{O}$ then by assumption it has a point — i.e., there is some $p \in \mathsf{pt}(L)$ with $\mathfrak{b}(p) \subseteq S \cap C^{\#}$. Then $\mathfrak{b}(p) \subseteq \mathsf{sp}(S) \subseteq C$, but also $\mathfrak{b}(p) \subseteq C^{\#}$, a contradiction. Hence $S \cap C^{\#} = \mathsf{O}$, or, equivalently, $S \subseteq C$ because of Properties 1.2.1 (2) and (3). By zero-dimensionality we conclude that $S \subseteq \bigcap\{C \mid \mathsf{sp}(S) \subseteq C,\ C \text{ complemented}\} = \mathsf{sp}(S)$ and so $S = \mathsf{sp}(S)$. Hence L is totally spatial. □

9.1.2 *D*-scatteredness

Recall that the notion of scatteredness for locales can be characterized as follows: a locale L is scattered if and only if $\mathsf{S}(L)$ is Boolean [100, Theorem 11]. It also makes sense to consider the analogue of scatteredness in the T_D-duality — i.e., to study locales L for which the system $\mathsf{S}_D(L)$ is Boolean. Accordingly, we shall call these locales *D-scattered*. By Theorem 8.5.2, totally T_D-spatial locales are D-scattered (more precisely, total T_D-spatiality is the conjunction of D-scatteredness and T_D-spatiality).

We have the following characterization:

Proposition 9.1.3. *The following are equivalent for a locale L:*

(i) *L is D-scattered;*

(ii) $\mathsf{S}_D(L) = \mathsf{S}_b(L)$*;*

(iii) *If S is a sublocale such that $\mathsf{pt}_D(S) = \varnothing$, then S is smooth.*

Proof. If (i) holds, that is, if $\mathsf{S}_D(L)$ is Boolean, then $\mathsf{S}_D(L)^{op}$ is a Boolean dense sublocale of $\mathsf{S}(L)^{op}$, hence it must coincide with the Booleanization of $\mathsf{S}(L)^{op}$. Moreover, (ii) clearly implies (iii) as a sublocale without covered primes is a D-sublocale for trivial reasons. Let us assume (iii) and let $S \in \mathsf{S}_D(L)$. Consider the decomposition $S = \mathsf{sp}_D(S) \vee (S \setminus \mathsf{sp}_D(S))$. Observe that by Proposition 8.2.5, $S \setminus \mathsf{sp}_D(S)$ is a D-sublocale as well — i.e., $\mathsf{pt}_D(S \setminus \mathsf{sp}_D(S)) \subseteq \mathsf{pt}_D(L)$. Assume that there exists some $p \in \mathsf{pt}_D(S \setminus \mathsf{sp}_D(S))$. Then $p \in \mathsf{pt}_D(L) \cap S \subseteq \mathsf{pt}_D(S)$ and hence $\mathfrak{b}(p) \subseteq \mathsf{sp}_D(S)$. Therefore, $\mathfrak{b}(p) \subseteq S \setminus \mathsf{sp}_D(S) \subseteq S \setminus \mathfrak{b}(p) \subseteq L \setminus \mathfrak{b}(p) = \mathfrak{b}(p)^{\#}$, which yields a contradiction because of Lemma 1.3.3 and the fact that $p \in \mathsf{pt}_D(L)$. Hence $\mathsf{pt}_D(S \setminus \mathsf{sp}_D(S)) = \varnothing$ and so $S \setminus \mathsf{sp}_D(S)$ is smooth in L. Moreover, $\mathsf{sp}_D(S) = \bigvee_{p \in \mathsf{pt}_D(S)} \mathfrak{b}(p)$ is also a smooth sublocale because $\mathsf{pt}_D(S) \subseteq \mathsf{pt}_D(L)$ and Lemma 1.3.3. Hence $S = \mathsf{sp}_D(S) \vee (S \setminus \mathsf{sp}_D(S))$ is smooth, as it is a join of smooth sublocales. □

Remark 9.1.4. Condition (iii) in the proposition is certainly simpler than (ii): being a D-sublocale is a relative condition with respect to the ambient locale, whereas the condition $\mathsf{pt}_D(S) = \varnothing$ (having no covered prime in itself) is absolute.

9.2 Some final remarks

Among the subsets of $\mathsf{S}(L)$ we studied, not all of them are of the same nature. Indeed, $\mathsf{sp}[\mathsf{S}(L)]$ and $\mathsf{S}_b(L)$ arise after applying a general locale-theoretic construction (spatialization and Booleanization, respectively) to the particular case of the locale $\mathsf{S}(L)^{op}$.

More precisely, recall that if L is a locale, then $B_L = \{a^* \mid a \in L\}$ and $\mathsf{sp}(L) = \{\bigwedge Y \mid Y \subseteq \mathsf{pt}(L)\}$ are the Booleanization and the spatialization of L, respectively. Thus they only depend on the lattice-theoretic structure of the assembly $\mathsf{S}(L)^{op}$ (and not on L itself, nor on the embedding $\mathfrak{c}_L \colon L \rightarrowtail \mathsf{S}(L)^{op}$).

The same applies to the case of $\mathsf{S}_D(L)$, even if its definition as given in Chapter 8 may slightly obscure this fact. Indeed, if L is a locale, set

$$L_D := \{a \in L \mid (p \in \mathsf{pt}(L),\ a \leq p,\ p \vee (p \to a) = 1) \implies p \text{ is complemented}\}.$$

This general locale-theoretic construction, when applied to the particular case of the locale $\mathsf{S}(L)^{op}$, leads to the equality $(\mathsf{S}(L)^{op})_D = \mathsf{S}_D(L)^{op}$ (by Property 1.2.3 (5), Lemma 1.3.3 and the fact that a sublocale S is complemented if and only if $S \cap S^{\#} = \mathsf{O}$). Consequently, if L and M are locales such that $\mathsf{S}(L) \cong \mathsf{S}(M)$ then one has $\mathsf{S}_D(L) \cong \mathsf{S}_D(M)$.

On the contrary, the construction $\mathsf{S}_c(L)$ does not depend only on the lattice structure of $\mathsf{S}(L)$. In fact, we may have locales L and M such that $\mathsf{S}(L) \cong \mathsf{S}(M)$ but $\mathsf{S}_c(L) \ncong \mathsf{S}_c(M)$.

Indeed, if $\mathbb{S}$ denotes de Sierpiński space and $\mathbb{B}_4$ is the 4-element Boolean algebra, one has $\mathsf{S}_c(\Omega(\mathbb{S})) \cong \Omega(\mathbb{S})$ and $\mathsf{S}_c(\mathbb{B}_4) \cong \mathbb{B}_4$. However $\mathsf{S}(\Omega(\mathbb{S})) \cong \mathsf{S}(\mathbb{B}_4) \cong \mathbb{B}_4$.

9.3 Summary

To sum up, we gather all the previously obtained results in the following table.

Table 9.1 Characterizations of locale-theoretic properties in terms of subsets of the assembly

Subsets of $\mathsf{S}(L)$	Property of L	Reference
$\mathsf{S}_b(L) \subseteq \mathsf{sp}[\mathsf{S}(L)]$	Spatial	Lemma 9.1.1
$\mathsf{S}_c(L) \subseteq \mathsf{sp}[\mathsf{S}(L)]$	Spatial	Lemma 9.1.1
$\mathsf{sp}[\mathsf{S}_b(L)] = \mathsf{S}_b(L)$	T_D-spatial	Theorem 7.5.2
$\mathsf{S}_b(L) = \mathsf{sp}[\mathsf{S}(L)]$	Strongly T_D-spatial	Lemma 9.1.1, Proposition 8.2.10, Lemma 1.3.9
$\mathsf{S}_D(L) = \mathsf{S}(L)$	T_D-locale	Proposition 8.2.10
$\mathsf{sp}[\mathsf{S}(L)] \subseteq \mathsf{S}_D(L)$	T_D-locale	Proposition 8.2.10
$\mathsf{sp}[\mathsf{S}(L)] \subseteq \mathsf{S}_b(L)$	T_D-locale	Proposition 8.2.10
$\mathsf{sp}[\mathsf{S}(L)] \subseteq \mathsf{S}_c(L)$	T_1-locale	[9, Lemma 5.2]
$\mathsf{S}(L) = \mathsf{sp}[\mathsf{S}(L)]$	Totally spatial	[89, Theorem 3.4]
$\mathsf{S}_D(L) \subseteq \mathsf{sp}[\mathsf{S}(L)]$	Totally spatial	Proposition 9.1.2
$\mathsf{S}_D(L) = \mathsf{sp}[\mathsf{S}_D(L)]$	Totally T_D-spatial	Theorem 8.5.2
$\mathsf{S}_b(L) = \mathsf{S}(L)$	Scattered	[100, Theorem 11]
$\mathsf{S}_c(L) = \mathsf{S}(L)$	Scattered and fit	[16, Theorem 3.6]
$\mathsf{S}_D(L) = \mathsf{S}_b(L)$	D-scattered	Proposition 9.1.3
$\mathsf{S}_D(L) = \mathsf{S}_c(L)$	D-scattered and subfit	Proposition 9.1.3, Theorem 7.2.1

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Appendix A

Relative variants of semicontinuities and relative insertion and extension results

In this appendix we continue the development of Section 3.1 and provide a brief account of the theory of relative real-valued functions in point-free topology (introduced by Gutiérrez García and Picado in [68]). All the concepts and results in [68] are stated for the frame of real numbers; however, for technical reasons, here we shall rather work with the frame of *extended* real numbers (clearly, all the results can be obtained from [68] with only minor modifications).

A.1 Relative semicontinuities

Let $\mathbb{F}$ be a sublocale selection. First, an $f \in \overline{\mathsf{F}}(L)$ is called

- *lower* $\mathbb{F}$*-semicontinuous* if for every $r < s$ in $\mathbb{Q}$ there is an $F_{r,s} \in \mathbb{F}(L)$ such that $f(s,—) \leq F_{r,s} \leq f(r,—)$;
- *upper* $\mathbb{F}$*-semicontinuous* if for every $r < s$ in $\mathbb{Q}$ there is a $F_{r,s} \in \mathbb{F}(L)$ such that $f(—,r) \leq F_{r,s} \leq f(—,s)$;
- $\mathbb{F}$*-continuous* if it is lower and upper $\mathbb{F}$-semicontinuous.

This defines the following subclasses of $\overline{\mathsf{F}}(L)$:

$$\overline{\mathsf{LSC}}^{\mathbb{F}}(L), \quad \overline{\mathsf{USC}}^{\mathbb{F}}(L), \quad \overline{\mathsf{C}}^{\mathbb{F}}(L) = \overline{\mathsf{LSC}}^{\mathbb{F}}(L) \cap \overline{\mathsf{USC}}^{\mathbb{F}}(L).$$

Importantly, in this setting lower semicontinuity and upper semicontinuity are dual notions (and hence $\mathbb{F}$-continuity is self dual):

Proposition A.1.1 ([68, Remark 5.2]). *Let $\mathbb{F}$ be a sublocale selection and $f \in \overline{\mathsf{F}}(L)$. Then f is lower $\mathbb{F}$-semicontinuous if and only if it is upper $\mathbb{F}^*$-semicontinuos.*

Corollary A.1.2 ([68, Remark 5.2]). *Let $\mathbb{F}$ be a sublocale selection and $f \in \overline{\mathsf{F}}(L)$. Then f is $\mathbb{F}$-continuous if and only if it is $\mathbb{F}^*$-continuous.*

In some cases continuity has a simpler form:

Lemma A.1.3. *Assume $\mathbb{F}$ is a sublocale selection closed under countable meets. Then:*

(1) $f \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ *if and only if* $f(r,\text{—}) \in \mathbb{F}(L)$ *for all* $r \in \mathbb{Q}$;

(2) $f \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ *if and only if* $f(\text{—},r) \in \mathbb{F}(L)$ *for all* $r \in \mathbb{Q}$;

(3) $f \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ *if and only if* $f(r,\text{—}) \in \mathbb{F}(L)$ *and* $f(\text{—},r) \in \mathbb{F}(L)$ *for all* $r \in \mathbb{Q}$.

Proof. The implications $\Leftarrow$ always hold. Further, (3) is an obvious consequence of (1) and (2). Assume that $f \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ and consider $r \in \mathbb{Q}$. Then by (r2)

$$f(r,\text{—}) = \bigsqcup_{s>r} f(s,\text{—}) \leq \bigsqcup_{s>r} F_{r,s} \leq f(r,\text{—})$$

for some $F_{r,s} \in \mathbb{F}(L)$ $(s > r)$. Hence $f(r,\text{—}) = \bigsqcup_{s>r} F_{r,s} = \bigcap_{s>r} F_{r,s} \in \mathbb{F}(L)$. The assertion in (2) may be proved in a similar way. ◻

Moreover, lower (resp. upper) $\mathbb{F}_{\mathfrak{c}}$-semicontinuity yields the usual notion of lower (resp. upper) semicontinuity for extended real valued localic maps (cf. [26]). The notions of semicontinuity obtained from the rest of the guiding examples (cf. Subsection 3.1.2) are described in Table A.1.

Table A.1 Examples of $\mathbb{F}$-semicontinuity and their duals

Selection	Lower $\mathbb{F}$-semicontinuous	Upper $\mathbb{F}$-semicontinuous	$\mathbb{F}$-continuous
$\mathbb{F}_{\mathfrak{c}}$	Lower semicontinuous	Upper semicontinuous	Continuous
$\mathbb{F}_{\mathrm{reg}}$	Normal lower semicontinuous	Normal upper semicontinuous	Continuous
$\mathbb{F}_{\mathrm{z}}$	Zero lower semicontinuous	Zero upper semicontinuous	Continuous
$\mathbb{F}_{\delta\mathrm{reg}}$	Regular lower semicontinuous	Regular upper semicontinous	Continuous

Remark A.1.4. The authors of [68] called $\mathbb{F}_z$-continuity *zero continuity* and $\mathbb{F}_{\delta\text{reg}}$-continuity *regular continuity*. However, they were not aware that they both reduce to ordinary continuity. Obviously, $\mathbb{F}_z$-continuity and $\mathbb{F}_{\delta\text{reg}}$-continuity imply ordinary continuity. Moreover, since $\mathbb{F}_z$ is closed under countable meets an $f \in \overline{\mathsf{F}}(L)$ is $\mathbb{F}_z$-continuous if and only if $f(r,—), f(—,r) \in \mathbb{F}_z(L)$ for all $r \in \mathbb{Q}$. Then, for an $f \in \overline{\mathsf{C}}(L)$, the sublocales $f(r,—), f(—,r)$ are always zero sublocales (one has $\mathrm{coz}((f - \boldsymbol{r})^+) = f(r,—)$ and $\mathrm{coz}((\boldsymbol{r} - f)^+) = f(—,r)$). For the case of $\mathbb{F}_{\delta\text{reg}}$, it suffices to observe that $\mathbb{F}_z(L) \subseteq \mathbb{F}_{\delta\text{reg}}(L)$.

Recall the extended characteristic function $\chi_S \in \overline{\mathsf{F}}(L)$ of a complemented sublocale $S \subseteq L$ from Subsection 5.3.1. Obviously, $\chi_S \in \overline{\mathsf{LSC}}(L)$ (resp. $\chi_S \in \overline{\mathsf{USC}}(L)$) if and only if S is an open (resp. closed) sublocale. More generally, we have the following:

Proposition A.1.5. *Let $\mathbb{F}$ be a sublocale selection and S a complemented sublocale of a locale L. Then*:

(1) $\chi_S \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ *if and only if* $S \in \mathbb{F}^*(L)$;

(2) $\chi_S \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ *if and only if* $S \in \mathbb{F}(L)$;

(3) $\chi_S \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ *if and only if* $S \in \mathbb{F}(L) \cap \mathbb{F}^*(L)$.

A.2 Relative insertion theorem

We now recall the relations $\Subset_{\mathbb{F}}$ in $\mathsf{S}(L)^{op}$ (for any L):

$$S \Subset_{\mathbb{F}} T \;\equiv\; \exists U \in \mathbb{F}(L),\ \exists V \in \mathbb{F}^*(L) \colon S \leq V \leq U \leq T.$$

We shall say that a sublocale selection $\mathbb{F}$ is a *Katětov selection on L* if for $S, S', T, T' \in \mathsf{S}(L)$,

(1) $S, S' \Subset_{\mathbb{F}} T$ implies $S \cap S' \Subset_{\mathbb{F}} T$, and

(2) $S \Subset_{\mathbb{F}} T, T'$ implies $S \Subset_{\mathbb{F}} T \vee T'$.

Example A.2.1. Almost all of the guiding examples are Katětov selections. Indeed, $\mathbb{F}_{\mathfrak{c}}$, $\mathbb{F}_{\mathrm{z}}$, $\mathbb{F}_{\delta\text{reg}}$, $\mathbb{F}^*_{\mathfrak{c}}$, $\mathbb{F}^*_{\mathrm{z}}$, $\mathbb{F}^*_{\delta\text{reg}}$ and $\mathbb{F}^*_{\text{reg}}$ are all Katětov selections on any locale while $\mathbb{F}_{\text{reg}}$ is a Katětov selection on any mildly normal locale.

Now, we state the main result of this section:

Theorem A.2.2 ([68, Theorem 7.1]). *Let $\mathbb{F}$ be a sublocale selection. The following are equivalent for any locale L such that $\mathbb{F}$ is a Katětov selection on L*:

(i) *L is $\mathbb{F}$-normal*;

(ii) *For every $f \in \overline{\mathsf{USC}}^{\mathbb{F}}(L)$ and $g \in \overline{\mathsf{LSC}}^{\mathbb{F}}(L)$ such that $f \leq g$, there exists an $h \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$ such that $f \leq h \leq g$.*

Applying the previous theorem to the various Katětov selections discussed above, several insertion theorems are unified within a single result. The reader may consult [68, Notes 7.3] and the references there for a detailed list of its corollaries.

A.3 Relative extension theorem

Under more restricted conditions, the authors of [68] also obtained a relative Tietze type extension theorem. In what follows we state it with some modifications (cf. Remarks A.3.2).

Recall from Section 3.1 that a sublocale selection $\mathbb{F}$ is hereditary on a locale L if for each $S \in \mathbb{F}(L)$ the equality

$$\mathbb{F}(S) = \{ S \cap T \mid T \in \mathbb{F}(L) \}$$

holds.

Theorem A.3.1 ([68, Theorem 8.6]). *Let $\mathbb{F}$ be a sublocale selection closed under countable meets and finite joins. The following are equivalent for any locale L such that $\mathbb{F}$ is a hereditary and Katětov selection on L:*

(i) *L is $\mathbb{F}$-normal;*

(ii) *For every $S \in \mathbb{F}(L)$, every $f \in \overline{\mathsf{C}}^{\mathbb{F}}(S)$ has an extension $\overline{f} \in \overline{\mathsf{C}}^{\mathbb{F}}(L)$.*

Remarks A.3.2. (1) The families $\mathbb{F}_{\mathfrak{c}}$, $\mathbb{F}_z$ and $\mathbb{F}_{\delta reg}$ are closed under countable meets and finite joins. However, their duals are not, hence one cannot use the extension theorem for $\mathbb{F}_{\mathfrak{c}}^*$, $\mathbb{F}_z^*$ and $\mathbb{F}_{\delta\mathrm{reg}}^*$. In one of the results of this thesis we fix this gap by proving a "dual" extension result which covers all of the dual cases (see Theorem 3.2.2).

(2) The term *hereditary* sublocale selection was not used in [68]. In fact, the authors of that paper use a different, more restrictive, notion of relative continuity for the function $f \in \overline{\mathsf{F}}(S)$ to be extended. This notion, however, does not generally coincide with the appropriate one, namely $\mathbb{F}$-continuity on S. The condition hereditary ensures that both continuities coincide. In any case, we note that the assumption on heredity cannot be dropped. For instance, if one takes the family $\mathbb{F}_{\mathrm{z}}$, condition (i) in Theorem A.3.1 is always satisfied (recall Table 3.1), whereas (ii) means that every zero sublocale is C^*-embedded, which is well known not to be generally satisfied (cf. [2]).

List of Symbols

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